Numerical Methods for Optimal Stopping Using Linear and Non-Linear Programming*

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Abstract. Computational methods for optimal stopping problems are presented. The first method to be described is based on a linear programming approach to exit time problems of Markov processes and is applicable whenever the objective function is a unimodal function of a threshhold parameter which specifies a stopping time. The second method, using linear and non-linear programming techniques, is a modification of a general linear programming approach to optimal stopping problems recently proposed by S. Röhl. Both methods are illustrated by solving Shiryaev's quickest detection problem for Brownian motion.

Key Words. Optimal stopping, linear programming, numerical methods for exit time problems, detection problem.

Subject Classification. *Primary* 60G40. *Secondary* 90C05, 90C30, 60G35, 93E20, 65C20.

1 Introduction

The purpose of the paper is to describe numerical methods based on linear programming and on non-linear optimization techniques for solving optimal stopping problems of Markov processes. We shall illustrate the power of these methods by (numerically) analyzing Shiryaev's quickest detection problem for a Wiener process.

The linear programming approach to optimal stopping and to stochastic control in general is an extension of work by Manne [13] who initiated the formulation of stochastic control problems as linear programs over a space of stationary distributions for the long-term average control of finite-state Markov chains, see Hernandez-Lerma et. al. [10] for details, generalizations and additional references. The generalization of the LP formulation for continuous time, general state and control spaces, and different objective functions has been established by Stockbridge [20], Kurtz and Stockbridge [11], [12], and Bhatt and Borkar [1]. The LP-formulation for stopping time problems and numerical methods for the solution of such problems have been presented by Cho [2], Cho and Stockbridge [3], and Röhl [16].

The basic idea of the LP-approach to stochastic control of Markov processes is to formulate such control problems as linear programs over a space of

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stationary distributions. Specifically, the variables in these infinite-dimensional linear programs are measures on the product of the state and control spaces and in the case of exit problems, each such variable is augmented by a second measure on the exterior of the state space. These variables are tied together by an adjoint equation involving the generator A of the Markov process and a family of test functions. Different numerical methods are determined by a judious choice of a finite set of test functions combined with a selection of a finite number of variables and/or restrictions imposed on the support of the occupation measure and the exterior measure. Such choices determine approximations of the infinite dimensional optimization problem by a finite dimensional one. The viability of these numerical methods has been demonstrated in an uncontrolled setting by Helmes et. al. [6] and Helmes and Stockbridge [8] by analyzing the distribution of the exit time for a variety of processes evolving on bounded intervals in one- and two-dimensions. In [7], [9] Helmes and Stockbridge have applied the method of moments, i. e. the test functions chosen are the monomials up to a given order, on a stochastic control problem and on a particular class of diffusion processes defined on a higher dimensional state space. In [14], [15] Mendiondo and Stockbridge have applied the discretization method, i. e. the support of all measures is restricted to a finite set of gridpoints, to the analysis of some stochastic control problems, while Cho [2] uses the discretization method to analyze a stopping time problem. Röhl [16] uses the method of moments and proposes an iterative scheme for the numerical analysis of one- and two-dimensional stopping problems.

In this note we shall built on [6] and [16] and exploit an advantage which the method of moments has over other variants of the LP technique. For exit time problems, cf. [6], the method naturally provides bounds on quantities of interest, e. g. upper and lower bounds on the mean exit time from a bounded interval, etc. So for optimal stopping problems whose objective function is a unimodal function of a threshhold value which specifies a stopping time, we can apply line search techniques, f. i. the Fibonacci or Golden Section rule, to the optimal values of such parametrized LP-problems. This way we obtain a range for the optimal stopping rule parameter and get bounds for the optimal value, see Section 3 below. Ideally, if the bounds converge to the same value we shall find an optimal solution to the original problem.

The second method which we propose is applicable to more general stopping problems and replaces Röhl's iteration technique by a pair of optimization problems, one being linear the other one being non-linear, and another LP-problem which constitutes the verification step of the procedure.

2 Formulation and Fundamental Theorems

We formulate optimal stopping problems in a restricted setting which fits the numerical methods which we propose and are adequate for the example to be analyzed. In order to keep the notation simple we shall also restrict the formulation to bounded intervals $I \subset \mathbb{R}$. Moreover, the Markov processes to be considered are diffusions with polynomial coefficients, i. e. the generator A has the form, $f \in \mathcal{D} := \text{domain}(A)$, $x \in I$,

$$Af(x) = \frac{a(x)}{2} f''(x) + b(x) f'(x), \tag{1}$$

where $a(x)$ and $b(x)$ are polynomials on I. For the quickest detection problem to be analyzed in Sections 3 and 4, see also Shiryaev [19], $I = [0, 1]$, $a(x) = \text{const} \cdot x^2(1-x)^2$, $b(x) = \tilde{\text{const}} \cdot (1-x)$, while for a stopping problem related to the valuation of a perpetual Russian option, cf. Shepp and Shiryaev [17], [18], $I = [1, K)$, $K \gg 1$, $a(x) = \text{const} \cdot x^2$, $b(x) = \tilde{\text{const}} \cdot x$, and $\mathcal{D} = \{g \in \mathcal{C}^2((1,\infty)) \mid g'(1+) = 0\}$.

The process $(X_t)_{t \geq 0}$ to be stopped is characterized as a solution to the martingale problem for the generator A and an initial position x, i. e. there exists a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ such that $(X_t)_t$ is $\{\mathcal{F}_t\}$-progressively measurable, $X_0 = x$, and for every $f \in \mathcal{D}$, $t \geq 0$, the expression

$$f(X_t) - f(X_0) - \int_0^t Af(X_s) ds$$

defines an $\{\mathcal{F}_t\}$-martingale.

