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Lectures 5 & 6 :
Difference Equations

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CONTENT

Part 1: Introduction

Part 2: First-Order Difference Equations

Part 3: First-Order Linear Difference Equations

1

Introduction

Part 1.1

An Example

Example 1 (Part 1)

Dagobert-

Example

cf. compound interest

Starting Point:

Given:

K_0 initial capital (in Euro)

p interest rate (in %)

r interest factor $r = \frac{p}{100}$

Objective:

Find ....

- **1.** The amount of capital after *1 year*.
- **2.** The amount of capital after *2 years*.

  ⋮
- **n.** The amount of capital after *n years*.

Solution:

After **one** year the amount of capital is:

$$\boxed{K_1 = K_0 + r \cdot K_0 = (1+r)K_0}$$

How much capital do we have after 2 years?

Solution:

After **one** year the amount of capital is:

$$K_1 = K_0 + r \cdot K_0 = (1+r)K_0$$

After **two** years the amount of capital is:

$$K_2 = K_1 + r \cdot K_1 = (1+r)K_1$$
$$= (1+r)(1+r)K_0$$

$$\boxed{K_2 = (1+r)^2 K_0}$$

Solution:

After **one** year the amount of capital is:

$$K_1 = K_0 + r \cdot K_0 = (1+r)K_0$$

After **two** years the amount of capital is:

$$\boxed{K_2 = (1+r)^2 K_0}$$

Solution:

After **n** years the amount of capital is:

$$K_1 = (1+r)^1 K_0$$

$$K_2 = (1+r)^2 K_0$$

⋮

$$\boxed{K_n = (1+r)^n K_0}$$

Observation:

The solution formula $K_n = (1+r)^n K_0$ can be rewritten in the following way:

recursion formula	*special difference equation*
$\boxed{K_n = (1+r)K_{n-1}}$	$\boxed{K_n - K_{n-1} = rK_{n-1}}$
K_0 is given, $n \geq 1$	K_0 is given, $n \geq 1$

Part 1.2

Difference Equations

Illustration:

A ***difference equation*** is a special system of equations, with

- (countably) **infinite many** equations,
- (countably) **infinite many** unknowns.

Hint:

The ***solution*** of a ***difference equation*** is a ***sequence*** (countably **infinite many** numbers).

How do we recognize a difference equation?

Definition: Difference Equation

An equation, that relates for any $n \geq k$ the ***n**^th^* **term** of a sequence to the (up to *k)* **preceding terms**, is called a (nonlinear) difference equation of *order k*.

Explicit form: $x_n = F(n,\ x_{n-1},\ x_{n-2},\ \dots,\ x_{n-k}),\quad n \geq k$

Implicit form: $0 = G(n,\ x_n,\ x_{n-1}, \dots, x_{n-k})$

2

First-Order Difference Equations

Part 2.1

A Model for the „Hog Cycle“

Difference Equations (Prof. Dr. K. Helmes)

Example 2

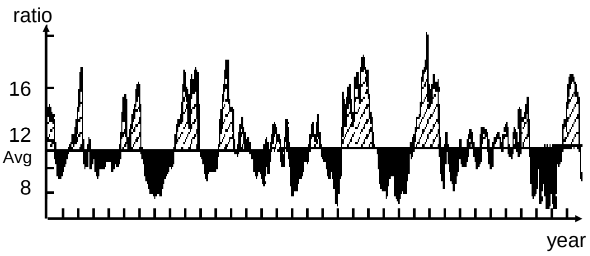
„Hog Cycle“

(Example)

cf. Microeconomic Theory

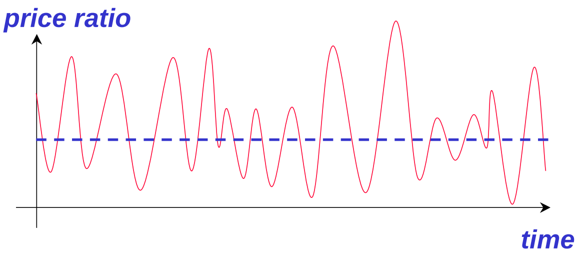
Starting Point:

Given: Hog-corn price ratio in Chicago in the period 1901-1935:



Starting Point:

Stylized:



Starting Point:

Find:

- A (first) model, which „explains“ / describes the cyclical fluctuations of the prices (ratio of prices).

Model (Part 1): *Supply and Demand*

The **suppply** of hogs:

- $s(t)$ in units at time t
- $s(t+1)$ in units at time $t+1$

The **demand** of hogs:

- $d(t)$ in units at time t
- $d(t+1)$ in units at time $t+1$

Model (Part 2): Supply and Price

Assumption:

The supply at time $(t+1)$ depends on the hog price $p(t)$ at time t.

Model (Part 2): Nature of the dependance

Assumption:

The *supply function* is linear:

$$s(t+1) = \alpha p(t) - \beta$$

i.e. it is determined by α and β, and $p(t)$.

