AA3 Networks

Tropical Mechanism Design

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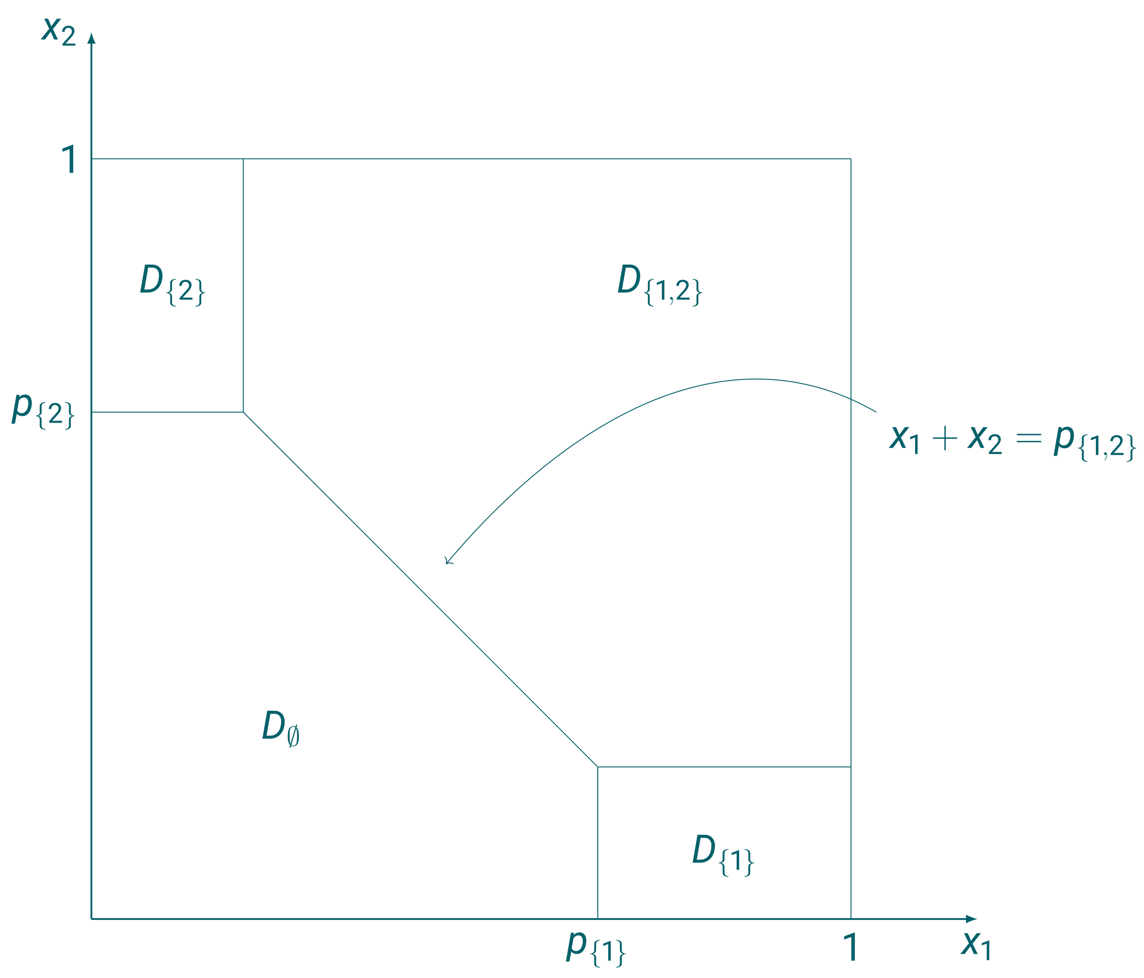
How to sell m items?

