Lectures 1 & 2 :

Sequences and Series

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Part 1: Sequences

Table of content

1 Sequences

Part 1.1

Definition of Sequences

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A sequence is a function, which is only “fed” with natural numbers:

$$\boxed{n \mapsto a(n) =: a_n} \quad n \in \mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$$

(a possibility: n is an element of an infinite subset of $\mathbb{N}$)

Remark:

In many economic and business type applications the variable $\mathbf{n}$ *is understood to refer to a time point/index.*

Ways to describe sequences:

- by a table of values:

$$\boxed{\begin{array}{l}\text{table of values}\\ (a_1, a_2, a_3, \ldots, a_n)\end{array}}$$

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- by a transformation rule

$\Downarrow$ $\Downarrow$

explicit recursive

Examples of explicit definitions:

$$a_n = a(n) = n^2, \quad n = 0, 1, 2, 3, \ldots$$

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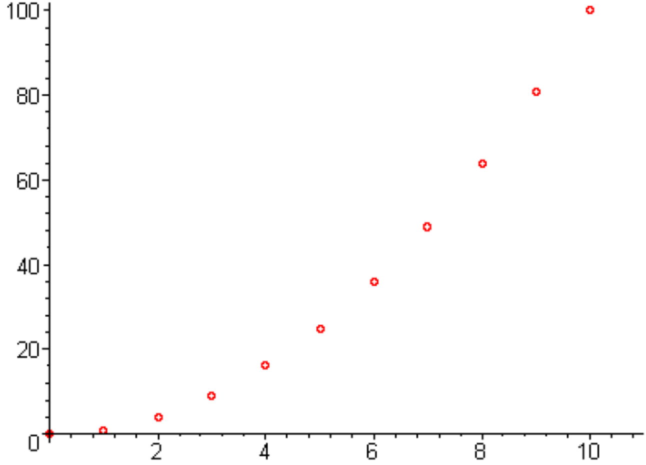

Growth of an **initial capital $\mathbf{K_0}$**, with **compound interest**; the annual interest rate is **p%**.

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Growth of an **initial capital $\mathbf{K_0}$**, with **compound interest**; the annual interest rate is **p%**.

$$\boxed{K_n = \left(1 + \frac{p}{100}\right)^n \cdot K_0, \quad n = 0, 1, 2, 3, \ldots}$$

Examples of explicit definitions:

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$$\boxed{\begin{array}{l} K_0 = 100 \\ p = 7\ \% \end{array}}$$

$$\downarrow$$

$$\boxed{\begin{array}{l} K_n = 100 \cdot 1,07^n \\ n = 0, 1, 2, ..., 24 \end{array}}$$

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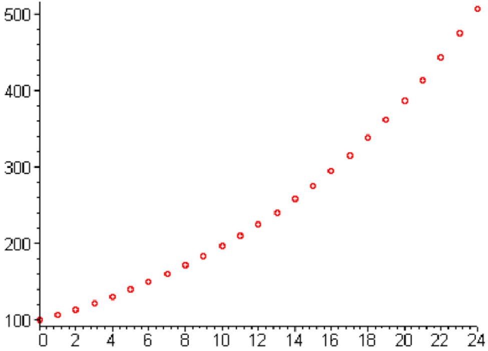

Examples of a recursive definition:

$$a_{n+1} = -0.9 a_n + 0.6, \quad a_0 = 1, \; n = 0, 1, 2, 3, \ldots$$

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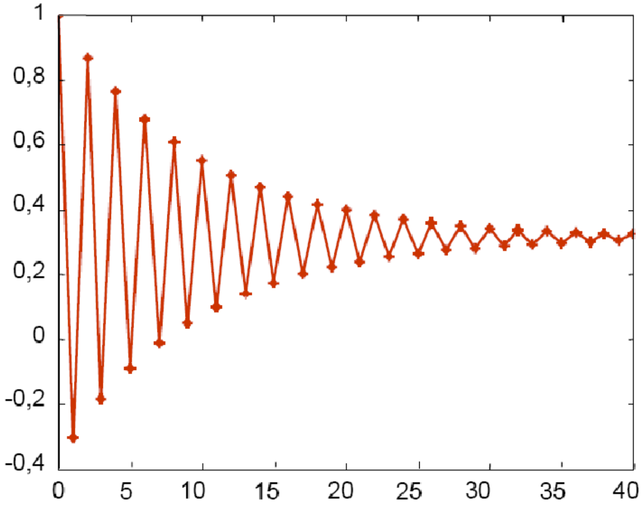

Examples of a recursive definition:

(see Newton's method, lectures 5&6):

The sequence of numbers $(a_n)_{n \geq 0}$ which is definded by the formula:

$$\boxed{a_{n+1} = \frac{a_n}{2} + \frac{1}{a_n}}$$

$a_0 \neq 0$ any fixed number

Examples of tables of values:

The gross national product of Germany 1980 - 1990:

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Year	real GDP (in Trillion DM)
1980	2.026
1981	2.026
1982	2.004
1983	2.045
1984	2.108
1985	2.149
1986	2.199
1987	2.233
1988	2.314
1989	2.411
1990	2.544

Source: Deutsche Bundesbank

Examples of tables of values:

The values of the German stock index for 1999;
d_1 = Jan.-DAX1999 is the first value of the finite sequence.

