Lectures 1 & 2 :

Sequences and Series

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1 Sequences

Part 1.1

Definition of Sequences

Definition: Sequences

A sequence is a function, which is only “fed” with natural numbers:

$$\boxed{n \mapsto a(n) =: a_n} \quad n \in \mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$$

(a possibility: n is an element of an infinite subset of $\mathbb{N}$)

Remark:

In many economic and business type applications the variable $\mathbf{n}$ *is understood to refer to a time point/index.*

Ways to describe sequences:

- by a table of values:

  table of values
  $(a_1, a_2, a_3, \ldots, a_n)$

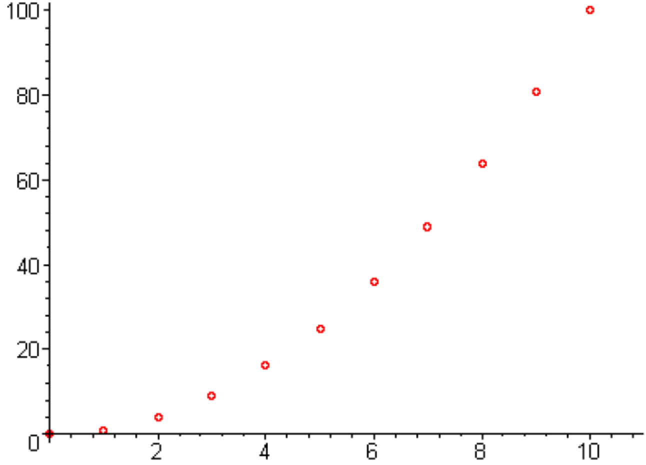
- by a transformation rule

  $\Downarrow$ explicit

  $\Downarrow$ recursive

Examples of explicit definitions:

$$a_n = a(n) = n^2, \quad n = 0, 1, 2, 3, \ldots$$



Examples of explicit definitions:

Growth of an **initial capital $\mathbf{K_0}$**, with **compound interest**; the annual interest rate is **$\mathbf{p}$%**.

$$K_n = \left(1 + \frac{p}{100}\right)^n \cdot K_0, \quad n = 0, 1, 2, 3, \ldots$$

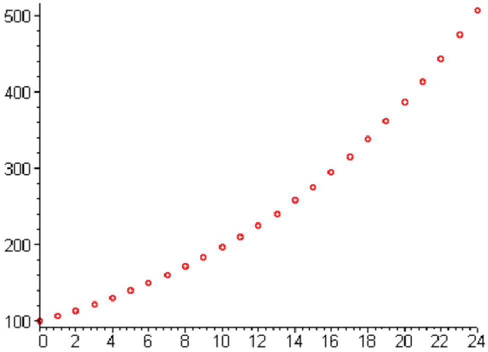
Examples of explicit definitions:

$$f.i.: K_n = \left(1 + \frac{p}{100}\right)^n \cdot K_0, \quad n = 0, 1, 2, 3, \ldots$$

$K_0 = 100$
$p = 7\ \%$

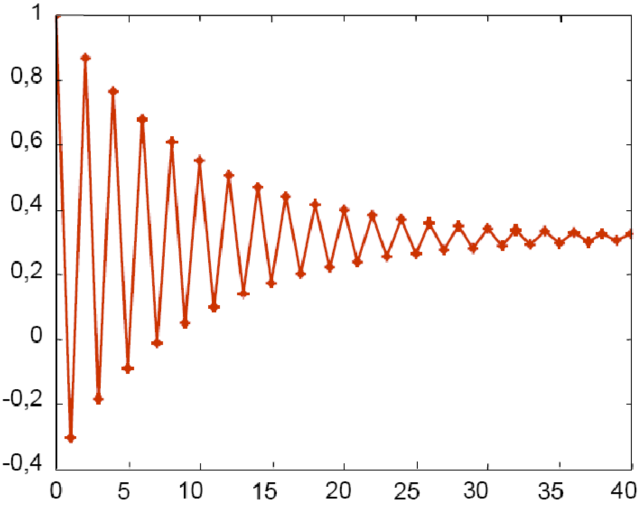
↓

$K_n = 100 \cdot 1,07^n$
$n = 0, 1, 2, ..., 24$



Examples of a recursive definition:

$$a_{n+1} = -0.9a_n + 0.6, \quad a_0 = 1, \; n = 0, 1, 2, 3, \ldots$$



Examples of a recursive definition:

(see Newton's method, lectures 5&6):

The sequence of numbers $(a_n)_{n\geq 0}$ which is definded by the formula:

$$a_{n+1} = \frac{a_n}{2} + \frac{1}{a_n}$$

$a_0 \neq 0$ any fixed number

Examples of tables of values:

The gross national product of Germany 1980 - 1990:

Year	real GDP (in Trillion DM)
1980	2.026
1981	2.026
1982	2.004
1983	2.045
1984	2.108
1985	2.149
1986	2.199
1987	2.233
1988	2.314
1989	2.411
1990	2.544

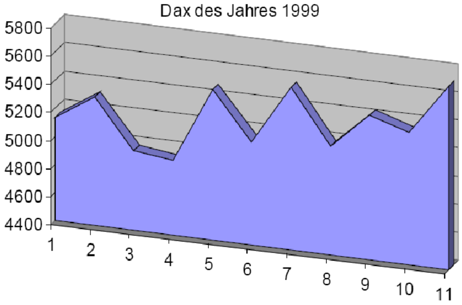
Source: Deutsche Bundesbank

Examples of tables of values:

The values of the German stock index for 1999;
$d_1 =$ Jan.-DAX1999 is the first value of the finite sequence.

