Numerical Evaluation of Resolvents and Laplace Transforms of Markov Processes Using Linear Programming [1]

Kurt Helmes[2] and Richard H. Stockbridge[3]

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Abstract. This paper uses linear programming to numerically evaluate the Laplace transform of the exit time distribution and the resolvent of the moments of various Markov processes in bounded regions. The linear programming formulation is developed from a martingale characterization of the processes and the use of occupation measures. The LP approach naturally provides both upper and lower bounds on the quantities of interest. The processes analyzed include the Poisson process, one-dimensional Brownian motion (with and without drift), an Ornstein-Uhlenbeck process and two-dimensional Brownian motion. The Laplace transform of the original Cameron-Martin formula is also numerically evaluated by reducing it to the analysis of an Ornstein-Uhlenbeck process.

Key words. Markov processes, Laplace transforms, resolvents, linear programming, Hausdorff moment conditions, Brownian motion, Ornstein-Uhlenbeck process, Cameron-Martin formula.

Abbreviated Title. Computing Laplace Transforms by Linear Programming.

AMS(MOS) subject classifications. 60G40, 60K99, 65N20, 90C05, 90C50, 93E25.

1 Introduction

The purpose of this paper is to provide a numerical method based on linear programming for computing both the resolvents and the Laplace transform of distributions associated with Markov processes. This numerical approach has the advantage over other approaches that it naturally provides bounds on the quantities of interest and excellent software is readily available.

We use a linear programming formulation for the evolution of Markov processes which involves solving a system of equations over measures on the states of the process. We characterize the set of feasible measures for the LP through linear conditions which must be satisfied. An important aspect of this approach is that we are directly analyzing the stochastic processes and not using some approximating processes.

In this paper we consider the special case of Markov processes which stop when they exit a bounded region. Specifically, we introduce the approach by considering a stopped Poisson process and analyze a stopped one-dimensional Brownian motion, drifted Brownian motion, an Ornstein-Uhlenbeck process, and a multidimensional Brownian motion process on the unit square, and we numerically evaluate the Laplace transform of the original Cameron-Martin formula. The linear programs for these problems are large and highly structured. This structure means that matrix generating languages like AMPL or GAMS are ideal as interfaces with the linear programming software. We illustrate the power and accuracy of the

[1]This research is partially supported by NSF under grant DMS 9803490.

[2]Institute of Operations Research, Humboldt University of Berlin, Berlin, Germany

[3]Department of Statistics, University of Kentucky, Lexington, KY 40506–0027

LP method by comparing the numerical results for the Laplace transforms of the exit time with analytical ones and by computing the resolvents of the processes and the transforms of the moments of the occupation measures.

Formulating the evolution of the Markov process through occupation measures developed from work by Young [21] for problems in the calculus of variations. The present approach is a special case of the more general situation of formulating stochastic control problems as linear programs over a space of stationary distributions studied in very general settings in [1, 14, 18, 19]. The variable in the infinite-dimensional linear program is a measure on the product of the state and control spaces, and in the case of exit problems, this variable is augmented by a second measure on the exterior of the state space (see [14]).

The linear programming approach to stochastic control at this level of generality had its foundations in the existence of stationary processes corresponding to the measure which satisfies a basic adjoint equation (the identity (3.5) of this paper). This was established in [18] and later extended to the more general adjoint equation for exit problems in [14]. Kurtz also applied these techniques in uncontrolled settings for patchwork and constrained martingale problems ([12, 13]) and, in joint work with Dai, used this in the analysis of Brownian networks [5]. The basic adjoint equation corresponding to the heavy traffic limit of queueing processes had previously been introduced by Harrison and Reiman [8]. Other papers using linear programming for stochastic control include [15, 17].

The processes in this paper are *not* controlled. However, the results cited above are applicable to these processes by taking the control space to consist of a single element. It then follows that the variables are measures solely on the states of the process (we include time as a state component).

The paper [9] uses a similar linear programming approach to the analysis of exit time distributions. However, the focus of that paper was on directly evaluating the moments of these distributions. In contrast we seek to evaluate in this paper the Laplace transforms of the exit time distribution, of the occupation measures of the process and the resolvents of the processes. This indirect approach to the computation of the distributions or the moments of these distributions has the advantage of reducing the dimension of the problems and, as a side effect, increases the numerical stability of the method. An example of the benefit of such a reduction in dimension is given in section 7 where the Laplace transform of the exit time distribution of two dimensional Brownian motion from the unit square is computed whereas only the mean exit time was computed in [9].

The fundamental idea of our numerical method is to analyze a time-space process for which the time process continues forever but the space process stops when it exits the bounded region and to start with the characterization of the problem as an infinite-dimensional linear program. The resolvents and Laplace transforms are obtained by making a judicious choice of test functions. The constraints are generated by these test functions via the martingale characterization of the processes (see (3.2)) and an associated identity (see (3.5)); see also [6] for the martingale characterization. The identity characterizes the processes in terms of the Laplace transform of the moments of measures which reduces the analysis from the space of measures to the space of moment sequences of the measures. Additional linear constraints (the Hausdorff conditions (2.6) and (3.10)) are added to ensure that the sequences, in fact, are the moments of the measures.

This paper is organized as follows. We motivate the linear programming approach by considering, in section 2, the example of a Poisson process. This example contains most of the key ideas in a simple form. Then in section 3, we state the general linear programming formulation in more general terms and, in particular, identify the identity (3.5) and the Hausdorff moment conditions (3.10) which are at the center of this approach. Beginning with section 4, we illustrate the accuracy of the linear programming approach with particular examples of continuous processes for which formulas for the "exact" values are known. In section 4, we analyze the simplest model of Brownian motion on the unit interval and provide numerical comparisons of the LP approach with exact values. Section 5 analyzes a Brownian motion process with drift on a general bounded interval. Though a simple model, it can be used to analyze the geometric Brownian motion process since taking the logarithm of geometric Brownian motion yields a drifted Brownian motion. In section 6 we examine the Ornstein-Uhlenbeck process to illustrate how state dependence in the coefficients can be handled. The exit time distribution of a two-dimensional Brownian motion process from the unit square is considered in section 7, where it is also necessary to use the Hausdorff moment conditions in a multidimensional setting. In section 8, we give numerical computation of the original Cameron-Martin formula for the Laplace transform of the squared L^2-norm of trajectories of Brownian motion in the interval $[0, t]$.

2 Poisson Process

We introduce the basic idea underlying the linear programming approach to the computation of the Laplace transform by considering one of the simplest examples. Let $N = (N_t : t \geq 0)$ be a Poisson process having parameter $\beta > 0$ and let τ denote the first time at which $N_t = K$, for a fixed positive integer K. Partition the states of the process $\{0, 1, 2, \ldots\}$ into a bounded region $E_0 = \{0, 1, \ldots, K-1\}$ and its complement and view the time τ as the first time N exits E_0. Note that when N exits, it does so by hitting the boundary state K. It is well-known that τ has a gamma (K, β) distribution so a simple calculation shows that the Laplace transform of the density, denoted ϕ, is

$$\int_0^\infty e^{-\lambda s} \phi(s)\, ds = \left(\frac{\beta}{\lambda + \beta} \right)^K .$$

Important to the linear programming formulation is the generator A of N. This operator acts on functions $f : \{0, 1, 2, \ldots\} \to \mathbb{R}$ and is given by

$$Af(x) = \beta(f(x+1) - f(x)).$$

We extend A to the generator of a time-space process by defining $\widehat{A}$ on functions $f : [0, \infty) \times \{0, 1, 2, \ldots\} \to \mathbb{R}$, with $f(t, x) \to 0$ as $t \to \infty$ and f continuously differentiable in t, by

$$\widehat{A} f(t, x) = \frac{\partial f}{\partial t}(t, x) + \beta(f(t, x+1) - f(t, x)) I_{E_0}(x).$$