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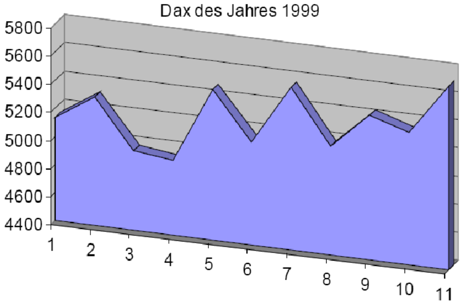
Januar	Februar	März	April	Mai	Juni	Juli	August	September	Oktober	November
5137	5313	4968	4936	5470	5132	5560	5170	5432	5343	5702

$\boxed{(d_i)_{i\geq 1}}$



Part 1.2

Arithmetic Sequences

Definition: Arithmetic sequences

An arithmetic sequence is a linear (affine) function whose domain is $\mathbb{N}_0$, $\mathbb{N}$ resp.

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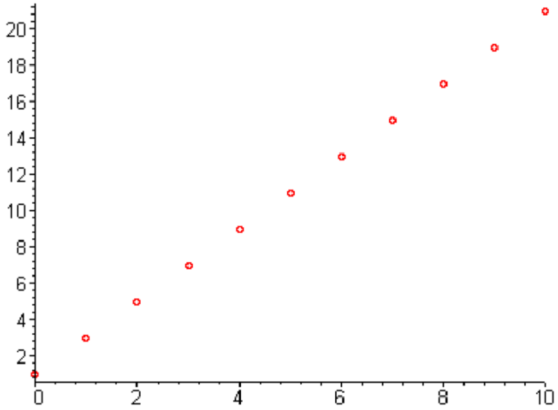
$$\boxed{a_n = a_0 + nd} \qquad n = 0, 1, 2, 3, \ldots,$$

where:

$$\begin{aligned} a_0 &: y-intercept \\ n &: variable \\ d &: slope \end{aligned}$$

Example: Arithmetic sequence



$$\boxed{a_n = 1 + 2n}$$

$$\downarrow$$

$$\begin{aligned} a_0 &= 1 \\ d &= 2 \\ n &= 0, 1, 2, \ldots \end{aligned}$$

Representation: Arithmetic sequences

Explicit form:	$a_n = a_0 + nd$

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Recursive form: (a_0 *is given*)	$a_{n+1} - a_n = d \iff a_{n+1} = a_n + d$

Part 1.3

Geometric Sequences

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Geometric sequences are exponential functions whose domain is restricted to $\mathbb{N}_0$, $\mathbb{N}$ resp.

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$$\boxed{a_n = a_0 q^n} \qquad n = 0, 1, 2, 3, \ldots$$

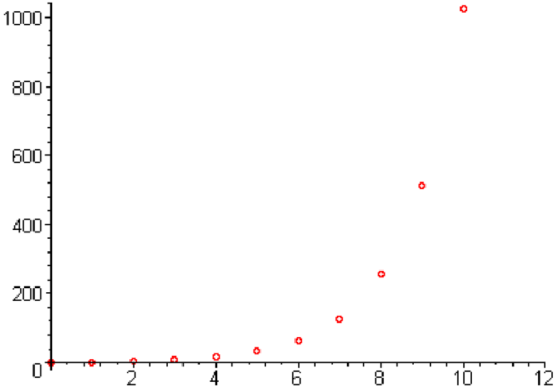
where:

$$a_0 : y-intercept$$

$$n : variable$$

$$q : \text{``}base\text{''}/multiplier$$

Example: Geometric sequence



$$\boxed{a_n = 2^n}$$

$$\downarrow$$

$$\boxed{\begin{aligned} a_0 &= 1 \\ q &= 2 \\ n &= 0, 1, 2, \ldots \end{aligned}}$$

Representation: Geometric Sequences

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Recursive form:	$a_{n+1} = q a_n$
$\frac{a_{n+1}}{a_n} = q$	$\iff$
$(a_0 \neq 0 \textit{ is given})$	$a_{n+1} - q a_n = 0$

Part 1.4

Properties of Sequences

Concept: Bounded Sequences

A sequence is called bounded if and only if ($\hat{=}$ iff):

$$\boxed{|a_n| \leq \; const., \quad n = 1, 2, 3, \ldots}$$

Example of a bounded sequence

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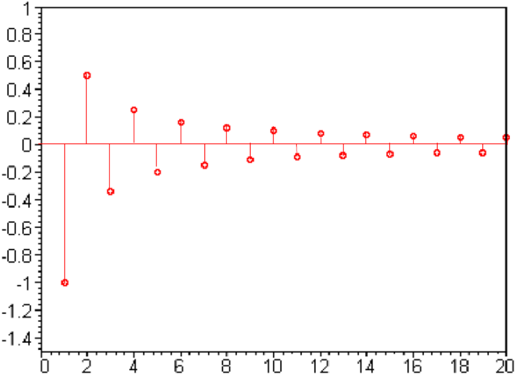

