Konrad-Zuse-Zentrum für Informationstechnik Berlin
Takustraße 7, D-14195 Berlin-Dahlem

Andreas Brandt
Manfred Brandt

On the $M(n)/M(m)/s$ Queue with impatient Calls

Preprint SC 97–19 (May 1997)

On the $M(n)/M(m)/s$ Queue with impatient Calls[1]

Andreas Brandt

Wirtschaftswissenschaftliche Fakultät, Humboldt-Universität Berlin,
Spandauer Str. 1, D-10178 Berlin, Germany
e-mail: brandt@wiwi.hu-berlin.de

Manfred Brandt

Konrad-Zuse-Zentrum für Informationstechnik Berlin (ZIB),
Takustr. 7, D-14195 Berlin, Germany
e-mail: brandt@zib.de

Abstract

The paper is concerned with the analysis of an s server queueing system in which the calls become impatient and leave the system if their waiting time exceeds their own patience. The individual patience times are assumed to be i.i.d. and arbitrary distributed. The arrival and service rate may depend on the number of calls in the system and in service, respectively. For this system, denoted by $M(n)/M(m)/s+GI$, where $m = \min(n, s)$ is the number of busy servers in the system, we derive a system of integral equations for the vector of the residual patience times of the waiting calls and their original maximal patience times. By solving these equations explicitly we get the stability condition and, for the steady state of the system, the occupancy distribution and various waiting time distributions. As an application of the $M(n)/M(m)/s+GI$ system we give a performance analysis of an Automatic Call Distributor system (ACD system) of finite capacity with outbound calls and impatient inbound calls, especially in case of patience times being the minimum of constant and exponentially distributed times.

Mathematics Subject Classification (MSC 1991): 60K25, 68M20, 45F99, 60G10.

Keywords: $M(n)/M(m)/s$ queue with impatient calls; state dependent arrivals and departures; finite capacity; integral equations; occupancy distribution; waiting time distribution; ACD system.

1 Introduction

In this paper we consider an s server queueing system with an unlimited waiting room with FCFS queueing discipline and where the calls waiting in the queue for service are impatient, cf. Fig. 1.1. The arrival and service process are allowed to be state dependent with respect to the number of calls in the system and busy servers, respectively. If n calls are in the system, i.e. $m = \min(n, s)$ are in service and $\ell = (n - s)_+$ are waiting, let $\lambda_n \geq 0$ be the arrival rate of calls and $\mu_m \geq 0$ the cumulative rate of finishing service by the servers. We assume that the sequence of the arrival rates λ_n is bounded and that $\lambda_n > 0$ for $n \geq 0$ or that there exists a non-negative integer k such that $\lambda_n > 0$ for $0 \leq n < s + k$ and $\lambda_n = 0$ for $n \geq s + k$. Concerning the cumulative rate of finishing service we assume that $\mu_0 = 0$, $\mu_s > 0$. Each call arriving at the system has a patience time U. If the virtual waiting time W^v (i.e. the time which a call would have to wait until service) exceeds U then the call departs from the system and

[1]This work was supported by a grant from the Siemens AG.

gets lost by impatience. The patience times are assumed to be i.i.d. with a general distribution $C(u) := P(U \le u)$, $u \in I\!R_+$ which may be defective, i.e. $P(U = \infty) > 0$ is not excluded. This system is denoted by $M(n)/M(m)/s + GI$, where $M(n)$ denotes the arrival process depending on the number of calls in the system, $M(m)$ the service process depending on the number of busy servers and GI stands for the i.i.d. patience times.

Remark 1.1. Note that if $\lambda_n > 0$ for $0 \le n < s + k$ and $\lambda_n = 0$ for $n \ge s + k$ then we have the case of a limited waiting room with k waiting places. In case of $\lambda_n = \lambda > 0$ for $n \ge 0$ and $\mu_m = m\mu$ for $0 \le m \le s$ the model corresponds to an $M/M/s + GI$ system, cf. [BH]. The more general case $\lambda_n = \lambda > 0$ for $n \ge s$ and $\mu_m = m\mu$ for $0 \le m \le s$ is treated in [Ju2]. Relations to the results of [BH] and [Ju2], which seem to be the mostly relevant papers to our, and further references are given below.

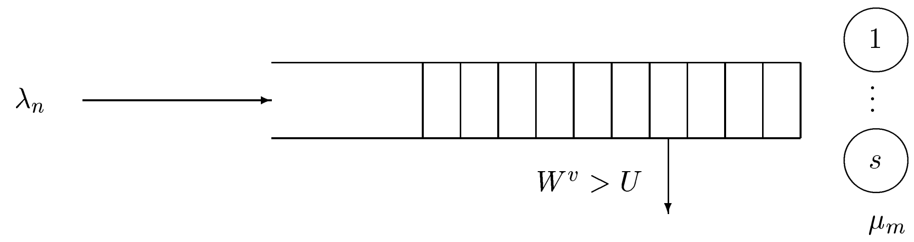

Fig. 1.1. The $M(n)/M(m)/s + GI$ system with impatient calls and state dependent arrival and departure rates, where n denotes the number of calls in the system, $m = \min(n, s)$ the number of calls in service, W^v the virtual waiting time and U the patience time.

The paper is organized as follows. In Sec. 2 a system of integral equations for the density of the stationary process of the vector of the number of calls, residual patience times and original maximal patience times of calls waiting for service in the system is derived. By a separation approach corresponding to local balance, the integral equations are solved explicitly; the densities are of an elementary structure. In Sec. 3 we derive the distribution of the number of calls in the system (occupancy distribution), the stability condition and performance measures as the impatience probability, the cumulative arrival rate and the waiting time distributions of the served calls as well as of the calls leaving by impatience. In Sec. 4 an application of the results to a performance analysis of an Automatic Call Distributor system (ACD system) with impatient inbound calls and outbound calls is given. In case of patience times being the minimum of constant and exponentially distributed times numerical algorithms and results are presented. The results of the paper can also be used for constructing system approximations for more complicated ACD systems. Such an approximation technique is given in [BB].

Let us now give some remarks concerning the literature and related papers. There is a lot of papers dealing with impatience phenomena. It seems that Barrer [Ba1], [Ba2] was the first one who dealt with the impatience problem; he analyzed the $M/M/1 + D$ system. Brodi [Br1], [Br2] derived and solved for the $M/M/1 + D$ system the corresponding integro differential equation. The general $GI/GI/1+GI$ system is treated in Daley [Dal]. The many server $M/M/s+D$ system was analyzed by [GK] by giving an explicit solution of the system of integro differential equations for the work load of the s servers, which yields formulas for the performance measures. This way was successfully proceeded by [Ju1] for impatience times being the minimum of a constant and an exponentially distributed time and [Ju2] for the general $M/M/s+GI$ system where the arrival intensities may depend on the number of busy servers (cf. also [GKo] p. 270). Later, independently [HSk] and [BH] derived results for the $M/M/s+GI$ system, too. Haugen and Skogan [HSk] started with a result of Wallstrom [W], cf. also [Hau] for an s server system with several Poisson input streams where each call type has an individual constant impatience time and where the call in the system (waiting and served) may become "exponentially" impatient. By an appropriate limiting construction they obtained a formula for the waiting time distribution for the $M/M/s+GI$ system and a generalization of this by allowing departures for the case of a limitation of the total time spent in the system (queue and server). In [BH] the Kolmogorov equations are derived and solved for the virtual waiting time of a call with an unlimited patience. By means of this quantity formulas for the relevant performance measures are derived. However, the distribution of the number of calls in the $M/M/s+GI$ system (those becoming impatient and those which will be served) and of the more general model in [Ju2] was not treated in the references mentioned above. Thus the results concerning the occupancy distribution given in Sec. 3 are new – as far as we know – also for the special case of an $M/M/s+GI$ system (treated in [BH]) and of the model in [Ju2]. Note, that our model includes the case of a finite waiting room, whereas this case is excluded in the models mentioned above.

In the literature, there are known several other mechanism where calls leave the system by impatience: If the call can calculate its prospective waiting time at its arrival instant then it leaves immediately if this time exceeds its patience. This strategy yields a better utilization of the waiting places because they will not be occupied by calls which later abandon by impatience. Also the patience may act on the sojourn times (waiting time plus service time). In this case not all work is useful because a call may leave the system by impatience during its service. For references and other more general models with impatience mechanism we refer to [BBH], [BH], [Ju3], [Sin], [Teg] and the references therein.

2 A system of integral equations and its solution

Throughout this section we assume that the queueing system is stable and that the distribution $C(u)$ of the patience times is non-defective and has a continuous density $c(u)$.

If $n > s$ calls are in the system then there are $\ell := (n-s)_+ > 0$ waiting calls in the system. (The notation $\ell := (n-s)_+$ will be used also in the following.) Accordingly to the FCFS discipline we number the waiting calls consecutively in such a way that the i-th call arrived at the system after the arrival of the $(i-1)$-th call, i.e. they are numbered accordingly to their positions in the queue. Let

$N(t)$	–	number of calls in the system at time t,
$M(t) := \min(N(t), s)$	–	number of busy servers at time t,
$L(t) := (N(t) - s)_+$	–	number of waiting calls at time t,
$(X_1(t), \ldots, X_{L(t)}(t))$	–	vector of the residual patience times of waiting calls ordered accordingly to their positions in the queue at t,
$(U_1(t), \ldots, U_{L(t)}(t))$	–	vector of the original maximal patience times of the waiting calls ordered accordingly to their positions in the queue at t,

$p(n) := P(N(t) = n)$ – stationary distribution of the number of calls in the system,

$P(n;\, x_1, \dots, x_\ell;\, u_1, \dots, u_\ell) := P(N(t) = n;\; X_1(t) \le x_1, \dots, X_\ell(t) \le x_\ell;$
$U_1(t) \le u_1, \dots, U_\ell(t) \le u_\ell)$

– stationary distribution on $N(t) = n$, where $\ell := (n-s)_+$.

Obviously, for fixed $n > s$ the support of $P(n; x_1, \dots, x_\ell; u_1, \dots, u_\ell)$ is contained in

$$\Omega_\ell := \{(x_1, \dots, x_\ell; u_1, \dots, u_\ell) \in I\!R_+^{2\ell} \,:\, u_1 - x_1 \ge \ldots \ge u_\ell - x_\ell \ge 0\}. \tag{2.1}$$

In view of the assumptions on $C(u)$ the density

$$p(n; x_1, \dots, x_\ell; u_1, \dots, u_\ell) := \frac{\partial^{2\ell}}{\partial x_1 \cdot \ldots \cdot \partial x_\ell \partial u_1 \cdot \ldots \cdot \partial u_\ell} P(n; x_1, \dots, x_\ell; u_1, \dots, u_\ell) \tag{2.2}$$

is continuous on Ω_ℓ; from a representation of this density given at the end of this section its continuity on Ω_ℓ follows, too.

