Characterization of Linear Mini-Max Estimators for Loss Functions of Arbitrary Power

by

K. Helmes[1)] *and C. Srinivasan*[2)]

1) Institute for Operations Research, Humboldt University of Berlin, e-mail: helmes@wiwi.hu-berlin.de; supported in part by NSF Grant DMS 98 03 490.
2) Department of Statistics, University of Kentucky, e-mail: srini@ms.uky.edu

Abstract

Let $Y(t)$, $t \in [0,1]$, be a stochastic process modeled as $dY_t = \theta(t)dt + dW(t)$, where $W(t)$ denotes a standard Wiener process, and $\theta(t)$ is an unknown function assumed to belong to a given set $\Theta \subset L^2[0,1]$. We consider the problem of estimating the value $\mathcal{L}(\theta)$, where $\mathcal{L}$ is a continuous linear function defined on Θ, using linear estimators of the form $\langle m, y\rangle = \int m(t)dY(t)$, $m \in L^2[0,1]$. The distance between the quantity $\mathcal{L}(\theta)$ and the estimated value is measured by a loss function. In this paper we consider the loss function to be an arbitrary even power function. We provide a characterization of the best linear mini-max estimator for a general power function which implies the characterization for two special cases which have previously been considered in the literature, viz. the case of a quadratic loss function and the case of a quartic loss function.

1 Introduction

We consider the following linear estimation problem. Let $Y(t)$, $t \in [0,1]$, be a stochastic process given by

$$dY(t) = \theta(t)dt + dW(t), \tag{1.1}$$

where W is a standard Wiener process and $\theta \in \Theta \subset L_2[0,1]$. A bounded continuous linear function $\mathcal{L}$ on Θ is given, and the statistical problem is to estimate $\mathcal{L}(\theta)$ by a linear estimator $\langle m, Y\rangle = \int_0^1 m(t)dY(t)$ for the case of a loss function assumed to be an arbitrary convex power function.

The problem for a *quadratic* loss function was first addressed by Ibragimov and Hasminski in [3] and [4]. They proved a minimax theorem and used it to derive a linear minimax estimator for this case. In [1] the authors used convex analysis to solve the quadratic case and in [2] used the same technique to solve the case of a *quartic* loss function. Here we show that the basic ideas of the proof carry over to the general case, although some of the details are slightly more intricate. For further references and applications of the theory we refer to the literature, see [2] and the references therein.

The paper is organized as follows. In Section 2 we state results from convex analysis which we need later on. In Section 3 we provide a characterization of the best linear mini-max estimator for a power loss function in a finite dimensional framework. Section 4 contains a characterization for the case of an infinite dimensional parameter θ. The infinite dimensional result is derived from the characterization for the finite dimensional parameter case derived in Section 3.

2 Preliminaries and Notations

Let f be a real-valued convex function defined on a locally convex topological vector space X. The subdifferential of the function f at x_0 is defined to be the set

$$\partial f(x_0) = \{x' \in X' \colon \langle x - x_0, x' \rangle \leq f(x) - f(x_0) \;\; \forall\, x \in X\}\,,$$

where X' denotes the dual of X. It is well-known that a convex function $f(x)$ attains a minimum at x^* if and only if $0 \in \partial f(x^*)$.

We will need the following results concerning the subdifferentials of convex functions. The first result is due to Moreau [6] and the second one, in its general form, is due to Ioffe and Levin [5].

Theorem 2.1 *Let f_1 and f_2 be two convex functions such that the relative interiors of their domains have a common point. Then*

$$\partial(f_1 + f_2)(x) = \partial f_1(x) + \partial f_2(x) \quad \textit{for all } x \in \textit{dom}\, f_1 \cap \textit{dom}\, f_2\,.$$

The next result is about the subdifferential of the pointwise maximum of convex functions. Let T be a compact space and $f_t(x)$ be a real function defined on $T \times X$ such that for each t, $f_t(x)$ is convex in x and for each x, continuous in t. Consider the convex function $H(x) := \max_{t \in T} f_t(x)$. We are interested in the subdifferential of H at x_0. Towards this end, let $T(x_0) = \{t \in T \colon f_t(x_0) = H(x_0)\}$, and define the set

$$W(x_0) = \{\pi : \pi \text{ is a probability measure on } T \text{ such that the support of } \pi \,\subset\, T(x_0)\}\,.$$

Theorem 2.2

$$\partial H(x_0) = \bigcup_{\pi \in W(x_0)} \left\{ \int_T \partial f_t(x_0) d\pi(t) \right\}$$

Proof: See Ioffe and Levin [5].