The generator $\widehat{A}$ implies that the time process always increases linearly but the space process stops when $N_t = K$. Define $\widehat{N}_t = N_{t \wedge \tau}$ for each t and note that $\widehat{N}$ and N are the same process up to the random time τ. Then for each function f in the domain of $\widehat{A}$,

$$f(t, \widehat{N}_t) - f(0, \widehat{N}_0) - \int_0^t \widehat{A} f(s, \widehat{N}_s) ds \tag{2.1}$$

is a martingale, which implies that for each $t > 0$,

$$E\left[f(t, \widehat{N}_t) - f(0, \widehat{N}_0) - \int_0^t \widehat{A} f(s, \widehat{N}_s) ds \right] = 0.$$

Since $f(t,x)$ is continuous in t for each x and $f(t, \widehat{N}_t) \to 0$ as $t \to \infty$, and $\widehat{N}_t \le K$ implies there are only a finite number of states x, letting $t \to \infty$ gives

$$E[f(0, \widehat{N}_0)] = -E\left[\int_0^\infty \widehat{A} f(s, \widehat{N}_s) ds \right]. \tag{2.2}$$

Now we consider a specific form for the test function f. Let $f(t,x) = e^{-\lambda t}(\frac{x}{K})^n$ with $\lambda > 0$ and observe that (2.2) can be written as

$$\begin{aligned}
0^n &= -E\left[\int_0^\infty e^{-\lambda s} \left((-\lambda) \left(\frac{\widehat{N}_s}{K} \right)^n + \beta \left(\left(\frac{\widehat{N}_s + 1}{K} \right)^n - \left(\frac{\widehat{N}_s}{K} \right)^n \right) I_{E_0}(\widehat{N}_s) \right) ds \right] \\
&= \lambda E\left[\int_0^\infty e^{-\lambda s} \left(\frac{\widehat{N}_s}{K} \right)^n I_{\{K\}}(\widehat{N}_s) ds \right] \\
&\quad + E\left[\int_0^\infty e^{-\lambda s} \left((\lambda + \beta) \left(\frac{\widehat{N}_s}{K} \right)^n - \beta \left(\frac{\widehat{N}_s + 1}{K} \right)^n \right) I_{E_0}(\widehat{N}_s) ds \right] \\
&= \lambda \int_0^\infty e^{-\lambda s} P(\tau \le s) ds \qquad (2.3) \\
&\quad + E\left[\int_0^\infty e^{-\lambda s} \left((\lambda + \beta) \left(\frac{\widehat{N}_s}{K} \right)^n - \beta \sum_{j=0}^n \binom{n}{j} \left(\frac{1}{K} \right)^{n-j} \left(\frac{\widehat{N}_s}{K} \right)^j \right) I_{E_0}(\widehat{N}_s) ds \right].
\end{aligned}$$

When $n = 0$, we take $0^0 = 1$. Notice that if we integrate by parts the first term of the last expression, we have

$$\lambda \int_0^\infty e^{-\lambda s} P(\tau \le s) ds = \int_0^\infty e^{-\lambda s} \phi(s) ds \tag{2.4}$$

in which ϕ denotes the density of the exit time τ. Thus the first term in (2.3) gives the Laplace transform of the density of the exit time τ or a factor times the Laplace transform of the distribution function of τ. The second term of the last expression of (2.3) is a linear combination of terms of the form

$$E\left[\int_0^\infty e^{-\lambda s} \left(\frac{\widehat{N}_s}{K} \right)^k I_{E_0}(\widehat{N}_s) ds \right]$$

which are the Laplace transforms of the (scaled) moments of $\widehat{N}$ during the period of time that $\widehat{N}$ occupies E_0. Designating the Laplace transform of the distribution function by q_λ and the Laplace transorm of these moments of $\widehat{N}$ by $\gamma_\lambda(k)$, (2.3) takes the form

$$0^n = \lambda q_\lambda + (\lambda+\beta)\gamma_\lambda(n) - \beta \sum_{j=0}^{n} \left(\begin{array}{c} n \\ j \end{array} \right) (1/K)^{n-j} \gamma_\lambda(j). \tag{2.5}$$

The identity (2.5) holds for each $n \geq 0$ which establishes linear conditions that q_λ and the sequence $\{\gamma_\lambda(j)\}$ must satisfy.

We now need to examine the sequence $\{\gamma_\lambda(j) : j = 0, 1, 2, \ldots\}$ more carefully. The elements of the sequence are the Laplace transforms of the (scaled) moments of $\widehat{N}$ while it occupies E_0. In this, the indicator $I_{E_0}(\widehat{N}_s)$ is the Radon-Nikodým derivative of the occupation measure. By considering a different Radon-Nikodým derivative, namely $e^{-\lambda s} I_{E_0}(\widehat{N}_s)$, $\gamma_\lambda(j)$ also gives the jth moment of $\widehat{N}$ under this new measure. Thus the sequence $\{\gamma_\lambda(j) : j = 0, 1, 2, \ldots\}$ must be a moment sequence. It is at this point that the scaling $(\frac{x}{K})$ becomes important. By adjusting the definition of the measure to incorporate this scaling into it, the measure will be concentrated on the interval $[0, 1]$. A necessary and sufficient condition for a sequence to be a moment sequence of a measure on $[0, 1]$ (see [7], p.62) is that the sequence satisfy the Hausdorff moment conditions

$$\sum_{i=0}^{k} \left(\begin{array}{c} k \\ i \end{array} \right) (-1)^i \gamma_\lambda(i+j) \geq 0, \qquad j = 0, 1, 2, \ldots, k = 0, 1, 2, \ldots. \tag{2.6}$$

These conditions can also be expressed in terms of iterative differences of the moment sequence (see [7]), which we now describe. Let $\Delta^0 \gamma_\lambda(n) = \gamma_\lambda(n)$, for $n = 0, 1, 2, \ldots$, and define recursively $\Delta^k \gamma_\lambda(n) = \Delta^{k-1}\gamma_\lambda(n+1) - \Delta^{k-1}\gamma_\lambda(n)$, for $n = 0, 1, 2, \ldots$ and $k = 1, 2, \ldots$. Then the Hausdorff conditions (2.6) are equivalent to

$$(-1)^k \Delta^k \gamma_\lambda(n) \geq 0, \qquad \forall k, n. \tag{2.7}$$

The approach to numerically computing, for example, the Laplace transform of the exit time distribution is to use the conditions (2.5) as the constraints of a linear program and to select λq_λ as the objective function. In theory, the linear program will only have one feasible solution since the conditions (2.5) characterize every moment of the process. In practice however, there can only be a finite number of constraints and thus the feasible region includes more than one solution. By both minimizing and maximizing the objective function, we obtain bounds on the value of the Laplace transform (see, for example, Table 1).

To illustrate the linear programming approach in more detail, we display the linear program when only zeroth, first and second moments are used:

Optimize $\quad \lambda q_\lambda$

S.t.
$$1 = \lambda q_\lambda + (\lambda+\beta)\gamma_\lambda(0) - \beta\gamma_\lambda(0) \tag{2.8}$$
$$0 = \lambda q_\lambda + (\lambda+\beta)\gamma_\lambda(1) - (\beta/K)\gamma_\lambda(0) - \beta\gamma_\lambda(1) \tag{2.9}$$

$$0 = \lambda q_\lambda + (\lambda+\beta)\gamma_\lambda(2) - (\beta/K^2)\gamma_\lambda(0) - (2\beta/K)\gamma_\lambda(1) - \beta\gamma_\lambda(2) \quad (2.10)$$

$$\gamma_\lambda(0), \gamma_\lambda(1), \gamma_\lambda(2) \geq 0 \quad (2.11)$$
$$\gamma_\lambda(0) - \gamma_\lambda(1), \gamma_\lambda(1) - \gamma_\lambda(2) \geq 0 \quad (2.12)$$
$$\gamma_\lambda(0) - 2\gamma_\lambda(1) + \gamma_\lambda(2) \geq 0. \quad (2.13)$$

In this linear program, the constraints (2.8) - (2.10) arise from the identity (2.5) and constraints (2.11) - (2.13) are the moment conditions (2.7) on the iterated differences. When M moments are included in the analysis, there will be M constraints arising from the identity (2.5) and $M+1$ *sets* of constraints from the moment conditions (2.7), *i.e.* $(M+1)(M+2)/2$ inequalities.