In case of $n \le s$ we have the balance equations

$$(\lambda_n + \mu_n)p(n) = 1\!\mathrm{I}\{n > 0\}\lambda_{n-1}p(n-1) + \mu_{n+1}p(n+1), \qquad n = 0, 1, \dots, s-1, \tag{2.3}$$

$$(\lambda_s + \mu_s)p(s) = \lambda_{s-1}p(s-1) + \int\limits_{I\!R_+} p(s+1; 0; u)\mathrm{d}u + \mu_s \int\limits_{I\!R_+^2} p(s+1; x; u)\mathrm{d}x\mathrm{d}u. \tag{2.4}$$

In view of $\mu_0 = 0$ by summing the first $n+1$ equations of (2.3) we obtain

$$\lambda_n p(n) = \mu_{n+1}p(n+1), \qquad n = 0, 1, \dots, s-1. \tag{2.5}$$

Therefore, (2.4) is equivalent to

$$\lambda_s p(s) = \int\limits_{I\!R_+} p(s+1; 0; u)\mathrm{d}u + \mu_s \int\limits_{I\!R_+^2} p(s+1; x; u)\mathrm{d}x\mathrm{d}u. \tag{2.6}$$

In case of $n > s$ and $(x_1, \dots, x_\ell; u_1, \dots, u_\ell) \in \Omega_\ell$, which in view of (2.1) especially implies $0 \le x_\ell \le u_\ell$, we get the balance conditions

$$\begin{aligned}
&p(n; x_1, \dots, x_\ell; u_1, \dots, u_\ell) \\
&\quad= p(n; x_1 + h, \dots, x_\ell + h; u_1, \dots, u_\ell)(1 - h\lambda_n - h\mu_s) \\
&\qquad + h\sum_{i=1}^{\ell+1} \int\limits_{I\!R_+} p(n+1; x_1, \dots, x_{i-1}, 0, x_i, \dots, x_\ell; u_1, \dots, u_{i-1}, u, u_i, \dots, u_\ell)\mathrm{d}u \\
&\qquad + h\mu_s \int\limits_{I\!R_+^2} p(n+1; x, x_1, \dots, x_\ell; u, u_1, \dots, u_\ell)\mathrm{d}x\mathrm{d}u + o(h), \quad h > 0, \quad x_\ell < u_\ell,
\end{aligned} \tag{2.7}$$

$$p(n;x_1,\dots,x_{\ell-1},u_\ell;u_1,\dots,u_\ell) = \lambda_{n-1}p(n-1;x_1,\dots,x_{\ell-1};u_1,\dots,u_{\ell-1})c(u_\ell). \tag{2.8}$$

In the following from (2.7), (2.8) we'll derive an equivalent system of integral equations for the density $p(n;x_1,\dots,x_\ell;u_1,\dots,u_\ell)$. Let $n>s$ and $(x_1,\dots,x_\ell;u_1,\dots,u_\ell)\in\Omega_\ell$ with $x_\ell<u_\ell$ be fixed. We define

$$\begin{aligned}
\varphi(t) := {} & p(n;x_1{+}t,\dots,x_\ell{+}t;u_1,\dots,u_\ell)e^{-\mu_s t} \\
& +\lambda_n\int_t^\infty p(n;x_1{+}\xi,\dots,x_\ell{+}\xi;u_1,\dots,u_\ell)e^{-\mu_s\xi}\mathrm{d}\xi \\
& -\sum_{i=1}^{\ell+1}\int_t^\infty\int_{I\!R_+} p(n{+}1;x_1{+}\xi,\dots,x_{i-1}{+}\xi,0,x_i{+}\xi,\dots,x_\ell{+}\xi; \\
& \qquad\qquad u_1,\dots,u_{i-1},u,u_i,\dots,u_\ell)e^{-\mu_s\xi}\mathrm{d}u\mathrm{d}\xi \\
& -\mu_s\int_t^\infty\int_{I\!R_+^2} p(n{+}1;x,x_1{+}\xi,\dots,x_\ell{+}\xi;u,u_1,\dots,u_\ell)e^{-\mu_s\xi}\mathrm{d}x\mathrm{d}u\mathrm{d}\xi, \quad t\in[0,u_\ell{-}x_\ell].
\end{aligned}$$

Because of the continuity of $p(n;x_1,\dots,x_\ell;u_1,\dots,u_\ell)$ on Ω_ℓ the function $\varphi(t)$ is continuous, too, and for $t\in[0,u_\ell{-}x_\ell)$, $h\in(0,u_\ell{-}x_\ell{-}t]$ by some algebra we get

$$\begin{aligned}
& (\varphi(t)-\varphi(t{+}h))e^{\mu_s t} \\
& = p(n;x_1{+}t,\dots,x_\ell{+}t;u_1,\dots,u_\ell) \\
& \quad -p(n;x_1{+}t{+}h,\dots,x_\ell{+}t{+}h;u_1,\dots,u_\ell)(1-h\lambda_n-h\mu_s) \\
& \quad -h\sum_{i=1}^{\ell+1}\int_{I\!R_+} p(n{+}1;x_1{+}t,\dots,x_{i-1}{+}t,0,x_i{+}t,\dots,x_\ell{+}t;u_1,\dots,u_{i-1},u,u_i,\dots,u_\ell)\mathrm{d}u \\
& \quad -h\mu_s\int_{I\!R_+^2} p(n{+}1;x,x_1{+}t,\dots,x_\ell{+}t;u,u_1,\dots,u_\ell)\mathrm{d}x\mathrm{d}u+o(h).
\end{aligned}$$

Applying now (2.7) at the point $(x_1{+}t,\dots,x_\ell{+}t;u_1,\dots,u_\ell)$ we obtain

$$\varphi(t{+}h)-\varphi(t)=o(h), \qquad t\in[0,u_\ell{-}x_\ell), \quad h\in(0,u_\ell{-}x_\ell{-}t].$$

By the continuity of $\varphi(t)$ therefore we conclude that $\varphi(t)$ is constant, especially we obtain the relation $\varphi(0)=\varphi(u_\ell{-}x_\ell)$. In view of the definition of $\varphi(t)$ and since the support of the density $p(n;x_1,\dots,x_\ell;u_1,\dots,u_\ell)$ is contained in Ω_ℓ the last relation reads

$$\begin{aligned}
& p(n;x_1,\dots,x_\ell;u_1,\dots,u_\ell) \\
& \quad +\lambda_n\int_{I\!R_+} p(n;x_1{+}\xi,\dots,x_\ell{+}\xi;u_1,\dots,u_\ell)e^{-\mu_s\xi}\mathrm{d}\xi
\end{aligned}$$

$$
\begin{aligned}
&-\sum_{i=1}^{\ell+1} \int\limits_{I\!R_+^2} p(n{+}1; x_1{+}\xi, \ldots, x_{i-1}{+}\xi, 0, x_i{+}\xi, \ldots, x_\ell{+}\xi; \\
&\qquad\qquad\qquad\qquad u_1, \ldots, u_{i-1}, u, u_i, \ldots, u_\ell) e^{-\mu_s \xi} \mathrm{d}u \mathrm{d}\xi \\
&-\mu_s \int\limits_{I\!R_+^3} p(n{+}1; x, x_1{+}\xi, \ldots, x_\ell{+}\xi; u, u_1, \ldots, u_\ell) e^{-\mu_s \xi} \mathrm{d}x \mathrm{d}u \mathrm{d}\xi \\
= \;& p(n; x_1{+}u_\ell{-}x_\ell, \ldots, x_{\ell-1}{+}u_\ell{-}x_\ell, u_\ell; u_1, \ldots, u_\ell) e^{-\mu_s (u_\ell - x_\ell)}.
\end{aligned}
$$

Because of the boundary condition (2.8) hence we get the following system of integral equations for $n > s$ and $(x_1, \ldots, x_\ell; u_1, \ldots, u_\ell) \in \Omega_\ell$:

$$
\begin{aligned}
&p(n; x_1, \ldots, x_\ell; u_1, \ldots, u_\ell) \\
= \;& \lambda_{n-1} p(n{-}1; x_1{+}u_\ell{-}x_\ell, \ldots, x_{\ell-1}{+}u_\ell{-}x_\ell; u_1, \ldots, u_{\ell-1}) c(u_\ell) e^{-\mu_s (u_\ell - x_\ell)} \\
&-\lambda_n \int\limits_{I\!R_+} p(n; x_1{+}\xi, \ldots, x_\ell{+}\xi; u_1, \ldots, u_\ell) e^{-\mu_s \xi} \mathrm{d}\xi \\
&+\sum_{i=1}^{\ell+1} \int\limits_{I\!R_+^2} p(n{+}1; x_1{+}\xi, \ldots, x_{i-1}{+}\xi, 0, x_i{+}\xi, \ldots, x_\ell{+}\xi; \\
&\qquad\qquad\qquad\qquad u_1, \ldots, u_{i-1}, u, u_i, \ldots, u_\ell) e^{-\mu_s \xi} \mathrm{d}u \mathrm{d}\xi \\
&+\mu_s \int\limits_{I\!R_+^3} p(n{+}1; x, x_1{+}\xi, \ldots, x_\ell{+}\xi; u, u_1, \ldots, u_\ell) e^{-\mu_s \xi} \mathrm{d}x \mathrm{d}u \mathrm{d}\xi. \qquad (2.9)
\end{aligned}
$$

On the other hand, from the system of integral equations (2.9) the balance conditions (2.7) and (2.8) may be derived.

In the following we'll solve the system of equations (2.5), (2.6), (2.9). From (2.5) we get

$$
p(n) = g \Big(\prod_{i=0}^{n-1} \lambda_i \Big) \Big(\prod_{i=n+1}^{s} \mu_i \Big), \qquad n \le s, \tag{2.10}
$$

where $g > 0$ is a normalizing factor.

Remark 2.1. In case of $\mu_m > 0$ for $m = 1, 2, \ldots, s$ the representation

$$
p(n) = g^* \prod_{i=0}^{n-1} \frac{\lambda_i}{\mu_{i+1}}, \qquad g^* = g \prod_{i=1}^{s} \mu_i
$$

is possible for $n \le s$, too, being more closely to the birth death process notations.