Januar	Februar	März	April	Mai	Juni	Juli	August	September	Oktober	November
5137	5313	4968	4936	5470	5132	5560	5170	5432	5343	5702

$$(d_i)_{i \geq 1}$$



Part 1.2

Arithmetic Sequences

Definition: Arithmetic sequences

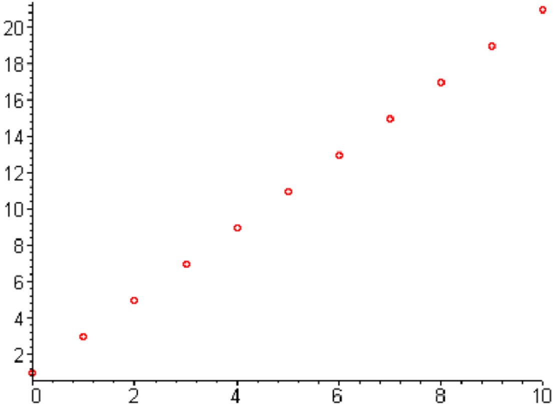
An arithmetic sequence is a linear (affine) function whose domain is $\mathbb{N}_0$, $\mathbb{N}$ resp.

$$\boxed{a_n = a_0 + nd} \quad n = 0, 1, 2, 3, \ldots,$$

where:

$$a_0 : y-intercept$$
$$n : variable$$
$$d : slope$$

Example: Arithmetic sequence



$$a_n = 1 + 2n$$

$\downarrow$

$a_0 = 1$
$d = 2$
$n = 0, 1, 2, \ldots$

Representation: Arithmetic sequences

Explicit form:	$a_n = a_0 + nd$
Recursive form: $(a_0 \textit{ is given})$	$a_{n+1} - a_n = d$ $\Longleftrightarrow$ $a_{n+1} = a_n + d$

Part 1.3

Geometric Sequences

Definition: Geometric sequences

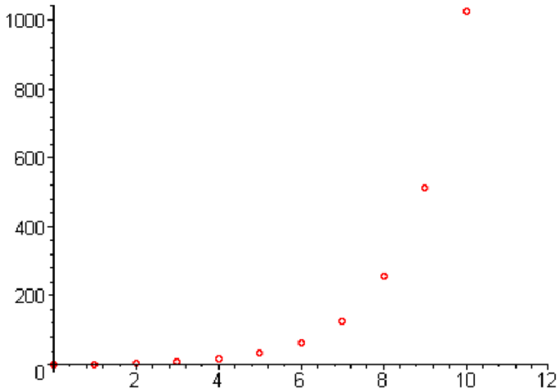
Geometric sequences are exponential functions whose domain is restricted to $\mathbb{N}_0$, $\mathbb{N}$ resp.

$$\boxed{a_n = a_0 q^n} \qquad n = 0, 1, 2, 3, \ldots$$

where:

$$a_0 : y-intercept$$
$$n : variable$$
$$q : \text{``}base\text{''}/multiplier$$

Example: Geometric sequence



$a_n = 2^n$

↓

$a_0 = 1$
$q = 2$
$n = 0, 1, 2, \ldots$

Representation: Geometric Sequences

Explicit form:	$a_n = a_0 q^n$
Recursive form:	$a_{n+1} = q a_n$
$\frac{a_{n+1}}{a_n} = q$	$\Longleftrightarrow$
$(a_0 \neq 0 \text{ is given})$	$a_{n+1} - q a_n = 0$

Part 1.4

Properties of Sequences

Concept: Bounded Sequences

A sequence is called **bounded** if and only if ($\hat{=}$ iff):

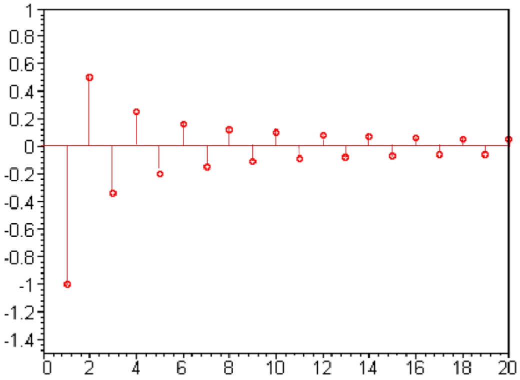
$$|a_n| \leq \ const., \quad n = 1, 2, 3, \ldots$$