Table 1 displays the results of running the linear program for a selection of values of λ. We used $M = 13$ moments and set $K = 7$ and $\beta = 2$.

Table 1
Laplace Transform of the Exit Time Distribution
$M = 13$, $K = 7$, $\beta = 2$

λ	Minimum	Exact	Maximum
0.01	0.96568963	0.96568963	0.96568963
0.05	0.84126524	0.84126524	0.84126524
0.10	0.71068133	0.71068133	0.71068133
0.50	0.20971520	0.20971520	0.20971520
0.75	0.10761706	0.10761706	0.10761706
0.80	0.09486451	0.09486451	0.09486451
1.00	0.05852766	0.05852766	0.05852766
2.00	0.00781234	0.00781250	0.00781263
3.00	0.00163740	0.00163840	0.00163970
4.00	0.00045346	0.00045725	0.00045989
5.00	0.00015106	0.00015543	0.00016145
6.00	0.00005672	0.00006104	0.00006581
7.00	0.00002282	0.00002676	0.00003033
8.00	0.00000934	0.00001280	0.00001541
9.00	0.00000000	0.00000657	0.00000847
10.00	0.00000000	0.00000357	0.00000496

At this point, it is instructive to further analyze the example involving only zeroth, first and second moments. Let

$$\tilde{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \tilde{1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \tilde{\gamma}_\lambda = \begin{pmatrix} \gamma_\lambda(0) \\ \gamma_\lambda(1) \\ \gamma_\lambda(2) \end{pmatrix}, \quad \tilde{P} = \begin{pmatrix} 1 & 0 & 0 \\ 1/K & 1 & 0 \\ 1/K^2 & 2/K & 1 \end{pmatrix}$$

and notice that $\tilde{P}$ is a lower diagonal matrix. The conditions (2.8) - (2.10) can then be written as

$$\tilde{e}_1 = \lambda q_\lambda \tilde{1} + (\lambda+\beta)\tilde{\gamma}_\lambda - \beta\tilde{P}\tilde{\gamma}_\lambda$$

or by rescaling

$$\frac{1}{\lambda+\beta}\,\tilde{e}_1 = \frac{\lambda}{\lambda+\beta}\,q_\lambda\tilde{1} + \left(I - \frac{\beta}{\lambda+\beta}\,\tilde{P}\right)\tilde{\gamma}_\lambda, \quad (2.14)$$

where I denotes the 3×3 identity matrix. Observe that (2.14) can be rewritten to express $\tilde{\gamma}_\lambda$ in terms of q_λ:

$$\tilde{\gamma}_\lambda = \left(I - \frac{\beta}{\lambda+\beta}\tilde{P}\right)^{-1} \left[\left(\frac{1}{\lambda+\beta}\right)\tilde{e}_1 - \frac{\lambda}{\lambda+\beta}\,\tilde{1}q_\lambda\right].$$

Letting

$$\tilde{b} = \frac{1}{\lambda+\beta}\left(I - \frac{\beta}{\lambda+\beta}\tilde{P}\right)^{-1}\tilde{e}_1 \qquad \text{and} \qquad \tilde{c} = \frac{\lambda}{\lambda+\beta}\left(I - \frac{\beta}{\lambda+\beta}\tilde{P}\right)^{-1}\tilde{1},$$

we have

$$\tilde{\gamma}_\lambda = \tilde{b} - q_\lambda \tilde{c}. \tag{2.15}$$

Thus the moments of $\tilde{\gamma}_\lambda$ are parametrized by the Laplace transform q_λ.

Now using (2.15) to substitute for γ_λ in (2.11) leads to the bounds

$$q_\lambda \le \frac{b_i}{c_i} \qquad \text{for every } i \text{ such that } c_i > 0$$

and

$$q_\lambda \ge \frac{b_i}{c_i} \qquad \text{for every } i \text{ such that } c_i < 0$$

and thus

$$\max_{i:c_i<0} \frac{b_i}{c_i} \le q_\lambda \le \min_{i:c_i>0} \frac{b_i}{c_i}.$$

Similarly, observing that $\Delta^k \gamma_\lambda(n) = \Delta^k \tilde{b} - q_\lambda \Delta^k \tilde{c}$, the conditions (2.12) and (2.13) imply

$$\max_{i:\Delta^1 c_i>0} \frac{\Delta^1 b_i}{\Delta^1 c_i} \le q_\lambda \le \min_{i:\Delta^1 c_i<0} \frac{\Delta^1 b_i}{\Delta^1 c_i}$$

and

$$\max_{i:\Delta^2 c_i<0} \frac{\Delta^2 b_i}{\Delta^2 c_i} \le q_\lambda \le \min_{i:\Delta^2 c_i>0} \frac{\Delta^2 b_i}{\Delta^2 c_i}.$$

From this analysis, it is clear how the linear program selects its optimum. Since we are only using $\gamma_\lambda(0), \gamma_\lambda(1)$ and $\gamma_\lambda(2)$, there are only a few bounds on q_λ from the Hausdorff conditions.

When more moments are allowed in the analysis, the dimensions of $\tilde{e}_1$, $\tilde{1}$, $\tilde{\gamma}_\lambda$ and $\tilde{P}$ increase but the analysis otherwise remains the same. Table 2 displays the effect of increasing the number of moments included in the linear program.

Table 2
Laplace Transform of the Exit Time Distribution
$K = 7$, $\beta = 2$, $\lambda = 1$

No. of Moments	Minimum	Exact	Maximum
3	0.01282051	0.05852766	0.08034745
4	0.04774775	0.05852766	0.06705245
5	0.05582329	0.05852766	0.06168753
6	0.05784762	0.05852766	0.05920951
7	0.05836848	0.05852766	0.05865447
8	0.05850473	0.05852766	0.05855123
9	0.05852429	0.05852766	0.05853168
10	0.05852717	0.05852766	0.05852815
11	0.05852760	0.05852766	0.05852772
12	0.05852766	0.05852766	0.05852767
13	0.05852766	0.05852766	0.05852766

3 The general formulation

We consider a general process $X = \{X_t : t \geq 0\}$, with $X_0 = x_0$ and generator A, and let $\mathcal{D}(A)$ denote the domain of the generator. We assume that X takes values in $\mathbb{R}^d$ for some value of d. The states are partitioned into two regions: a bounded open region E_0 and its complement. For simplicity of exposition, we assume that $E_0 \subset [0,1]^d$. This restriction can be accomplished by a change in scale and location, as we illustrate in some of the examples. Let τ denote the first time X hits E_0^c.

As in the case of the Poisson process, we consider a time-space generator $\widehat{A}$ defined by

$$\widehat{A}f(t,x) = \frac{\partial f}{\partial t}(t,x) + Af(t,x)I_{E_0}(x) \tag{3.1}$$

in which the generator A is applied to $f(t,\cdot)$ for each fixed t and we assume that f is continuously differentiable in t and vanishes at ∞. This generator implies that the time component always increases linearly at rate 1 whereas the spacial process stops when it leaves the region E_0. Define $\widehat{X}_t = X_{t\wedge\tau}$ for $t \geq 0$. It follows that for every f in the domain of $\widehat{A}$,

$$f(t,\widehat{X}_t) - f(0,\widehat{X}_0) - \int_0^t \widehat{A}f(s,\widehat{X}_s)ds \tag{3.2}$$

is a martingale.