The notation on the right hand side of the formula in Theorem 2.2 is to be understood as the union of sets defined as set valued integrals: for details see [5].

3 The Finite Dimensional Case

In this section we prove the following minimax theorem for the finite dimensional case. Let $y = (y_1, \ldots, y_p)$ be a p-dimensional Gaussian random vector with mean $\theta \in \Theta \subset \mathbb{R}^p$ and variance-covariance matrix $I_{p\times p}$. Consider estimating the quantity $\langle l, \theta \rangle = \sum_1^p l_i \theta_i$, where $l \in \mathbb{R}^p$, by a linear estimator of the form $\langle m, y \rangle = \sum_i^p m_i y_i$ given the loss function $L(\theta, m) = (\langle \ell, \theta \rangle - \langle m, y \rangle)^{2r}$, $r \geq 1$. Since $\left\langle \frac{m}{\|m\|}, \varepsilon \right\rangle$, $\varepsilon = y - \theta$, is $\mathcal{N}(0,1)$-distributed the risk of any such linear estimator is given by

$$R(\theta, m) = E\left[L(\theta, m)\right] = \sum_{j=0}^{r} \langle \ell - m, \theta \rangle^{2j} \|m\|^{2(r-j)} \binom{2r}{2j} \frac{[2(r-j)]!}{(r-j)! 2^{(r-j)}}\,. \tag{3.1}$$

When Θ is a symmetric (around zero), compact and convex set we shall show that there is a linear estimator $\langle m^*, y\rangle$ minimax in the class of all linear estimators of the form $\langle m, y\rangle$, $m \in \mathbb{R}^p$.

Theorem 3.1 *Suppose $\Theta \subset \mathbb{R}^p$ is compact, convex and symmetric (around zero). Then*

$$\min_{m\in\mathbb{R}^p}\max_{\theta\in\Theta} R(\theta, m) = \max_{\theta\in\Theta}\min_{m\in\mathbb{R}^p} R(\theta, m)\,.$$

Proof: It suffices to prove that

$$\min_{m}\max_{\theta\in\Theta} R(\theta, m) \le \max_{\theta\in\Theta}\min_{m} R(\theta, m)\,. \tag{3.2}$$

Let us first analyze the r.h.s. of Ineq. (3.2). For the time being assume $\theta \in \Theta$ to be fixed. First, we solve the unrestricted minimization problem

$$\min_{m} R(\theta, m),$$

where $R(\theta, m)$ is given by Eq. (3.1). To this end, we exploit the necessary optimality condition $(\mathrm{grad}_m R(\theta, m))^T =: \frac{\partial R}{\partial m} = 0$. Note that $m \mapsto R(\theta, m)$ is a convex function. Setting $\beta_j(r) := \binom{2r}{2j}\frac{[2(r-j)]!}{(r-j)!2^{(r-j)}}$ we have

$$\begin{aligned}
\frac{\partial R}{\partial m} &= \frac{(2r)!}{r!2^r} r\left(\|m\|^2\right)^{r-1} 2m + r\left(\langle \ell - m, \theta\rangle^2\right)^{r-1} 2\langle \ell - m, \theta\rangle(-\theta) \\
&\quad + \sum_{j=1}^{r-1} \beta_j(r)(r-j)\left(\|m\|^2\right)^{r-j-1} 2m\, c^{2j} \\
&\quad + \sum_{j=1}^{r-1} j\beta_j(r)\langle \ell - m, \theta\rangle^{j-1} 2\langle \ell - m, \theta\rangle(-\theta)\left(\|m\|^2\right)^{r-j}
\end{aligned} \tag{3.3}$$

and the optimality condition yields

$$m_\theta = \frac{c_\theta}{w_\theta}\theta, \tag{3.4}$$

where $c_\theta = \langle \ell - m_\theta, \theta\rangle$, and

$$\frac{1}{w_\theta} = \frac{\left[r\left(\langle \ell - m, \theta\rangle^2\right)^{r-1} + \sum_{j=1}^{r-1} j\beta_j(r)\left(\|m\|^2\right)^{(r-j)}\left(\langle \ell - m, \theta\rangle^2\right)^{j-1}\right]}{\left[\frac{(2r)!}{(r-1)!2^r}\left(\|m\|^2\right)^{r-1} + \sum_{j=1}^{r-1}(r-j)\beta_j(r)\left(\|m\|^2\right)^{r-j-1}\left(\langle \ell - m, \theta\rangle^2\right)^{j}\right]}\,. \tag{3.5}$$