Example of a bounded sequence

$a_n = (-1)^n \frac{1}{n}, \quad n = 1, 2, \ldots$

$\downarrow$

is a bounded sequence
$|(-1)^n \frac{1}{n}| \leq 1$

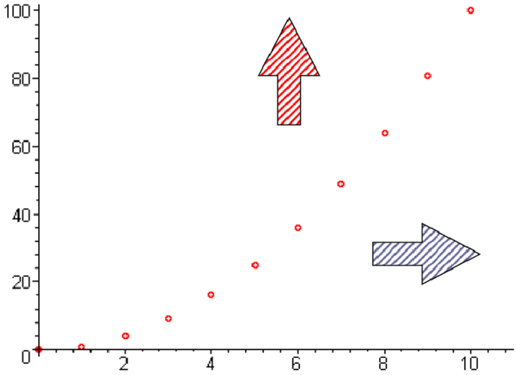


Example of an unbounded sequence:

$a_n = n^2, \quad n = 1, 2, \ldots$

↓

is an unbounded sequence



Monotone increasing / decreasing sequences:

monotone increasing:

$\uparrow$

$$a_n \leq a_{n+1}$$

$$\Leftrightarrow$$

$$a_{n+1} - a_n \geq 0$$

$n = 1, 2, \ldots$

---

monotone decreasing:

$\downarrow$

$$a_n \geq a_{n+1}$$

$$\Leftrightarrow$$

$$a_{n+1} - a_n \leq 0$$

$n = 1, 2, \ldots$

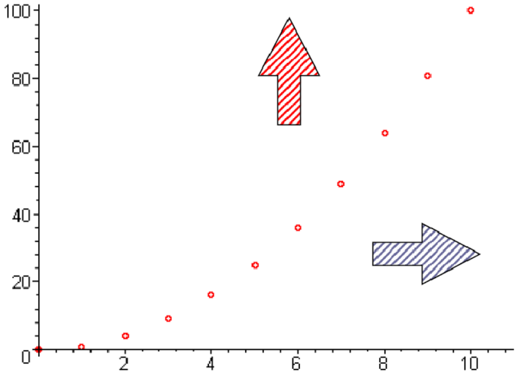
Example: Monotone increasing

$a_n = n^2, \quad n = 1, 2, \dots$

$\downarrow$

is a monotone increasing sequence

$(n+1)^2 = n^2 + \boxed{2n+1} \geq n^2$



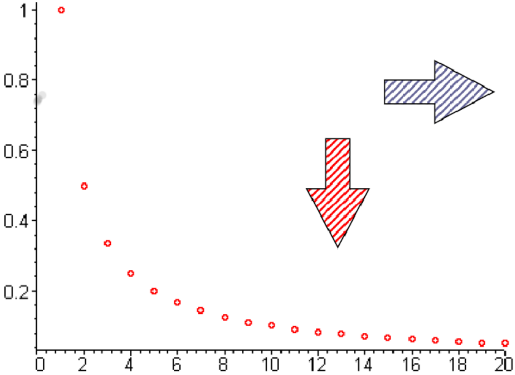
Example: Monotone increasing

$$a_n = \frac{1}{n}, \quad n = 1, 2, \ldots$$

$\downarrow$

is a
monotone
decreasing
sequence

$$\frac{1}{(n+1)} = \frac{n}{n+1} \cdot \frac{1}{n} + \boxed{2n+1} \leq \frac{1}{n}$$

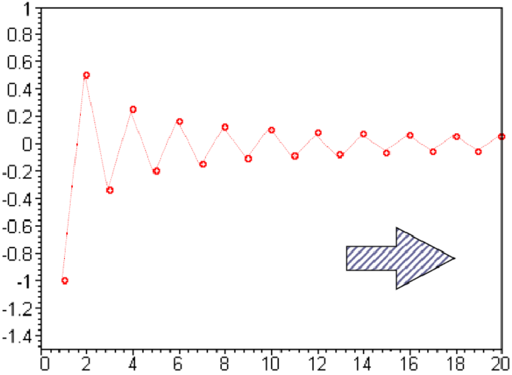


Example:

$a_n = (-1)^n \frac{1}{n}, \quad n = 1, 2, \ldots$

$\downarrow$

is neither
monotone
decreasing nor
monotone
increasing



Concept: Alternating sequences

A sequence is called alternating iff:

$$a_n \neq a_{n+1}, \qquad n = 1, 2, \ldots$$

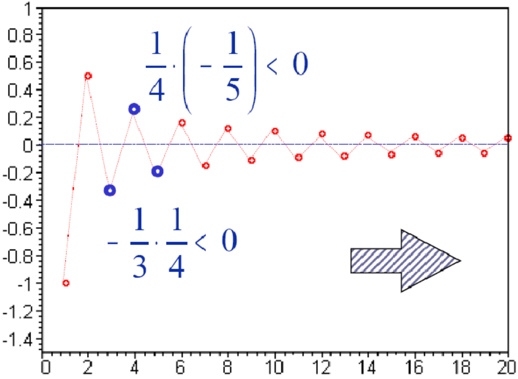
and: $a_n \cdot a_{n+1} < 0$

Example: Alternating sequence

$a_n = (-1)^n \frac{1}{n}, \quad n = 1, 2, \ldots$

↓

is an alternating sequence



Concept: Convergent/divergent sequences

- $\lim_{n\longrightarrow\infty} a_n$ *exists*

  $\Leftrightarrow$ $(a_n)_n$ *is convergent*

- $\lim_{n\longrightarrow\infty} a_n$ *does not exist*

  $\Leftrightarrow$ $(a_n)_n$ *is divergent*

Definition: $\lim_{n\to\infty}(a_n) = A$

For any $\epsilon > 0$ there is an integer N_ϵ such that for all $n \geq N_\epsilon$:

$$|a_n - A| \leq \epsilon$$

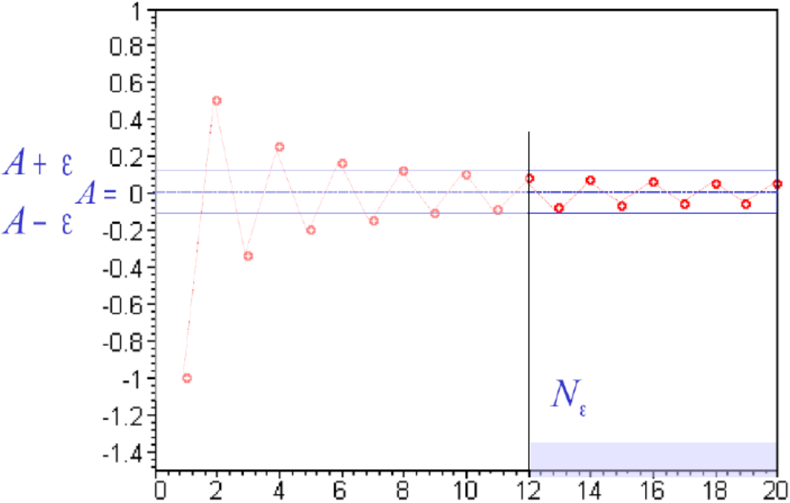


Figure 1: Illustration of the definition of the limit

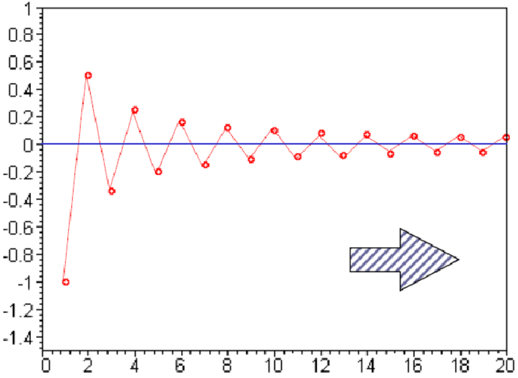
Example: A convergent sequence

$a_n = (-1)^n \frac{1}{n}, \quad n = 1, 2, \ldots$

$\downarrow$

limit
exists

$lim(a_n) \; = \; 0$

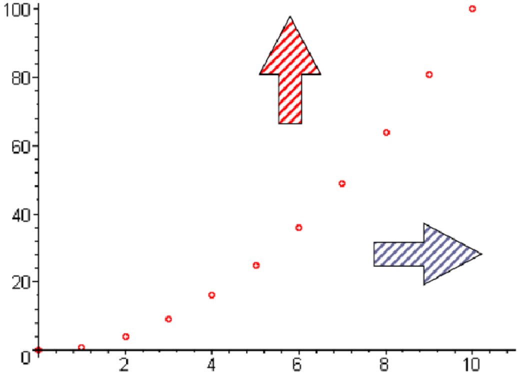


Example: A divergent sequence

$a_n = n^2, \quad n = 1, 2, \ldots$

$\downarrow$

limit does not exist



A convergence theorem

Every bounded and monotone increasing sequence does converge

Every bounded and monotone decreasing sequence does converge

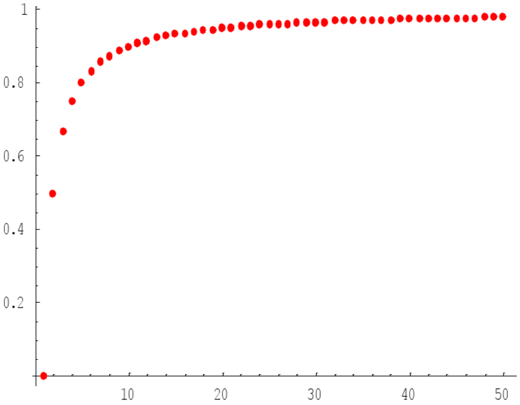
Example: bounded and monotone increasing

$a_n = 1 - \frac{1}{n}, \quad n = 1, 2, \ldots$

$\downarrow$

limit
exists

$\lim_{n\to\infty} (a_n) \;=\; 1$



2 Series

Part 2.1

Definition: Series

Definition: A finite series

A finite series is a finite sum of numbers.