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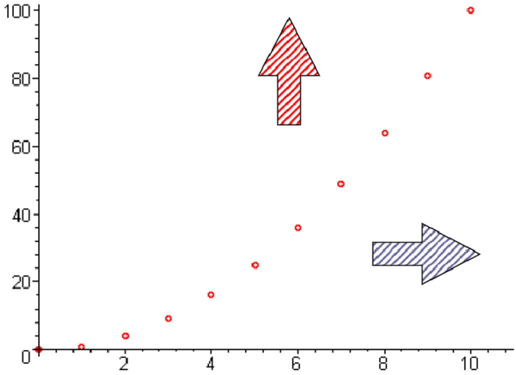

Monotone increasing / decreasing sequences:

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monotone increasing:

$$\boxed{a_n \;\leq\; a_{n+1}}$$

$\uparrow$

$$\Leftrightarrow$$

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$n = 1, 2, \ldots$

Monotone increasing / decreasing sequences:

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---

monotone decreasing:

$$\boxed{a_n \geq a_{n+1}}$$

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$$\Leftrightarrow$$

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$(n+1)^2 =$
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Example: Monotone increasing

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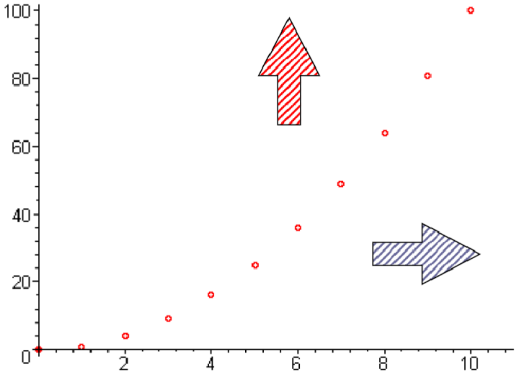

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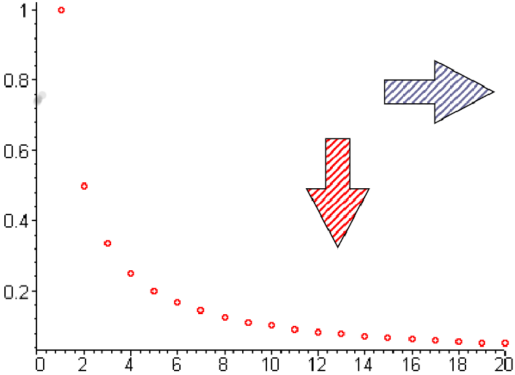

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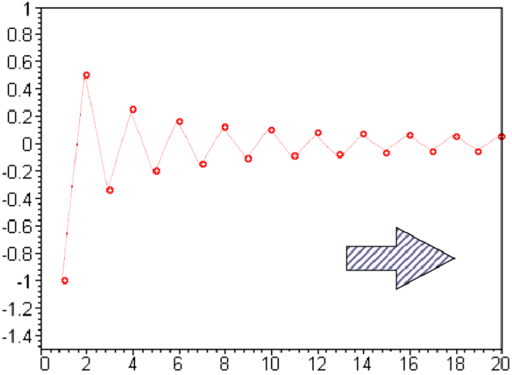

A sequence is called alternating iff:

$$a_n \neq a_{n+1}, \qquad n = 1, 2, \ldots$$

and: $a_n \cdot a_{n+1} < 0$

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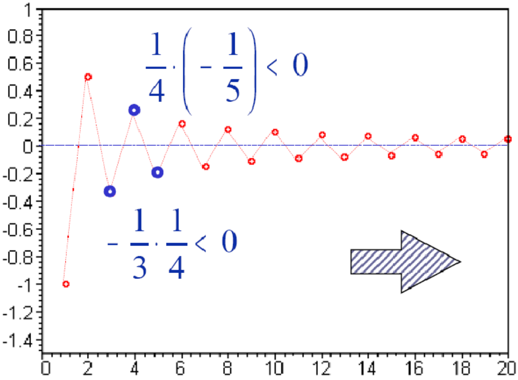

Concept: Convergent/divergent sequences

- $\boxed{\lim_{n\to\infty} a_n \quad \textit{exists}}$

$$\Leftrightarrow \boxed{(a_n)_n \quad \textit{is convergent}}$$

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  $\Leftrightarrow \boxed{(a_n)_n \quad \textit{is convergent}}$

