How to... Integrate by parts

Given: A function f that can be integrated by parts.
Wanted: The (indefinite) integral of f.

Example

We want to compute the integral

$$\int_0^\pi x^2 \sin(x)\,dx$$

1 Find partial functions u and v'

Define functions $u(x)$ and $v'(x)$ such that the integral can be written as

$$\int_a^b u(x)\,v'(x)\,dx.$$

Important: The choice of $u(x)$ and $v'(x)$ is crucial for the method of integration by parts. If u and v' are not chosen correctly, integration by parts will not yield a (meaningful) solution.

When defining $u(x)$ and $v'(x)$ you should keep the following things in mind:

a) You must be able to compute the derivative of $u(x)$ (normally not a problem)

b) You must be able to find an anti-derivative of $v'(x)$ (Keep in mind that not all functions have an explicit anti-derivative!)

c) The remainder integral containing the product of the derivative of $u(x)$ and the anti-derivative $v'(x)$ should (in general) be simpler then the one you started with. A good rule of thumb is to choose $u(x)$ as a polynomial whenever possible since a polynomial vanishes if you differentiate it multiple times.

We choose $u(x) = x^2$ and $v'(x) = \sin(x)$ because $u(x)$ is a polynomial this way. This will lead to a simplification of the remainder integral. Furthermore, the anti-derivative of $v'(x) = \sin(x)$ can be computed easily. Hence, our integral has now the form

$$\int_0^\pi x^2 \sin(x)\,dx = \int_0^\pi u(x)\,v'(x)\,dx$$

2 Compute the missing derivative and anti-derivative

Compute $u'(x)$, the derivative of the function $u(x)$ and $v(x)$, the anti-derivative of the function $v'(x)$.

We compute the missing functions (red entries in the following table) as

$u(x) = x^2$	$v(x) = -\cos(x)$
$u'(x) = 2x$	$v'(x) = \sin(x)$

3 Use the formula

Use the formula for integration by parts

$$\int_a^b u(x)v'(x)\,dx = \big[u(x)v(x)\big]_a^b - \int_a^b u'(x)v(x)\,dx$$

with the functions u, u', v, v' from above.
If you are computing the indefinite integral the first part of the right hand side is already part of the anti-derivative you are computing. You may need to add the constant $+c$.

We use the formula and obtain

$$\int_0^\pi x^2\sin(x)\,dx = \big[-x^2\cos(x)\big]_0^\pi + \int_0^\pi 2x\cos(x)\,dx = \pi^2 + \int_0^\pi 2x\cos(x)\,dx$$

or

$$\int x^2\sin(x)\,dx = -x^2\cos(x) + c - \int 2x\cos(x)\,dx$$

if we want to compute the indefinite integral. (Note that the remainder integral has a positive sign as we factored out the -1 factor of $v(x) = -\cos(x)$).

4 Solve the remainder integral

Finally, solve the remainder integral. If $u(x)$ and $v'(x)$ were chosen incorrectly you might encounter an unsolvable integral at this point, or at least an integral that is much harder to solve.
It may happen that you have to use integration by parts again (maybe multiple times).

The remainder integral in our case is $\int_0^\pi x\sin(x)dx$. We can solve this, by again using integration by parts. This time, we use

$$\begin{array}{ll} u(x) = 2x & v(x) = \sin(x) \\ u'(x) = 2 & v'(x) = \cos(x) \end{array}$$

where the black entries represent our choice of $u(x)$ and $v'(x)$ and the red entries are the computed derivative and anti-derivative. Using the integration by parts formula again we obtain

$$\int_0^\pi 2x\cos(x)dx = \left[2x\sin(x)\right]_0^\pi - \int_0^\pi 2\sin(x) = 0 - \left[-2\cos(x)\right]_0^\pi = -4.$$

Hence, the complete integral is

$$\int_0^\pi x^2\sin(x)dx = \pi^2 + \int_0^\pi 2x\cos(x)dx = \pi^2 - 4.$$

The indefinite integral is

$$\int x^2\sin(x)dx = -x^2\cos(x) + 2x\sin(x) + 2x\cos(x) + c.$$