Definition: Infinite series

An **infinite series** is the *limit of the sequence of partial sums* associated with a given sequence $(a_i)_{i\geq 1}$.

$$\sum_{i=1}^{\infty} a_i := \lim_{n\to\infty} \left\{ \sum_{i=1}^{n} a_i \right\} = \lim_{n\to\infty} \{s_n\}$$

Sequence of partial sums:

Given: A sequence $(a_i)_{i \geq 1}$:

$$
\begin{array}{llllll}
a_1 & & & & & \longrightarrow s_1 = a_1 \\
a_1 & a_2 & & & & \longrightarrow s_2 = a_1 + a_2 \\
a_1 & a_2 & a_3 & & & \longrightarrow s_3 = a_1 + a_2 + a_3 \\
\vdots & & & & & \\
a_1 & a_2 & a_3 & \cdots & a_n & \longrightarrow s_n = a_1 + a_2 + a_3 + \cdots + a_n
\end{array}
$$

Sequence of partial sums:

i.e., the sequence $(a_i)_{i \geq 1}$ generates, by the operation of summation, the partial sums $(s_n)_{n \geq 1}$:

$$s_n := a_1 + a_2 + \ldots + a_n = \sum_{i=1}^{n} a_i$$

Notation:

$$\sum_{i=1}^{\infty} a_i = \lim_{n\to\infty} \{s_n\} = \lim_{n\to\infty} \left\{ \sum_{i=1}^{n} a_i \right\}$$

If this limit exists, i.e. is a finite number,

we call $\sum_{i=1}^{\infty} a_i$ a *convergent* series which is associated with the sequence $(a_i)_{i\geq 1}$

.

Notation:

$$\sum_{i=1}^{\infty} a_i = \lim_{n\to\infty}\{s_n\} = \lim_{n\to\infty}\left\{\sum_{i=1}^{n} a_i\right\}$$

If this limit does not exist we call the *series to be divergent*.

Example 1:

The given sequence $(a_i)_{i\geq 1}$:
$a_n = \frac{1}{n^2}, \quad n = 1, 2, 3, \ldots$

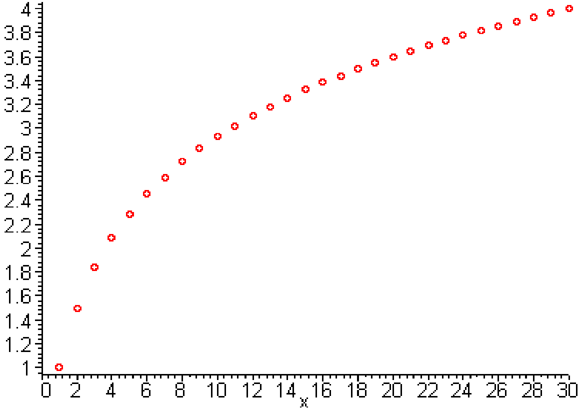
Question 1: Does $\sum_{n=1}^{\infty} a_n$ exist ???

$$1 + \frac{1}{4} + \frac{1}{9} + \ldots = \sum_{n=1}^{\infty} \frac{1}{n^2} \quad \textit{converges}$$

Question 2: Find the value of $\sum_{n=1}^{\infty} a_n$?

Example 1:

Graph of the sequence of partial sums $\left(\sum_{i=1}^{n} \frac{1}{i^2}\right)_{n=1...30}$



Example 2:

The given sequence $(h_n)_{n \geq 1}$:

$$h_n = \frac{1}{n}, \quad n = 1, 2, 3, \ldots$$

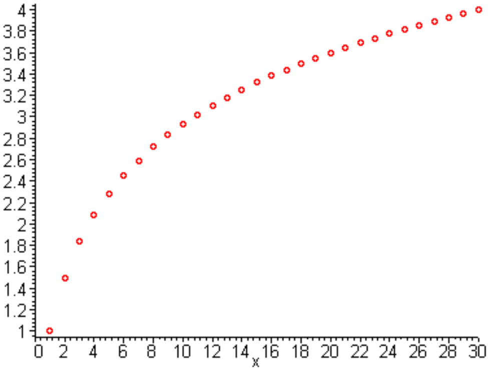
$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \ldots \boxed{= \sum_{n=1}^{\infty} \frac{1}{n}}$$

harmonic series

$\longrightarrow$ *diverges*

Example 2:

Graph of the sequence of partial sums $\left(\sum_{i=1}^{n} \frac{1}{i}\right)_{n=1\ldots30}$



Part 2.2

Arithmetic Series

Defintion: Arithmetic Series

Given: An *arithmetic sequence* $(a_i)_i$:

$$\sum_{i=1}^{\infty} a_i = \lim_{n\to\infty} \left(\sum_{i=1}^{n} a_i \right)$$

$$\downarrow$$

arithmetic series

→ Example 1:

Gauß as a "schoolboy"
("little" Gauß)

cf. the sum of all natural numbers from 1 to 100

Carl Friedrich Gauß (1777-1855)

Carl Friedrich Gauß, *was born in Brunswick; he was a working class kid.*

He is considered by many the greatest mathematican of his time (of all times ???).