In order not to overburden the notation we shall suppress the subscript θ whenever we are using the parameters c_θ and w_θ in the next couple of lines. Multiplying Eq. (3.4) by $\frac{1}{\|m\|}$ and then taking the norm on both sides of the equation we see from (3.5) that w solves the equation

$$1 = w \, \frac{\displaystyle\sum_{j=0}^{r-1} (j+1)\beta_{j+1}(r)\|\theta\|^{-2j} w^{2j}}{\displaystyle\sum_{j=0}^{r-1} (r-j)\beta_j(r)\|\theta\|^{-2j} w^{2j}} . \tag{3.6}$$

Equation (3.6) can be written as

$$-\beta_1 \left(w - \frac{r\beta_0}{\beta_1} \right) = \sum_{j=1}^{r-1} (j+1)\beta_{j+1}(r) \left[w - \frac{(r-j)}{(j+1)} \frac{\beta_j(r)}{\beta_{j+1}(r)} \right] \|\theta\|^{-2j} w^{2j}. \tag{3.7}$$

It follows from the definition of $\beta_j(r)$, $0 \le j \le r-1$, that

$$\frac{(r-j)}{(j+1)} \frac{\beta_j(r)}{\beta_{j+1}(r)} = 2j+1.$$

Analyzing the polynomial equation (3.7) we see that there is a unique real root w_θ of (3.7). Hence, by (3.4) and the abbreviations used, $\min\limits_m R(\theta, m)$, for fixed θ, is attained at

$$m_\theta = \frac{\langle \ell, \theta \rangle}{w_\theta + \|\theta\|^2} \, \theta, \tag{3.8}$$

where w_θ solves Eq. (3.7). Substituting the expression on the r.h.s. of Eq. (3.8) in (3.1) we obtain, after some trivial modifications,

$$\max_{\theta \in \Theta} \min_m R(\theta, m) = \max_{\theta \in \Theta} \left\{ \left(\langle \ell, \theta \rangle \frac{\|\theta\|}{w_\theta + \|\theta\|^2} \right)^{2r} \sum_{j=0}^{r} \left(\frac{w_\theta}{\|\theta\|} \right)^{2r} \beta_j(r) \right\} . \tag{3.9}$$

Now we consider the l.h.s. of Eq. (3.2). It is easily checked that $R(\theta, m)$, as given by Eq. (3.1), is a coercive, convex function in m. Consequently, $\min\limits_m \max\limits_{\theta \in \Theta} R(\theta, m)$ is attained. We denote a minimum point by m^*, i.e.

$$\min_m \max_{\theta \in \Theta} R(\theta, m) = \max_{\theta \in \Theta} R(\theta, m^*) . \tag{3.10}$$

Since m^* is a minimum point, we must have

$$0 \in \partial \left[\max_{\theta \in \Theta} R(\theta, m^*) \right] . \tag{3.11}$$

Condition (3.11) implies, exploiting Theorems 2.2 and 2.1, and using the fact that the subdifferential of a Gateaux-differentiable function is a singleton set, we conclude:

$$0 \;\in\; \bigcup_{\pi\in W(m^*)} \left\{ \int \partial\Big[\langle \ell - m^*, \theta\rangle^{2r} + \sum_{j=1}^{r-1} \langle \ell - m*, \theta\rangle^{2j} \|m^*\|^{2(r-j)}\beta_j(r)\Big]\, d\pi(\theta) \right\} + 2r\beta_0(r)\|m^*\|^{2(r-1)} m^*, \tag{3.12}$$

where

$$W(m^*) = \{\pi\colon \pi \text{ is a probability measure on } \Theta \text{ with the support of } \pi\ \subset T(m^*)\}$$

and

$$T(m^*) \;=\; \left\{ \theta\in\Theta \;\middle|\; \langle \ell - m^*, \theta\rangle^{2r} + \sum_{j=1}^{r-1} \langle \ell - m^*, \theta\rangle^{2j} \|m^*\|^{2(r-j)}\beta_j(r) = \max_{\eta\in\Theta} \left[\langle \ell - m^*, \eta\rangle^{2r} + \sum_{j=1}^{r-1} \langle \ell - m^*, \eta\rangle^{2j} \|m^*\|^{2(r-j)}\beta_j(r) \right] \right\}.$$