+ Define prices p_J for each bundle of items $J \subseteq \{1, ..., m\}$
+ Buyer has valuation $x_j \in [0, 1]$ according to a probability function f_j for each item $j \in \{1, ..., m\}$
+ Buyer takes bundle which maximizes her *utility*:

$$u(\mathbf{x}) = \max_{J \subseteq \{1,...,m\}} \left(\sum_{j \in J} x_j - p_J \right)$$

The set of all $\mathbf{x}$ for which the buyer takes bundle J is called D_J
$\rightarrow$ u is a *tropical polynomial* and the regions D_J are separated by its vanishing locus

Geometric representation for $m = 2$:



Our Goal: Choose optimal prices to maximize the *expected revenue*:

$$Rev(\mathbf{p}) = \sum_{J \subseteq \{1,...,m\}} \int_{D_J} p_J \, \mathbf{d}f(\mathbf{x}) = \int_{[0,1]^m} \Big(\nabla u(\mathbf{x}) \cdot \mathbf{x} - u(\mathbf{x}) \Big) \mathbf{d}f(\mathbf{x})$$

Duality

+ Use the utility u instead of $\mathbf{p}$ as optimization variable and define the optimization program:

$$\sup_u \int_{[0,1]^m} \Big(\nabla u(\mathbf{x}) \cdot \mathbf{x} - u(\mathbf{x}) \Big) \mathbf{d}f(\mathbf{x})$$

$$\text{s.t.: } 0 \le \frac{\partial u(\mathbf{x})}{\partial x_j} \le 1, \quad u : [0,1]^m \rightarrow \mathbb{R} \text{ convex}$$

+ **Remark**: This generalizes the framework to lotteries - we sell tickets for the objects and the buyer gets object j with a probability of $\frac{\partial u(\mathbf{x})}{\partial x_j}$ (see [3])
+ The dual optimization problem (by [1]):

$$\inf_{z_1,...,z_m} \int_{[0,1]^m} \sum_{j=1}^m z_j(\mathbf{x}) \mathbf{d}\mathbf{x}$$

$$\text{s.t.: } \sum_{j=1}^m \frac{\partial z_j(\mathbf{x})}{\partial x_j} \le (m+1) f(\mathbf{x}) + \mathbf{x} \cdot \nabla f(\mathbf{x})$$

$$z_j(0, \mathbf{x}_{-j}) = 0, \quad z_j(1, \mathbf{x}_{-j}) \ge 1, \quad z_j(\mathbf{x}) \ge 0, \qquad j \in \{1, ..., m\}$$

where $z_j(\lambda, \mathbf{x}_{-j}) = z_j((x_1, ..., x_{j-1}, \lambda, x_{j+1}, ..., x_m))$