$\alpha, \beta > 0, \quad s(0)$ is given

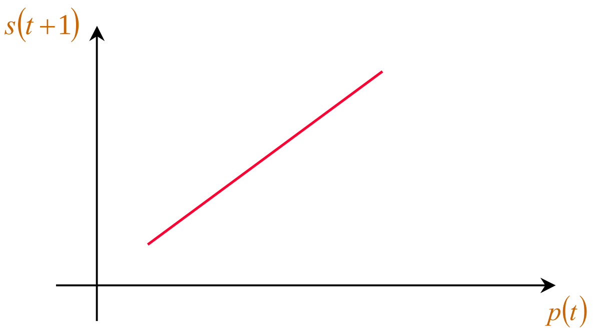


Figure 1: *Graphical representation of the supply function*

Model (Part 3): Demand and Price

Assumption:

For the demand we assume: If the hog price increases, the demand will decrease, thus:

$$d(t) = -\gamma p(t) + \delta$$

$\gamma, \delta > 0$ parameter

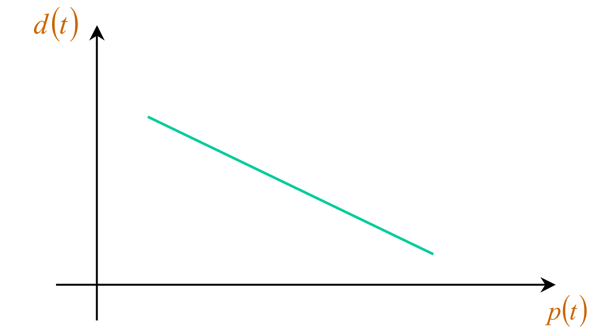


Figure 2: *Graphical representation of the demand function*

Model (Part 4): Equilibrium

Postulate: Supply equals demand at any time:

$$s(t+1) = d(t+1)$$

for all $t \geq 0$

Model (Part 4): Equilibrium

The *equilibrium relation* yields a defining equation for the price function:

$$\boxed{\underbrace{\alpha p(t)-\beta}_{s(t+1)} = \underbrace{-\gamma p(t+1)+\delta}_{d(t+1)}}$$

Solution (Part 4): Equilibrium

Thus we obtain the following difference equation:

$$\boxed{p(t+1) = -\frac{\alpha}{\gamma}\,p(t) + \frac{\delta+\beta}{\gamma}}$$

$t = 0, 1, 2, \ldots,\ p(0) = p_0\,,\ p_0$ is given

Model (Part 4): Equilibrium

$$p(t+1) = -\frac{\alpha}{\gamma}\,p(t) + \frac{\delta+\beta}{\gamma}$$

This difference equation is:

- *first-order*
- *linear*
- *inhomogeneous*

Model (Part 5): Analysis

$$p(t+1) = \underbrace{\boxed{-\frac{\alpha}{\gamma}}}_{=\,m\,<\,0}\,p(t) + \underbrace{\boxed{\frac{\delta+\beta}{\gamma}}}_{=\,\xi}$$

solution formula: $t = 0,1,2,\ldots;\quad p(0) = p_0$

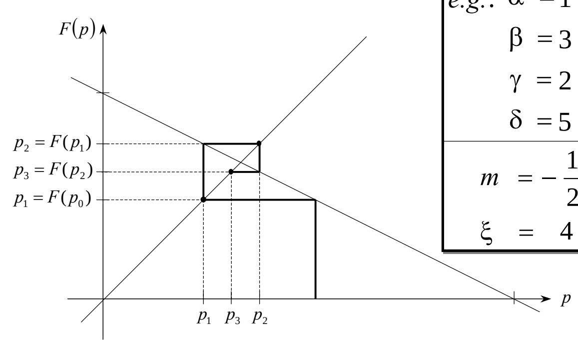
$$\boxed{p(t) = m^t\left(p_0 - \frac{\xi}{1-m}\right) + \frac{\xi}{1-m}}$$

$m \neq 1$

Deriving the Solution Formula:

$$\begin{aligned} p_{t+1} &= mp_t + \xi \\ &= m\,(mp_{t-1} + \xi) + \xi \\ &= m^3 p_{t-2} + m^2\xi + \mathrm{m}\xi + \xi \\ &\ \vdots \\ &= m^{t+1} p_0 + \xi\,\underbrace{(m^t + m^{t-1} + \ldots + 1)}_{\frac{1-m^{t+1}}{1-m}} \end{aligned}$$

$$\boxed{p_{t+1} = m^{t+1}\left(p_0 - \frac{\xi}{1-m}\right) + \frac{\xi}{1-\mathrm{m}}}$$



e.g.:

α	$=$	1
β	$=$	3
γ	$=$	2
δ	$=$	5
m	$=$	$-\frac{1}{2}$
ξ	$=$	4

Figure 2: *Iteration rule* $F(p) = -(1/2)p + 4$

Model (Part 5): Analysis

Results:

$\boxed{-1 < m < 0}$ The equation / solution is ***stable***.

„<u>stable</u>": The values $p(t)$ converge to the equilibrium state when $t \to \infty$.

$\boxed{m \leq -1}$ The equation / solution is ***unstable***.