- $\boxed{\lim_{n\to\infty} a_n \quad \textit{does not exist}}$

  $\Leftrightarrow \boxed{(a_n)_n \quad \textit{is divergent}}$

Definition: $\lim_{n \longrightarrow \infty} (a_n) = A$

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For any $\epsilon > 0$ there is an integer N_ϵ such that for all $n \geq N_\epsilon$:

$$\boxed{|a_n - A| \leq \epsilon}$$

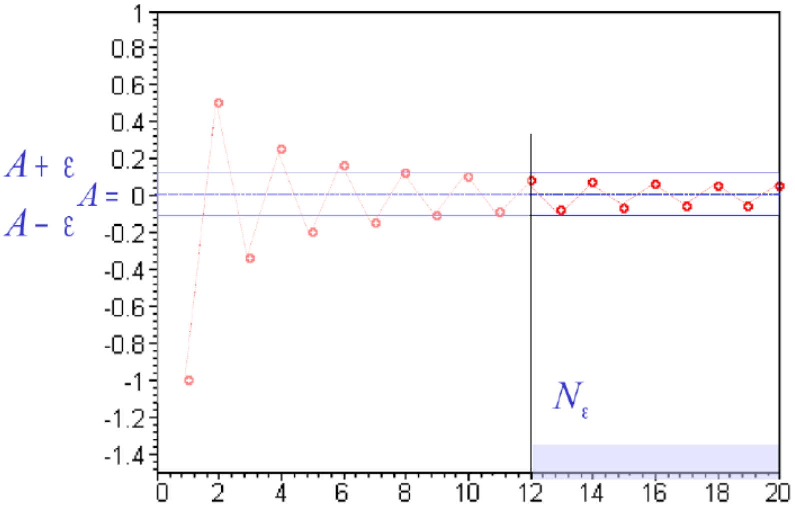

Figure 1: Illustration of the definition of the limit

Example: A convergent sequence

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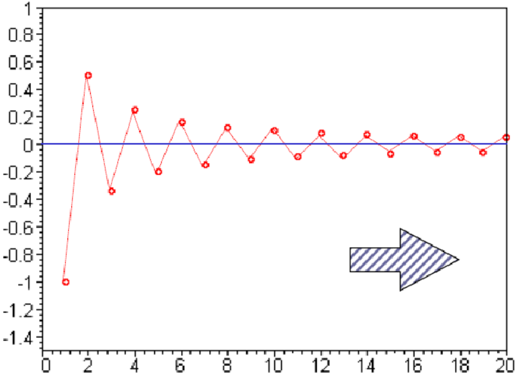

Example: A divergent sequence

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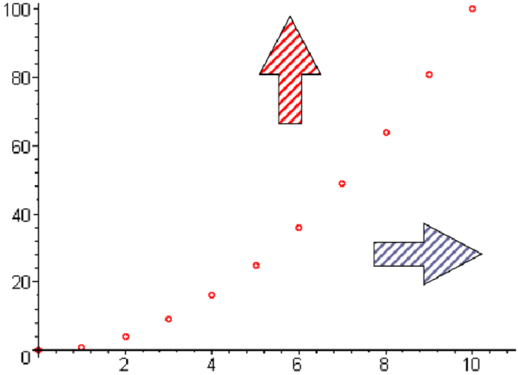

A convergence theorem

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Every bounded and monotone increasing sequence does converge

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Every bounded and monotone increasing sequence does converge

Every bounded and monotone decreasing sequence does converge

Example: bounded and monotone increasing

$a_n = 1 - \frac{1}{n}, \quad n = 1, 2, \ldots$

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limit
exists

$$\lim_{n \to \infty} (a_n) = 1$$

Example: bounded and monotone increasing

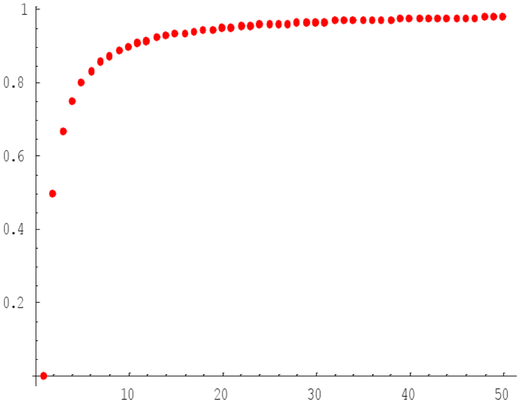
$a_n = 1 - \frac{1}{n}, \quad n = 1, 2, \ldots$

$\downarrow$

limit

exists

$\lim_{n\to\infty} (a_n) \; = \; 1$



2 Series

Part 2.1

Definition: Series

Definition: A finite series

A finite series is a finite sum of numbers.