His contributions include results in mathematics, astronomy, statistic, physics, etc.

Exercise:

1

Find the *explicit representation* of the **recursively defined** arithmetic sequence $(a_n)_n$, where
$a_{n+1} - a_n = 1, \quad a_0 = 0$.

2

Find the *value (a formula)* of the n-th component of the corresponding sequence of partial sums.

Exercise:

3

Decide whether or not

$\sum_{i=0}^{\infty} a_i$ *converges*.

Solution: Question 1

$$\begin{aligned}
a_0 &= 0 \\
a_1 - a_0 &= 1 &\implies&& a_1 &= 1 \\
a_2 - a_1 &= 1 &\implies&& a_2 &= 1 + a_1 = 2 \\
&\vdots &&& &\vdots \\
a_n - a_{n-1} &= 1 &\implies&& a_n &= \cdots = n
\end{aligned}$$

i.e.: $(a_n)_{n\geq 0} = (0, 1, 2, 3, 4, 5, \ldots)$

Solution: Question 2

$$s_n = 1 + 2 + 3 + 4 + 5 + \ldots + n$$

$$+$$

$$s_n = n + n - 1 + n - 2 + \ldots + 1$$

$$\Downarrow$$

$$2s_n = \underbrace{(n+1) + (n+1) + \ldots + (n+1)}_{n-times} \Longrightarrow s_n = \frac{1}{2}n(n+1)$$

Solution: Question 3

The *limit* of this sequence of partial sums *does not exist*.

The components/elements of the *sequence of partial sums* do *not stabilize (around a finite value)*. The sequence $(s_n)_n$ is *unbounded*.

$$\sum_{i=0}^{\infty} a_i = \lim_{n\to\infty} \left(\sum_{i=0}^{n} a_i \right) = \lim_{n\to\infty} \left\{ \frac{1}{2} n(n+1) \right\} = \infty$$

Applications
of
Arithmetic Sequences & Series:

- Linear depreciation of capital goods
- Simple interest calculations
- Annuities
- Inventory problems

Example: linear depreciation

R_0 — *cost/value of the capital good at time $n = 0$ (brand-new)*

R_n — *value (bookvalue) at the end of year n*

---

r — *constant rate of depreciation*

Example: linear depreciation

Bookvalue after the *1st year*:

$$R_1 = R_0 - r$$

Bookvalue after the *2nd year*:

$$R_2 = R_1 - r \quad \Leftrightarrow \quad R_2 - R_1 = d = -r$$

Bookvalue after n^{th} *year*:

$$R_n = R_0 - nr$$

Example: linear depreciation

Find r so that the bookvalue after *5 years* is zero,

i.e. satisfy the requirement $R_5 = 0$.

Idea: Choose

$$r = \frac{R_0}{5} \quad \hat{=} \quad \left(\frac{\textit{purchasing cost}}{\textit{useful lifetime}} \right)$$

Recall: Bookvalue after *n years*:

$$R_n = R_0 - nr$$

Part 2.3

Geometric Series

Defintion: Geometric Series

Given: A geometric series $(a_i)_{i \geq 0}$, i.e. $a_i = a_0\, q^i,\ q \in \mathbb{R}$:

$$\sum_{i=0}^{\infty} a_i = \lim_{n \to \infty} \left(\sum_{i=0}^{n} a_i \right) = a_0 \sum_{i=0}^{\infty} q^i$$

$$\downarrow$$

geometric series

Example 2

"Big" Gauß

cf. compound interest and annuities

Imagine that at the time when Christ was born the roman emperor Augustus had been able to invest

$\$ 1.23$

in a bank account and had been guaranteed an annual interest rate of 3%; assume interest payments to be compounded every year.

What was the balance account at the end of the first year of the new millenium, i.e. after 2000 years of compounded interest payments ?