In view of the representation (2.10) of $p(n)$ for $n \le s$, in case of $n \ge s$ we choose the substitution

$$
p(n; x_1, \ldots, x_\ell; u_1, \ldots, u_\ell) = g \Big(\prod_{i=0}^{n-1} \lambda_i \Big) q(n; x_1, \ldots, x_\ell; u_1, \ldots, u_\ell). \tag{2.11}
$$

This substitution is verified by the fact that obviously we have $p(n;x_1,\ldots,x_\ell;u_1,\ldots,u_\ell)=0$ in case of $\lambda_0\cdot\ldots\cdot\lambda_{n-1}=0$. From (2.10), (2.11) it follows $q(s)=1$. Hence equation (2.6) now reads

$$1=\int\limits_{I\!R_+} q(s+1;0;u)\mathrm{d}u+\mu_s\int\limits_{I\!R_+^2} q(s+1;x;u)\mathrm{d}x\mathrm{d}u, \tag{2.12}$$

and for $n>s$, $(x_1,\ldots,x_\ell;u_1,\ldots,u_\ell)\in\Omega_\ell$ equations (2.9) read

$$\begin{aligned}
&q(n;x_1,\ldots,x_\ell;u_1,\ldots,u_\ell)\\
&\quad= q(n-1;x_1+u_\ell-x_\ell,\ldots,x_{\ell-1}+u_\ell-x_\ell;u_1,\ldots,u_{\ell-1})c(u_\ell)e^{-\mu_s(u_\ell-x_\ell)}\\
&\qquad-\lambda_n\int\limits_{I\!R_+} q(n;x_1+\xi,\ldots,x_\ell+\xi;u_1,\ldots,u_\ell)e^{-\mu_s\xi}\mathrm{d}\xi\\
&\qquad+\lambda_n\sum_{i=1}^{\ell+1}\int\limits_{I\!R_+^2} q(n+1;x_1+\xi,\ldots,x_{i-1}+\xi,0,x_i+\xi,\ldots,x_\ell+\xi;\\
&\qquad\qquad\qquad u_1,\ldots,u_{i-1},u,u_i,\ldots,u_\ell)e^{-\mu_s\xi}\mathrm{d}u\mathrm{d}\xi\\
&\qquad+\lambda_n\mu_s\int\limits_{I\!R_+^3} q(n+1;x,x_1+\xi,\ldots,x_\ell+\xi;u,u_1,\ldots,u_\ell)e^{-\mu_s\xi}\mathrm{d}x\mathrm{d}u\mathrm{d}\xi.
\end{aligned}\tag{2.13}$$

In case of a finite waiting room with k waiting places, i.e. $\lambda_n>0$ for $0\le n<s+k$ and $\lambda_n=0$ for $n\ge s+k$, equation (2.12) is relevant only in case of $k>0$ and the equations (2.13) are relevant only for $s<n\le s+k$. The structure of (2.12) and (2.13) suggests to conjecture that $q(n;x_1,\ldots,x_\ell;u_1,\ldots,u_\ell)$ is independent of λ_n. Assuming this independence then for $n>s$, $(x_1,\ldots,x_\ell;u_1,\ldots,u_\ell)\in\Omega_\ell$ from (2.13) it follows

$$\begin{aligned}
&q(n;x_1,\ldots,x_\ell;u_1,\ldots,u_\ell)\\
&\quad= q(n-1;x_1+u_\ell-x_\ell,\ldots,x_{\ell-1}+u_\ell-x_\ell;u_1,\ldots,u_{\ell-1})c(u_\ell)e^{-\mu_s(u_\ell-x_\ell)},
\end{aligned}\tag{2.14}$$

$$\begin{aligned}
&\int\limits_{I\!R_+} q(n;x_1+\xi,\ldots,x_\ell+\xi;u_1,\ldots,u_\ell)e^{-\mu_s\xi}\mathrm{d}\xi\\
&\quad=\sum_{i=1}^{\ell+1}\int\limits_{I\!R_+^2} q(n+1;x_1+\xi,\ldots,x_{i-1}+\xi,0,x_i+\xi,\ldots,x_\ell+\xi;\\
&\qquad\qquad\qquad u_1,\ldots,u_{i-1},u,u_i,\ldots,u_\ell)e^{-\mu_s\xi}\mathrm{d}u\mathrm{d}\xi\\
&\qquad+\mu_s\int\limits_{I\!R_+^3} q(n+1;x,x_1+\xi,\ldots,x_\ell+\xi;u,u_1,\ldots,u_\ell)e^{-\mu_s\xi}\mathrm{d}x\mathrm{d}u\mathrm{d}\xi.
\end{aligned}\tag{2.15}$$

In view of $q(s)=1$ and of the definition (2.1) of Ω_ℓ from (2.14) by induction over $n>s$ we get

$$q(n;x_1,\ldots,x_\ell;u_1,\ldots,u_\ell)$$

$$= \quad \mathrm{I\!I}\{(x_1,\ldots,x_\ell;u_1,\ldots,u_\ell)\in\Omega_\ell\}\Big(\prod_{i=1}^{\ell} c(u_i)\Big)e^{-\mu_s(u_1-x_1)}, \qquad n>s. \tag{2.16}$$

Since the distribution $C(u)$ of the patience times is non-defective by integration it follows that the function $q(s+1;x_1;u_1)$ defined by (2.16) satisfies (2.12). Moreover, using the convention $x_{\ell+1}=u_{\ell+1}=0$ for the functions $q(n;x_1,\ldots,x_\ell;u_1,\ldots,u_\ell)$ defined by (2.16) for $n>s$, $(x_1,\ldots,x_\ell;u_1,\ldots,u_\ell)\in\Omega_\ell$ we obtain

$$\begin{aligned}
&\sum_{i=1}^{\ell+1}\int_{I\!R_+} q(n+1;x_1,\ldots,x_{i-1},0,x_i,\ldots,x_\ell;u_1,\ldots,u_{i-1},u,u_i,\ldots,u_\ell)\mathrm{d}u\\
&\qquad+\mu_s\int_{I\!R_+^2} q(n+1;x,x_1,\ldots,x_\ell;u,u_1,\ldots,u_\ell)\mathrm{d}x\mathrm{d}u\\
&= \mathrm{I\!I}\{(x_1,\ldots,x_\ell;u_1,\ldots,u_\ell)\in\Omega_\ell\}\Big(\prod_{i=1}^{\ell} c(u_i)\Big)\Big(\int_{u_1-x_1}^{\infty} c(u)e^{-\mu_s u}\mathrm{d}u\\
&\qquad+e^{-\mu_s(u_1-x_1)}\sum_{i=2}^{\ell+1}\int_{u_i-x_i}^{u_{i-1}-x_{i-1}} c(u)\mathrm{d}u+\mu_s\int_{u_1-x_1}^{\infty}\int_0^{u-(u_1-x_1)} c(u)e^{-\mu_s(u-x)}\mathrm{d}x\mathrm{d}u\Big)\\
&= q(n;x_1,\ldots,x_\ell;u_1,\ldots,u_\ell)e^{\mu_s(u_1-x_1)}\Big(\int_{u_1-x_1}^{\infty} c(u)e^{-\mu_s u}\mathrm{d}u\\
&\qquad+e^{-\mu_s(u_1-x_1)}\int_0^{u_1-x_1} c(u)\mathrm{d}u+\int_{u_1-x_1}^{\infty} c(u)\Big(e^{-\mu_s(u_1-x_1)}-e^{-\mu_s u}\Big)\mathrm{d}u\Big)\\
&= q(n;x_1,\ldots,x_\ell;u_1,\ldots,u_\ell).
\end{aligned}$$

From this equation applied at the point $(x_1+\xi,\ldots,x_\ell+\xi;u_1,\ldots,u_\ell)$ it follows that the functions $q(n;x_1,\ldots,x_\ell;u_1,\ldots,u_\ell)$ defined by (2.16) satisfy (2.15), too. Hence these functions solve the system of integral equations (2.12) and (2.13). Since the density $p(n;x_1,\ldots,x_\ell;u_1,\ldots,u_\ell)$ is uniquely determined as the normalized solution of (2.7) and (2.8) this density is given by (2.11) and (2.16). Summarizing we get the representation

$$\begin{aligned}
&p(n;x_1,\ldots,x_\ell;u_1,\ldots,u_\ell)\\
&= \mathrm{I\!I}\{(x_1,\ldots,x_\ell;u_1,\ldots,u_\ell)\in\Omega_\ell\}g\Big(\prod_{i=0}^{n-1}\lambda_i\Big)\Big(\prod_{i=1}^{\ell} c(u_i)\Big)e^{-\mu_s(u_1-x_1)}, \qquad n>s.
\end{aligned}\tag{2.17}$$

3 Stability condition, occupancy and waiting time distribution

As in Section 2 let us assume that the queueing system is stable and that the distribution $C(u)$ of the patience times is non-defective and has a continuous density $c(u)$. The stationary probability $p(n)$ that n calls are in the system is given by (2.10) for $n \le s$. In case of $n > s$ from (2.17) it follows

$$\begin{aligned} p(n) &= \int_{\mathbb{R}_+^{2\ell}} p(n; x_1, \ldots, x_\ell; u_1, \ldots, u_\ell) \mathrm{d}x_1 \ldots \mathrm{d}x_\ell \mathrm{d}u_1 \ldots \mathrm{d}u_\ell \\ &= g\Big(\prod_{i=0}^{n-1} \lambda_i\Big) \int_{\Omega_\ell} \Big(\prod_{i=1}^{\ell} c(u_i)\Big) e^{-\mu_s(u_1-x_1)} \mathrm{d}x_1 \ldots \mathrm{d}x_\ell \mathrm{d}u_1 \ldots \mathrm{d}u_\ell, \end{aligned}$$

where $\ell := (n-s)_+$. In view of the definition (2.1) of Ω_ℓ the substitution $u_i = \xi_i + x_i$ for $i = 1, \ldots, \ell$ yields

$$\begin{aligned} p(n) &= g\Big(\prod_{i=0}^{n-1} \lambda_i\Big) \int_{\mathbb{R}_+^{2\ell}} 1\!\mathrm{I}\{\xi_1 \ge \xi_2 \ge \ldots \ge \xi_\ell\} \Big(\prod_{i=1}^{\ell} c(\xi_i + x_i)\Big) e^{-\mu_s \xi_1} \mathrm{d}x_1 \ldots \mathrm{d}x_\ell \mathrm{d}\xi_1 \ldots \mathrm{d}\xi_\ell \\ &= g\Big(\prod_{i=0}^{n-1} \lambda_i\Big) \int_{\mathbb{R}_+^{\ell}} 1\!\mathrm{I}\{\xi_1 \ge \xi_2 \ge \ldots \ge \xi_\ell\} \Big(\prod_{i=1}^{\ell} (1 - C(\xi_i))\Big) e^{-\mu_s \xi_1} \mathrm{d}\xi_1 \ldots \mathrm{d}\xi_\ell \\ &= g\Big(\prod_{i=0}^{n-1} \lambda_i\Big) \int_0^\infty \frac{1}{(\ell-1)!} \Big(\int_0^\xi (1 - C(\eta)) \mathrm{d}\eta\Big)^{\ell-1} (1 - C(\xi)) e^{-\mu_s \xi} \mathrm{d}\xi \\ &= g\Big(\prod_{i=0}^{n-1} \lambda_i\Big) \frac{\mu_s}{\ell!} \int_0^\infty \Big(\int_0^\xi (1 - C(\eta)) \mathrm{d}\eta\Big)^{\ell} e^{-\mu_s \xi} \mathrm{d}\xi, \qquad n > s. \end{aligned} \tag{3.1}$$

