Exercises

Integration

Exercise 1. Compute the following integrals:

(a) $\displaystyle\int_0^2 x^2 - 3x \, dx$ (b) $\displaystyle\int_1^2 \sqrt{2x} + \frac{3}{x} \, dx$ (c) $\displaystyle\int_0^{\pi} 2\cos(2x) \, dx$

(d) $\displaystyle\int_0^{\frac{\pi}{4}} 1 + \tan^2(x) \, dx$ (e) $\displaystyle\int_{-1}^{0} \frac{1}{1+x^2} \, dx$

Hint: For (d) and (e) use exercise 5 (c) from the exercise sheet on derivatives.

Exercise 2. Compute the following improper integrals if they exist.

(a) $\displaystyle\int_1^{\infty} \frac{1}{x^2} dx$ (b) $\displaystyle\int_1^2 \frac{1}{x-2} dx$ (c) $\displaystyle\int_0^{\frac{\pi}{2}} 1 + \tan^2(x) dx$

Exercise 3. A random variable X is called **continuous random variable** if there is some function $f(t) \geq 0$ such that

$$\mathbb{P}[X \leq x] = \int_{-\infty}^{x} f(t) dt$$

i.e. the probabilities are the integral of the so-called **density function** f.
The expectation of a continuous random variable can be computed as

$$\mathbb{E}[X] := \int_{-\infty}^{\infty} t f(t) dt$$

and the variance as

$$\mathrm{Var}[X] := \int_{-\infty}^{\infty} t^2 \cdot f(t) dt - (\mathbb{E}[X])^2.$$

(a) Consider a **uniformly distributed** random variable on some interval $[a, b]$. This is a random variable with density function

$$f(t) = \begin{cases} 0 & t < a \\ \frac{1}{b-a} & a \leq b \\ 0 & t > b \end{cases}$$

(i) Sketch the density function for some interval $[a, b]$.

(ii) Consider the interval $[0, 1]$. Compute $\mathbb{P}[X \le 1/2]$.

(iii) Compute $\mathbb{E}[X]$ and $\mathrm{Var}[X]$ for $[a, b] = [0, 1]$.

(b) Consider an **exponentially distributed** random variable. This is a random variable with density function

$$f(t) = \begin{cases} 0 & t < 0 \\ \lambda e^{-\lambda t} & t \ge 0 \end{cases}$$

where $\lambda > 0$ is some parameter.

(i) Let $\lambda = 4$. Compute $\mathbb{P}[X \le 1]$.

(ii) Compute $\mathbb{E}[X]$ in dependence of λ.
Hint: Use $\int te^{-t} = -e^{t}(t + 1) + c$ to figure out the antiderivative of $\int_0^\infty \lambda te^{-\lambda t} dt$ (or use partial integration).