Some additional topics

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Inductive Proofs

Theorem (Induction)

Let $P(n)$ be some property depending on $n \in \mathbb{N}$. If

1. *$P(n_0)$ is true and*
2. *for every $n \in \mathbb{N}$ $P(n) \Rightarrow P(n+1)$ is true*

then $P(n)$ holds for all $n \in \mathbb{N}$.

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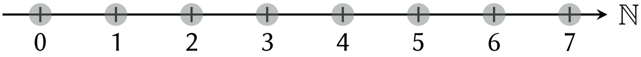

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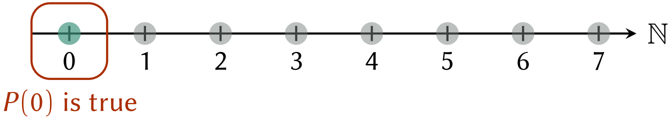

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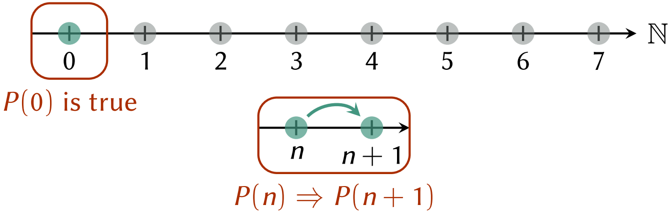

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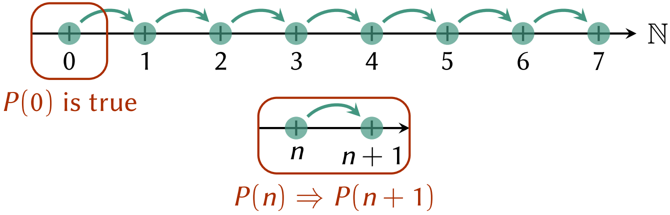

Optimization with side constraints

The Karush-Kuhn-Tucker conditions

Optimization with side constraints

- With Lagrange Optimization we can solve problems of the form

$$\begin{aligned} &\min f(x) \\ &\text{s.t. } g_i(x) = 0 \quad \text{for } i = 1, \ldots, k \end{aligned}$$

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- Many problems have *inequalities* as side constrains, e.g.
  - budget constraints
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  - capacity constraints

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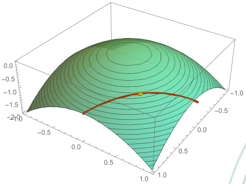
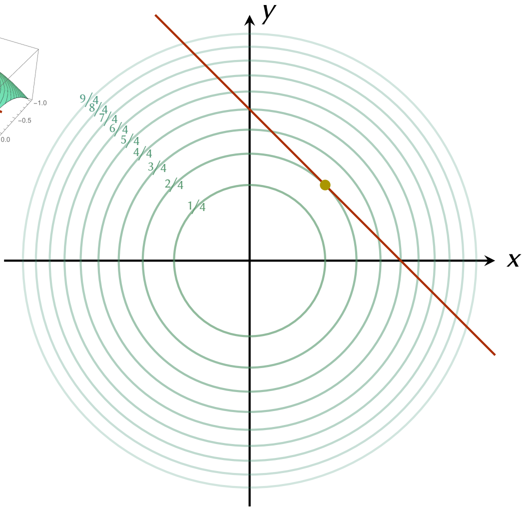
  We can optimize a function subject to *equality*-constraints
- Many problems have *inequalities* as side constrains, e.g.
  - budget constraints
  - time constraints
  - capacity constraints
- We want to solve problems of the form

$$\begin{aligned} &\min f(x) \\ &\text{s.t. } g_i(x) \leq 0 \quad \text{for } i = 1, \dots, k \\ &\qquad h_j(x) = 0 \quad \text{for } j = 1, \dots, l \end{aligned}$$

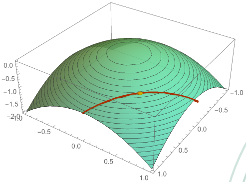
Lagrange Multipliers

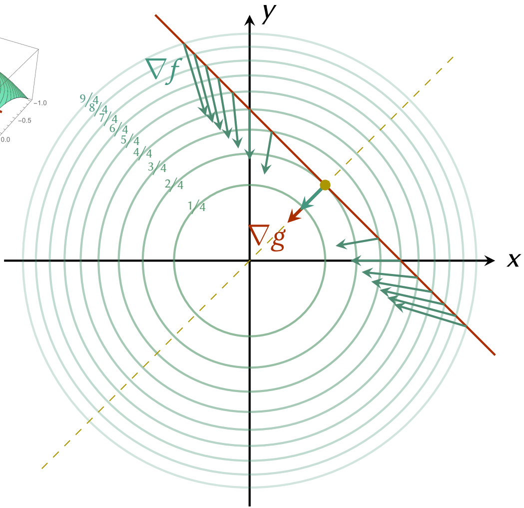




$$\max f(x, y) = -x^2 - y^2$$

$$\text{s.t. } g(x, y) = -x - y + 1 = 0$$





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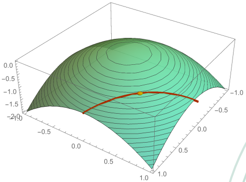