From (2.10) and (3.1) we get the normalizing factor g:

$$g^{-1} = \sum_{j=0}^{s-1} \Big(\prod_{i=0}^{j-1} \lambda_i\Big)\Big(\prod_{i=j+1}^{s} \mu_i\Big) + \mu_s \sum_{j=0}^{\infty} \Big(\prod_{i=0}^{s+j-1} \lambda_i\Big) \frac{1}{j!} \int_0^\infty \Big(\int_0^\xi (1 - C(\eta)) \mathrm{d}\eta\Big)^{j} e^{-\mu_s \xi} \mathrm{d}\xi. \tag{3.2}$$

Obviously, the stability of the system is equivalent to the finiteness of the right-hand side of (3.2). Therefore, the stability condition reads

$$\sum_{j=0}^{\infty} \Big(\prod_{i=0}^{s+j-1} \lambda_i\Big) \frac{1}{j!} \int_0^\infty \Big(\int_0^\xi (1 - C(\eta)) \mathrm{d}\eta\Big)^{j} e^{-\mu_s \xi} \mathrm{d}\xi < \infty. \tag{3.3}$$

The case of a general distribution $C(u)$ of the patience times is obtained by considering $C(u)$ as the limit in distribution of a sequence of non-defective distributions $C_\nu(u)$ with continuous density. From (3.3), (2.10) and (3.1) applied to $C_\nu(u)$ by arguments of continuity we get

Theorem 3.1. Let the patience times be i.i.d. with a general distribution $C(u)$. Then the system is stable iff

$$\sum_{j=0}^{\infty}\left(\prod_{i=0}^{s+j-1}\lambda_i\right)\frac{1}{j!}\int_0^{\infty}\left(\int_0^{\xi}(1-C(\eta))\mathrm{d}\eta\right)^j e^{-\mu_s\xi}\mathrm{d}\xi<\infty. \tag{3.4}$$

In case of a stable system for the stationary occupancy distribution it holds

$$p(n)=\begin{cases} g\left(\prod\limits_{i=0}^{n-1}\lambda_i\right)\left(\prod\limits_{i=n+1}^{s}\mu_i\right), & n=0,1,\ldots,s,\\[2ex] g\left(\prod\limits_{i=0}^{n-1}\lambda_i\right)\frac{\mu_s}{(n-s)!}\int\limits_0^{\infty}\left(\int\limits_0^{\xi}(1-C(\eta))\mathrm{d}\eta\right)^{n-s}e^{-\mu_s\xi}\mathrm{d}\xi, & n=s+1,s+2,\ldots, \end{cases} \tag{3.5}$$

where the normalizing factor g is given by (3.2).

Let the system be stable, i.e. let (3.4) be satisfied. For the investigation of the various waiting time distributions of a typical arriving call we need some notations:

Λ	–	cumulative arrival intensity of the calls in the steady state,
p_W	–	probability that a typical arriving call has to wait for service,
p_I	–	probability that a typical arriving call will leave the system by impatience later,
$W_S(x)=P(W_S\le x)$	–	distribution function of the waiting time W_S of a typical arriving call under the condition that it will be served,
$W_I(x)=P(W_I\le x)$	–	distribution function of the waiting time W_I of a typical arriving call under the condition that it will leave the system by impatience later,
$W(x)=P(W\le x)$	–	distribution function of the (unconditional) waiting time W of a typical arriving call.

The cumulative arrival intensity of the calls in the steady state Λ is given by

$$\Lambda=\sum_{n=0}^{\infty}\lambda_n p(n). \tag{3.6}$$

Since the sequence of the arrival intensities λ_n is bounded Λ is finite.

In the following we will make use of the Palm-distribution with respect to arrival epochs of calls (cf. e.g. [FKAS], (1.2.6)) and of the conservation principle for stationary point processes with respect to arrival epochs of calls and the epochs where the calls leave the waiting room, respectively. (If a call finds at its arrival a free server, then it leaves the queue immediately, i.e. its arrival epoch coincides with the epoch of leaving the queue.) The corresponding rigorous proofs are not outlined here because the results are part of folklore. (However, exact proofs can be given e.g. by means of Campbell's formula along the lines as e.g. in [BFL], Sec. 6.4, (6.4.1).)

The probability $1-p_W$ is given by the ratio of the arrival intensity of calls finding at their arrival a free server and of the total arrival intensity Λ, i.e.

$$1 - p_W = \frac{1}{\Lambda} \sum_{n=0}^{s-1} \lambda_n p(n). \tag{3.7}$$

Hence in view of (2.5) for the probability p_W that a typical arriving call has to wait for service it holds

$$p_W = 1 - \frac{1}{\Lambda} \sum_{n=1}^{s} \mu_n p(n). \tag{3.8}$$

(Note, that by the intensity conservation principle we have that the intensity of calls finding at their arrival at least one free server equals to the intensity of served calls leaving behind at least one free server. The latter intensity is just the sum in (3.8).)

The probability $1 - p_I$ is given by the ratio of the intensity $\lambda^{(S)}$ of arriving calls which will be served (immediately or after waiting in the queue) and of the total arrival intensity Λ. By the conservation principle $\lambda^{(S)}$ is the intensity of calls leaving the waiting room for starting service. This yields

$$1 - p_I = \frac{1}{\Lambda}\Big(\sum_{n=0}^{s-1} \lambda_n p(n) + \sum_{n=s+1}^{\infty} \mu_s p(n) \Big) = \frac{1}{\Lambda}\Big(\sum_{n=1}^{s} \mu_n p(n) + \sum_{n=s+1}^{\infty} \mu_s p(n) \Big), \tag{3.9}$$

where the last equality is again a consequence of (2.5). Note, that the expression in the bracket of the last equation is just the intensity of served calls, which by the conservation principle is equal to the intensity of calls leaving the waiting room for service (immediately after their arrival or after waiting). Hence for the probability p_I that a typical arriving call will leave the system by impatience later we get

$$p_I = 1 - \frac{1}{\Lambda}\Big(\mu_s + \sum_{n=0}^{s-1} (\mu_n - \mu_s) p(n) \Big). \tag{3.10}$$

Next we are interested in the waiting time distributions $W_S(x)$, $W_I(x)$ and $W(x)$ of a typical arriving call. Let us in the following assume that the queueing system is stable and that the distribution $C(u)$ of the patience times is non-defective and has a continuous density $c(u)$. For fixed $x \in \mathbb{R}_+$ the probability $P(W_S > x) = 1 - W_S(x)$ is just the ratio of the intensity $\lambda^{(S,x)}$ of arriving calls which will be served and whose waiting times up to service are larger than x to the intensity $\lambda^{(S)} = (1-p_I)\Lambda$ of all arriving calls which will be served. By the intensity conservation principle $\lambda^{(S,x)}$ equals to the intensity of time instants where calls leave the waiting room for starting service and which have waited longer than x. Hence we get

$$\begin{aligned} &1 - W_S(x) \\ &= \frac{1}{(1-p_I)\Lambda} \sum_{n=s+1}^{\infty} \mu_s \int_{\mathbb{R}_+^{2\ell}} \mathbb{1}\{x < u_1 - x_1\} p(n; x_1, \dots, x_\ell; u_1, \dots, u_\ell) \mathrm{d}x_1 \dots \mathrm{d}x_\ell \mathrm{d}u_1 \dots \mathrm{d}u_\ell. \end{aligned}$$

Because of (2.1), (2.17) for $n > s$ the substitution $u_i = \xi_i + x_i$ yields

$$\begin{aligned}
&\int\limits_{\mathbb{R}_+^{2\ell}} 1\!\!\mathrm{I}\{x < u_1 - x_1\} p(n; x_1, \ldots, x_\ell; u_1, \ldots, u_\ell) \mathrm{d}x_1 \ldots \mathrm{d}x_\ell \mathrm{d}u_1 \ldots \mathrm{d}u_\ell \\
&= g\Big(\prod_{i=0}^{n-1} \lambda_i\Big) \int\limits_{\Omega_\ell} 1\!\!\mathrm{I}\{x < u_1 - x_1\} \Big(\prod_{i=1}^{\ell} c(u_i)\Big) e^{-\mu_s(u_1 - x_1)} \mathrm{d}x_1 \ldots \mathrm{d}x_\ell \mathrm{d}u_1 \ldots \mathrm{d}u_\ell \\
&= g\Big(\prod_{i=0}^{n-1} \lambda_i\Big) \int\limits_{\mathbb{R}_+^{\ell}} 1\!\!\mathrm{I}\{x < \xi_1\} 1\!\!\mathrm{I}\{\xi_1 \geq \xi_2 \geq \ldots \geq \xi_\ell\} \Big(\prod_{i=1}^{\ell} (1 - C(\xi_i))\Big) e^{-\mu_s \xi_1} \mathrm{d}\xi_1 \ldots \mathrm{d}\xi_\ell \\
&= g\Big(\prod_{i=0}^{n-1} \lambda_i\Big) \frac{1}{(\ell-1)!} \int\limits_x^\infty \Big(\int\limits_0^\xi (1 - C(\eta)) \mathrm{d}\eta\Big)^{\ell-1} (1 - C(\xi)) e^{-\mu_s \xi} \mathrm{d}\xi, \qquad x \geq 0.
\end{aligned}$$

Thus for $x \in \mathbb{R}_+$ it holds

$$1 - W_S(x) = \frac{g}{(1-p_I)\Lambda} \sum_{j=0}^{\infty} \Big(\prod_{i=0}^{s+j} \lambda_i\Big) \frac{\mu_s}{j!} \int\limits_x^\infty \Big(\int\limits_0^\xi (1 - C(\eta)) \mathrm{d}\eta\Big)^j (1 - C(\xi)) e^{-\mu_s \xi} \mathrm{d}\xi. \tag{3.11}$$