Now consider test functions of the form

$$f(t,x) = e^{-\lambda t}g(x) \tag{3.3}$$

with $\lambda > 0$ and $g \in \mathcal{D}(A)$. Since $\widehat{X}_t$ is bounded for all t and (3.2) is a martingale, taking expectations and letting $t \to \infty$ we obtain

$$\begin{aligned} g(x_0) &= -E\left[\int_0^\infty \widehat{A}f(s,\widehat{X}_s)ds\right] \\ &= -E\left[\int_0^t [(-\lambda)e^{-\lambda s}g(\widehat{X}_s) + e^{-\lambda s}Ag(\widehat{X}_s)I_{E_0}(\widehat{X}_s)]ds\right] \end{aligned}$$

$$
\begin{aligned}
&= \lambda \int_0^\infty e^{-\lambda s} E[g(\widehat{X}_s) I_{\{\tau \le s\}}] ds \\
&\quad + E\left[\int_0^\infty e^{-\lambda s} [\lambda g(\widehat{X}_s) - Ag(\widehat{X}_s)] I_{E_0}(\widehat{X}_s) ds \right]. \qquad (3.4)
\end{aligned}
$$

Note that we can justify the interchange of the expectation and limit in the above argument due to the particular test functions. Observe that when $g(x) = 1$ for all $x \in E_0^c$ (and integrating by parts as before) the first term of the last expression is

$$
\lambda \int_0^\infty e^{-\lambda s} P(\tau \le s) ds = \int_0^\infty e^{-\lambda s} \phi(s) ds \qquad (3.5)
$$

again where ϕ denotes the density of the exit time τ. The expressions in (3.5) are the (scaled and unscaled by λ) Laplace transforms of the distribution and the density functions of τ, respectively.

We now consider $g(x) = \prod_{i=1}^d x_i^{n_i}$, with $n_1, \ldots, n_d \ge 0$. For the models in this paper, the generator A applied to g results in a linear combination of functions having the same form. As a result, we will be able to rewrite the second term in (3.4) as a linear combination of terms of the form

$$
E\left[\int_0^\infty e^{-\lambda s} \prod_{i=1}^d \widehat{X_{is}}^{m_i} I_{E_0}(\widehat{X}_s)\, ds \right], \qquad (3.6)
$$

where $\widehat{X_{is}}$ denotes the ith coordinate of $\widehat{X}_s$.

We now consider these terms more carefully. Define the "occupation measure" α of the process X in E_0 by

$$
\alpha(\Gamma) = E\left[\int_0^\infty I_{\Gamma \cap E_0}(\widehat{X}_s) ds \right], \qquad \forall \Gamma \subset \mathbb{R}^d.
$$

Then (3.6) can be viewed as the Laplace transform of the joint $(m_1, \ldots, m_d)$th moment of the occupation measure α. For numerical purposes however, we include the exponential term in the definition of a different measure γ_λ (parametrized by λ) on E_0 and let

$$
\gamma_\lambda(\Gamma) = E\left[\int_0^\infty e^{-\lambda s} I_{\Gamma \cap E_0}(\widehat{X}_s) ds \right], \qquad \forall \Gamma \subset \mathbb{R}^d. \qquad (3.7)
$$

The term (3.6) then represents the $(m_1, \ldots, m_d)$th moment of γ_λ, which, as for the Poisson process and slightly abusing notation, we denote $\gamma_\lambda(m_1, \ldots, m_d)$.

Now consider the general form of the first term of the last expression of (3.4). Again define a new measure q_λ on E_0^c by

$$
q_\lambda(\Gamma) = E\left[\int_0^t e^{-\lambda s} I_{\Gamma \cap E_0^c}(\widehat{X}_s) ds \right], \qquad \forall \Gamma \subset \mathbb{R}^d. \qquad (3.8)
$$

Then the first terms in (3.4) give the $(n_1, \ldots, n_d)$th moments of the measure q_λ and, by (3.5) the Laplace transform of the exit time distribution is given by

$$
\lambda q_\lambda(0, \ldots, 0) = \int_0^\infty e^{-\lambda s} \phi(s) ds. \qquad (3.9)
$$

Note that in the (one-dimensional) Poisson example of the previous section, q_λ is concentrated at the value K so the measure is simply a point mass and $q_\lambda(n) = q_\lambda$, after rescaling so that $E_0 \subset [0,1]$.

We, thus, see that the conditions (3.4) can be expressed in terms of the sequences $\{q_\lambda(n_1,\ldots,n_d) : n_1,\ldots,n_d \geq 0\}$ and $\{\gamma_\lambda(n_1,\ldots,n_d) : n_1,\ldots,n_d \geq 0\}$.

Again as in the case of the Poisson process, the sequences must be joint moment sequences. The multidimensional Hausdorff conditions are necessary and sufficient condition for them to be moment sequences (see [16]). Since the examples in this paper are one and two dimensional and the one dimensional conditions were given in (2.6), we state the two-dimensional form for the Hausdorff conditions:

$$\sum_{k=0}^{n}\sum_{l=0}^{m}\binom{m}{k}\binom{n}{l}(-1)^{k+l}\gamma(i+k,j+l) \geq 0 \tag{3.10}$$

for all $m,n = 1,2,3,\ldots$ and $i,j = 0,1,2,\ldots$, where

$$\gamma(i,j) = \int_{(0,1)\times(0,1)} x^i y^j \gamma_\lambda(dx \times dy) \tag{3.11}$$

denotes the i-jth moment of the measure γ_λ on $E_0 = (0,1)\times(0,1)$.

The linear expressions (3.4), along with the Hausdorff conditions (3.10) for each of the sequences, form the constraints of the linear program. The objective function is chosen as the quantity of interest so that bounds are obtained by minimizing and maximizing the objective function.

We defer a precise statement of the linear program to the discussion of each of the models since the specific forms of q_λ and Ag will vary from model to model.

4 Brownian motion on the interval $[0,1]$

We begin our numerical examples for continuous processes with one of the simplest stochastic processes, Brownian motion. The process $\widehat{X}$ is started at x_0, with $0 < x_0 < 1$, and runs until it hits either 0 or 1. The generator of Brownian motion is $Af(x) = 1/2f''(x)$ so the time-space generator is

$$\widehat{A}f(t,x) = \frac{\partial f}{\partial t}(t,x) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,x)I_{(0,1)}(x)$$

acting on test functions $f \in C^{1,2}(\mathbb{R})$ which are bounded on $\mathbb{R}^+ \times [0,1]$ and vanish as $t \to \infty$. For $f(t,x) = e^{-\lambda t}x^n$ with $\lambda > 0$, the identity (3.4) becomes

$$\begin{aligned} x_0^n &= E\left[\int_0^\infty e^{-\lambda s}\left(\lambda \widehat{X}_s^n - \frac{n(n-1)}{2}\widehat{X}_s^{n-2}I_{(0,1)}(\widehat{X}_s)\right)ds\right] \\ &= \lambda E\left[\int_0^\infty e^{-\lambda s}\widehat{X}_s^n I_{\{0,1\}}(\widehat{X}_s)ds\right] \\ &\quad + E\left[\int_0^\infty e^{-\lambda s}\left(\lambda \widehat{X}_s^n - \frac{n(n-1)}{2}\widehat{X}_s^{n-2}\right)I_{(0,1)}(\widehat{X}_s)ds\right] \\ &= \lambda 0^n q_\lambda^{(0)} + \lambda q_\lambda^{(1)} + \lambda\gamma_\lambda(n) - \frac{n(n-1)}{2}\gamma_\lambda(n-2), \end{aligned} \tag{4.1}$$

where $q_\lambda^{(0)}$ and $q_\lambda^{(1)}$ denote the masses of the boundary measure q_λ at 0 and 1, respectively and 0^0 is defined to be 1, as before.

The linear program for this model is

$$\begin{array}{ll} \text{Optimize} & \lambda q_\lambda^{(0)} + \lambda q_\lambda^{(1)} \\ \text{Subj. to} & \bullet\ x_0^n = \lambda 0^n q_\lambda^{(0)} + \lambda q_\lambda^{(1)} + \lambda \gamma_\lambda(n) - \dfrac{n(n-1)}{2}\gamma_\lambda(n-2), \qquad n \geq 0 \\ & \bullet\ \displaystyle\sum_{i=0}^{k} \binom{k}{i} (-1)^i \gamma_\lambda(i+j) \geq 0, \qquad j = 0,1,2,\ldots, k = 0,1,2,\ldots \\ & \bullet\ q_\lambda^{(0)}, q_\lambda^{(1)} \geq 0. \end{array}$$

We provide two displays of results from solving the linear program. Table 3 displays the Laplace transform of the exit time distribution for various values of λ. The exact values are based on the formula (see, for instance, [10, p.100])

$$E_{x_0}\left[e^{-\lambda\tau}\right] = \frac{\cosh\left(\left(x_0 - \frac{1}{2}\right)\sqrt{2\lambda}\right)}{\cosh\left(\frac{1}{2}\sqrt{2\lambda}\right)}. \tag{4.2}$$

Note that when $x_0 = 1/2$, as it does for Table 3, this reduces to

$$E\left[e^{-\lambda\tau}\right] = \frac{1}{\cosh\left(\frac{1}{2}\sqrt{2\lambda}\right)}.$$

Table 3
Laplace Transform of the Exit Time of Brownian Motion from $[0,1]$ as a Function of λ
$x_0 = 1/2$; $M = 12$.