Note, $T(m^*)$ is non empty since Θ is compact. So there exists a probability measure $\pi_0 \in W(m^*)$ such that

$$0 = \int \frac{\partial R}{\partial m}\,(\theta, m^*) d\pi_0(\theta), \tag{3.13}$$

where $\frac{\partial R}{\partial m}\,(\theta, m^*)$ is given by Eq. (3.3), with m replaced by m^* on the r.h.s. of the equation. Next, for m^* and any $\theta\in T(m^*)$,

$$R(\theta, m^*) - \beta_0\|m^*\|^{2r} = \langle \ell - m^*, \theta\rangle^2 \sum_{j=1}^{r} \langle \ell - m^*, \theta\rangle^{2(j-1)} \|m^*\|^{r-j}\beta_j(r) = \overset{\approx}{\text{const}}$$

$$\Longrightarrow\quad \langle \ell - m^*, \theta\rangle^2 = \overset{\sim}{\text{const}} \qquad \Longrightarrow\quad |\langle \ell - m^*, \theta\rangle| = \text{const} =: c. \tag{3.14}$$

Let $s(\theta)$ denote the sign of $\langle \ell - m^*, \theta\rangle$. By Eq. (3.13) and the definition of c,

$$\left[\sum_{j=0}^{r-1} (r-j)\beta_j(r)\left(c^{2j}\|m^*\|^{2(r-j-1)}\right)\right] = \left[\sum_{j=0}^{r-1} (j+1)\beta_{j+1}(r) c^{2j} \|m^*\|^{2(r-j-1)}\right] \int s(\theta)\theta d\pi_0(\theta). \tag{3.15}$$

Since Θ is a convex and symmetric (around zero) set,

$$\theta^* = \int s(\theta)\theta \, d\pi_0(\theta) \ \in \Theta. \tag{3.16}$$

Exploiting the linearity and continuity of $\langle \ell - m^*, \theta\rangle$, we obtain

$$\begin{aligned} c = \int c \, d\pi_0(\theta) &= \int |\langle \ell - m^*, \theta\rangle| d\pi_0(\theta) \\ &= \int \langle \ell - m^*, \theta\rangle s(\theta) d\pi_0(\theta) \\ &= \int \langle \ell - m^*, s(\theta)\theta\rangle d\pi_0(\theta) \\ &= \langle \ell - m^*, \theta^*\rangle \, . \end{aligned}$$

Hence, cf. Eqs. (3.4) and (3.5), we get

$$m^* = \frac{c^*}{w^*}\,\theta^*, \tag{3.17}$$

where $c^* = c$, c defined by Eq. (3.14), and w^* is given by Eq. (3.5) with θ replaced by θ^*.

Repeating the arguments following Eq. (3.5), and with a slight abuse of notation, we finally get

$$R(\theta^*, m^*) = \left(\frac{\langle \ell, \theta^*\rangle ||\theta^*||}{w^* + ||\theta^*||^2}\right)^{2r} \sum_{j=0}^{r} \beta_j(r) \left(\frac{w^*}{||\theta^*||}\right)^{2j} . \tag{3.18}$$

Therefore,

$$\begin{aligned} \min_m \max_{\theta\in\Theta} R(\theta, m) &= \max_{\theta\in\Theta} R(\theta, m^*) \\ &= \max_{\theta\in\Theta} \left\{ \sum_{j=0}^{r} \langle \ell - m^*, \theta\rangle^{2j} ||m^*||^{2(r-j)} \binom{2r}{2j} \frac{[2(r-j)]!}{(r-j)!2^{(r-j)}} \right\} \\ &= R\left(\theta^*, m^*\right) \\ &= \left(\frac{\langle \ell, \theta^*\rangle ||\theta^*||}{w^* + ||\theta^*||^2}\right)^{2r} \sum_{j=0}^{r} \beta_j(r) \left(\frac{w^*}{||\theta^*||}\right)^{2j} \\ &\le \max_{\theta\in\Theta} \left\{ \left(\frac{\langle \ell, \theta\rangle ||\theta||}{w_\theta + ||\theta||^2}\right)^{2r} \sum_{j=0}^{r} \beta_j(r) \left(\frac{w_\theta}{||\theta||}\right)^{2j} \right\} \\ &= \max_{\theta\in\Theta} \min_m R(\theta, m) \, . \end{aligned}$$

$\Diamond$

As an immediate consequence of the preceding theorem and its proof we have the following representation result.