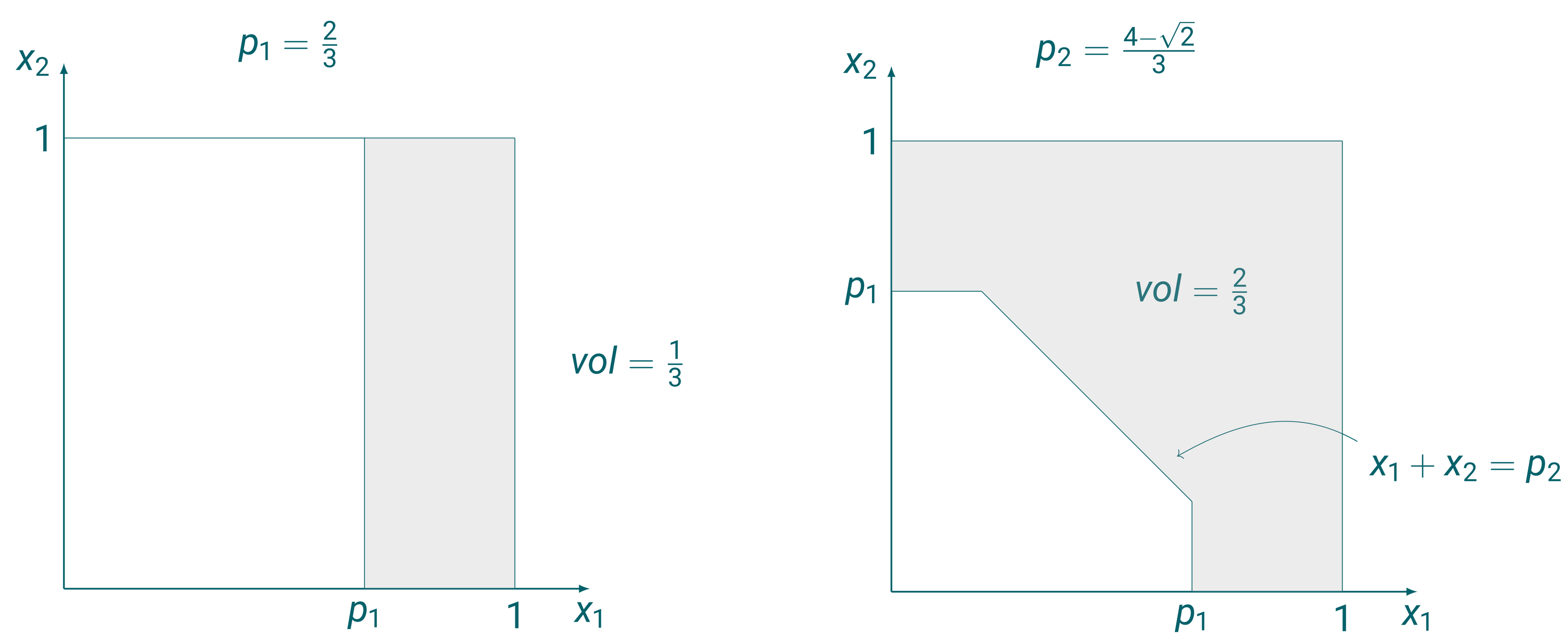
SJA-prices

+ Consider the case, where $f_j \equiv 1$ for all $j \in \{1, ..., m\}$, so $\mathbf{x}$ is drawn uniformly
+ Choose prices depending on the cardinality of the bundle, that is if $|J_1| = |J_2|$, then $p_{J_1} = p_{J_2} =: p_{|J_1|}$. If $p_1, ..., p_{r-1}$ are already defined, define p_r s.t.:

$$vol \left(\bigcup_{J \subseteq \{1,...,r\}} \left\{ \mathbf{x} \in [0,1]^m \mid \sum_{j \in J} x_j \ge p_{|J|} \right\} \right) = \frac{r}{m+1} \qquad (1)$$

+ Proven to be optimal for $m \le 6$ (see [1])

SJA-prices for $m = 2$:



Duality for SJA

+ Consider the dual problem for uniform distributions of $\mathbf{x}$:

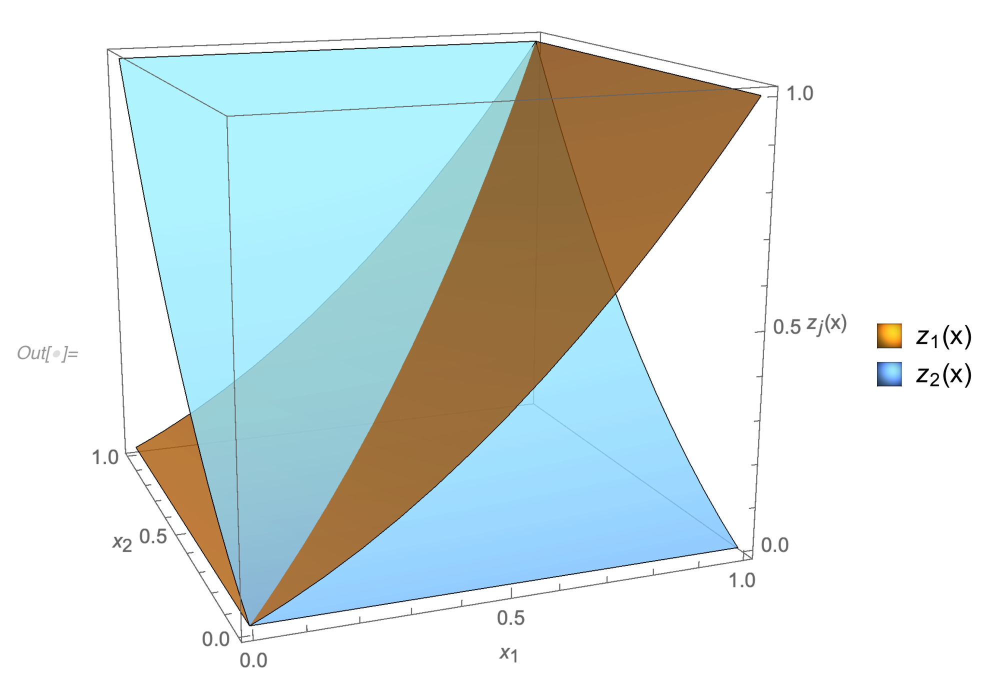
$$\inf_{z_1,...,z_m} \int_{[0,1]^m} \sum_{j=1}^m z_j(\mathbf{x}) \mathbf{d}\mathbf{x}$$