λ	Minimum	Exact Values	Maximum
0.100	0.97551045	0.97551045	0.97551045
0.200	0.95200191	0.95200191	0.95200191
0.300	0.92941796	0.92941796	0.92941796
0.400	0.90770639	0.90770639	0.90770639
0.500	0.88681888	0.88681888	0.88681888
0.600	0.86671057	0.86671057	0.86671057
0.700	0.84733980	0.84733980	0.84733980
0.800	0.82866780	0.82866780	0.82866780
0.900	0.81065848	0.81065848	0.81065848
1.000	0.79327818	0.79327818	0.79327818
2.000	0.64805427	0.64805427	0.64805427
3.000	0.54096010	0.54096010	0.54096010
4.000	0.45909811	0.45909813	0.45909815
5.000	0.39477089	0.39477097	0.39477105
6.000	0.34310267	0.34310291	0.34310314
7.000	0.30085664	0.30085723	0.30085779
8.000	0.26580098	0.26580223	0.26580342
9.000	0.23634785	0.23635024	0.23635256
10.000	0.21133746	0.21134172	0.21134587

Table 4 compares the results of the linear program with the exact value for a selection of different initial values. Again, the exact values are based on the formula (4.2).

Table 4
Laplace Transform of the Exit Time of Brownian Motion from $[0,1]$
as a Function of the Initial Position x_0
$\lambda = 1$: $M = 12$.

x_0	Minimum	Exact Values	Maximum
0.1	0.92362365	0.92362365	0.92362366
0.2	0.86575059	0.86575059	0.86575059
0.3	0.82522141	0.82522141	0.82522141
0.4	0.80122419	0.80122419	0.80122419
0.5	0.79327818	0.79327818	0.79327818
0.6	0.80122419	0.80122419	0.80122419
0.7	0.82522141	0.82522141	0.82522141
0.8	0.86575059	0.86575059	0.86575059
0.9	0.92362365	0.92362365	0.92362365

5 Drifted Brownian motion on the interval $[a, b]$

The next example, Brownian motion with drift, is only slightly more complicated but can also be used to analyze the geometric Brownian motion process which is of importance in the modelling of stock prices. Let $a, b \in \mathbb{R}$ be given. The drifted Brownian motion starts at x_0, with $a < x_0 < b$, satisfies

$$dx_t = \mu\, dt + \sigma\, dW_t$$

and runs until it hits either a or b. The generator of the drifted Brownian motion process is

$$Af(x) = \frac{\sigma^2}{2} f''(x) + \mu f'(x)$$

acting on $f \in C^2(\mathbb{R})$. The generator of the time-space process is defined by (3.1). For $f(t,x) = e^{-\lambda t}(\frac{x-a}{b-a})^n$ with $\lambda > 0$, the identity (3.4) is

$$\begin{aligned}
\left(\frac{x_0 - a}{b-a}\right)^n &= -E\left[\int_0^\infty e^{-\lambda s}\left(-\lambda\left(\frac{\widehat{X}_s - a}{b-a}\right)^n + \left(\frac{\mu\, n}{b-a}\left(\frac{\widehat{X}_s - a}{b-a}\right)^{n-1}\right.\right.\right. \\
&\qquad \left.\left.\left. + \frac{n(n-1)\sigma^2}{2(b-a)^2}\left(\frac{\widehat{X}_s - a}{b-a}\right)^{n-2}\right) I_{(a,b)}(\widehat{X}_s)\right) ds\right] \\
&= \lambda E\left[\int_0^\infty e^{-\lambda s}\left(\frac{\widehat{X}_s - a}{b-a}\right)^n \left(I_{\{a\}}(\widehat{X}_s) + I_{\{b\}}(\widehat{X}_s)\right) ds\right] \\
&\quad + E\left[\int_0^\infty e^{-\lambda s}\left(\lambda\left(\frac{\widehat{X}_s - a}{b-a}\right)^n - \frac{\mu\, n}{b-a}\left(\frac{\widehat{X}_s - a}{b-a}\right)^{n-1}\right.\right. \\
&\qquad \left.\left. - \frac{n(n-1)\sigma^2}{2(b-a)^2}\left(\frac{\widehat{X}_s - a}{b-a}\right)^{n-2}\right) I_{(a,b)}(\widehat{X}_s)\, ds\right]
\end{aligned}$$

$$
\begin{aligned}
= \quad & \lambda 0^n \int_0^\infty e^{-\lambda s} P(\widehat{X}_s = a)\, ds + \lambda \int_0^\infty e^{-\lambda s} P(\widehat{X}_s = b)\, ds \\
& + \lambda E\left[\int_0^\infty e^{-\lambda s} \left(\frac{\widehat{X}_s - a}{b-a}\right)^n I_{(a,b)}(\widehat{X}_s)\, ds\right] \\
& - \frac{\mu\, n}{b-a} E\left[\int_0^\infty e^{-\lambda s} \left(\frac{\widehat{X}_s - a}{b-a}\right)^{n-1} I_{(a,b)}(\widehat{X}_s)\, ds\right] \\
& - \frac{n(n-1)\sigma^2}{2(b-a)^2} E\left[\int_0^\infty e^{-\lambda s} \left(\frac{\widehat{X}_s - a}{b-a}\right)^{n-2} I_{(a,b)}(\widehat{X}_s)\, ds\right]. \qquad (5.1)
\end{aligned}
$$

Now let $q_\lambda^{(a)} = \int_0^\infty e^{-\lambda s} P(\widehat{X}_s = a) ds$ and similarly for $q_\lambda^{(b)}$. Also making the substitution $y = \frac{x-a}{b-a}$, the definition of γ_λ in (3.7) leads to a measure on $[0,1]$ for which the terms in (5.1) are in the moment set of γ_λ.

The linear program to compute the Laplace transform of the exit time distribution is

$$
\begin{array}{ll}
\text{Optimize} & \lambda q_\lambda^{(a)} + \lambda q_\lambda^{(b)} \\
\text{Subj. to} & \left(\dfrac{x_0 - a}{b-a}\right)^n = \lambda 0^n q_\lambda^{(a)} + \lambda q_\lambda^{(b)} + \lambda \gamma_\lambda(n) \\
& \qquad\qquad - \dfrac{\mu\, n}{b-a} \gamma_\lambda(n-1) - \dfrac{n(n-1)}{2(b-a)^2} \gamma_\lambda(n-2), \qquad n \geq 0 \\
& \displaystyle\sum_{i=0}^{k} \binom{k}{i} (-1)^i \gamma_\lambda(i+j) \geq 0, \qquad j = 0,1,2,\ldots, k = 0,1,2,\ldots \\
& q_\lambda^{(a)}, q_\lambda^{(b)} \geq 0.
\end{array}
$$

We can provide more detail by considering each term of the objective function individually. Table 5 displays the results of running the linear program for computing $\lambda q_\lambda^{(b)}$ and compare these to the exact values.

