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Numerical Bounds on Moments of Distributions of low-dimensional Diffusion Processes by Linear Programming

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Abstract

We present a numerical method, based on linear programming (LP), to compute upper and lower bounds on (higher) moments of distributions of low-dimensional diffusion processes, e. g. 2- or 3-dimensional processes. The method relies on a LP-formulation of the evolution of time- and space truncated versions of such processes. It leads to finite dimensional LP problems on the space of Hausdorff sequences of finite order where the objective is to maximize and minimize moments of distributions under consideration. Since Hausdorff sequences of finite order comprise all moment sequences this approach provides upper as well as lower bounds on the quantity of interest. For the 1-dimensional case the set of Hausdorff sequences of order M is defined by $\frac{M(M+1)}{2}$ nontrival linear inequalities. We provide a characterization of the set of such sequences based on just M linear inequalities. This characterisation, expecially for 2- and 3-dimensional process, yields a substantial reduction in the dimensions of the LP problems to be solved.

We shall illustrate the method by looking at examples which range from exponential functionals of Brownian motion related to Asian options to exit time distributions of squares of Bessel processes.

1 Introduction

The numerical method to be described below is based on the fact that the evolution of a Markov process can be described through the occupation measure of the process. This description involves a system of equations over measures on the state space of the Markov process, cf. e. g. Bhatt and Borkar [1], Kurtz and Stockbridge [7], Hernandez-Lerma et al. [6]. The system of equations is indexed by test functions to be chosen from a class rich enough to specify the generator of the Markov process. Depending on the set of test functions used this system of equations, see Eqn. (2.1) below, determines uncountably many linear constraints for the measures involved. If the state space is a closed bounded interval these measures are uniquely characterized by their moments, and these moments, due to the Hausdorff conditions, cf. e. g. [8], are uniquely characterized by countably many linear inequalities. Thus, if we choose as test functions the monomials restricted to a closed bounded interval the evolution of the process is captured by countably many linear equations together with countably many linear inequalities in as many unknowns. Any quantity of interest which can be expressed as a linear form of these variables can therefore be characterized as the value of an infinite dimensional linear program. For numerical computation, any such problem is reduced to finite dimensions by considering only a finite number of moments and finitely many constraints. To analyze processes on an unbounded state space we employ a truncation technique. Since the set of feasible points of the finite dimensional problem is larger than the set of feasible points of the original infinite dimensional problem maximizing and minimizing the individual moments subject to the constraints yields upper and lower bounds for these moments.

The paper is organized as follows. In section 2 we state some theoretical results which form the basis of the numerical method. In section 3 we illustrate the method by computing higher moments of the exit time of squared Bessel processes from a bounded domain. Squared Bessel processes are related, among other things, to a particular interest rate model. The exit time problem is important for the analysis of a class of barrier options. In section 4 we report on numerical results concerning some exponential functionals of Brownian motion which are of interest in the context of Asian options. Since analytical results are known for these problems they are good test cases for judging the accuracy of the method proposed.

2 The LP formulation

Let $(Y_t)_{t\geq 0}$ be a Markov process on $\mathbb{R}^d$ with generator $\mathcal{A}$ and initial point y_0. Let its state space be written as the disjoint union of a bounded set G and part or all of its boundary ∂G. The following result is an immediate consequence of the martingale characterization of Markov processes, see [2], and the optional sampling theorem.

Theorem 2.1 *Let τ denote the first exit time of $(Y_t)_t$ from G. Let ∂G denote the exit region. Assume $E[\tau] < \infty$. Then there exists a probability measure $\mu_{\partial G}$ on ∂G and a measure μ_G on G such that for every f in the domain of the generator $\mathcal{A}$ (a test function) the following equation holds:*

$$\int\limits_{\partial G} f(y)\mu_{\partial G}(dy) - f(y_0) = \int\limits_{G} Af(y)\mu_G(dy). \tag{2.1}$$

It is shown in [1] and [7] that the converse of Theorem 2.1 holds, i. e. given measures $\mu_{\partial G}$ and μ_G which satisfy Eqn. (2.1) these measures are the exit distribution and the occupation measure of a Markov process with generator $\mathcal{A}$.

Theorem 2.1 and its converse are the basis of the LP formulation and the numerical method to be described below.