Theorem 3.2 *Let $r \geq 1$ be given. Let the maximum of the r.h.s. of Eq.* (3.9) *be attained at θ^*. Let w^* denote the (unique) real root of the polynomial equation* (3.7) *with $||\theta||$ replaced by $||\theta^*||$. The linear minimax estimator for the case of the $2r$-power loss function is given by*

$$m^* = \frac{\langle \ell, \theta^* \rangle}{w^* + ||\theta^*||^2} \, \theta^*. \tag{3.19}$$

Example. Let Θ be the unit ball in $\mathbb{R}^p$, i.e. $\Theta = \{\theta \in \mathbb{R}^p \mid ||\theta|| \leq 1\}$, and assume, without loss of generality, $||\ell|| = 1$. Let m be fixed. Since Θ is the unit ball the maximum (with respect to θ) of $R(\theta, m)$ is attained at

$$\theta = \frac{\ell - m}{||\ell - m||} \, .$$

Thus

$$\max_{\theta \in \Theta} R(\theta, m) = ||\ell - m||^{2r} + \sum_{j=1}^{r-1} ||\ell - m||^{2j} ||m||^{2(r-j)} \beta_j(r) + \beta_0(r) ||m||^{2r}. \tag{3.20}$$

It is easy to see that the minimum (with respect to m) of the right hand side of (3.20) is attained for

$$m = \alpha \ell,$$

where α is the minimum of

$$q(\alpha) := (1-\alpha)^{2r} + \sum_{j=1}^{r-1} (1-\alpha)^{2j} \alpha^{2(r-j)} \beta_j(r) \; + \; \beta_0(r) \alpha^{2r}.$$

The following table provides the α values for some r values, e.g. $r = 1, 2, \ldots, 8$. The numbers can be obtained, for any r, by using a non-linear optimization routine.

Table. The optimal scaling factors $\alpha^*(r)$ (up to the first 4 digits) for the best linear minimax estimators as a function of r and the corresponding minimax risks; Θ is the unit ball in $\mathbb{R}^p$, $p \geq 1$.

Loss function r	Scaling factor $\alpha^*(r)$	Minimax risk $R(\theta^*, m^*)$
1	0.5	0.5
2	0.3067	0.5288
3	0.1934	0.5546
4	0.1403	0.5677
5	0.1098	0.5755
6	0.0902	0.5808
7	0.0765	0.5845
8	0.0663	0.5873

4 The Infinite Dimensional Case

We consider the infinite dimensional problem in this section. So in contrast to Section 2 $Y(t)$, $t \in [0,1]$, is a stochastic process given by the stochastic differential equation

$$dY(t) = \theta(t)dt + dW(t), \quad Y(0) = 0,$$

where $W(t)$ is a standard Wiener process, and the unknown parameter θ is assumed to lie in a set $\Theta \subset L_2[0,1]$. A continuous linear function $\mathcal{L}$ defined on Θ is given, and the problem is to estimate $\mathcal{L}(\theta)$ by a linear estimator $\langle m, Y\rangle = \int_0^1 m(t)dY_t$ with the loss measured by $L(\theta, m) := (\mathcal{L}(\theta) - \langle m, Y\rangle)^{2r}$, $r \geq 1$. The risk (i.e., expected loss) of any such estimator is given by, $\theta \in \Theta$, $m \in L_2$

$$\begin{aligned} R(\theta, m) \;=\; E\left[L(\theta, m)\right] \;=\; & (\mathcal{L}(\theta) - \langle m, \theta\rangle)^{2r} \\ & + \sum_{j=1}^{r-1} (\mathcal{L}(\theta) - \langle m, \theta\rangle)^{2j} \|m\|^{2(r-j)} \beta_j(r) \\ & + \beta_0(r)\|m\|^{2r}, \end{aligned} \tag{4.1}$$

where $\langle m, \theta\rangle = \int_0^1 m(t)\theta(t)dt$ and $\|m\|$ is the L_2 norm of m.