Definition: Infinite series

An **infinite series** is the *limit of the sequence of partial sums* associated with a given sequence $(a_i)_{i \geq 1}$.

$$\boxed{\sum_{i=1}^{\infty} a_i := \lim_{n \to \infty} \left\{ \sum_{i=1}^{n} a_i \right\} = \lim_{n \to \infty} \{ s_n \}}$$

Sequence of partial sums:

Given: A sequence $(a_i)_{i \geq 1}$:

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$$a_1 \quad a_2 \qquad\qquad \longrightarrow s_2 = a_1 + a_2$$

Sequence of partial sums:

Given: A sequence $(a_i)_{i \geq 1}$:

$$
\begin{array}{llll}
a_1 & & & \longrightarrow s_1 = a_1 \\
a_1 & a_2 & & \longrightarrow s_2 = a_1 + a_2 \\
a_1 & a_2 & a_3 & \longrightarrow s_3 = a_1 + a_2 + a_3
\end{array}
$$

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\vdots & & & & & \\
a_1 & a_2 & a_3 & \cdots & a_n & \longrightarrow s_n = a_1 + a_2 + a_3 + \cdots + a_n
\end{array}
$$

Sequence of partial sums:

i.e., the sequence $(a_i)_{i\geq 1}$ generates, by the operation of summation, the partial sums $(s_n)_{n\geq 1}$:

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$$\boxed{s_n := a_1 + a_2 + \ldots + a_n = \sum_{i=1}^{n} a_i}$$

Notation:

$$\sum_{i=1}^{\infty} a_i = \lim_{n\to\infty}\{s_n\} = \lim_{n\to\infty}\left\{\sum_{i=1}^{n} a_i\right\}$$

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If this limit exists, i.e. is a finite number,

we call $\sum_{i=1}^{\infty} a_i$ a *convergent* series which is associated with the sequence $(a_i)_{i\geq 1}$

.

Notation:

$$\boxed{\sum_{i=1}^{\infty} a_i = \lim_{n\to\infty} \{s_n\} = \lim_{n\to\infty} \left\{ \sum_{i=1}^{n} a_i \right\}}$$

If this limit does not exist we call the

series to be divergent.

Example 1:

The given sequence $(a_i)_{i\geq 1}$:

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$$1 + \frac{1}{4} + \frac{1}{9} + \ldots = \sum_{n=1}^{\infty} \frac{1}{n^2} \quad \textit{converges}$$

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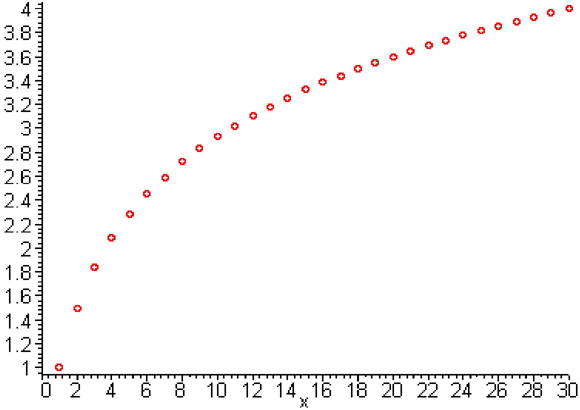
$$1 + \frac{1}{4} + \frac{1}{9} + \ldots = \sum_{n=1}^{\infty} \frac{1}{n^2} \quad \textit{converges}$$

Question 2: Find the value of $\sum_{n=1}^{\infty} a_n$?

Example 1:

Graph of the sequence of partial sums $\left(\sum_{i=1}^{n} \frac{1}{i^2}\right)_{n=1\ldots30}$



Example 2:

The given sequence $(h_n)_{n\geq 1}$:

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harmonic series

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harmonic series

$\longrightarrow$ *diverges*

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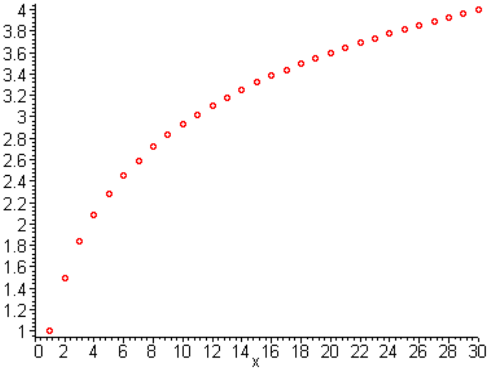

Part 2.2

Arithmetic Series

Defintion: Arithmetic Series

Given: An *arithmetic sequence* $(a_i)_i$:

$$\boxed{\sum_{i=1}^{\infty} a_i = \lim_{n\to\infty}\left(\sum_{i=1}^{n} a_i\right)}$$

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$$\downarrow$$

arithmetic series

→ Example 1:

Gauß as a "schoolboy"

("little" Gauß)

cf. the sum of all natural numbers from 1 to 100

Carl Friedrich Gauß (1777-1855)

Carl Friedrich Gauß, *was born in Brunswick; he was a working class kid.*

He is considered by many the greatest mathematican of his time (of all times ???).

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His contributions include results in mathematics, astronomy, statistic, physics, etc.

$\boxed{1}$

Find the *explicit representation* of the **recursively defined** arithmetic sequence $(a_n)_n$, where
$a_{n+1} - a_n = 1, \quad a_0 = 0.$

Exercise:

$\boxed{1}$

Find the *explicit representation* of the **recursively defined** arithmetic sequence $(a_n)_n$, where $a_{n+1} - a_n = 1, \quad a_0 = 0.$

$\boxed{2}$

Find the *value (a formula)* of the n-th component of the corresponding sequence of partial sums.