For getting an explicit expression for $W_I(x)$ we proceed analogously to $W_S(x)$. The probability $P(W_I > x) = 1 - W_I(x)$ is the ratio of the intensity $\lambda^{(I,x)}$ of arriving calls departing from the system by impatience later and whose time spent in the queue is larger than x to the intensity $\lambda^{(I)} = p_I \Lambda$ of all arriving calls becoming impatient later. By the conservation principle $\lambda^{(I,x)}$ equals to the intensity of time instants where calls leave the waiting room by impatience and which have waited longer than x. This yields

$$\begin{aligned}
1 - W_I(x) = \frac{1}{p_I \Lambda} \sum_{n=s+1}^{\infty} \sum_{i=1}^{\ell} \int\limits_{\mathbb{R}_+^{2\ell-1}} & 1\!\!\mathrm{I}\{x < u_i\} p(n; x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_\ell; u_1, \ldots, u_\ell) \\
& \mathrm{d}x_1 \ldots \mathrm{d}x_{i-1} \mathrm{d}x_{i+1} \ldots \mathrm{d}x_\ell \mathrm{d}u_1 \ldots \mathrm{d}u_\ell.
\end{aligned}$$

In view of (2.1), (2.17) and using the substitution $u_i = \xi_i + x_i$ we get for $n > s$:

$$\begin{aligned}
&- \frac{\mathrm{d}}{\mathrm{d}x} \Big(\sum_{i=1}^{\ell} \int\limits_{\mathbb{R}_+^{2\ell-1}} 1\!\!\mathrm{I}\{x < u_i\} p(n; x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_\ell; u_1, \ldots, u_\ell) \\
&\qquad \mathrm{d}x_1 \ldots \mathrm{d}x_{i-1} \mathrm{d}x_{i+1} \ldots \mathrm{d}x_\ell \mathrm{d}u_1 \ldots \mathrm{d}u_\ell \Big) \\
&= \sum_{i=1}^{\ell} \int\limits_{\mathbb{R}_+^{2\ell-2}} p(n; x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_\ell; u_1, \ldots, u_{i-1}, x, u_{i+1}, \ldots, u_\ell) \\
&\qquad \mathrm{d}x_1 \ldots \mathrm{d}x_{i-1} \mathrm{d}x_{i+1} \ldots \mathrm{d}x_\ell \mathrm{d}u_1 \ldots \mathrm{d}u_{i-1} \mathrm{d}u_{i+1} \ldots \mathrm{d}u_\ell \\
&= g\Big(\prod_{i=0}^{n-1} \lambda_i\Big) c(x) \Big(e^{-\mu_s x} \int\limits_{\mathbb{R}_+^{\ell-1}} 1\!\!\mathrm{I}\{x \geq \xi_2 \geq \ldots \geq \xi_\ell\} \Big(\prod_{j=2}^{\ell} (1 - C(\xi_j))\Big) \mathrm{d}\xi_2 \ldots \mathrm{d}\xi_\ell
\end{aligned}$$

$$+\sum_{i=2}^{\ell}\int_{\mathbb{R}_+^{i-1}} \mathrm{1\!I}\{\xi_1 \geq \ldots \geq \xi_{i-1} \geq x\}\Big(\prod_{j=1}^{i-1}(1-C(\xi_j))\Big)e^{-\mu_s\xi_1}\mathrm{d}\xi_1\ldots\mathrm{d}\xi_{i-1}$$

$$\cdot\int_{\mathbb{R}_+^{\ell-i}} \mathrm{1\!I}\{x \geq \xi_{i+1} \geq \ldots \geq \xi_\ell\}\Big(\prod_{j=i+1}^{\ell}(1-C(\xi_j))\Big)\mathrm{d}\xi_{i+1}\ldots\mathrm{d}\xi_\ell\Big)$$

$$= g\Big(\prod_{i=0}^{n-1}\lambda_i\Big)c(x)\sum_{i=1}^{\ell}\frac{\mu_s}{(i-1)!}\int_x^\infty\Big(\int_x^\xi(1-C(\eta))\mathrm{d}\eta\Big)^{i-1}e^{-\mu_s\xi}\mathrm{d}\xi$$

$$\cdot\frac{1}{(\ell-i)!}\Big(\int_0^x(1-C(\eta))\mathrm{d}\eta\Big)^{\ell-i}$$

$$= g\Big(\prod_{i=0}^{n-1}\lambda_i\Big)c(x)\frac{\mu_s}{(\ell-1)!}\int_x^\infty\Big(\int_0^\xi(1-C(\eta))\mathrm{d}\eta\Big)^{\ell-1}e^{-\mu_s\xi}\mathrm{d}\xi, \qquad x\geq 0.$$

Hence by integration it follows

$$\sum_{i=1}^{\ell}\int_{\mathbb{R}_+^{2\ell-1}}\mathrm{1\!I}\{x<u_i\}p(n;x_1,\ldots,x_{i-1},0,x_{i+1},\ldots,x_\ell;u_1,\ldots,u_\ell)$$
$$\mathrm{d}x_1\ldots\mathrm{d}x_{i-1}\mathrm{d}x_{i+1}\ldots\mathrm{d}x_\ell\mathrm{d}u_1\ldots\mathrm{d}u_\ell$$

$$= g\Big(\prod_{i=0}^{n-1}\lambda_i\Big)\frac{\mu_s}{(\ell-1)!}\int_x^\infty c(\vartheta)\int_\vartheta^\infty\Big(\int_0^\xi(1-C(\eta))\mathrm{d}\eta\Big)^{\ell-1}e^{-\mu_s\xi}\mathrm{d}\xi\mathrm{d}\vartheta$$

$$= g\Big(\prod_{i=0}^{n-1}\lambda_i\Big)\frac{\mu_s}{(\ell-1)!}\int_x^\infty\Big(\int_0^\xi(1-C(\eta))\mathrm{d}\eta\Big)^{\ell-1}(C(\xi)-C(x))e^{-\mu_s\xi}\mathrm{d}\xi, \qquad x\geq 0.$$

Thus for $x\in\mathbb{R}_+$ it holds

$$1-W_I(x)=\frac{g}{p_I\Lambda}\sum_{j=0}^{\infty}\Big(\prod_{i=0}^{s+j}\lambda_i\Big)\frac{\mu_s}{j!}\int_x^\infty\Big(\int_0^\xi(1-C(\eta))\mathrm{d}\eta\Big)^j(C(\xi)-C(x))e^{-\mu_s\xi}\mathrm{d}\xi. \tag{3.12}$$

The waiting time distribution $W(x)$ is given by

$$W(x)=(1-p_I)W_S(x)+p_IW_I(x). \tag{3.13}$$

The case of a general distribution $C(u)$ of the patience times is obtained again by considering $C(u)$ as the limit in distribution of a sequence of non-defective distributions $C_\nu(u)$ with continuous density. In view of (3.5), from (3.11), (3.12) and (3.13) applied to $C_\nu(u)$ by arguments of continuity we obtain

Theorem 3.2. Let the system be stable with a general distribution $C(u)$ of the i.i.d. patience times. Then for the waiting time distributions for $x \in I\!R_+$ it holds

$$1 - W_S(x) = \frac{p(s)}{(1-p_I)\Lambda} \sum_{j=0}^{\infty} \left(\prod_{i=s}^{s+j} \lambda_i \right) \frac{\mu_s}{j!} \int_x^{\infty} \left(\int_0^{\xi} (1 - C(\eta)) \mathrm{d}\eta \right)^j (1 - C(\xi)) e^{-\mu_s \xi} \mathrm{d}\xi, \quad (3.14)$$

$$1 - W_I(x) = \frac{p(s)}{p_I \Lambda} \sum_{j=0}^{\infty} \left(\prod_{i=s}^{s+j} \lambda_i \right) \frac{\mu_s}{j!} \int_x^{\infty} \left(\int_0^{\xi} (1 - C(\eta)) \mathrm{d}\eta \right)^j (C(\xi) - C(x)) e^{-\mu_s \xi} \mathrm{d}\xi, \quad (3.15)$$

$$1 - W(x) = \frac{p(s)}{\Lambda} (1 - C(x)) \sum_{j=0}^{\infty} \left(\prod_{i=s}^{s+j} \lambda_i \right) \frac{\mu_s}{j!} \int_x^{\infty} \left(\int_0^{\xi} (1 - C(\eta)) \mathrm{d}\eta \right)^j e^{-\mu_s \xi} \mathrm{d}\xi, \quad (3.16)$$

where the probability $p(s)$ that exactly s calls are in the system, the probability p_I that a typical arriving call will leave the system by impatience later and the cumulative arrival intensity Λ in the steady state are given by (3.5), (3.10) and (3.6), respectively.

Especially from (3.14) and (3.15) we get by integration over $x \in I\!R_+$:

$$EW_S = \frac{p(s)}{(1-p_I)\Lambda} \sum_{j=1}^{\infty} \left(\prod_{i=s}^{s+j-1} \lambda_i \right) \frac{\mu_s}{j!} \int_0^{\infty} \left(\int_0^{\xi} (1 - C(\eta)) \mathrm{d}\eta \right)^j (\mu_s \xi - 1) e^{-\mu_s \xi} \mathrm{d}\xi, \quad (3.17)$$

$$EW_I = \frac{p(s)}{p_I \Lambda} \sum_{j=1}^{\infty} \left(\prod_{i=s}^{s+j-1} \lambda_i \right) \frac{\mu_s}{j!} \int_0^{\infty} \left(\int_0^{\xi} (1 - C(\eta)) \mathrm{d}\eta \right)^j (j + 1 - \mu_s \xi) e^{-\mu_s \xi} \mathrm{d}\xi. \quad (3.18)$$

From Little's formula we obtain for the (unconditional) mean waiting time EW of a typical arriving call

$$EW = \frac{1}{\Lambda} \sum_{n=s+1}^{\infty} (n - s) p(n). \quad (3.19)$$

4 Application: Performance analysis of an ACD system with outbound calls and impatient inbound calls

In this section we apply the results of Section 3 to a performance analysis of the following Automatic Call Distributor system (ACD system). At a call center, consisting of a finite number s of agents and a finite number k of waiting places (i.e. $s+k$ lines), there arrive calls from outside (inbound calls) accordingly to a Poisson process of intensity λ. An arriving inbound call requires an exponentially distributed service time with parameter μ. If there is at least one free line, i.e. server or waiting place, then it will be accepted; otherwise it gets lost. An accepted call will be served immediately by one of the agents if there is anyone free. If all agents are busy then it begins to wait until one of the agents becomes free; the queueing discipline is FCFS. But the inbound calls are impatient, i.e. if the (virtual) waiting time until service exceeds a random time (patience), then it gets lost. The maximal patience times are assumed to be i.i.d. with a general distribution $C(u) = P(U \leq u)$, where U denotes a typical patience time. The patience times may correspond to real patience times or/and to a special management for those inbound calls whose waiting time exceeds an acceptable amount of time. Of special interest is the case that $U = \min(X, \tau)$, where X is exponentially distributed describing the individual patience time of an inbound call and τ is deterministic describing the "technical" impatience of the system, i.e. τ may be the deterministic time after that the inbound call is routed e.g. to another call center of the company.