The following variant of Theorem 2.1 and its converse is applied in section 4. For another example see [5].

Corollary 2.2 *Let $\mathcal{A}^\lambda$ denote the generator of the process defined by killing a Markov process with generator $\mathcal{A}$ at rate λ and restarting the process at the initial position. There exists a measure μ on the state space $G \cup \partial G$ such that for every test function f*

$$\int\limits_{G\cup\partial G} \mathcal{A}^\lambda f(y)\mu(dy) = 0, \tag{2.2}$$

and for any measurable real-valued function φ

$$\lambda \int\limits_0^\infty e^{-\lambda t} E[\varphi(Y_t)]dt = \int\limits_{G\cup\partial G} \varphi(y)\mu(dy).$$

From now on we shall assume that G is a rectangle and, without loss of generality, see examples in section 3 and 4 for details, that G is a hypercube.

The measures $\mu_{\partial G}$ and μ_G are uniquely characterized by their moment sequences, $z_{\partial G}$ and z_G, i. e. $n = (n_1, \ldots, n_d)$, $n_i \in \mathbb{N}_0$,

$$z_{\partial G}(n) = \int\limits_{\partial G} z_1^{n_1} \cdot \ldots \cdot z_d^{n_d} \mu_{\partial G}(dz)$$

and

$$z_G(n) = \int\limits_{G} z_1^{n_1} \cdot \ldots \cdot z_d^{n_d} \mu_G(dz).$$

Since G is assumed to be a hypercube $(z_G(n))_n$ and $(z_{\partial G}(n))_n$ are sequences characterized by linear inequalities, viz. by the Hausdorff conditions, cf. e. g. [8]. In order to simplify the exposition we shall only state the conditions for the 1-dimensional case and the 2-dimensional one, i. e. if $G = [0, 1]$,

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} z_G(j+k) \geq 0, \qquad (2.3)$$

for $n, k = 0, 1, 2, \ldots$, and, if $G = [0, 1] \times [0, 1]$,

$$\sum_{j_1=0}^{n_1} \sum_{j_2=0}^{n_2} \binom{n_1}{j_1} \binom{n_2}{j_2} (-1)^{k_1+k_2} \cdot z_G(j_1 + k_1, j_2 + k_2) \geq 0, \quad (2.4)$$

for all $n_1, n_2, k_1, k_2 = 0, 1, 2, \ldots$; the conditions for dimensions 3 and up are straightforward generalizations of inequalities (2.4).

Any infinite sequence which satisfies the inequalities (2.3) is called a 1-dimensional *moment sequence*. A finite sequence $(z_0, z_1, \ldots, z_M)$ is called a 1-dimensional *Hausdorff sequence* of order M iff inequalities (2.3) hold for all $k = 0, 1, 2, \ldots, M$ and $n = 0, 1, 2, \ldots, M - k$. There is obviously a natural extension of the terminology for multi-dimensional sequences.

In many applications the generator $\mathcal{A}$ is composed of differential and/or difference operators. Choosing monomials as test functions in such cases, $\mathcal{A}f$ becomes a polynomial and the (adjoint) Eqn. (2.1) can be phrased in terms of moments. Choosing only finitely many such test functions the following finite dimensional LP problems provide upper and lower bounds for, e. g. the mean exit time of τ.

Theorem 2.3 *Let $\mathcal{A}$ be a differential operator composed of a differential and/or difference operator. Let $\bar{v}(M)$ be the optimal value of*

$$\max \overset{!}{=} z_G(0)$$

subject to z_G, $z_{\partial G}$ resp., Hausdorff-sequences of order M of dimension d, $d-1$ resp., which satisfy the adjoint equation (2.1) if f is chosen to be any monomial of order less or equal to M. Let $\underline{v}(M)$ be the minimum value of the same LP problem. Then

$$\underline{v}(M) \leq E[\tau] \leq \bar{v}(M).$$

Corollary 2.4 *Let T_λ be an exponential random variable independent of $(Y_t)_t$. Let $(\eta_t)_t$ denote the first component of Y. Then for every n and M*

$$\underline{v}(M, n) \leq E[\eta_{T_\lambda}^n] \leq \bar{v}(M, n),$$

where $\bar{v}(M, n)$ is the value of the LP-problem:

$$\max \overset{!}{=} z_G(n, 0, \ldots, 0),$$

subject to all Hausdorff sequences of order M which satisfy Eqn. (2.2) if f is a monomial of order less or equal to M.