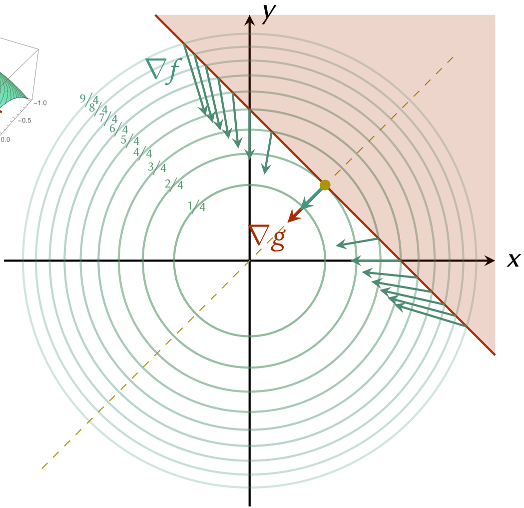

$$\max f(x, y) = -x^2 - y^2$$

$$\text{s.t. } g(x, y) = -x - y + 1 \leq 0$$

Theorem

Let $x^{(0)} = (x_n^{(0)}, \dots, x_n^{(0)})$ *be an optimal solution of*

$$\begin{aligned} \max \quad & f(x) \\ s.t. \quad & g_i(x) \geq 0 \quad \forall i = 1, \dots, k \\ & h_j(x) = 0 \quad \forall j = 1, \dots, l \end{aligned}$$

and f, g, h *partially differentiable then there are* $\lambda_1, \dots, \lambda_k \in \mathbb{R}$ *and* $\mu_1, \dots, \mu_l \in \mathbb{R}$ *such that*

$$\nabla f(x^{(0)}) - \sum_{i=1}^{k} \lambda_j \nabla g_i(x^{(0)}) - \sum_{j=1}^{l} \mu_j \nabla h_j(x^{(0)}) = 0$$

and

$$\begin{aligned} \lambda_j \geq 0 \quad & \forall j = 1, \dots, l \\ \lambda_j h_j(x^{(0)}) = 0 \quad & \forall j = 1, \dots, l. \end{aligned}$$

Optimization with side constraints

The Simplex-Algorithm

- An interesting special case of optimization is the following:

$$\begin{aligned} &\max f(x) \\ &\text{s.t. } g_i(x) \leq 0 \quad \text{for } i = 1, \dots, k \end{aligned}$$

where f and g are *linear* functions.

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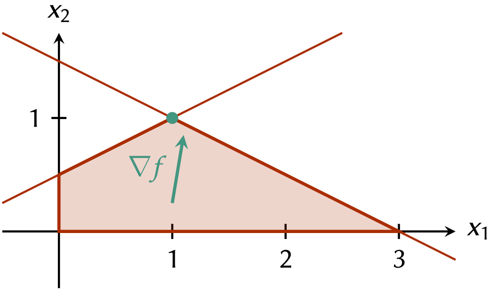
where f and g are *linear* functions.

- Example:

$$\begin{aligned} \max \quad & x_1 + 6x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 \leq 3 \\ & -x_1 + 2x_2 \leq 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$



The simplex method

$$
\begin{aligned}
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\end{aligned}
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$$\begin{aligned} \max \quad & x_1 + 6x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 + y_1 = 3 \\ & -x_1 + 2x_2 + y_2 = 1 \\ & x_1, x_2, y_1, y_2 \geq 0 \end{aligned}$$

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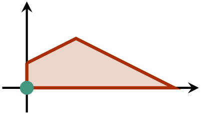
0	-1	-6	0	0
3	1	2	1	0
1	-1	2	0	1



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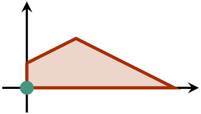
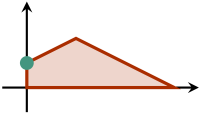


3	-4	0	0	3
2	2	0	1	-1
$1/2$	$-1/2$	1	0	$1/2$



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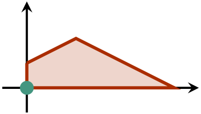

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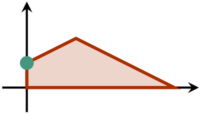
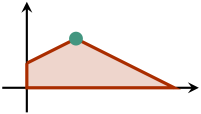


7	0	0	2	1
1	1	0	$1/2$	$-1/2$
1	0	1	$1/4$	$1/4$



Conclusion

Things you...	... should know	... should keep in mind
Linear Algebra	Basic vector and matrix computations Solving systems of linear equations	Eigenvalues
Analysis	Derivatives Basic integration	Induction Sequences & Limits
Optimization	Computing extrema of uni-/multivariate functions Lagrange Optimization	KKT conditions Simplex Algorithm

Thank you!

And good look with your studies at HU Berlin!