For an improvement of the efficiency of the call center the agents dial outbound calls, too. Specifically, we introduce a parameter, $a \in \{1, 2, \ldots, s\}$, such that if more than a agents are idle (and therefore no calls are in the queue) then one of the idle agents will dial an outbound call rather than wait for an inbound call. Thus at each time instant with probability one at least $(s - a)$ agents are busy, c.f. [DPW]. We assume that there is an infinite reservoir of possible outbound calls (list of customers) and that the service times of the outbound calls are exponentially distributed with the same parameter μ as the inbound calls, cf. Fig. 4.1.

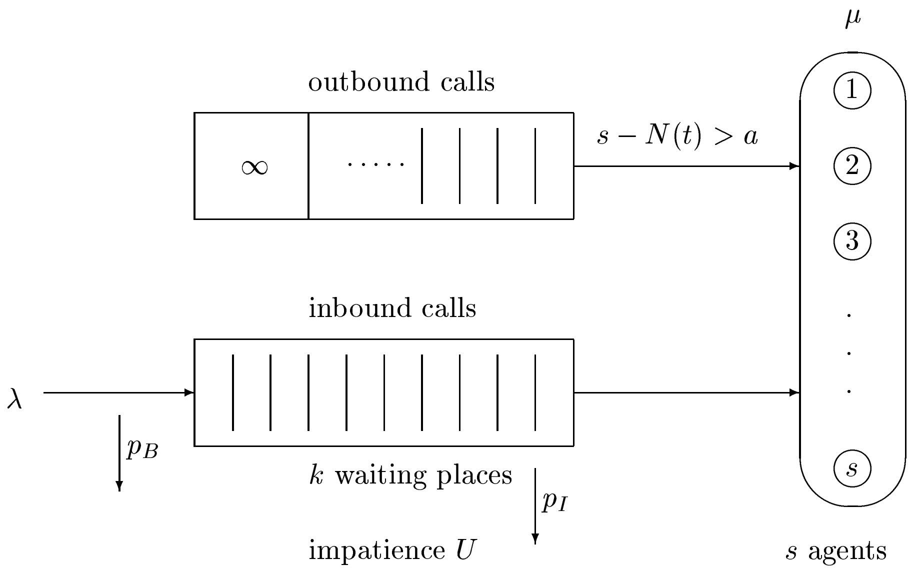

Fig. 4.1. **A**utomatic **C**all **D**istributor system: combined inbound-outbound call center with impatient inbound calls, s agents and k waiting places.

For the ACD system the following performance characteristics are of interest:

Λ_{in}	–	rate of accepted inbound calls,
Λ_{out}	–	rate of dialed outbound calls,
p_B	–	probability that a typical arriving inbound call finds no free waiting place or server (blocking probability, $1 - p_B$ is the acceptance probability),
p_W	–	probability that a typical accepted inbound call has to wait for service,
p_I	–	probability that a typical accepted inbound call gets lost because of impatience later (impatience probability),

$W_S(x)$	–	waiting time distribution of a typical accepted inbound call up to its service under the condition that it will be served,
$W_I(x)$	–	waiting time distribution of a typical accepted inbound call under the condition that it will leave the system by impatience later,
$W(x)$	–	(unconditional) waiting time distribution of a typical accepted inbound call.

The ACD system may be modeled by an $M(n)/M(m)/s+GI$ queueing system, where n corresponds to the number of inbound and served outbound calls in the ACD system and m to the number of busy agents, cf. Fig. 1.1. The state dependent arrival intensity λ_n is given by

$$\lambda_n = \mathbb{1}\{n < s+k\}\lambda, \qquad n = 0, 1, \ldots \tag{4.1}$$

and the state dependent service intensity μ_m by

$$\mu_m = \mathbb{1}\{m > s-a\}m\mu, \qquad m = 0, 1, \ldots, s. \tag{4.2}$$

(Note, that the time instants where a new outbound call is started doesn't correspond to changes of the system state. However, for getting the relevant performance measures this doesn't matter as seen below.)

In view of Theorem 3.1 the system is stable and for the stationary occupancy distribution it holds

$$p(n) = \begin{cases} g\, s!\mu^s \frac{(\lambda/\mu)^n}{n!}, & s-a \le n \le s, \\ g\, \lambda^s \frac{\mu_s}{(n-s)!} \int\limits_0^\infty \left(\lambda \int\limits_0^\xi (1-C(\eta))\mathrm{d}\eta\right)^{n-s} e^{-\mu_s \xi}\mathrm{d}\xi, & s < n \le s+k, \\ 0 & \text{elsewhere.} \end{cases} \tag{4.3}$$

From (3.2), (4.1) and (4.2) we get the normalizing factor

$$g^{-1} = s!\mu^s \sum_{j=s-a}^{s-1} \frac{(\lambda/\mu)^j}{j!} + \lambda^s \sum_{j=0}^{k} \frac{\mu_s}{j!} \int\limits_0^\infty \left(\lambda \int\limits_0^\xi (1-C(\eta))\mathrm{d}\eta\right)^j e^{-\mu_s \xi}\mathrm{d}\xi. \tag{4.4}$$

For the ACD system $p(n)$ is the stationary probability that n inbound and served outbound calls are in the system. The cumulative arrival intensity Λ in the $M(n)/M(m)/s+GI$ queueing system in the steady state corresponds to the rate Λ_{in} of accepted inbound calls in the ACD system. Hence from (3.6), (4.1) and (4.3) it follows

$$\Lambda_{in} = \Lambda = \lambda \sum_{n=s-a}^{s+k-1} p(n) = (1 - p(s+k))\lambda. \tag{4.5}$$

The acceptance probability $1-p_B$ for inbound calls is given by the ratio of the rate of accepted inbound calls and of the arrival intensity λ in the ACD system, i.e.

$$\Lambda_{in} = \Lambda = (1-p_B)\lambda. \tag{4.6}$$

Consequently, in view of (4.5) for the blocking probability it holds

$$p_B = p(s+k). \tag{4.7}$$

The service of an outbound call is started if $n = s - a$ agents are busy and one of the service times just finishes. The probability of being in the state $n = s-a$ is $p(s-a)$ and for the ACD system the cumulative intensity of finishing service in this state is $(s-a)\mu$. Therefore, for the rate Λ_{out} of dialed outbound calls we get

$$\Lambda_{out} = (s-a)\mu p(s-a). \tag{4.8}$$

From (3.7), (3.8), (4.2) and (4.6) we obtain the probability p_W that an accepted inbound call has to wait for service

$$p_W = 1 - \frac{1}{1-p_B}\sum_{n=s-a}^{s-1} p(n) = 1 - \frac{\mu}{(1-p_B)\lambda}\sum_{n=s-a+1}^{s} np(n) \tag{4.9}$$

and from (3.9), (3.10), (4.2), (4.3) and (4.6) the probability p_I that an accepted inbound call gets lost because of impatience later

$$\begin{aligned} p_I &= 1 - \frac{1}{(1-p_B)\lambda}\left(\sum_{n=s-a}^{s-1} \lambda p(n) + \sum_{n=s+1}^{\infty} \mu_s p(n)\right) \\ &= 1 - \frac{\mu}{(1-p_B)\lambda}\left(s - sp(s-a) - \sum_{n=s-a+1}^{s-1} (s-n)p(n)\right). \end{aligned} \tag{4.10}$$

From (3.14), (4.1) and (4.6) for the waiting time distribution of an accepted inbound call up to its service under the condition that it will be served it follows for $x \in \mathbb{R}_+$

$$1 - W_S(x) = \frac{p(s)}{(1-p_B)(1-p_I)}\sum_{j=0}^{k-1} \frac{\mu_s}{j!}\int_x^{\infty}\left(\lambda\int_0^{\xi}(1-C(\eta))\mathrm{d}\eta\right)^j (1-C(\xi))e^{-\mu_s\xi}\mathrm{d}\xi. \tag{4.11}$$

From (3.15), (4.1) and (4.6) for the waiting time distribution of an accepted inbound call under the condition that it will leave the system by impatience later we get for $x \in \mathbb{R}_+$

$$1 - W_I(x) = \frac{p(s)}{(1-p_B)p_I}\sum_{j=0}^{k-1} \frac{\mu_s}{j!}\int_x^{\infty}\left(\lambda\int_0^{\xi}(1-C(\eta))\mathrm{d}\eta\right)^j (C(\xi)-C(x))e^{-\mu_s\xi}\mathrm{d}\xi. \tag{4.12}$$

From (3.16), (4.1) and (4.6) for the unconditional waiting time distribution of an accepted inbound call we obtain for $x \in \mathbb{R}_+$

$$1 - W(x) = \frac{p(s)}{1-p_B}(1-C(x))\sum_{j=0}^{k-1} \frac{\mu_s}{j!}\int_x^{\infty}\left(\lambda\int_0^{\xi}(1-C(\eta))\mathrm{d}\eta\right)^j e^{-\mu_s\xi}\mathrm{d}\xi. \tag{4.13}$$

The corresponding expectations result from (4.1), (4.6) and (3.17), (3.18), respectively:

$$EW_S = \frac{p(s)}{(1-p_B)(1-p_I)\lambda}\sum_{j=1}^{k}\frac{\mu_s}{j!}\int_0^\infty\left(\lambda\int_0^\xi(1-C(\eta))\mathrm{d}\eta\right)^j(\mu_s\xi-1)e^{-\mu_s\xi}\mathrm{d}\xi, \tag{4.14}$$

$$EW_I = \frac{p(s)}{(1-p_B)p_I\lambda}\sum_{j=1}^{k}\frac{\mu_s}{j!}\int_0^\infty\left(\lambda\int_0^\xi(1-C(\eta))\mathrm{d}\eta\right)^j(j+1-\mu_s\xi)e^{-\mu_s\xi}\mathrm{d}\xi. \tag{4.15}$$

Little's formula yields for the expectation EW of the unconditional waiting time of an arriving inbound call

$$EW = \frac{1}{(1-p_B)\lambda}\sum_{n=s+1}^{s+k}(n-s)p(n). \tag{4.16}$$

Remark 4.1. For $k \to \infty$ we obtain the case of an unlimited waiting room. From (4.4) for the normalizing factor we get

$$g^{-1} = s!\mu^s\sum_{j=s-a}^{s-1}\frac{(\lambda/\mu)^j}{j!} + \lambda^s\mu_s\int_0^\infty\exp\left(\lambda\int_0^\xi(1-C(\eta))\mathrm{d}\eta-\mu_s\xi\right)\mathrm{d}\xi.$$

The stability condition corresponds to $g^{-1} < \infty$ which is equivalent to $(\lambda/\mu_s)\lim_{u\to\infty}(1-C(u)) < 1$, cf. [BH].