Table 5
Laplace Transform of the Exit Time at $b = 1$ of drifted Brownian Motion
$\mu = 1$, $\sigma = 1$, $a = -20$; $M = 53$

λ	Minimum	Exact	Maximum
0.1	0.90896825	0.90896825	0.90896825
0.2	0.83258833	0.83258833	0.83258833
0.3	0.76727418	0.76727419	0.76727419
0.4	0.71060340	0.71060342	0.71060345
0.5	0.66085961	0.66085980	0.66085996
0.6	0.61678146	0.61678196	0.61678251
0.7	0.57741381	0.57741540	0.57741674
0.8	0.54201770	0.54202045	0.54202455
0.9	0.51000582	0.51001250	0.51002043
1.0	0.48091069	0.48092170	0.48093647
2.0	0.29035330	0.29052432	0.29119066
3.0	0.19215311	0.19286760	0.19453453
4.0	0.13446763	0.13533528	0.13574555
5.0	0.09580964	0.09860584	0.10202859
6.0	0.07243503	0.07386241	0.07731938
7.0	0.05425288	0.05653003	0.05999360
8.0	0.03802487	0.04402025	0.04650826
9.0	0.02924622	0.03477353	0.03760293
10.0	0.02311307	0.02780399	0.03032581

In this example the LP bounds provide excellent accuracy when λ is small whereas the accuracy lessens for large values of λ.

6 The Ornstein-Uhlenbeck process

We now consider an Ornstein-Uhlenbeck process to illustrate the method on a process for which there is state dependence in the drift coefficient. This example examines a process on an interval which is symmetric about zero for two reasons. The exposition is simplified and this model will be applied to the study of the Cameron-Martin formula.

Let $b \in \mathbb{R}$ be given. The stopped Ornstein-Uhlenbeck process $\widehat{X}$ starts at 0, satisfies

$$dx_t = \rho x_t \, dt + \sigma \, dW_t,$$

and stops when it hits either b or $-b$. The generator $\widehat{A}$ of the (stopped) time-space Ornstein-Uhlenbeck process is

$$\widehat{A}f(x) = \frac{\partial f}{\partial t} + \left(\rho x \frac{\partial f}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2}\right) I_{(-b,b)}(x) \tag{6.1}$$

acting on $f \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R})$ with $f(t,x) \to 0$ as $t \to \infty$ for all x.

Considering $f(t,x) = e^{-\lambda t}(\frac{x}{b})^n$ with $\lambda > 0$, the identity (3.4) is

$$0^n \;=\; (-1)^n \lambda \int_0^\infty e^{-\lambda s} P(\widehat{X}_s = -b)\, ds + \lambda \int_0^\infty e^{-\lambda s} P(\widehat{X}_s = b)\, ds$$

$$
\begin{aligned}
&+(\lambda-\rho n)E\left[\int_0^\infty e^{-\lambda s}\left(\frac{\widehat{X}_s}{b}\right)^n I_{(-b,b)}(\widehat{X}_s)\,ds\right] \\
&-\frac{n(n-1)\sigma^2}{2b^2}E\left[\int_0^\infty e^{-\lambda s}\left(\frac{\widehat{X}_s}{b}\right)^{n-2} I_{(-b,b)}(\widehat{X}_s)\,ds\right] \\
= \;&(-1)^n\lambda q_\lambda^{(-b)}+\lambda q_\lambda^{(b)}+(-1)^n(\lambda-\rho n)\gamma_\lambda^-(n)-(-1)^{n-2}\frac{n(n-1)\sigma^2}{2b^2}\gamma_\lambda^-(n-2) \\
&+(\lambda-\rho n)\gamma_\lambda^+(n)-\frac{n(n-1)\sigma^2}{2b^2}\gamma_\lambda^+(n-2)
\end{aligned}
$$

where $\gamma_\lambda^-(n)$ is the nth (discounted, scaled) moment on $(-b,0)$:

$$
\gamma_\lambda^-(n)=E\left[\int_0^\infty e^{-\lambda s}\left(\frac{\widehat{X}_s}{b}\right)^n I_{(-b,0)}(\widehat{X}_s)\,ds\right]
$$

and similarly $\gamma_\lambda^+(n)$ is the nth moment of $\widehat{X}$ on $(0,b)$. Note that in this example, we are splitting the symmetric interval $(-b,b)$ into $(-b,0)$ and $(0,b)$ and rescaling each interval, whereas in the previous example, we rescaled the spacial dimension so that the entire interval (a,b) mapped onto $(0,1)$.

For this example, we illustrate the computation of the Laplace transform of the moments of the occupation measure until the process stops. We refer to these moments as interior moments. The linear program to compute the Laplace transform of the nth interior moment is

$$
\begin{array}{ll}
\text{Optimize} & \gamma_\lambda^-(n)+\gamma_\lambda^+(n) \\
\text{Subj. to} & 0^n=\lambda 0^n q_\lambda^{(-b)}+\lambda q_\lambda^{(b)}+(-1)^n(\lambda-\rho n)\gamma_\lambda^-(n) \\
& \qquad -(-1)^{n-2}\dfrac{n(n-1)\sigma^2}{2b^2}\gamma_\lambda^-(n-2)+(\lambda-\rho n)\gamma_\lambda^+(n) \\
& \qquad -\dfrac{n(n-1)\sigma^2}{2b^2}\gamma_\lambda^+(n-2), \qquad n\ge 0 \\
& \displaystyle\sum_{i=0}^k \binom{k}{i}(-1)^i\gamma_\lambda^-(i+j)\ge 0, \qquad j=0,1,2,\ldots,k=0,1,2,\ldots \\
& \displaystyle\sum_{i=0}^k \binom{k}{i}(-1)^i\gamma_\lambda^+(i+j)\ge 0, \qquad j=0,1,2,\ldots,k=0,1,2,\ldots \\
& q_\lambda^{(-b)},q_\lambda^{(b)}\ge 0.
\end{array}
$$

Table 6 gives the results of solving the linear program for $n=2,4,6,\ldots,20$ and compares the values with the Laplace transforms (using $\lambda=1$) of moments of a normal distribution with mean zero and variance $(1-e^{-2t})/2$. This normal distribution is the law at time t of the unstopped Ornstein-Uhlenbeck process with drift -1.

Table 6
Laplace Transform of Moments of the Stopped Ornstein-Uhlenbeck Process
$\lambda = 1$, $\rho = -1$, $b = 5$, $x_0 = 0$; $M = 23$

No.	Transform of Interior Moments for Stopped Process by LP	Transform of Moments for Unstopped Process
2	0.33333325	0.33333333
4	0.39999867	0.40000000
6	0.85711812	0.85714285
8	2.66616400	2.66666667
10	10.89832713	10.90909091
12	55.14577271	55.38461539
14	330.56011244	336.00000000
16	2245.33074941	2371.764706
18	16111.62293873	19098.94737
20	101230.43513094	172800.00000000

7 Two-dimensional Brownian motion process

In this section we look at a multivariate example, the two-dimensional Brownian motion process in the unit square.

Let $(X_s, Y_s) = (x_0 + W_s^1, y_0 + W_s^2)$ where $x_0, y_0 \in (0,1)$ and $W = (W_s^1, W_s^2)$ is a two-dimensional standard Brownian motion process. The generator of the process (X_s, Y_s) is given by

$$Af(x,y) = \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x,y) + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(x,y).$$

Define the time-space generator $\widehat{A}$ by (3.1).

The model considers the exit of the process from the bounded domain $E_0 = (0,1) \times (0,1)$ and the boundary consists of four parts $F_{bottom} = [0,1] \times \{0\}$, $F_{top} = [0,1] \times \{1\}$, $F_{left} = \{0\} \times (0,1)$ and $F_{right} = \{1\} \times (0,1)$. The measure γ_λ given by (3.7) is defined on the unit square and the boundary measure q_λ can be broken into four measures $(q_\lambda^{(b)}, q_\lambda^{(t)}, q_\lambda^{(l)}, q_\lambda^{(r)})$ corresponding to the parts of the boundary. Each boundary measure is characterized by its one-dimensional Hausdorff conditions (2.6) while the measure γ_λ on E_0 is characterized by the two-dimensional conditions (3.10).

