Exercises

Integration – Solutions

Exercise 1.

(a)

$$\int_0^2 x^2 - 3x dx = \left[\frac{1}{3}x^3 - \frac{3}{2}x^2\right]_0^2 = \frac{1}{2}\cdot 8 - \frac{3}{2}\cdot 4 - (0-0) = -\frac{10}{3}$$

(b)

$$\begin{aligned}\int_1^2 \sqrt{2x} + \frac{3}{x} dx &= \int_1^2 \sqrt{2}\cdot x^{\frac{1}{2}} + 3\cdot\frac{1}{x} dx = \left[\sqrt{2}\cdot\frac{2}{3}x^{\frac{3}{2}} + 3\ln(|x|)\right]_1^2 \\ &= \left[\sqrt{2}\cdot\frac{2}{3}\sqrt{x}\cdot x + 3\ln(|x|)\right]_1^2 \\ &= \sqrt{2}\cdot\frac{2}{3}\cdot\sqrt{2}\cdot 2 + 3\ln(2) - (\sqrt{2}\cdot\frac{2}{3} + 3\ln(1)) \\ &= \frac{8}{3} - \frac{2}{3}\sqrt{2} + 3\ln(2)\end{aligned}$$

(c)

$$\int_0^\pi 2\cos(2x) dx = [\sin(2x)]_0^\pi = 0 - 0 = 0$$

(d) We have $(\tan(x))' = 1 + \tan^2(x)$ (c.f. derivatives exercise sheet). Thus

$$\int_0^{\frac{\pi}{4}} 1 + \tan^2(x) dx = [\tan(x)]_0^{\frac{\pi}{4}} = \tan\left(\frac{\pi}{4}\right) - \tan(0) = 1$$

(e) We have $(\arctan(x))' = \frac{1}{1+x^2}$ (c.f. derivatives exercise sheet). Thus

$$\int_{-1}^0 \frac{1}{1+x^2} = [\arctan(0)]_{-1}^0 = \arctan(0) - \arctan(-1) = 0 - \left(-\frac{\pi}{4}\right) = \frac{\pi}{4}$$

Exercise 2.

(a)

$$\int_1^\infty \frac{1}{x^2} dx = \lim_{\alpha\to\infty}\int_1^\alpha \frac{1}{x^2} = \lim_{\alpha\to\infty}\left(\left[-\frac{1}{x}\right]_1^\alpha\right) = \lim_{\alpha\to\infty}\left(-\frac{1}{\alpha} - (-1)\right) = 0 + 1 = 1$$

(b)

$$\int_1^2 \frac{1}{x-2}dx = \lim_{\alpha\uparrow 2}\int_1^\alpha \frac{1}{x-2}dx = \lim_{\alpha\uparrow 2}[\ln(|x-2|)]_1^\alpha = \lim_{\alpha\uparrow 2}\ln(|\alpha-2|) - \ln(1) = -\infty$$

The improper integral doesn't exist.

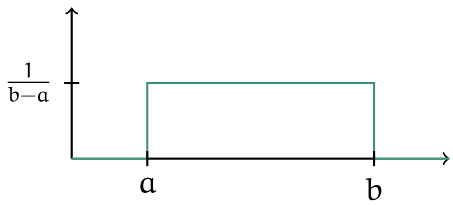
(c)

$$\int_0^{\frac{\pi}{2}} 1+\tan^2(x)dx = \lim_{\alpha\uparrow\frac{\pi}{2}}\int_0^\alpha 1+\tan^2(x)dx = \lim_{\alpha\uparrow\frac{\pi}{2}}[\arctan(x)]_0^\alpha = \lim_{\alpha\uparrow\frac{\pi}{2}}\arctan(x) = \infty$$

The improper integral doesn't exist.

Exercise 3.

1. (a) Sketch of the density function:



(b) For X uniformly distributed on $[a,b]=[0,1]$ the density function is

$$f(t) = \begin{cases} 0 & \text{if } t<0, \\ 1 & \text{if } 0\leq 1, \\ 0 & \text{if } t>1. \end{cases}$$

We can compute:

$$\mathbb{P}\left[X\leq\frac{1}{2}\right] = \int_{-\infty}^{\frac{1}{2}} f(t)dt = \int_0^{\frac{1}{2}} 1dt = [t]_0^{\frac{1}{2}} = \frac{1}{2}$$

(c) We compute the expectation as

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} tf(t)dt = \int_0^1 t\cdot 1dt = \left[\frac{1}{2}t^2\right]_0^1 = \frac{1}{2}-0 = \frac{1}{2}.$$

We have

$$\int_{-\infty}^{\infty} t^2f(t)dt = \int_0^1 t^2 = \left[\frac{1}{3}x^3\right]_0^1 = \frac{1}{3}-0 = \frac{1}{3}$$

and thus we get

$$\mathrm{Var}[X] = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{12}.$$

2. (a)

$$\int_{-\infty}^{1} f(t)dt = \int_{0}^{1} 4e^{-4t}dt = \left[-e^{-4t}\right]_0^1 = -e^{-4} - (-e^0) = 1 - e^{-4} \approx 0.981$$

(b)

$$\begin{aligned}
\mathbb{E}[X] &= \int_{-\infty}^{\infty} tf(t)dt = \int_{0}^{\infty} te^{-t}dt \\
&= \lim_{\alpha\to\infty} \int_{0}^{\alpha} te^{-t}dt \\
&= \lim_{\alpha\to\infty} \left[-e^{-t}(t+1)\right]_0^{\alpha} \\
&= \lim_{\alpha\to\infty} -e^{-\alpha}(\alpha+1) - \left(e^0 \cdot 1\right) \\
&= \lim_{\alpha\to\infty} -\alpha e^{-\alpha} - \underbrace{\lim_{\alpha\to\infty} e^{-\alpha}}_{=0} + 1 \\
&= \lim_{\alpha\to\infty} -\frac{\alpha}{e^{\alpha}} + 1 \\
&\overset{L'H}{=} \lim_{\alpha\to\infty} \frac{1}{e^{\alpha}} + 1 \\
&= 0 + 1 = 1
\end{aligned}$$