$$\text{s.t.: } \sum_{j=1}^m \frac{\partial z_j(\mathbf{x})}{\partial x_j} \le (m+1)$$

$$z_j(0, \mathbf{x}_{-j}) = 0, \quad z_j(1, \mathbf{x}_{-j}) \ge 1, \quad z_j(\mathbf{x}) \ge 0, \qquad j \in \{1, ..., m\}$$

+ **Intuitive description**: We search functions $z_1, ..., z_m$ s.t. z_j starts with 0 at the "lower" facet (according to the j-th coordinate) of the $[0, 1]$ m-dimensional hypercube and ends with at least 1 at the "upper" facet, while the sum of all z_j cannot grow too steeply (in their respective direction)

A feasible solution for $m = 2$:



Min-Cost-Flow

+ We want to discretize $\frac{\partial z_j(\mathbf{x})}{\partial x_j}$ on the m-dimensional $(N \times ... \times N)$-grid
+ **Construction of graph**:

Nodes: $V = U \cup B \cup \{s, t\}$, with
$U = \{u_\alpha | \ \alpha \in \{1, ..., N\}^m\}$ for the cells of the grid
$B = \left\{ b^j_{\alpha_{-j}} | \ \alpha_{-j} \in \{1, ..., N\}^{m-1}, j \in \{1, ..., m\} \right\}$ as supplementary nodes on the boundary.