The collection of test functions we use is $\mathcal{D}_0 = \{f(t,x,y) = e^{-\lambda t} x^m y^n; m,n \geq 0\}$. The multidimensional identities (3.4) can be expressed as follows. Let $\gamma(i,j)$ be defined in (3.11), and $q_\lambda^{(b)}(i)$, $q_\lambda^{(t)}(i)$, $q_\lambda^{(l)}(i)$ and $q_\lambda^{(r)}(i)$ denote the ith moment of the measures defined by (3.8) restricted to F_{bottom}, F_{top}, F_{left} and F_{right}, respectively. If we also define $0^0 = 1$, then for $m, n \geq 0$,

$$\begin{aligned} x_0^m y_0^n \;=\; & \lambda(\gamma(m,n) + 0^n q_\lambda^{(b)}(m) + q_\lambda^{(t)}(m) + 0^m q_\lambda^{(l)}(n) + q_\lambda^{(r)}(n)) \\ & - \frac{m(m-1)}{2}\gamma(m-2,n) - \frac{n(n-1)}{2}\gamma(m,n-2). \end{aligned}$$

Notice that when m and n both equal 0, this condition reduces to the requirement that the total mass of all the measures is $1/\lambda$.

To these equality constraints we add the one-dimensional Hausdorff conditions (2.6) for each of the four sets of moments on the boundary measures and the two-dimensional Hausdorff conditions (3.10) on the moments of the measure in the interior in order to characterize the feasible moment sequences.

The Laplace transform of the exit time distribution (see (3.9)) is given by

$$\lambda[q_\lambda^{(b)}(0) + q_\lambda^{(t)}(0) + q_\lambda^{(l)}(0) + q_\lambda^{(r)}(0)].$$

The linear program for numerically evaluating this Laplace transform is then

$$\begin{array}{ll}
\text{Optimize} & \lambda[q_\lambda^{(b)}(0) + q_\lambda^{(t)}(0) + q_\lambda^{(l)}(0) + q_\lambda^{(r)}(0)] \\[2ex]
\text{Subj. to} & \begin{aligned} x_0^m y_0^n = {} & \lambda(\gamma(m,n) + 0^n q_\lambda^{(b)}(m) + q_\lambda^{(t)}(m) + 0^m q_\lambda^{(l)}(n) + q_\lambda^{(r)}(n)) \\ & -\frac{m(m-1)}{2}\gamma(m-2,n) - \frac{n(n-1)}{2}\gamma(m,n-2), \end{aligned} \\
& \displaystyle\sum_{k=0}^{n}\sum_{l=0}^{m} \binom{m}{k}\binom{n}{l}(-1)^{k+l}\gamma(i+k,j+l) \geq 0, \\
& \displaystyle\sum_{i=0}^{k} \binom{k}{i}(-1)^i q_\lambda^{(b)}(i+j) \geq 0, \\
& \displaystyle\sum_{i=0}^{k} \binom{k}{i}(-1)^i q_\lambda^{(t)}(i+j) \geq 0, \\
& \displaystyle\sum_{i=0}^{k} \binom{k}{i}(-1)^i q_\lambda^{(l)}(i+j) \geq 0, \\
& \displaystyle\sum_{i=0}^{k} \binom{k}{i}(-1)^i q_\lambda^{(r)}(i+j) \geq 0.
\end{array}$$

This linear program was solved for several values of λ with a number of different initial positions. Table 7 provides the results.

Table 7
Upper and lower bounds for the Laplace transform of the exit time of 2-dimensional Brownian motion from the unit square
$\lambda = 0.1$, 0.5 and 1.0,
initial positions, $(0.1, 0.1 - 0.5)$ and $(0.5, 0.1 - 0.5)$; $M = 15$

$\lambda = 0.1$

	$x_0 = 0.1$		$x_0 = 0.5$	
y_0	lower	upper	lower	upper
0.1	0.99642159	0.99846427	0.99279036	0.99546061
0.2	0.99466893	0.99717630	0.98842238	0.99156405
0.3	0.99363165	0.99621155	0.98577229	0.98869997
0.4	0.99325860	0.99590192	0.98433581	0.98724313
0.5	0.99285136	0.99528419	0.98394051	0.98640935

$\lambda = 0.5$

	$x_0 = 0.1$		$x_0 = 0.5$	
y_0	lower	upper	lower	upper
0.1	0.98260028	0.99206697	0.96878635	0.97745116
0.2	0.97404771	0.98547008	0.94928432	0.95879148
0.3	0.96946046	0.98086364	0.93623788	0.94580226
0.4	0.96932006	0.97860490	0.92912650	0.93771325
0.5	0.96611242	0.97880194	0.92728349	0.93482609

$\lambda = 1.0$

	$x_0 = 0.1$		$x_0 = 0.5$	
y_0	lower	upper	lower	upper
0.1	0.96621338	0.98511674	0.93463965	0.95622908
0.2	0.95066467	0.97152443	0.89476555	0.92047336
0.3	0.94203634	0.96397177	0.87815896	0.89585653
0.4	0.93641863	0.96047900	0.85874601	0.88102095
0.5	0.94215604	0.95700921	0.85996724	0.87663823

8 Cameron-Martin formula

The examples we have presented so far have defined the process and then simply applied the identity (3.4) and the Hausdorff moment conditions (3.10) to obtain the constraints of the linear program. In this section, we show that the method also provides good numerical results when we use stochastic analysis to first reformulate the process and then characterize the process using (3.4) and the Hausdorff conditions.

Let w be a one-dimensional Brownian motion process and define

$$y_t = \int_0^t w_s^2 ds, \qquad t \geq 0.$$

Cameron and Martin ([3]) showed that for each $\mu \in \mathbb{R}$,

$$E\left[e^{-\frac{\mu^2}{2} y_t}\right] = \frac{1}{\sqrt{\cosh(\mu t)}}. \tag{8.1}$$

In this section we seek to compute the Laplace transforms of the left hand side of (8.1) using the linear programming approach. We compare the results with numerical computation of the Laplace transforms of the right hand side of (8.1).

We reformulate the problem to one which involves the analysis of an Ornstein-Uhlenbeck process and apply the approach of section 6, though the particular choice of test functions will be different leading to a different set of conditions based on the identity (3.4). Observe that

$$\begin{aligned} E\left[e^{-\frac{\mu^2}{2}\int_0^t w_s^2 ds}\right] &= E\left[e^{\mu\int_0^t w_s dw_s - \frac{\mu^2}{2}\int_0^t w_s^2 ds - \mu\int_0^t w_s dw_s}\right] \\ &= E\left[e^{\left(\mu\int_0^t w_s dw_s - \frac{\mu^2}{2}\int_0^t w_s^2 ds\right)} e^{-\mu\left(\frac{w_t^2 - t}{2}\right)}\right] \\ &= \tilde{E}\left[e^{-\mu\left(\frac{w_t^2 - t}{2}\right)}\right], \end{aligned}$$

where $\tilde{E}$ denotes expectation with respect to the measure $\tilde{P}$ obtained by a Girsanov transformation having density

$$\frac{d\tilde{P}}{dP} = \exp\left(\mu\int_0^t w_s\, dw_s - \frac{\mu^2}{2}\int_0^t w_s^2\, ds\right)$$

on the σ-algebra generated by w up to time t. Note that under the measure $\tilde{P}$, the process w satisfies the stochastic differential equation

$$dw_t = \mu w_t dt + d\tilde{w}_t, \qquad w(0) = 0,$$

in which $\tilde{w}$ is a Brownian motion under $\tilde{P}$; that is, w is an Ornstein-Uhlenbeck process under $\tilde{P}$.