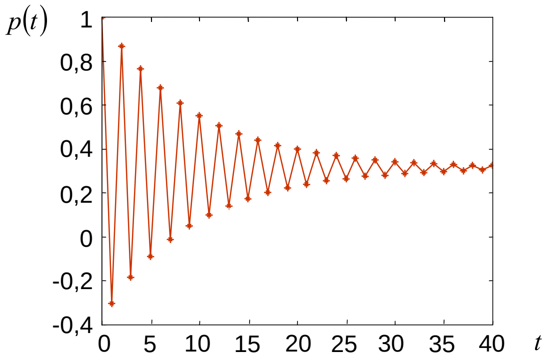


Figure 3: Price development for: $m = -0{,}9; \quad p_0 = 1; \quad \xi = 0{,}6$

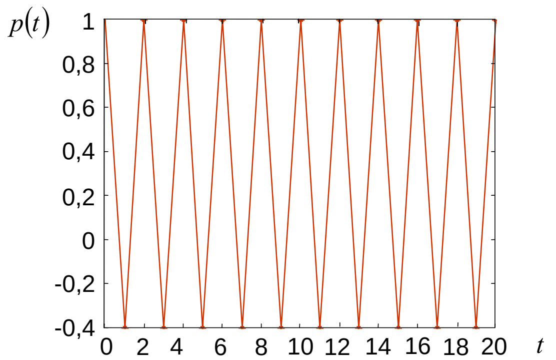


Figure 4: Price development for: $m = -1; \quad p_0 = 1; \quad \xi = 0{,}6$

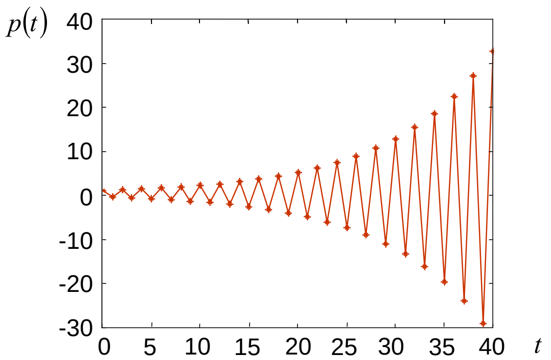


Figure 5: Price development for: $m = -1{,}1; \quad p_0 = 1; \quad \xi = 0{,}6$

Summary:

- The given difference equation has a unique solution; it can be solved explicitly.
- The price is the sum of a constant and a power function.
- The term $\left(-\frac{\alpha}{\gamma}\right)^t = (-1)^t \left(\frac{\alpha}{\gamma}\right)^t$ has an alternating sign, $\alpha/\gamma > 0$

CONCLUSION:

We can model and analyze *dynamic processes* with *difference equations*.

Part 2.2

Definitions und Concepts for First-Order Difference Equations

Multivariable Calculus: The Implicit Function Theorem (Prof. Dr. K. Helmes)

Definition:

A (general) first-order nonlinear difference equation has the form :

$$x_{t+1} = F(t, x_t) \qquad t = 0, 1, 2, \ldots$$

(**F** is defined for all values of the variables.)

Important Questions:

- Does at least one solution exist?
- Is there a unique solution?
- How many solutions do exist?
- How does the solution change, if „parameters“ of the system of equations are changed (*sensitivity analysis*)?

Important Questions:

- Do explicit *formulae* for the solution exist?
- How do we *calculate* the solution?
- Does the system of equations has a special *structure* ?

  e.g.: a) *linear* or *nonlinear*,

  b) *one-* or *multidimensional* ?

Remark:

If the ***initial value*** of the solution (sequence) of a difference equation is given, i.e.

$x_0 =$ ”fixed number”,

then we call our problem an

” initial value problem ”

related to a first-order difference equation.

Remark:

- The *initial value problem* of a first-order difference equation has a *unique* solution.
- If x_0 is an arbitrary fixed number, then there exists a uniquely determined function/sequence $x_t,\, t \geq 0$, that is a solution of the equation and has the given value x_0 for $t = 0$.

Remark:

➡ In general there exists for each choice of x_0 a ***different (corresponding) unique* solution sequence**.

Definition: Invariant Points

For *time homogeneous* ***nonlinear*** difference equations $x_{t+1} = F(x_t)$ we call points which satisfy the equation

$$\boxed{x = F(x)}$$

invariant points.

F "right-hand side".

Invariant Points:

For *time homogeneous* ***linear*** difference equations $x_{t+1} = ax_t + \xi, \quad (a \neq 1)$, an *invariant point* is characterized by:

$$\boxed{\underbrace{x = ax + \xi}_{F(x)} \Leftrightarrow x = \frac{\xi}{1-a}}$$

Invariant Points:

ATTENTION:

If the solution of a difference equation "starts" at an **invariant point**, it stays there, i.e. if x_0 is an *invariant point* then

$$\Rightarrow \quad x_0 = x_1 = x_2 = \ldots$$

Chaotic Systems (an Example)

$$x_{t+1} = \frac{1}{\alpha} x_t - \frac{1}{\alpha} x_t^2$$

x_0 is given, $\frac{1}{4} < \alpha$

Chaotic Systems (an Example)

Iteration rule:

$$F(t, x) = \frac{x}{\alpha}(1-x)$$

x_0 is given, $\frac{1}{4} < \alpha$

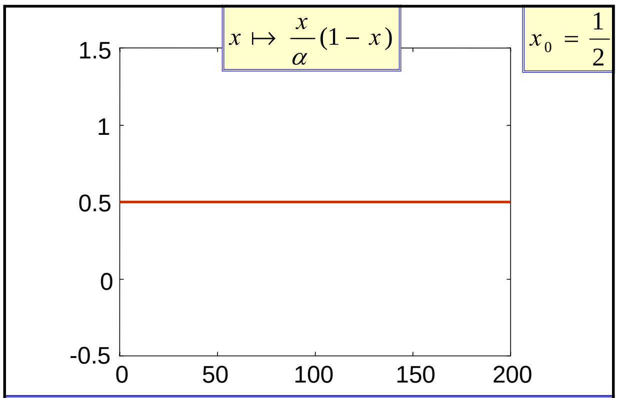


Figure 6: *Solution* (x_t) *for* $\alpha = 0{,}5$

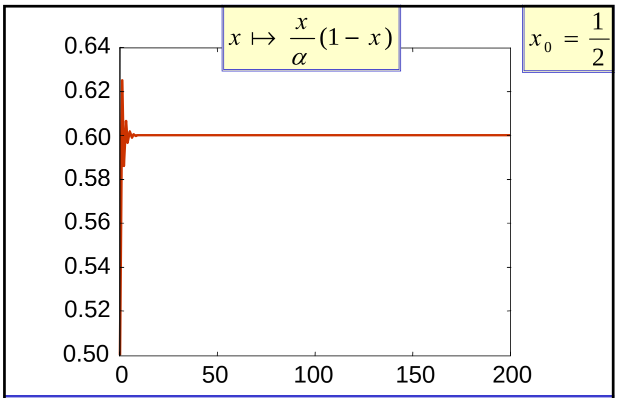


Figure 7: *Solution* (x_t) *for* $\alpha = 0{,}4$

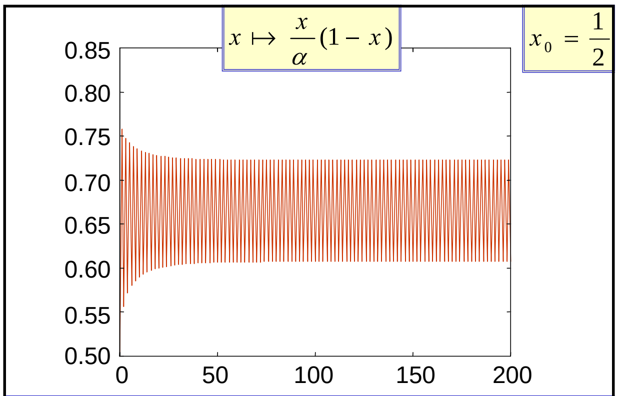


Figure 8 : *Solution* (x_t) *for* $\alpha = 0{,}33$

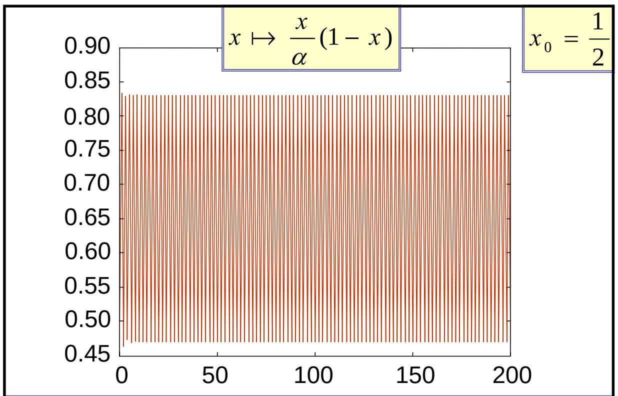


Figure 9: *Solution* (x_t) *for* $\alpha = 0{,}3$

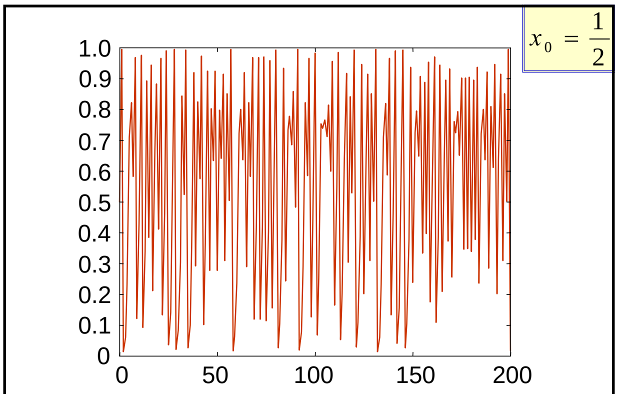


Figure 10: *Solution* (x_t) *for* $\alpha = 0{,}251$

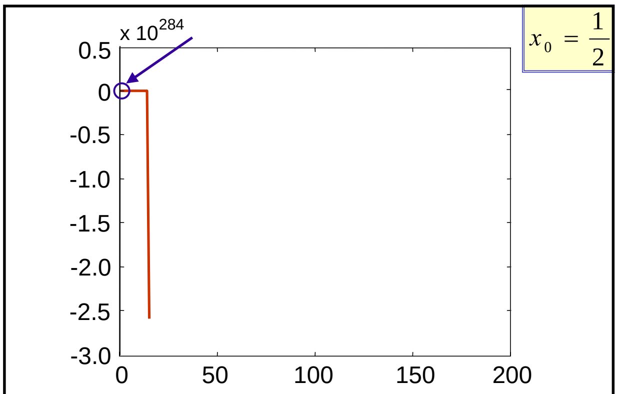


Figure 11a: *Solution* (x_t) *for* $\alpha = 0{,}2499$

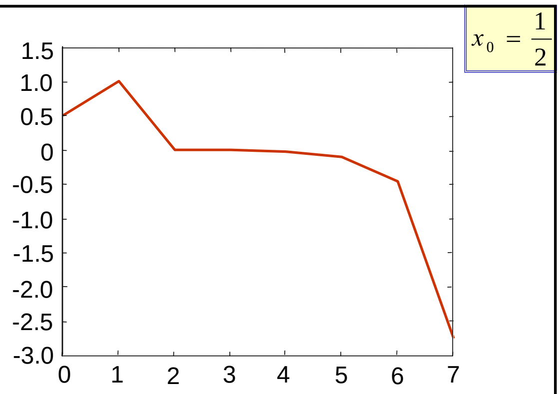


Figure 11b: *Solution* (x_t) *for* $\alpha = 0{,}2499$

Example 3

„Newton´s-Method "