4.1 Special case: Impatience time as the minimum of a constant and an exponentially distributed time

In this subsection we consider the special case that the typical maximal patience time U is the minimum of a constant and an exponentially distributed time, i.e. $U = \min(X, \tau)$, where X is exponentially distributed with parameter α describing the individual patience time of an inbound call and τ is deterministic describing the "technical" impatience of the ACD system. The distribution of U is given by

$$C(u;\alpha,\tau) := \begin{cases} 1-e^{-\alpha u}, & 0 \le u < \tau, \\ 1, & u \ge \tau. \end{cases} \tag{4.17}$$

From (4.17) for $\xi \ge 0$ we get

$$\int_0^\xi(1-C(\eta;\alpha,\tau))\mathrm{d}\eta = \int_0^{\min(\xi,\tau)}e^{-\alpha\eta}\mathrm{d}\eta = \frac{1-e^{-\alpha\min(\xi,\tau)}}{\alpha}. \tag{4.18}$$

Lemma 4.2. Let α, β, x be positive real numbers and j a non-negative integer.

Then it holds

$$\frac{1}{j!}\int_0^x \left(\frac{1-e^{-\alpha\xi}}{\alpha}\right)^j e^{-\beta\xi}\mathrm{d}\xi$$

$$= \left(\prod_{i=0}^{j}\frac{1}{\beta+i\alpha}\right)\left(1-e^{-\beta x}\sum_{i=0}^{j}\binom{-\beta/\alpha}{i}\left(e^{-\alpha x}-1\right)^i\right) \tag{4.19}$$

$$= \left(\prod_{i=0}^{j}\frac{1}{\beta+i\alpha}\right)e^{-\beta x}\sum_{i=j+1}^{\infty}\binom{-\beta/\alpha}{i}\left(e^{-\alpha x}-1\right)^i, \tag{4.20}$$

$$\frac{1}{j!}\int_0^x \left(\frac{1-e^{-\alpha\xi}}{\alpha}\right)^j e^{-\beta\xi}\xi\mathrm{d}\xi - \left(\sum_{\ell=0}^{j}\frac{1}{\beta+\ell\alpha}\right)\frac{1}{j!}\int_0^x \left(\frac{1-e^{-\alpha\xi}}{\alpha}\right)^j e^{-\beta\xi}\mathrm{d}\xi$$

$$= -\left(\prod_{i=0}^{j}\frac{1}{\beta+i\alpha}\right)e^{-\beta x}\sum_{i=0}^{j}\left(x-\sum_{\ell=0}^{i-1}\frac{1}{\beta+\ell\alpha}\right)\binom{-\beta/\alpha}{i}\left(e^{-\alpha x}-1\right)^i \tag{4.21}$$

$$= \left(\prod_{i=0}^{j}\frac{1}{\beta+i\alpha}\right)e^{-\beta x}\sum_{i=j+1}^{\infty}\left(x-\sum_{\ell=0}^{i-1}\frac{1}{\beta+\ell\alpha}\right)\binom{-\beta/\alpha}{i}\left(e^{-\alpha x}-1\right)^i. \tag{4.22}$$

Proof. For $j=1,2,\dots$ integration by parts yields

$$\frac{1}{j!}\int_0^x \left(\frac{1-e^{-\alpha\xi}}{\alpha}\right)^j e^{-\beta\xi}\mathrm{d}\xi = \frac{1}{j!}\int_0^x \left(\frac{e^{\alpha\xi}-1}{\alpha}\right)^j e^{-(\beta+j\alpha)\xi}\mathrm{d}\xi$$

$$= \frac{1}{(j+1)!}\Bigg(\frac{(j+1)!}{(\beta+j\alpha)\dots(\beta+\alpha)\beta}$$

$$-\sum_{i=0}^{j}\frac{(j+1)\dots(i+1)}{(\beta+j\alpha)\dots(\beta+i\alpha)}\left(\frac{e^{\alpha x}-1}{\alpha}\right)^i e^{-(\beta+i\alpha)x}\Bigg).$$

Since this equation also holds in case of $j=0$ we get (4.19). Using the binomial series from (4.19) it follows (4.20). Multiplication of (4.19), (4.20) by $\beta(\beta+\alpha)\dots(\beta+j\alpha)$ and differentiation with respect to β yields the representations (4.21) and (4.22), respectively.

Firstly, we investigate the integrals occurring in the representation (4.3) of the stationary occupancy distribution. In view of (4.18) for $j=1,2,\dots,k$ integration by parts yields

$$I_j(\tau) := \frac{\mu_s}{j!}\int_0^\infty \left(\lambda\int_0^\xi (1-C(\eta;\alpha,\tau))\mathrm{d}\eta\right)^j e^{-\mu_s\xi}\mathrm{d}\xi$$

$$= \frac{\lambda^j}{(j-1)!}\int_0^\tau \left(\frac{1-e^{-\alpha\xi}}{\alpha}\right)^{j-1} e^{-(\mu_s+\alpha)\xi}\mathrm{d}\xi. \tag{4.23}$$

Hence from Lemma 4.2, (4.19) and (4.20), for $j = 1, 2, \ldots, k$ we get

$$I_j(\tau) = \left(\prod_{i=1}^{j} \frac{\lambda}{\mu_s + i\alpha}\right)\left(1 - e^{-(\mu_s+\alpha)\tau} \sum_{i=0}^{j-1} \binom{-\mu_s/\alpha - 1}{i} \left(e^{-\alpha\tau} - 1\right)^i\right) \tag{4.24}$$

$$= \left(\prod_{i=1}^{j} \frac{\lambda}{\mu_s + i\alpha}\right) e^{-(\mu_s+\alpha)\tau} \sum_{i=j}^{\infty} \binom{-\mu_s/\alpha - 1}{i} \left(e^{-\alpha\tau} - 1\right)^i . \tag{4.25}$$

Obviously, equations (4.24) and (4.25) also hold in case of $j = 0$. The first equation (4.24) allows to compute $I_j(\tau)$ recursively for $j = 0, 1, \ldots, k$. But this recursion may become numerically unstable. The second equation (4.25) yields convergent series with positive members and therefore a stable method for computing $I_j(\tau)$ in case of $j = k$. After the computation of the integral for $j = k$ the integrals for $j = k-1, k-2, \ldots, 0$ may be computed recursively by means of (4.25).

In view of (4.17), (4.18) and (4.23) for the integrals occurring in the representation (4.11) of the waiting time distribution $W_S(x)$ of an accepted inbound call up to its service under the condition that it will be served for $x \in [0, \tau)$ and $j = 0, 1, \ldots, k-1$ it follows

$$\frac{\mu_s}{j!} \int_x^\infty \left(\lambda \int_0^\xi (1 - C(\eta; \alpha, \tau))\mathrm{d}\eta\right)^j (1 - C(\xi; \alpha, \tau))e^{-\mu_s \xi}\mathrm{d}\xi$$

$$= \frac{\mu_s \lambda^j}{j!} \int_x^\tau \left(\frac{1 - e^{-\alpha\xi}}{\alpha}\right)^j e^{-(\mu_s+\alpha)\xi}\mathrm{d}\xi = \frac{\mu_s}{\lambda} \left(I_{j+1}(\tau) - I_{j+1}(x)\right). \tag{4.26}$$

In view of (4.17), (4.18) and (4.23) for the integrals occurring in the representation (4.13) of the unconditional waiting time distribution $W(x)$ of an accepted inbound call for $x \in [0, \tau)$ and $j = 0, 1, \ldots, k-1$ integration by parts yields

$$\frac{\mu_s}{j!} \int_x^\infty \left(\lambda \int_0^\xi (1 - C(\eta; \alpha, \tau))\mathrm{d}\eta\right)^j e^{-\mu_s \xi}\mathrm{d}\xi$$

$$= \frac{\mu_s \lambda^j}{j!} \left(\int_x^\tau \left(\frac{1 - e^{-\alpha\xi}}{\alpha}\right)^j e^{-\mu_s\xi}\mathrm{d}\xi + \int_\tau^\infty \left(\frac{1 - e^{-\alpha\tau}}{\alpha}\right)^j e^{-\mu_s\xi}\mathrm{d}\xi\right)$$

$$= \frac{\lambda^j}{j!} \left(\frac{1 - e^{-\alpha x}}{\alpha}\right)^j e^{-\mu_s x} + \left(I_j(\tau) - I_j(x)\right). \tag{4.27}$$

Using (3.13) the waiting time distribution $W_I(x)$ of an accepted inbound call under the condition that it will leave the system by impatience later may be computed from the waiting time distributions $W_S(x)$ and $W(x)$. Analogously, the corresponding expectation EW_I may be computed from the expectations EW_S and EW. Because of (4.16) the expectation EW of the unconditional waiting time is given by the stationary occupancy distribution. In view of (4.18) for the integrals occurring in the representation (4.14) of the expectation EW_S for $j = 1, 2, \ldots, k$

integration by parts yields

$$
\begin{aligned}
&\frac{\mu_s}{j!}\int\limits_0^\infty\left(\lambda\int\limits_0^\xi(1-C(\eta;\alpha,\tau))\mathrm{d}\eta\right)^j(\mu_s\xi-1)e^{-\mu_s\xi}\mathrm{d}\xi\\
&= \frac{\mu_s\lambda^j}{(j-1)!}\int\limits_0^\tau\left(\frac{1-e^{-\alpha\xi}}{\alpha}\right)^{j-1}e^{-(\mu_s+\alpha)\xi}\xi\mathrm{d}\xi.
\end{aligned}
$$

Hence from Lemma 4.2, (4.21) and (4.22), and from (4.23), (4.24), (4.25) for $j=1,2,\ldots,k$ we get

$$
\begin{aligned}
&\frac{\mu_s}{j!}\int\limits_0^\infty\left(\lambda\int\limits_0^\xi(1-C(\eta;\alpha,\tau))\mathrm{d}\eta\right)^j(\mu_s\xi-1)e^{-\mu_s\xi}\mathrm{d}\xi\\
&= \left(\sum_{\ell=1}^j\frac{\mu_s}{\mu_s+\ell\alpha}\right)I_j(\tau)\\
&\quad -\left(\prod_{i=1}^j\frac{\lambda}{\mu_s+i\alpha}\right)e^{-(\mu_s+\alpha)\tau}\sum_{i=0}^{j-1}\left(\mu_s\tau-\sum_{\ell=1}^i\frac{\mu_s}{\mu_s+\ell\alpha}\right)\binom{-\mu_s/\alpha-1}{i}\left(e^{-\alpha\tau}-1\right)^i\\
&= \mu_s\tau I_j(\tau)-\left(\prod_{i=1}^j\frac{\lambda}{\mu_s+i\alpha}\right)\\
&\quad\left(\mu_s\tau-\sum_{\ell=1}^j\frac{\mu_s}{\mu_s+\ell\alpha}\left(1-e^{-(\mu_s+\alpha)\tau}\sum_{i=0}^{\ell-1}\binom{-\mu_s/\alpha-1}{i}\left(e^{-\alpha\tau}-1\right)^i\right)\right) \qquad (4.28)\\
&= \left(\sum_{\ell=1}^j\frac{\mu_s}{\mu_s+\ell\alpha}\right)I_j(\tau)\\
&\quad +\left(\prod_{i=1}^j\frac{\lambda}{\mu_s+i\alpha}\right)e^{-(\mu_s+\alpha)\tau}\sum_{i=j}^{\infty}\left(\mu_s\tau-\sum_{\ell=1}^i\frac{\mu_s}{\mu_s+\ell\alpha}\right)\binom{-\mu_s/\alpha-1}{i}\left(e^{-\alpha\tau}-1\right)^i\\
&= \mu_s\tau I_j(\tau)\\
&\quad -\left(\prod_{i=1}^j\frac{\lambda}{\mu_s+i\alpha}\right)e^{-(\mu_s+\alpha)\tau}\sum_{i=j+1}^{\infty}\left(\sum_{\ell=j+1}^i\frac{\mu_s}{\mu_s+\ell\alpha}\right)\binom{-\mu_s/\alpha-1}{i}\left(e^{-\alpha\tau}-1\right)^i. \qquad (4.29)
\end{aligned}
$$