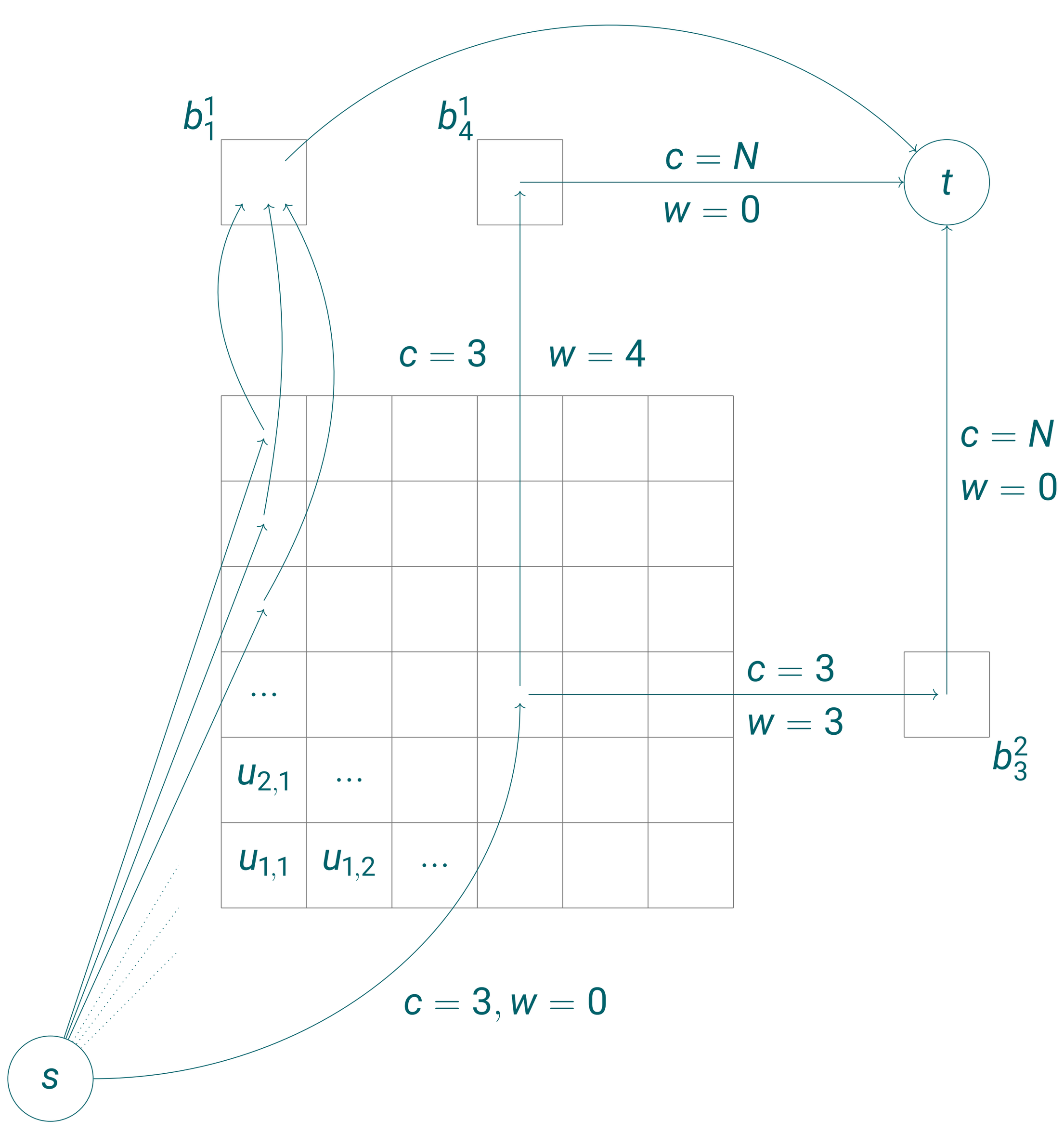
Edges: $E = S \cup F \cup T$, with
$S = \{(s, u) | \ u \in U\}$,
$F = \{(u_\alpha, b^j_{\alpha_{-j}}) | \ j \in \{1, ..., m\}\}$,
$T = \{(b, t) | \ b \in B\}$

Capacities: $c(e) = \begin{cases} m+1, & e \in S \cup F \\ N, & e \in T \end{cases}$ Weights: $w(e) = \begin{cases} 0, & e \in S \cup T \\ N + 1 - \alpha_j, & e = (u_\alpha, b^j_{\alpha_{-j}}) \end{cases}$

$\rightarrow$ Edge-capacities simulate the constraints, weights simulate the objective value

+ Every $(s - t)$-flow of capacity mN^m represents a feasible solution for the dual program
+ Get an optimal solution by searching a min-cost-flow for $N \rightarrow \infty$

Graph for $m = 2, N = 6$:



Advantages:

+ Simpler proof for the optimality of SJA-prices in the 2-item case than existing one
+ May be adaptable for an arbitrary amount of items (instead of just $m \le 6$)
+ May help to find a proof for a greater class of distributions and thus fix a flaw in [2]

References

[1] Y. Giannakopoulos and E. Koutsoupias.
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[2] Y. Giannakopoulos and E. Koutsoupias.
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[3] J. Rochet.
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