(Example)

Starting Point:

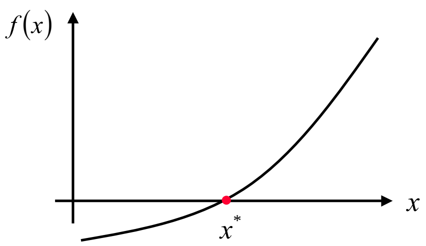
Finding the **roots** of a nonlinear function „analytically" is rarely possible.

Therefore we have to use ***numerical methods***.

Starting Point:

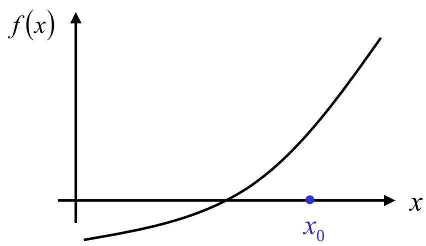
For differentiable functions a numerical root-finding algorithm exists. It goes back to ***Isaac Newton*** (1643 – 1727).

Goal:



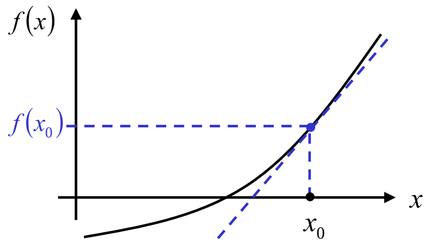
Find an x^* with $f(x^*) = 0$.

Idea:



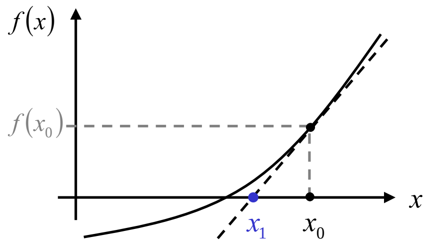
Choose an initial *value* x_0 reasonably „*close* to" the true root.

Idea:



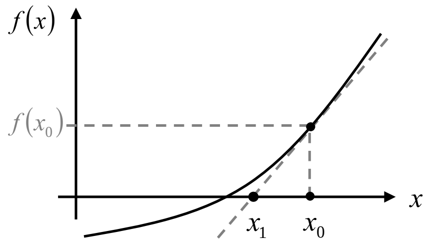
➡ *Determine* the tangent to the graph of the function f at the point $(x_0, f(x_0))$.

Idea:



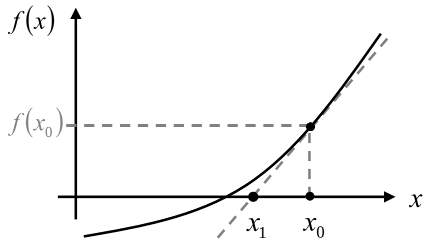
➡ *Determine* the *intersection* of the tangent with the x-axis.

Idea:



➡ Take this intersection x_1 as a new approximation to the function's root.

Idea:



➡ *Repeat* this operation „many times“.

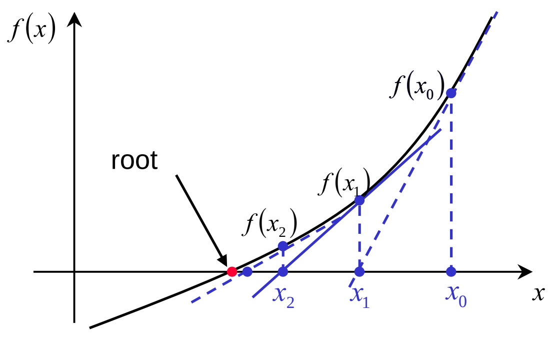


Figure 12: *Schematic representation of Newton's – Method*

Solution:

Let x_1 denote the root of the tangent and $f'(x_0)$ the slope of f at x_0. Then, if $f'(x_0) \neq 0$, the following formula holds:

$$\boxed{\frac{f(x_0)-0}{x_0-x_1} = f'(x_0)}$$

$$f(x_0) = f'(x_0)(x_0 - x_1) \Leftrightarrow \frac{f(x_0)}{f'(x_0)} = (x_0 - x_1)$$

Solution:

Let x_1 denote the root of the tangent and $f'(x_0)$ the slope of f at x_0. Then, if $f'(x_0) \neq 0$, the following formula holds:

$$\frac{f(x_0)-0}{x_0-x_1} = f'(x_0)$$

$$\boxed{x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}} \quad \Leftrightarrow \frac{f(x_0)}{f'(x_0)} = (x_0 - x_1)$$

Solution:

By the same idea we compute $x_2, x_3, \ldots$ as (difference equation):

$$\boxed{x_{t+1} = x_t - \frac{f(x_t)}{f'(x_t)}}$$

$t = 0, 1, 2, \ldots, \quad x_0 =$ fixed initial value

Solution:

This is a (nonlinear) *first-order difference equation*, and:

$$F(x) = x - \frac{f(x)}{f'(x)}$$

Numerical Example:

Consider the problem of finding the root of

$$f(x) = x^2 - 3 \ .$$

The difference equation according to Newton´s Method is:

$$x_{t+1} = x_t - \frac{f(x_t)}{f'(x_t)} = x_t - \frac{x_t^2 - 3}{2x_t} = \frac{x_t}{2} + \frac{3}{2x_t}$$

x_0 chosen initial value

3

First-Order Linear Difference Equations

Part 3.1

First-Order Linear Difference Equations with a „Constant a-Term“

Definition:

Time-dependent, ***inhomogeneous linear*** *difference equations of* ***first order*** *with* ***constant "a-term"*** take the form *:*

Equation:

$$x_t = ax_{t-1} + \xi_t$$

$(t = 1, 2, \ldots)$, x_0 (is given)

Definition:

Time-dependent, ***inhomogeneous linear*** *difference equations of* ***first order*** *with* ***constant "a-term"*** take the form *:*

Equation:

Iteration Rule:

$$F(t, x) = ax + \xi_t$$

$(t = 1, 2, \ldots)$, x_0 (is given)

Lösungsformel:

Time-dependent, ***inhomogeneous linear*** *difference equations of* ***first order*** *with* ***constant "a-term"*** have the solution *:*

Solution formula:

$$x_t = a^t x_0 + \sum_{s=1}^{t} a^{t-s} \xi_s$$

$$(t = 0, 1, 2, \ldots)$$

Deriving the Solution Formula:

$$\begin{aligned} x_t &= ax_{t-1} + \xi_t \\ &= a(a\, x_{t-2} + \xi_{t-1}) + \xi_t \\ &= a^2 x_{t-2} + a\, \xi_{t-1} + \xi_t \\ &\vdots \\ &= a^t x_0 + a^{t-1}\xi_1 + a^{t-2}\xi_2 + \ldots + a\xi_{t-1} + \xi_t \end{aligned}$$

$$\boxed{x_t = a^t x_0 + \sum_{s=1}^{t} a^{t-s} \xi_s}$$

Special Case: $x_t = a^t x_0 + \sum_{s=1}^{t} a^{t-s} \xi_s$

For first-order linear difference equations with ***constant coefficients*** *it holds:*

$$x_t = ax_{t-1} + \xi$$

$$\boxed{x_t = a^t\left(x_0 - \frac{\xi}{1-a}\right) + \frac{\xi}{1-a}}$$

$a \neq 1$, x_0 (is given) , $t = 0, 1, 2, 3, \ldots$

Example of an Exam Exercise:

1 Solve the difference equation:

$$x_n = \frac{1}{2} x_{n-1} + 3, \quad n \geq 1$$

Solution: Backwards iteration yields:

$$x_n = \frac{1}{2} x_{n-1} + 3$$

Example of an Exam Exercise:

1 Solve the difference equation:

$$x_n = \frac{1}{2}x_{n-1} + 3, \quad n \geq 1$$

Solution: *Inserting* the predecessor of x_{n-1}:

$$x_n = \frac{1}{2}x_{n-1} + 3 = \frac{1}{2}\left[\frac{1}{2}x_{n-2} + 3\right] + 3$$

Example of an Exam Exercise:

1 Solve the difference equation:

$$x_n = \frac{1}{2}x_{n-1} + 3, \quad n \geq 1$$

Solution: *Inserting* the predecessor of x_{n-2} etc. yields:

$$\frac{1}{2}\left[\frac{1}{2}x_{n-2} + 3\right] + 3 = \frac{1}{2}\left[\frac{1}{2}\left[\frac{1}{2}x_{n-3} + 3\right] + 3\right] + 3$$

Example of an Exam Exercise:

1 Solve the difference equation:

$$x_n = \frac{1}{2}x_{n-1} + 3, \quad n \geq 1$$

Intermediate Calculation: *Expanding* the equation:

$$x_n = \frac{1}{2}\left[\frac{1}{2}\left[\frac{1}{2}x_{n-3} + 3\right] + 3\right] + 3$$

Example of an Exam Exercise:

1 Solve the difference equation:

$$x_n = \frac{1}{2}x_{n-1} + 3, \quad n \geq 1$$

Solution: *Expanding* the equation

$$x_n = \frac{1}{2^3}x_{n-3} + 3 + \frac{1}{2}\cdot 3 + \frac{1}{2^2}\cdot 3$$

Example of an Exam Exercise:

1 Solve the difference equation:

$$x_n = \frac{1}{2}x_{n-1} + 3, \quad n \geq 1$$

Solution: General *condensation* of the terms:

$$\boxed{x_n = \frac{1}{2^n}x_0 + 3\sum_{k=0}^{n-1}\left(\frac{1}{2}\right)^k}$$

Example of an Exam Exercise:

2 Solve the associated ***initial value problem*** with $x_0 = 6$

Example of an Exam Exercise:

2 Solve the associated ***initial value problem*** with $x_0 = 6$

i.e., find a solution $x = (x_0, x_1, x_2 \ldots)$

according to the formula

$x_n = \frac{1}{2} x_{n-1} + 3,\ n \geq 1$ with $x_0 = 6$

Example of an Exam Exercise:

2 Solve the associated ***initial value problem*** with $x_0 = 6$

The initial value problem can be solved either *directly by using the solution formula*, i.e.

$$\boxed{x_n = \frac{6}{2^n} + 3\sum_{k=0}^{n-1}\left(\frac{1}{2}\right)^k}$$

Example of an Exam Exercise:

2 Solve the associated ***initial value problem*** with $x_0 = 6$

$$\boxed{x_n = \frac{6}{2^n} + 3\sum_{k=0}^{n-1}\left(\frac{1}{2}\right)^k} = \frac{6}{2^n} + 3\left(\frac{1-\left(\frac{1}{2}\right)^n}{1-\left(\frac{1}{2}\right)}\right)$$

$$\boxed{x_n = 6} = \frac{6}{2^n} + 6\left(1 - \frac{1}{2^n}\right)$$

Example of an Exam Exercise:

2 Solve the associated ***initial value problem*** with $x_0 = 6$

or by *forward iteration*:

$$x_0 = 6 \quad \Rightarrow \quad x_1 = \frac{1}{2}x_0 + 3 = \frac{6}{2} + 3 = 6$$

Example of an Exam Exercise:

2 Solve the associated ***initial value problem*** with $x_0 = 6$

Continuing with *forward iteration* :

$$x_1 = 6 \quad \Rightarrow \quad x_2 = \frac{1}{2}x_1 + 3 = 6$$

and in general : $\boxed{x_n = \frac{1}{2}x_{n-1} + 3 = 6}$

Example of an Exam Exercise:

The simplest way to solve this special initial value problem is to identify $x_0 = 6$ as an invariant point of the function $F(x) = \frac{1}{2}x + 3$, i.e.

Example of an Exam Exercise:

$$\boxed{F(6)=6}$$

and it holds:

Does the system starts at an **invariant** point, i.e. $F(x)=x$, it stays there, i.e. $x_n=6$ for all $n\geq 0$.

Example 1 (Part 2)

Dagobert-

Example

(with deposits and payments)

Starting Point:

Consider the following dynamic of Trick's account balance ω_t; at time t it holds:

$$\boxed{\omega_t=(1+r)\omega_{t-1}+(y_t-c_t)}$$

interest factor

deposits at time t

payments at time t

Moreover, we know the capital stock ω_0.

Find: Formula for the account balance

$$\omega_t=(1+r)\omega_{t-1}+(y_t-c_t)$$

Solution formula:

$$\boxed{\omega_t=(1+r)^t\omega_0+\sum_{s=1}^{t}(1+r)^{t-s}(y_s-c_s)}$$

$$t=1,2,\dots$$

Formula for the account balance:

$$\omega_t=(1+r)\omega_{t-1}+(y_t-c_t)$$

The *discounted capital flow* is:

$$\boxed{(1+r)^{-t}\omega_t=\omega_0+\sum_{s=1}^{t}(1+r)^{-s}(y_s-c_s)}$$

$$t=1,2,\dots$$

Summary:

➡ The **discounted capital stock** at time t equals the **capital stock** at time $t=0$ plus the sum of the **discounted deposits** minus the sum of the **discounted payments** up to time t.

Part 3.2

First-Order Linear Difference Equations with Variable Coefficients

Difference Equations (Prof. Dr. K. Helmes)

Definition:

*First-order linear difference equations with **variable coefficients** take the form:*

$$x_t = a_t x_{t-1} + \xi_t$$

$(t = 1, 2, \ldots)$, x_0 (is given)

Solution formula:

The solution of first-order linear difference equations with variable coefficients is given by:

$$x_t = \left(\prod_{s=1}^{t} a_s\right) x_0 + \sum_{s=1}^{t} \left(\prod_{k=s+1}^{t} a_k\right) \xi_s$$

$(t = 1, 2, \ldots)$

Example 1 (Part 3)

Dagobert-Example

with variable interest rate and proportional deposits and payments

Starting Point:

Consider a capital model with ***time-dependent*** interest factor:

$$\omega_t = (1 + r_t)\omega_{t-1} + (y_t - c_t)$$

interest factor at time t

deposits at time t

payments at time t

Moreover, we know the capital stock ω_0.

Starting Point:

Special Case: Capital model with proportional deposits and payments:

$$\omega_t = (1 + r_t)\omega_{t-1} + (y_t - c_t)$$

$$y_t = \alpha_t \omega_{t-1} \qquad c_t = \beta_t \omega_{t-1}$$

$$0 < \alpha_t, \beta_t < 1$$

Proportional In- and Outpayments:

$$\omega_t = (1 + r_t + \alpha_t - \beta_t)\omega_{t-1},$$

ω_0 given

$$\boxed{\omega_t = \left[\prod_{s=1}^{t}(1 + r_s + \alpha_s - \beta_s)\right]\omega_0}$$

$t = 1, 2, \ldots$

Numerical Example:

Capital stock: $\omega_0 =$ 1000 Euro

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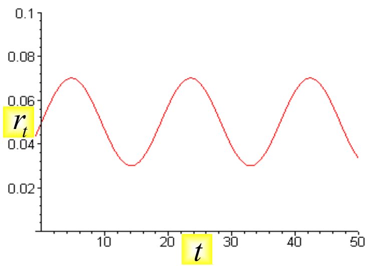
Capital stock: $\omega_0 =$ 1000 Euro

Interest factor: $r_t = 0{,}05 + 0{,}02\sin\left(\frac{t}{3}\right)$



Numerical Example:

Capital stock : $\omega_0 =$ 1000 Euro

Interest factor: $r_t = 0{,}05 + 0{,}02\sin\left(\frac{t}{3}\right)$

Rate of deposits: $\alpha_t = \begin{cases} 0.1, & 26 \le t \le 64 \\ 0, & \text{else} \end{cases}$