$\boxed{3}$

Decide whether or not

$\sum_{i=0}^{\infty} a_i$ *converges*.

Solution: Question 1

$$a_0 = 0$$

$$a_1 - a_0 = 1 \quad \Longrightarrow \quad a_1 = 1$$

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\begin{aligned}
a_0 &= 0 \\
a_1 - a_0 &= 1 \quad \Longrightarrow \quad a_1 = 1 \\
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\end{aligned}
$$

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&\vdots & & & &\vdots \\
a_n - a_{n-1} &= 1 & \Longrightarrow & & a_n &= \cdots = n
\end{aligned}
$$

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&\quad\vdots \qquad\qquad\qquad\qquad\qquad \vdots \\
&a_n - a_{n-1} = 1 \quad \Longrightarrow \quad a_n = \cdots = n
\end{aligned}
$$

i.e.: $(a_n)_{n \geq 0} = (0, 1, 2, 3, 4, 5, \ldots)$

Solution: Question 2

$$s_n = 1 + 2 + 3 + 4 + 5 + ... + n$$

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$$\Downarrow$$

$$\boxed{2s_n = \underbrace{(n+1) + (n+1) + ... + (n+1)}_{n-times}}$$

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$$\Downarrow$$

$$\boxed{2s_n = \underbrace{(n+1) + (n+1) + \ldots + (n+1)}_{n-times}} \Longrightarrow \boxed{s_n = \frac{1}{2}n(n+1)}$$

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The *limit* of this sequence of partial sums *does not exist*.

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The components/elements of the *sequence of partial sums* do *not stabilize (around a finite value)*. The sequence $(s_n)_n$ is *unbounded*.

$$\boxed{\sum_{i=0}^{\infty} a_i = \lim_{n\to\infty}\left(\sum_{i=0}^{n} a_i\right) = \lim_{n\to\infty}\left\{\frac{1}{2}n(n+1)\right\} = \infty}$$

Applications
of
Arithmetic Sequences & Series:

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- Linear depreciation of capital goods

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- Simple interest calculations

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Example: linear depreciation

$\boxed{R_0}$ *cost/value of the capital good at time $n = 0$ (brand-new)*

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$\boxed{R_n}$ *value (bookvalue) at the end of year n*

$\boxed{r}$ *constant rate of depreciation*

Example: linear depreciation

Bookvalue after the *1st year*:

$$\boxed{R_1 = R_0 - r}$$

Example: linear depreciation

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Bookvalue after n^{th} *year*:

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Find r so that the bookvalue after *5 years* is zero, i.e. satisfy the requirement $R_5 = 0$.

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Idea: Choose

$$\boxed{r = \frac{R_0}{5} \quad \hat{=} \quad \left(\frac{\textit{purchasing cost}}{\textit{useful lifetime}}\right)}$$

Recall: Bookvalue after n *years*:

$$\boxed{R_n = R_0 - nr}$$

Part 2.3

Geometric Series

Defintion: Geometric Series

Given: A geometric series $(a_i)_{i \geq 0}$, i.e. $a_i = a_0\, q^i$, $q \in \mathbb{R}$:

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$$\downarrow$$

geometric series

Example 2

“Big” Gauß

cf. compound interest and annuities

Imagine that at the time when Christ was born the roman emperor Augustus had been able to invest

$$\boxed{\$\,1.23}$$

in a bank account and had been guaranteed an annual interest rate of *3%*; assume interest payments to be compounded every year.

Question: Part 1

What was the balance account at the end of the first year of the new millenium, i.e. after 2000 years of compounded interest payments ?

Solution: Balance account

Initial amount: $a_0 = 1.23$

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$\vdots$

After n years: $\boxed{a_n = q a_{n-1} = a_0 q^n}$

and $n = 2000$

Solution: Balance account

After 2000 years:

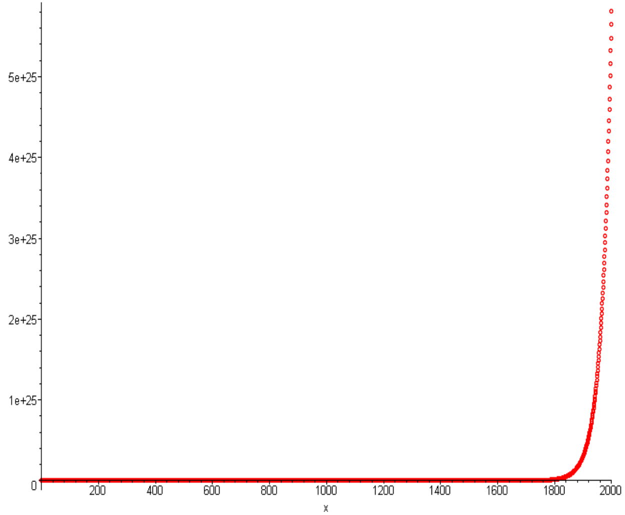
$$\boxed{\approx \$\,5.8123 \cdot 10^{25}}$$

