Advanced Integration Techniques

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Reversing derivation rules?

- We have seen the following rules for integration:

$$\int_a^b c \cdot f(x) dx = c \cdot \int_a^b f(x) dx$$
$$\int_a^b f(x) + g(x) dx = \int_a^b f(x) dx + \int_a^b g(x) dx$$

- The latter is the inversion of the sum-rule for derivation
- What about inverse rules for product and chain rule?

Integration by parts

Theorem

Let u, v be two functions with the derivatives u', v'. Then

$$\int_a^b u(x)v'(x)dx = [u(x)v(x)]_a^b - \int_a^b u'(x)v(x)dx.$$

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- Inversion of the product rule

Theorem

Let f be a continuous function, ϕ a differentiable function. Then

$$\int_{\phi^{-1}(a)}^{\phi^{-1}(b)} f(\phi(x)) \cdot \phi'(x) dx = \int_a^b f(x) dx$$

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Example:

$$\int_1^4 \exp(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}} dx$$

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Example:

$$\int_1^4 \underbrace{\exp}_{=f}\big(\underbrace{\sqrt{x}}_{=\phi}\big) \cdot \underbrace{\frac{1}{2\sqrt{x}}}_{=\phi'} dx$$

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$$\int_1^4 \underbrace{\exp}_{=f}(\underbrace{\sqrt{x}}_{=\phi}) \cdot \underbrace{\frac{1}{2\sqrt{x}}}_{=\phi'} dx = \int_1^2 \exp(x) dx = [e^x]_1^4 = e^4 - e^1$$

Example:

$$\int_0^{\pi} cos(x)\sqrt{\sin(x)}dx$$

Using the substitution rule

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$$t := sin(x)$$

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$$\int_0^{\pi} cos(x)\sqrt{\sin(x)}dx = \int \cancel{\cos(x)}\sqrt{t}\frac{dt}{\cancel{\cos(x)}}$$

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Adjust Endpoints:

$$\tilde{a} = t(a) = sin(0) = 0$$

$$\tilde{b} = t(b) = sin(\pi) = 1$$

Example:

$$\int_0^{\pi} cos(x)\sqrt{\sin(x)}dx = \int_0^1 \cancel{\cos(x)}\sqrt{t}\frac{dt}{\cancel{\cos(x)}}$$

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