Numerical Example:

Capital stock : $\omega_0 =$ 1000 Euro

Interest factor: $r_t = 0{,}05 + 0{,}02\sin\left(\frac{t}{3}\right)$

Rate of deposits: $\alpha_t = \begin{cases} 0.1, & 26 \le t \le 64 \\ 0, & \text{else} \end{cases}$

Rate of payments: $\beta_t = \begin{cases} 0, & t < 65 \\ 0.07, & 65 \le t \le 74 \end{cases}$

Numerical Example:

$$\omega_t = \begin{cases} 1000 \cdot \prod_{s=1}^{t}(1 + r_s), & 0 \le t \le 25 \\ \omega_{25} \cdot \prod_{s=26}^{t}(1 + r_s + 0.1), & 26 \le t \le 64 \\ \omega_{65} \cdot \prod_{s=65}^{t}(1 + r_s - 0.07), & 65 \le t \le 74 \end{cases}$$

Numerical Example:

$\omega_0 = 1.000$

$\vdots$

$\omega_{25} = 3.700,58$

$\vdots$

$\omega_{50} = 120.527,19$

$\vdots$

$\omega_{65} = 860.246,44$

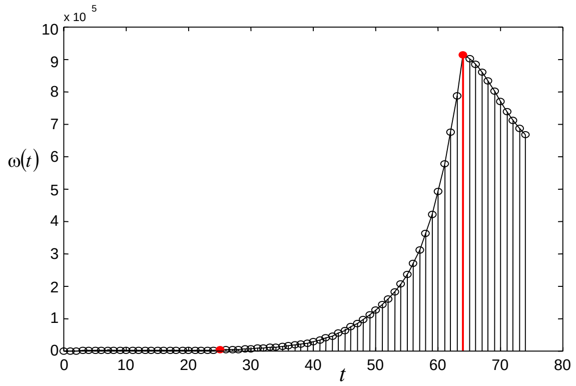
$\omega_{66} = 842.889,23$

$\omega_{67} = 820.374,88$

$\vdots$

$\omega_{74} = 635.803,00$

Numerical Example:



Part 3.3

Stability of First-Order Linear Difference Equations

Difference Equations (Prof. Dr. K. Helmes)

Definition: Stability

A first-order difference equation is called **stable**, if

- the solution of the **homogeneous** equation converges *for any initial value to zero*.

cf.. 1) unstable

2) chaotic

Stability Conditions:

- A linear difference equation x_t with *constant* coefficient is a **stable**, iff $|a| < 1$

- $$x_t \to x^* = \frac{\xi}{1-a}$$

Stability Conditions: Remark 1

If

$$x_s = \frac{\xi}{1-a}$$

holds for one time point s, then for all $t \geq s$:

$$x_t = x_s = x^* = \frac{\xi}{1-a}$$

Stability Conditions: *Remark 2*

Stability comes along in different forms:

Example

1 $0 < a < 1, \ x_0 > x^*$

➡ x_t converges *monotonically decreasing* to the equilibrium state x^*.

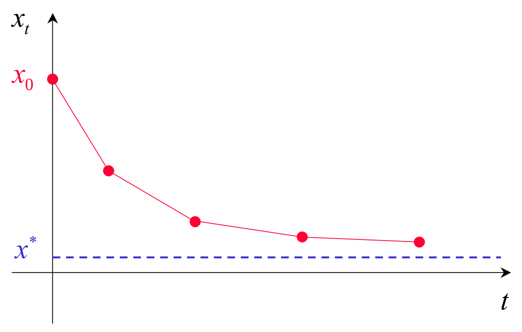


Figure 13: *Schematic representaion of stability - **Case A***

Stability Conditions: *Remark 2*

Stability comes along in different forms:

Example

2 $-1 < a < 0, \ x_0 > x^*$

➡ x_t shows *decreasing "fluctuations"* around the equilibrium state x^* (*damped oscillations*)

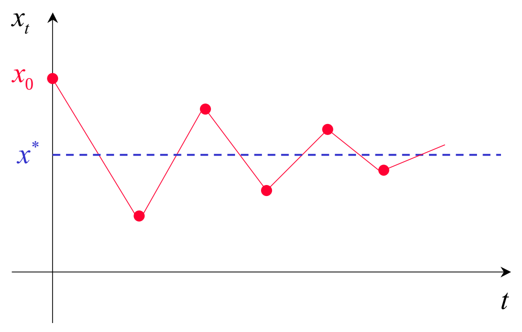


Figure 14: *Schematic representaion of stability - **Case B***

Stability Conditions: *Remark 3*

If $|a| > 1$, then the equation x_t is *not stable*, i.e. x_t moves farther and farther away from the equilibrium state x^*.

Exception: $x_0 = \frac{\xi}{1-a} = x^*$

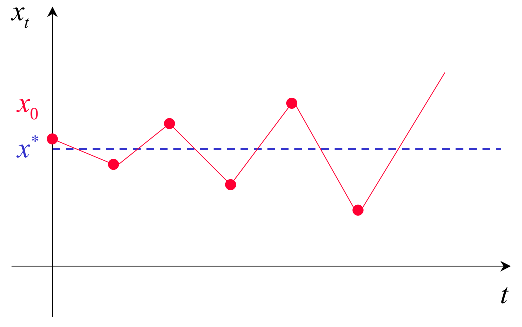


Figure 15: *Schematic representaion of stability - **Case C***