The first representation (4.28) allows to compute the integrals at the left-hand side recursively for $j=1,2,\ldots,k$. But this recursion may become numerically unstable. The second representation (4.29) yields convergent series with positive members and therefore a stable method for the computation of the integral at the left-hand side in case of $j=k$. After the computation of the integral for $j=k$ the integrals for $j=k-1,k-2,\ldots,1$ may be computed recursively by means of (4.29).

Remark 4.3. The limiting case $k\to\infty$ with $\lambda_n=\lambda$ for $n\geq s$ and $\mu_m=m\mu$ for $0\leq m\leq s$ was earlier analyzed in [Ju1]. For several performance measures, basing on the waiting time vector process, in this case formulas are available. However, they don't cover the occupancy distribution.

Remark 4.4. The considered case covers two special cases. For $\tau \to \infty$ the process $N(t)$ of the number of calls in the system converges to a birth death process which can be analyzed in the usual manner. For $\alpha \to 0$ we get a system with constant patience times; for earlier results we refer to [GK].

4.2 Numerical results

In this subsection we give some numerical results. The basic parameter set is as follows:

- mean service time $1/\mu$: 120 seconds,
- mean individual patience time $1/\alpha$: 90 seconds,
- deterministic patience time τ: 60 seconds.

In table 4.2 the offered load λ/μ is 10 Erl, in table 4.3 the offered load is 100 Erl. The number s of agents, the number k of waiting places and the outbound parameter a vary in these tables. The blocking probability p_B, the impatience probability p_I, the expectations EW_S and EW_I of the conditional waiting time distributions and the rate of dialed outbound calls Λ_{out} are presented. In the last four lines of the tables the corresponding performance measures for an ACD system without outbound calls are given.

s	k	a	λ/μ	$1/\mu$	$1/\alpha$	τ	p_B	p_I	EW_S	EW_I	Λ_{out}
8	3	3	10	120	90	60	0.137	0.170	11.472	22.286	0.003
12	3	3	10	120	90	60	0.049	0.061	4.729	14.258	0.015
16	3	3	10	120	90	60	0.016	0.024	1.955	9.988	0.039
20	3	3	10	120	90	60	0.006	0.011	0.884	7.687	0.067
8	6	3	10	120	90	60	0.024	0.254	15.696	26.739	0.002
12	6	3	10	120	90	60	0.006	0.088	6.341	17.738	0.015
16	6	3	10	120	90	60	0.001	0.031	2.446	11.960	0.038
20	6	3	10	120	90	60	0.000	0.013	1.037	8.769	0.067
8	3	6	10	120	90	60	0.131	0.162	10.769	22.286	0.000
12	3	6	10	120	90	60	0.034	0.042	3.174	14.258	0.004
16	3	6	10	120	90	60	0.006	0.009	0.723	9.988	0.019
20	3	6	10	120	90	60	0.001	0.002	0.173	7.687	0.045
8	3	8	10	120	90	60	0.131	0.162	10.758	22.286	0.000
12	3	12	10	120	90	60	0.031	0.039	2.931	14.258	0.000
16	3	16	10	120	90	60	0.003	0.005	0.388	9.988	0.000
20	3	20	10	120	90	60	0.000	0.000	0.023	7.687	0.000

Tab. 4.2: Blocking probability p_B, impatience probability p_I, the expectations EW_S and EW_I of the conditional waiting time distributions and the rate of dialed outbound calls Λ_{out} for an offered load of 10 Erl.

s	k	a	λ/μ	$1/\mu$	$1/\alpha$	τ	p_B	p_I	EW_S	EW_I	Λ_{out}
90	15	10	100	120	90	60	0.037	0.081	7.365	6.568	0.006
100	15	10	100	120	90	60	0.012	0.042	3.681	5.328	0.025
110	15	10	100	120	90	60	0.003	0.018	1.551	4.307	0.067
120	15	10	100	120	90	60	0.001	0.007	0.608	3.501	0.127
90	30	10	100	120	90	60	0.002	0.111	10.155	8.817	0.005
100	30	10	100	120	90	60	0.000	0.050	4.442	6.466	0.024
110	30	10	100	120	90	60	0.000	0.020	1.702	4.827	0.066
120	30	10	100	120	90	60	0.000	0.007	0.634	3.726	0.127
90	15	20	100	120	90	60	0.037	0.079	7.146	6.568	0.000
100	15	20	100	120	90	60	0.010	0.036	3.117	5.328	0.004
110	15	20	100	120	90	60	0.002	0.011	0.913	4.307	0.022
120	15	20	100	120	90	60	0.000	0.002	0.200	3.501	0.065
90	15	90	100	120	90	60	0.036	0.079	7.138	6.568	0.000
100	15	100	100	120	90	60	0.010	0.035	3.053	5.328	0.000
110	15	110	100	120	90	60	0.001	0.009	0.776	4.307	0.000
120	15	120	100	120	90	60	0.000	0.001	0.102	3.501	0.000

Tab. 4.3: Blocking probability p_B, impatience probability p_I, the expectations EW_S and EW_I of the conditional waiting time distributions and the rate of dialed outbound calls Λ_{out} for an offered load of 100 Erl.

The numerical results show the impact of the operational strategy (outbound parameter a) on the system performance. Also they illustrate the tradeoff between the blocking probability p_B and the impatience probability p_I if k varies. There is no obvious relation between the mean conditional waiting times EW_S and EW_I.

References

[BBH] Baccelli, F., Boyer, P., Hebuterne, G., Single server queues with impatient customers. Adv. Appl. Prob. 16 (1984) 887–905.

[BH] Baccelli, F., Hebuterne, G., On queues with impatient customers. Performance '81. F.J. Kylstra (editor). North-Holland Publishing Company, (1981) 159–179.

[Ba1] Barrer, D.Y., Queueing with impatient customers and indifferent clerks. Operations Research 5 (1957) 644–649.

[Ba2] Barrer, D.Y., Queueing with impatient customers and ordered service. Operations Research 5 (1957) 650–656.

[BB] Brandt, A., Brandt, M., Dimensioning of ACD systems, I–IV (in German). Working papers, Humboldt-University of Berlin, Berlin, 1994–1996. Preprint in preparation.

[BFL] Brandt, A., Franken, P., Lisek, B., Stationary Stochastic Models. Akademie-Verlag, Berlin; Wiley, Chichester, 1990.

[Br1] Brodi, S.M., On an integro-differential equation for systems with τ-waiting (in Ukrainian). Dop. AN URSR No. 6 (1959) 571–573.

[Br2] Brodi, S.M., On a service problem (in Russian). Tr. V. Vsesojuznogo soveščanija po teorii verojatnostej i matematičeskoj statistike, Izd-vo AN Arm. SSR, Erevan (1960) 143–147.

[Dal] Daley, D.J., General customer impatience in the queue GI/G/1. J. Appl. Prob. 2 (1965) 186–205.

[DPW] Dumas, G., Perkins, M., White, C., Improving efficiency of PBX-based call centers: Combining inbound and outbound agents with automatic call sharing. Proc. 15th International Switching Symposium Berlin, (1995) 346–350.

[FKAS] Franken, P., König, D., Arndt, U., Schmidt, V., Queues and Point Processes. Akademie-Verlag, Berlin; Wiley, Chichester, 1982.

[GKo] Gnedenko, B.W., König, D., Handbuch der Bedienungstheorie, Vol. II. Akademie-Verlag, Berlin, 1984.

[GK] Gnedenko, B.W., Kowalenko, I.N., Einführung in die Bedienungstheorie. (first edition in Russian, Nauka, 1966) Akademie-Verlag, Berlin, 1974.

[Hau] Haugen, F.R.B., Queueing Systems with several input streams and time out. Telektronikk nr. 2 (1978) 100–106.

[HSk] Haugen, R.B., Skogan, E., Queueing Systems with Stochastic Time Out. IEEE Trans. Commun. vol. COM-28 (1980) 1984–1989.

[Ju1] Jurkevič, O.M., On the investigation of many-server queueing systems with bounded waiting time (in Russian). Izv. Akad. Nauk SSSR Techničeskaja kibernetika No. 5 (1970) 50–58.

[Ju2] Jurkevič, O.M., On many-server systems with stochastic bounds for the waiting time (in Russian). Izv. Akad. Nauk SSSR Techničeskaja kibernetika No. 4 (1971) 39–46.

[Ju3] Jurkevič, O.M., On many-server systems with general service time distribution of "impatient customers" (in Russian). Izv. Akad. Nauk SSSR Techničeskaja kibernetika No. 6 (1973) 80–89.

[Per] Perry, M., Performance Modelling of Automatic Call Distributors. Ph. D. Thesis, North Carolina State University, 1991.

[PN] Perry, M., Nilsson, A., Performance Modelling of Automatic Call Distributors: Assignable Grade of Service Staffing. Proc. 14th Int. Switching Symposium Yokohama (1992) 294–298.

[Sin] Singh, M., Steady state behaviour of serial queueing processes with impatient customers. Math. Operationsforsch. ser. statist. 15 (2) (1984) 289–298.

[ST] So, K.C., Tang, C., Operational strategies for managing congestion in service systems. Working paper, Graduate School of Management University of California, Irvine, California, 1993.

[Teg] Teghem, J., Use of discrete transforms for the study of a GI/M/s queue with impatient customer phenomena. Zeitschrift für Operations Research 23 (1979) 95–106.

[W] Wallstrom, B., A queueing system with time-outs and random departure. Proc. ITC 8 Melbourne (1976), paper 231.