Solution: Balance account

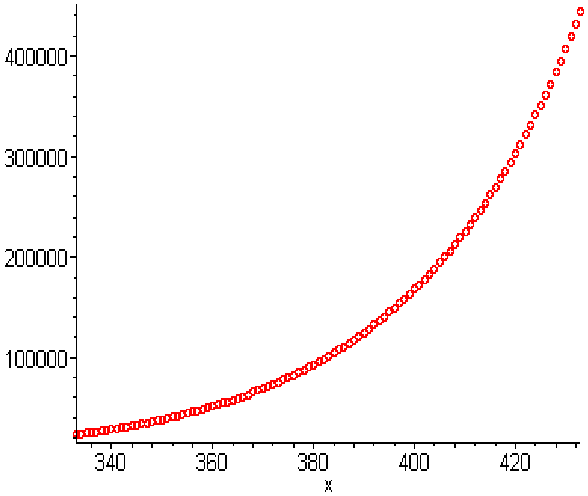
After 2000 years:

$$\boxed{\$\,58,123,869,869,669,184,628,080,369.86}$$

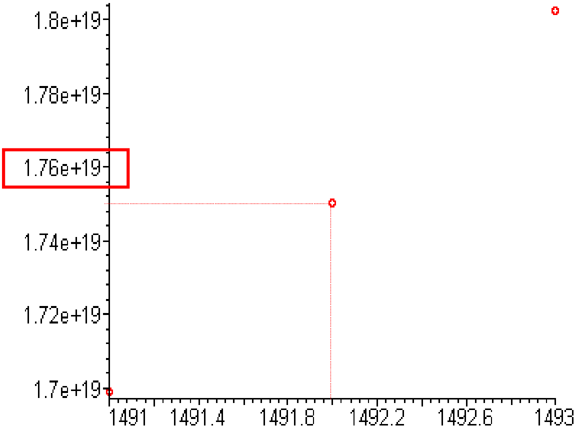
... *approx. \$ 58 septillions*



Balance account over the years $n = 1, \ldots, 2000$



Balance account over the *years* $n = 333, \ldots, 433$



Balance account when Columbus discovered Cuba

Assume that besides the initial deposit

"relatives" of the emperor had since then deposited

$ 1.23

in that very account *at the beginning of each new year.*

What was the balance of the account

on December 31, 2000 ?

Solution: Part 2

The total value of all deposits together with their compounded interest is given by (n=2000):

$$
\begin{aligned}
s_n &= \Big((\text{deposit on } 01.01.2000) + \text{its interest}\Big) \\
&+ \Big((\text{deposit on } 01.01.1999) + \text{its compound interest}\Big) \\
&+ \Big((\text{deposit on } 01.01.1998) + \text{its compound interest}\Big) \\
&\vdots \\
&+ \Big((\text{deposit when Christ was born}) + \text{its compound interest}\Big)
\end{aligned}
$$

Solution:

The balance s_n *,n=2000, after 2000 deposits* and (compounded) interest payments:

$$
\begin{aligned}
s_n &= qa_0 + q^2 a_0 + \cdots + a_0 q^{2000} \\
&= q\, a_0 (1 + q + q^2 + \cdots + q^{1999}) \\
&= q\, a_0 \sum_{i=0}^{1999} q^i
\end{aligned}
$$

Problem:

$$\boxed{\sum_{i=0}^{1999} q^i =: Q_k = ???, \qquad k = 1999}$$

Solution:

$$Q_k = 1 + q + q^2 + q^3 + \ldots + q^k$$

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$$qQ_k = q + q^2 + q^3 + ... + q^k + q^{k+1}$$

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$$\boxed{Q_k - qQ_k = 1 - q^{k+1}}$$

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$$\boxed{Q_k - qQ_k = 1 - q^{k+1}} \Longrightarrow \boxed{(1-q)Q_k = 1 - q^{k+1}}$$

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$$Q_k - qQ_k = 1 - q^{k+1} \quad \Rightarrow \quad (1-q)Q_k = 1 - q^{k+1}$$

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$$\Downarrow$$

$$\boxed{Q_k = \frac{1 - q^{k+1}}{1-q} = \frac{q^{k+1} - 1}{q - 1} \quad \textit{if} \quad q \neq 1}$$

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$$\Downarrow$$

for the special parameter values

$$\boxed{Q_k = \sum_{i=0}^{1999} q^i = \frac{(1.03)^{2000}-1}{(1.03)-1} = \frac{100}{3}((1.03)^{2000}-1)}$$

Solution:

The solution of the 2^{nd} part of the problem is given by:

$$\boxed{s_n = q\, a_0\, Q_{1999}, \quad n = 2000}$$

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The solution of the 2nd part of the problem is given by:

$$\boxed{s_n = q\, a_0\, Q_{1999}, \quad n = 2000}$$

and:

$$q\, a_0\, Q_{1999} = q\, a_o \sum_{i=0}^{1999} q^i \approx 1.99559 \cdot 10^{27}$$

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Example: Geometric depreciation

Decreasing amounts of **depreciation** *for using of a capital good;*

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Examples: Geometric Depreciation

National accounting rules, f.i. the rate of depreciation satisfies:

- $p\% \leq \dfrac{200}{lifetime}\%$

and

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and

- $p\% \leq 20\%$

Example: Geometric Depreciation

Formula:

$$\boxed{\begin{aligned} &R_0 \mathrel{\hat{=}} \textit{initial value (purchasing price)} \\ &A_1 = \frac{p}{100} R_0 \mathrel{\hat{=}} 1^{st}\ \textit{amount of depreciation} \end{aligned}}$$

Example: Geometric Depreciation

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$$A_1 = \frac{p}{100} R_0 \mathrel{\hat{=}} 1^{st}\ amount\ \ of\ \ depreciation$$

$$\Rightarrow \qquad R_1 = R_0 - A_1 = R_0 - \frac{p}{100} R_0 = \left(1 - \frac{p}{100}\right) R_0$$

Example: Geometric Depreciation

Formula:

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$$\boxed{A_2 = \frac{p}{100} R_1 \mathrel{\hat{=}} 2^{nd} \; amount \;\; of \;\; depreciation}$$

$$R_2 \;=\; R_1 - A_2 \;=\; \left(1 - \frac{p}{100}\right) R_0 - \frac{p}{100}\left(1 - \frac{p}{100}\right) R_0$$

Example: Geometric Depreciation

Formula:

$$\boxed{A_2 = \frac{p}{100} R_1 \mathrel{\hat{=}} 2^{nd} \; amount \;\; of \;\; depreciation}$$

$$\begin{aligned} R_2 \;=\; R_1 - A_2 \;&=\; \left(1 - \frac{p}{100}\right) R_0 - \frac{p}{100}\left(1 - \frac{p}{100}\right) R_0 \\ &=\; \left(1 - \frac{p}{100}\right)^2 R_0 \;=\; q^2 R_0, \end{aligned}$$

where $q = \left(1 - \frac{p}{100}\right)$

Example: Geometric Depreciation

Formula:

$$\boxed{A_n = \frac{p}{100} R_{n-1} \mathrel{\hat{=}} n^{th}\ amount\ \ of\ \ depreciation}$$

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Bookvalue at the end of the n^{th} year:

$$R_n = q^n R_0$$

Example: Geometric Depreciation

Table:

Year	bookvalue at the beginning of the year	amount of depreciation	bookvalue at the end of the year
1	460000	92000	368000
2	368000	73600	294400
3	294400	58880	235520
4	235520	47104	188416
5	188416	37683	150733

Part 2.4

Some Properties of Series

Criteria of convergence:

Criteria of convergence:

Condition on q so that

geometric series

$$\sum_{i=0}^{\infty} q^i$$

does converge.

Criteria of convergence:

A simple idea:

Let $q \neq 1$, then

$$\sum_{i=0}^{\infty} q^i = \lim_{n\to\infty} \left\{ \sum_{i=0}^{n} q^i \right\} = \lim_{n\to\infty} \left\{ \frac{1-q^{n+1}}{1-q} \right\}$$

Criteria of convergence:

A simple idea:
Let $q \neq 1$, then

$$\sum_{i=0}^{\infty} q^i = \lim_{n\to\infty} \left\{ \sum_{i=0}^{n} q^i \right\} = \lim_{n\to\infty} \left\{ \frac{1-q^{n+1}}{1-q} \right\}$$

$$= \frac{1}{1-q} - \boxed{\lim_{n\to\infty} q^{n+1}}$$

Criteria of convergence:

Hence,

- $\sum_{i=0}^{\infty} q^i = \dfrac{1}{1-q}$ converges if $|q| < 1$
- $\sum_{i=0}^{\infty} q^i$ diverges if $|q| \geq 1$

Criteria of convergence:

(a special case of the dominating principle)

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Assumption:

$(a_i)_{i\geq 0}$ is a sequence such that:

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Criteria of convergence:

(a special case of the dominating principle)

Assumption:

$(a_i)_{i\geq 0}$ is a sequence such that:

$$|a_i| \leq q^i \text{ für } 0 < q < 1 \text{ und } i \geq i_0$$

Claim: $\sum_{i=0}^{\infty} a_i$ *is a convergent series*

Example of the criterium:

Let $a_i = \frac{i}{2^i}$; the series $\sum_{i=0}^{\infty} \frac{i}{2^i}$ *converges*

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Proof: $\frac{i}{2^i} \leq \left(\frac{3}{4}\right)^i$, if $i \geq 1$, i.e. $q = \frac{3}{4}$ and $i_0 = 1$

Finally!!! ;)

The End