Solution: Balance account

Initial amount: $a_0 = 1.23$

After 1 year:

$$a_1 = \left(1 + \frac{p}{100}\right) a_0 = q a_0, \quad \textit{where } q = 1.03$$

After 2 years: $a_2 = q a_1 = a_0 q^2$

$\vdots$

After n years: $\boxed{a_n = q a_{n-1} = a_0 q^n}$

and $n = 2000$

Solution: Balance account

After 2000 years:

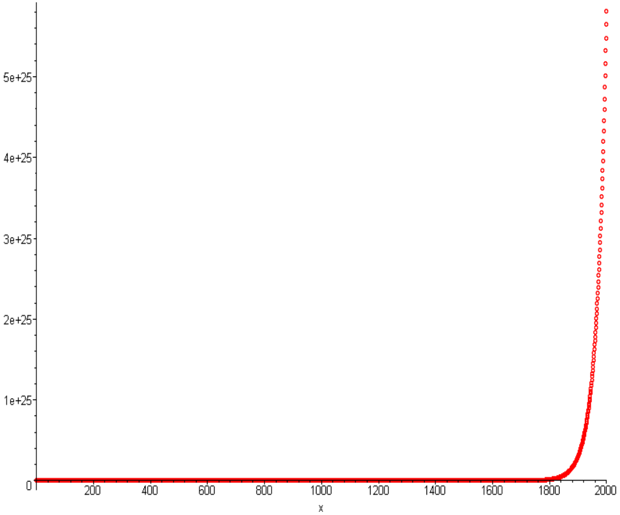
$$\boxed{\approx \$\,5.8123 \cdot 10^{25}}$$

Solution: Balance account

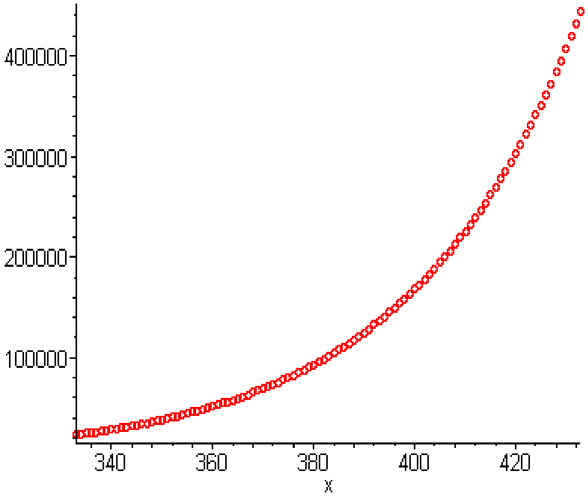
After 2000 years:

$$\$\,58,123,869,869,669,184,628,080,369.86$$

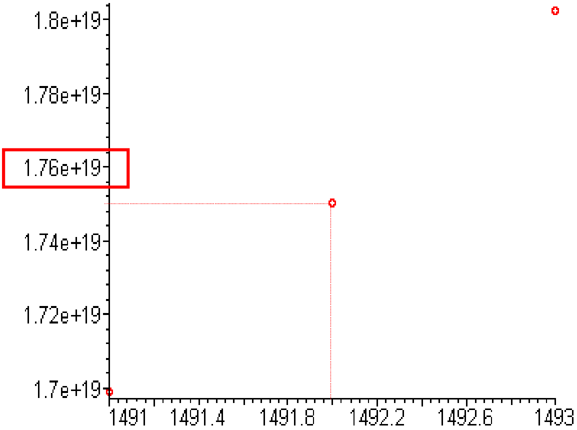
... *approx. $ 58 septillions*



Balance account over the years $n = 1, \ldots, 2000$



Balance account over the $\color{red}{\textit{years } n = 333, \ldots, 433}$



Balance account when Columbus discovered Cuba

Story: Part 2

Assume that besides the initial deposit

"relatives" of the emperor had since then deposited

$ 1.23

in that very account *at the beginning of each new year.*

Question: Part 2

What was the balance of the account on December 31, 2000 ?

Solution: Part 2

The total value of all deposits together with their compounded interest is given by (n=2000):

$$
\begin{aligned}
s_n &= \Big(\text{(deposit on 01.01.2000)} + \text{its interest}\Big) \\
&+ \Big(\text{(deposit on 01.01.1999)} + \text{its compound interest}\Big) \\
&+ \Big(\text{(deposit on 01.01.1998)} + \text{its compound interest}\Big) \\
&\vdots \\
&+ \Big(\text{(deposit when Christ was born)} + \text{its compound interest}\Big)
\end{aligned}
$$

Solution:

The balance s_n ,n=2000, after 2000 deposits and (compounded) interest payments:

$$\begin{aligned} s_n &= qa_0 + q^2 a_0 + \cdots + a_0 q^{2000} \\ &= q\, a_0 (1 + q + q^2 + \cdots + q^{1999}) \\ &= q\, a_0 \sum_{i=0}^{1999} q^i \end{aligned}$$