To find bounds on higher moments of exit time distributions we reformulate Theorems 2.1 and 2.2 and the Corallaries for the case of time-space processes $(t, Y_t)_t$ whose generator equals $\frac{\partial}{\partial t} + \mathcal{A}$; see section 3 for an example. Additional examples can be found in [3] and [4].

3 Squares of Bessel processes

Let $(B_t)_{t\geq 0}$ be a d-dimensional, $d \in \mathbb{N}$, Brownian motion starting from a, and let $Y_t := |B_t|^2$ denote the square of the corresponding Bessel process. Then $(Y_t)_{t\geq 0}$ satisfies the following stochastic differential equation, $Y(0) = y_0 = |a|^2$,

$$dY_t = d \cdot dt + 2\sqrt{Y_t}d\beta_t, \tag{3.1}$$

where $(\beta_t)_{t\geq 0}$ is a 1-dimensional Brownian motion starting at zero. Eqn. (3.1) has a (strong) solution not only for non-negative integers but for any non-negative real number d; we call the corresponding solution of Eqn. (3.1) the (generalized) squared Bessel process of "dimension" d.

The generator of $(X_t) = (t, Y_t)_t$ is given by, f a test function,

$$\mathcal{A}f(t,y) = \frac{\partial f}{\partial t}(t,y) + 2y\frac{\partial^2 f}{\partial y^2}(t,y).$$

For given b and $T > 0$ let τ denote the first exit time of $(X_t)_{t\geq 0}$ from the domain $G = (0,b) \times (0,T)$; the (relevant) boundary ∂G consists of three parts $G_{\text{bottom}} = [0,T] \times \{0\}$, $G_{\text{top}} = [0,T] \times \{b\}$ and $G_{\text{right}} = \{T\} \times [0,b]$. Thus we shall work with the 2-dimensional (occupation) measure μ_G defined on $(0,T) \times (0,b)$ and the probability measure $\mu_{\partial G}$ defined on $G_{\text{bottom}} \cup G_{\text{top}} \cup G_{\text{right}}$. We split the exit distribution $\mu_{\partial G}$ into three parts defined on the three different pieces which form the boundary. Let $\mu_{ij} = \int_G \left(\frac{t}{T}\right)^i \left(\frac{y}{b}\right)^j \mu_G(dt,dy)$ denote the (scaled) i,jth moment of the occupation measure μ_G and let us denote by $\mu_i^{(b)}$, $\mu_i^{(t)}$ and $\mu_i^{(r)}$ resp. the (similarly) scaled ith moment of the boundary measures on G_b, G_t and G_r resp. The feasible set of the finite dimensional LP problem, cf. section 2, is described by the linear inequalities,

$$\sum_{k=0}^{m}\sum_{\ell=0}^{n}\binom{m}{k}\binom{n}{\ell}(-1)^{k+\ell}\mu_{i+k,j+\ell} \geq 0,$$

for all $m,n \in \{0,1,\ldots,M\}$ and for all $0 \leq i \leq M-m$, $0 \leq j \leq M-n$,

$$\sum_{k=0}^{m}\binom{m}{k}(-1)^k\mu_{i+k}^{(s)} \geq 0,$$

for $s \in \{\text{bottom}, \text{top}, \text{right}\}$, and the linear equations

$$\begin{aligned}
0 &= \frac{m}{T}\mu_{m-1,n} + \frac{2n(n-1)}{b^2}\mu_{m,n-2} \\
&\qquad + \frac{nd}{b}\mu_{m,n-1} - \mu_m^{(t)} - \mu_n^{(r)}, \\
&\qquad\qquad \text{for } m \geq 1, n \geq 2, \\
0 &= \frac{m}{T}\mu_{m-1,1} + \frac{d}{b}\mu_{m,0} - \mu_m^{(t)} - \mu_n^{(r)}, \\
&\qquad\qquad \text{for } m \geq 1, n = 1, \\
0 &= \frac{m}{T}\mu_{m-1,0} - \mu_m^{(b)} - \mu_m^{(t)} - \mu_0^{(r)}, \\
&\qquad\qquad \text{for } m \geq 1, n = 0, \\
0 &= \frac{2n(n-1)}{b^2}\mu_{0,n-2} + \frac{nd}{b}\mu_{0,n-1} + \\
&\qquad \left(\frac{y_0}{b}\right)^n - \mu_0^{(t)} - \mu_n^{(r)}, \\
&\qquad\qquad \text{for } m = 0, n \geq 1.
\end{aligned}$$