A careful analysis of the proofs of the various linear minimax theorems for estimating $\mathcal{L}(\theta)$ in the infinite dimensional case given for the quadratic and the quartic cases in [1] and [2] shows that because of Theorem 3.1 these theorems also hold for a general loss function. Here we state one of these theorems.

Theorem 4.1 *Let Θ be a symmetric (around zero) closed and bounded set in L_2. Suppose $\mathcal{L}$ is weakly continuous on Θ. For each $\theta \in \Theta$ let w_θ denote the (unique) real root of Eq. (3.7). Then*

$$\begin{aligned} \min_{m \in L_2} \max_{\theta \in \Theta} R(\theta, m) &= \max_{\theta \in \Theta} \min_{m \in L_2} R(\theta, m) \\ &= \max_{\theta \in \Theta} \left\{ \left(\mathcal{L}(\theta) \frac{\|\theta\|}{w_\theta + \|\theta\|^2} \right)^{2r} \sum_{j=0}^{r} \left(\frac{w_\theta}{\|\theta\|} \right)^{2r} \beta_j(r) \right\}. \end{aligned} \tag{4.2}$$

Moreover, if $\max_{\theta\in\Theta} \left\{ \left(\mathcal{L}(\theta) \frac{\|\theta\|}{w_\theta + \|\theta\|^2} \right)^{2r} \sum_{j=0}^{r} \left(\frac{w_\theta}{\|\theta\|} \right)^{2r} \beta_j(r) \right\}$ *attains its maximum at θ^*, then the linear minimax estimator is given by*

$$\langle m^*, Y\rangle = \int_0^1 m^*(t)\, dY(t)$$

where, $w^* := w_{\theta^*}$,

$$m^* = \frac{\mathcal{L}(\theta^*)}{w^* + \|\theta^*\|^2}\,\theta^*. \tag{4.3}$$

5 Summary

We have dealt with the following statistical estimation problem, $r \geq 1$,

$$\min_{m \in L_2[0,1]} \max_{\theta \in \Theta} R(\theta, m), \quad \text{where} \quad R(\theta, m) = E\Big[\Big(\mathcal{L}(\theta) - \langle m, Y\rangle\Big)^{2r}\Big],$$

and Y is given by Eq. (1.1). Using convex analysis we have shown that under appropriate conditions on Θ a mini-max theorem holds (see Theorem 3.1 for the finite-dimensional version and Theorem 4.1 for an infinite-dimensional version of the theorem). The mini-max result leads to a characterization of the linear mini-max estimator in terms of the solution of a non-linear optimization problem, see Theorems 3.2 and 4.1. As an example of the theory we have computed the best linear mini-max estimator when Θ is a unit ball for several values of r. The cases $r = 1$ and $r = 2$ had previously been considered in the literature. The results for larger values of r are new. The example clearly shows the effect the loss function has on the form of the estimator. The larger the exponent of the loss function the smaller becomes the "shrinkage" factor in the estimator. The shrinkage factor depends in a highly non-linear way on the norms of the unknown parameters.

References

[1] Helmes, K. and Srinivasan, C. (1992). Linear minimax estimation, *Lecture Notes in Economics and Mathematics,* 389, 9–23.

[2] Helmes, K. and Srinivasan, C. (2000). A linear mini-max estimator for the case of a quartic loss function, to appear in *Random Oper. and Stoch. Equ.,* vol. 8, No. 3.

[3] Ibragimov, A.D. and Hasminski, R.Z. (1984). On non-parametric estimation of the value of a linear function in Gaussian white noise. *Theory of Probability and Its Applications,* Vol. XXIX, No. 1, 18–32.

[4] Ibragimov, A.D. and Hasminski, R.Z. (1987). Estimation of linear functionals in Gaussian noise. *Theory Prob. Appl.,* vol. 32, No. 1, 30–39.

[5] Ioffe, A.D. and Levin, V.L. (1974). Subdifferentials of convex functions. *Transactions of the Moscow Mathematical Society,* vol. 26.

[6] Moreau, J.J. (1964). Sur la fonction polair d'une fonction semi-continue superieurment, *C.R. Acad. Sci. Paris,* 258, 1128–1130.