Problem:

$$\sum_{i=0}^{1999} q^i =: Q_k = ???, \quad k = 1999$$

Solution:

$$Q_k = 1 + q + q^2 + q^3 + \dots + q^k$$

$$-$$

$$qQ_k = q + q^2 + q^3 + \dots + q^k + q^{k+1}$$

$$\Downarrow$$

$$Q_k - qQ_k = 1 - q^{k+1} \quad \Longrightarrow \quad (1-q)Q_k = 1 - q^{k+1}$$

Solution:

$$Q_k - qQ_k = 1 - q^{k+1} \quad \Rightarrow \quad (1-q)Q_k = 1 - q^{k+1}$$

$$\Downarrow$$

$$Q_k = \frac{1-q^{k+1}}{1-q} = \frac{q^{k+1}-1}{q-1} \quad \text{if} \quad q \neq 1$$

Solution:

$$Q_k = \frac{1 - q^{k+1}}{1 - q} = \frac{q^{k+1} - 1}{q - 1} \quad if \quad q \neq 1$$

$$\Downarrow$$

for the special parameter values

$$Q_k = \sum_{i=0}^{1999} q^i = \frac{(1.03)^{2000} - 1}{(1.03) - 1} = \frac{100}{3}((1.03)^{2000} - 1)$$

Solution:

The solution of the 2^{nd} part of the problem is given by:

$$s_n = q\, a_0\, Q_{1999}, \quad n = 2000$$

and:

$$q\, a_0\, Q_{1999} = q\, a_o \sum_{i=0}^{1999} q^i \approx 1.99559 \cdot 10^{27}$$

Applications of geometric Sequences & Series:

- geometric depreciation
- compound interest calculations
- annuities
- production theory
- dynamical systems

Example: Geometric depreciation

Decreasing amounts of **depreciation** *for using of a capital good;*

the amounts are a fixed percentage of the remaining value

Examples: Geometric Depreciation

National accounting rules, f.i. the rate of depreciation satisfies:

- $p\% \leq \dfrac{200}{lifetime}\%$

and

- $p\% \leq 20\%$

Example: Geometric Depreciation

Formula:

$$R_0 \mathrel{\hat{=}} initial\ \ value\ \ (purchasing\ \ price)$$
$$A_1 = \frac{p}{100} R_0 \mathrel{\hat{=}} 1^{st}\ amount\ \ of\ \ depreciation$$

$$\Rightarrow \qquad R_1 = R_0 - A_1 = R_0 - \frac{p}{100} R_0 = \left(1 - \frac{p}{100}\right) R_0$$

Example: Geometric Depreciation

Formula:

$$A_2 = \frac{p}{100} R_1 \mathrel{\hat{=}} 2^{nd} \; amount \;\; of \;\; depreciation$$

$$\begin{aligned} R_2 = R_1 - A_2 &= \left(1 - \frac{p}{100}\right) R_0 - \frac{p}{100}\left(1 - \frac{p}{100}\right) R_0 \\ &= \left(1 - \frac{p}{100}\right)^2 R_0 = q^2 R_0, \end{aligned}$$

where $q = \left(1 - \frac{p}{100}\right)$

Example: Geometric Depreciation

Formula:

$$A_n = \frac{p}{100} R_{n-1} \hat{=} n^{th} \; amount \;\; of \;\; depreciation$$

Bookvalue at the end of the n^{th} year:

$$R_n = q^n R_0$$

Example: Geometric Depreciation

Table:

Year	bookvalue at the beginning of the year	amount of depreciation	bookvalue at the end of the year
1	460000	92000	368000
2	368000	73600	294400
3	294400	58880	235520
4	235520	47104	188416
5	188416	37683	150733

Part 2.4

Some Properties of Series

Criteria of convergence:

Condition on q so that

geometric series

$$\sum_{i=0}^{\infty} q^i$$

does converge.

Criteria of convergence:

A simple idea:

Let $q \neq 1$, then

$$\sum_{i=0}^{\infty} q^i = \lim_{n\to\infty} \left\{ \sum_{i=0}^{n} q^i \right\} = \lim_{n\to\infty} \left\{ \frac{1-q^{n+1}}{1-q} \right\}$$

$$= \frac{1}{1-q} - \boxed{\lim_{n\to\infty} q^{n+1}}$$

Criteria of convergence:

Hence,

- $\sum_{i=0}^{\infty} q^i = \dfrac{1}{1-q}$ converges if $|q| < 1$
- $\sum_{i=0}^{\infty} q^i$ diverges if $|q| \geq 1$

Criteria of convergence:

(a special case of the dominating principle)

Assumption:

$(a_i)_{i \geq 0}$ is a sequence such that:

$$|a_i| \leq q^i \text{ für } 0 < q < 1 \text{ und } i \geq i_0$$

Claim: $\sum_{i=0}^{\infty} a_i$ *is a convergent series*

Example of the criterium:

Let $a_i = \frac{i}{2^i}$; the series $\sum_{i=0}^{\infty} \frac{i}{2^i}$ *converges*

Proof: $\frac{i}{2^i} \leq \left(\frac{3}{4}\right)^i$, if $i \geq 1$, i.e. $q = \frac{3}{4}$ and $i_0 = 1$

Finally!!! ;)

The End