We obtain bounds on, for instance, the mth moment of the exit time distribution of τ by maximizing/minimizing the expression $mT^{m-1}\mu_{m-1,0}$ subject to the linear constraints described above. If we let T become large we get bounds on the moments of the exit distribution of the squared Bessel process from a strip $(0,b)$.

In the talk we shall present numerical results based on numerous computations.

4 Exponential functionals of Brownian motion

In section 4 $(B_t)_{t\geq 0}$ denotes a 1-dimensional Brownian motion starting at zero. Let, $\varrho \in \mathbb{R}$,

$$A_t := \int_0^t \exp[2(Bs + \varrho s)]ds.$$

The process (A_t) is of interest in the context of Asian options, cf. [9]. Yor, see [9], p. 69, derived the following formula for the nth moment of the random variable A_{T_λ}, where T_λ denotes an exponential variable with parameter λ independent of (B_t):

$$E\left[A_{T_\lambda}^n\right] = \frac{n!}{\prod_{j=1}^{n} (\lambda - 2(j^2 + j\varrho))}. \quad (4.1)$$

The following method by which these moments can be numerically computed is based on Corollary 2.4. To begin with, we define the 2-dimensional Markov process $Y_t = \left(Z_t, \int_0^t Z_s ds\right)$, where $Z_t = \exp[2(B_t + \varrho t)]$. Obviously, $A_t = \int_0^t Z_s ds$, $t \geq 0$, and $(Z_t)_t$ satisfies the stochastic differential equation, $Z(0) = 1$,

$$dZ_t = (\varrho + 2)Z_t dt + 2Z_t dB_t.$$

The process $(Y_t)_t$ evolves on the nonnegative orthant in $\mathbb{R}^2$. Its generator is given by, f a test function, $0 \leq y = (y_1, y_2) \in \mathbb{R}^2$,

$$\mathcal{A}f(y) = 2y_1^2 \frac{\partial^2 f}{\partial y_1^2} + (\varrho + 2)y_1 \frac{\partial f}{\partial y_1} + y_1 \frac{\partial f}{\partial y_2}.$$

To compute the right-hand side of Eqn. (4.1) we kill the process $(Y_t)_t$ at rate λ and restart it at $y_0 = (1, 0)$. The generator $\mathcal{A}^\lambda$ of the modified process has the form, f a test function,

$$\mathcal{A}^\lambda f(y) = \mathcal{A}f(y) + \lambda[f(y_0) - f(y)].$$

To obtain a LP problem we further modify the process as follows. We choose a "large" number b and we restrict the killed and restarted process to $G = (0, b] \times [0, b)$. Whenever one of the components hits the level b the process jumps back to the starting point $(1, 0)$ after it spent an exponential (with parameter ν) random time at the hitting point. Since the killing mechanism operates independently of the jump mechanism the generator of the process becomes

$$\mathcal{A}^\lambda f(y) I_{\{y\in G\}} + (\lambda+\nu)[f(y_0) - f(y)] I_{\{y\in\partial G\}},$$

where $\partial G = \{b\} \times [0, b] \cup (0, b] \times \{b\}$. The linear programming formulation (see [7]) which yields bounds for the left-hand side of Eqn. (4.1) reads:

$$\max / \min \int_G y^n \mu_G(dy) + \int_{\partial G} y^n \mu_{\partial G}(dy)$$

subject to

$$\int_G \mathcal{A}^\lambda f(y)\mu_G(dy) + \int_{\partial G} \mathcal{A}^\lambda f(y)\mu_{\partial G}(dy) = 0,$$

$\mu_G + \mu_{\partial G}$ is a probability measure.

We shall report on the stability and, in light of the analytical results, on the accuracy of the method.

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