We seek to compute

$$E\left[\int_0^\infty e^{-\lambda t} e^{-\frac{\mu^2}{2} y_t} dt\right] = \tilde{E}\left[\int_0^\infty e^{-\left(\lambda - \frac{\mu}{2}\right)t} e^{-\frac{\mu}{2} w_t^2} dt\right]$$

for $\lambda > \frac{\mu}{2}$. Recall the generator of the time-space Ornstein-Uhlenbeck process given in (6.1). Now using test functions of the form $f(t,x) = e^{-\left(\lambda - \frac{\mu}{2}\right)t} e^{-\frac{\mu}{2}x^2}\left(\frac{x}{b}\right)^n$, the identity (3.4) becomes

$$\begin{aligned} 0^n = {} & \left(\lambda - \frac{\mu}{2}\right) E\left[\int_0^\infty e^{-(\lambda-\mu/2)t} e^{-\mu b^2/2}\left((-1)^n I_{\{w_s=-b\}} + I_{\{w_s=b\}}\right) ds\right] \\ & + \lambda E\left[\int_0^\infty e^{-(\lambda-\mu/2)t} e^{-(\mu/2)w_s^2}\left(\frac{w_s}{b}\right)^n I_{(-b,b)}(w_s) ds\right] \\ & + \frac{\mu^2 b^2}{2} E\left[\int_0^\infty e^{-(\lambda-\mu/2)t} e^{-(\mu/2)w_s^2}\left(\frac{w_s}{b}\right)^{n+2} I_{(-b,b)}(w_s) ds\right] \\ & - \frac{n(n-1)}{2b^2} E\left[\int_0^\infty e^{-(\lambda-\mu/2)t} e^{-(\mu/2)w_s^2}\left(\frac{w_s}{b}\right)^{n-2} I_{(-b,b)}(w_s) ds\right] \end{aligned}$$

$$
\begin{aligned}
= \; & \left(\lambda - \frac{\mu}{2}\right) e^{-\mu b^2/2} \left((-1)^n q^{(-b)}_{\lambda-\mu/2} + q^{(b)}_{\lambda-\mu/2}\right) \\
& + \lambda \left((-1)^n \gamma^-_{\lambda-\mu/2}(n) + \gamma^+_{\lambda-\mu/2}(n)\right) \\
& + \left(\frac{\mu^2 b^2}{2}\right) \left((-1)^n \gamma^-_{\lambda-\mu/2}(n+2) + \gamma^+_{\lambda-\mu/2}(n+2)\right) \\
& - \left(\frac{n(n-1)}{2b^2}\right) \left((-1)^n \gamma^-_{\lambda-\mu/2}(n-2) + \gamma^+_{\lambda-\mu/2}(n-2)\right)
\end{aligned}
$$

where $\gamma^-_{\lambda-\mu/2}(n) = E\left[\int_0^\infty e^{-(\lambda-\mu/2)t} e^{-(\mu/2)w_s^2} \left(\frac{|w_s|}{b}\right)^n I_{(-b,0)}(w_s)\, ds\right]$ and similarly for $\gamma^+_{\lambda-\mu/2}(n)$.

To compute the Laplace transforms of the Cameron-Martin formula (8.1), we use the following linear program

$$
\begin{aligned}
\text{Optimize} \quad & \gamma^-_{\lambda-\mu/2}(0) + \gamma^+_{\lambda-\mu/2}(0) \\
\text{Subj. to} \quad & 0^n = (\lambda - \mu/2)\, e^{-\mu b^2/2} (-1)^n q^{(-b)}_{\lambda-\mu/2} + (\lambda - \mu/2)\, e^{-\mu b^2/2} q^{(b)}_{\lambda-\mu/2} \\
& \qquad + (-1)^n \lambda \gamma^-_{\lambda-\mu/2}(n) + \lambda \gamma^+_{\lambda-\mu/2}(n) \\
& \qquad + (-1)^{n+2} \frac{\mu^2 b^2}{2} \gamma^-_{\lambda-\mu/2}(n+2) + \frac{\mu^2 b^2}{2} \gamma^+_{\lambda-\mu/2}(n+2) \\
& \qquad - (-1)^{n-2} \frac{n(n-1)}{2b^2} \gamma^-_{\lambda-\mu/2}(n-2) - \frac{n(n-1)}{2b^2} \gamma^+_{\lambda-\mu/2}(n-2), n \geq 0 \\
& \sum_{i=0}^{k} \binom{k}{i} (-1)^i \gamma^-_{\lambda-\mu/2}(i+j) \geq 0, \qquad j = 0, 1, 2, \ldots, k = 0, 1, 2, \ldots \\
& \sum_{i=0}^{k} \binom{k}{i} (-1)^i \gamma^+_{\lambda-\mu/2}(i+j) \geq 0, \qquad j = 0, 1, 2, \ldots, k = 0, 1, 2, \ldots \\
& q^{(-b)}_{\lambda-\mu/2}, q^{(b)}_{\lambda-\mu/2} \geq 0.
\end{aligned}
$$

Note that in this example we are actually approximating the Cameron-Martin formula since this analysis has the process stopping when it hits either boundary $\{\pm b\}$. The value of b is chosen so that very little mass is placed on the boundary for the measures $q^{(\pm b)}_{\lambda-\mu/2}$.

Table 8 displays the results of running the linear program for two choices of parameter sets. Included in the table are the values of the Laplace transform of (8.1) as numerically evaluated using Maple and the total mass of the measures at the locations where the process is stopped. As can be seen, for large values of λ the total mass on the boundary is zero (to six decimal places) and the minimum and maximum values from the LP for the stopped process provide bounds on the Laplace transform of (8.1). For small values of λ the stopped process places small positive mass on the boundary and therefore the numerical results are approximations to the exact values.

Table 8
Laplace Transform of the Cameron-Martin Formula

$\mu = 0.1,\ b = 10.9$

λ	Minimum	Maximum	$q^{(-b)}_{\lambda-\mu/2} + q^{(b)}_{\lambda-\mu/2}$	Laplace Transform of $1/\sqrt{\cosh(\mu t)}$
0.1	8.01713556	8.01713850	0.009850	8.0174341
0.2	4.59287638	4.59287669	0.001035	4.5931398
0.3	3.18877655	3.18877663	0.000191	3.1888414
0.4	2.43345144	2.43345147	0.000047	2.4334704
0.5	1.96422823	1.96422824	0.000014	1.9642344
0.6	1.64534017	1.64534019	0.000004	1.6453425
0.7	1.41487766	1.41487767	0.000002	1.4148785
0.8	1.24070287	1.24070287	0.000001	1.2407032
0.9	1.10451878	1.10451879	0.000000	1.1045189
1.0	0.99516020	0.99516020	0.000000	0.9951603
2.0	0.49938034	0.49938034	-0.000000	0.4993803
3.0	0.33314886	0.33314886	0.000000	0.3331489
4.0	0.24992204	0.24992204	0.000000	0.2499220
5.0	0.19996006	0.19996006	0.000000	0.1999601
6.0	0.16664354	0.16664354	0.000000	0.1666435
7.0	0.14284258	0.14284258	0.000000	0.1428426
8.0	0.12499024	0.12499024	0.000000	0.1249902
9.0	0.11110426	0.11110426	0.000000	0.1111043
10.0	0.09999500	0.09999500	0.000000	0.0999950

$\mu = 1.0,\ b = 4.0$

λ	Minimum	Maximum	$q^{(-b)}_{\lambda-\mu/2} + q^{(b)}_{\lambda-\mu/2}$	Laplace Transform of $1/\sqrt{\cosh(\mu t)}$
0.1	2.15940227	2.15963847	0.009756	2.1596626
0.2	1.83107719	1.83117350	0.003625	1.8312355
0.3	1.58624764	1.58630261	0.001837	1.5863491
0.4	1.39694064	1.39697278	0.001067	1.3970129
0.5	1.24639749	1.24641864	0.000670	1.2464505
0.6	1.12396023	1.12397330	0.000444	1.1240000
0.7	1.02253418	1.02254179	0.000306	1.0225641
0.8	0.93721348	0.93722079	0.000217	0.9372373
0.9	0.86450324	0.86450984	0.000157	0.8645225
1.0	0.80184489	0.80184814	0.000116	0.8018598
2.0	0.45931146	0.45931244	0.000011	0.4593142
3.0	0.31888348	0.31888374	0.000002	0.3188841
4.0	0.24334691	0.24334692	0.000000	0.2433470
5.0	0.19642341	0.19642342	0.000000	0.1964234
6.0	0.16453423	0.16453424	0.000000	0.1645343
7.0	0.14148783	0.14148786	-0.000000	0.1414879
8.0	0.12407032	0.12407034	-0.000000	0.1240703
9.0	0.11045188	0.11045190	-0.000000	0.1104519
10.0	0.09951603	0.09951647	-0.000001	0.0995160

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