The objective of the decision maker is to minimize (or maximize) an expected pay-off,

$$E_x \left[R(X_\tau) + \int_0^\tau \ell(X_s) ds \right], \tag{2}$$

over all $\{\mathcal{F}_t\}$-stopping times τ for which $E_x[\tau] < \infty$, where R and ℓ are polynomial functions on I. For instance, for the detection problem $R(x) = 1 - x$ and $\ell(x)$ is proportional to x, and (2) is to be minimized.

There are two well known methods which can be employed to solve the optimal stopping problem

$$\inf_{\tau, E_x[\tau] < \infty} \left\{ E_x \left[R(X_\tau) + \int_0^\tau \ell(X_s) ds \right] \right\} =: v^*(x), \tag{3}$$

viz. the supermartingale characterization or the variational inequality approach, cf. Shiryaev [19]. The LP-approach to exit time problems and the LP-approach to stopping provide an alternative to these methods and are particularly important from the point of view of numerical computations.

In the sequel we shall repeatedly use the following shorthand writing

$$\langle \mu, f \rangle := \int_I f(x) \mu(dx),$$

where μ denotes a non-negative ($\mu \geq 0$) measure on I and f is any (Borel)-measurable and μ-integrable function defined on the interval. We formally define the adjoint operator A^* of A, i. e. A^* is applied to measures μ, by the equation

$$\langle A^*\mu, f\rangle := \langle \mu, Af\rangle$$

for all $f \in \mathcal{D}$. We let δ_x denote the Dirac measure at x. So the equation

$$\mu_1 - \delta_x - A^*\mu_0 = 0, \tag{4}$$

where μ_0 and μ_1 are non-negative measures on I, is to be understood as shorthand writing for the family of equations

$$\forall\, f \in \mathcal{D}, \qquad \langle \mu_1, f\rangle - f(x) - \langle \mu_0, Af\rangle = 0.$$

The symbol $\mathbb{1}$ stands for the constant function identical to one.

2.1 The exit time approach (cf. Method I)

Let $(X_t)_{t\geq 0}$ denote the Markov process to be stopped so to minimize the expected pay-off (2). By definition, the quantity

$$f(X_t) - f(x) - \int_0^t Af(X_s)ds$$

is a martingale for each $f \in \mathcal{D}$ and thus it follows by the optional sampling theorem that for each admissible stopping time τ (note that $E_x[\tau] < \infty$)

$$E_x[f(X_\tau)] - f(x) - E_x\left[\int_0^\tau Af(X_s)ds\right] = 0. \tag{5}$$

Define the occupation measure μ_0 and exit distribution μ_1 by

$$\mu_0(\Gamma) = E_x\left[\int_0^\tau I_\Gamma(X_s)ds\right] \qquad \text{and} \qquad \mu_1(\Gamma) = P_x[X_\tau \in \Gamma]$$

for Borel sets $\Gamma \subset I$. It then follows that (5) can be written as (4).

We refer to (4) as the basic adjoint equation. In Kurtz and Stockbridge [11] it is shown for very general (controlled) models which include our model as a special case that for each μ_0 and μ_1 satisfying (4) there is a process $X = (X_t)_t$ and a stopping time τ for which (5) is satisfied, and τ is essentially the first exit time of X. Thus the basic adjoint equation characterizes the occupation measure μ_0 and the exit distribution μ_1 of a Markov process defined on I having generator A. Applying this reasoning to any subinterval $B \subset I$ we obtain the following result.

Theorem 1. For each measure μ_0 and μ_1 restricted to $B = [a, b] \subset I$ and satisfying the basic adjoint equation, the expression

$$\langle \mu_1, R \rangle + \langle \mu_0, \ell \rangle =: \Psi(a, b)$$

equals the expected pay-off of a Markov process with generator A which is stopped when hitting a or b.

2.2 A general LP-approach to optimal stopping (cf. Method II)

The exit time approach will lead to a numerical method which is only applicable to a restricted class of stopping problems. Moreover, it is an indirect approach by which optimal solutions can be found. A direct method based on linear programming has recently been proposed in two theses, cf. Cho [2] and Röhl [16]. The following result is a special case of general theorems proved by these authors and provides the analytical underpinning for Method II, see Section 4.

Theorem 2. Consider the optimal stopping time problem (3). Then $v^*(x)$ equals the optimal value of the infinite-dimensional linear program

$$\inf_{\mu_\tau, \mu \geq 0} \{ \langle \mu_\tau, R \rangle + \langle \mu, \ell \rangle \mid \langle \mu_\tau, \mathbb{1} \rangle = 1, \;\; \mu_\tau - \delta_x - A^* \mu = 0 \} . \tag{6}$$

3 Method I

To further simplify the exposition we shall – without loss of generality – assume from now on that I equals the unit interval $[0\,1]$; the change of variable $x \mapsto (x - a)/(b - a)$ will transform general cases to this special one.

In Section 2.1 we have seen that each exit time problem is equivalent to a particular infinite-dimensional linear program. Since measures on bounded intervals are determined by their moments and since the generator A is assumed to have polynomial coefficients, choosing finitely many moments as variables we can associate with each stopping time $\tau_b := \inf\{t \mid X_t \geq b\}$, $0 \leq b \leq 1$, i. e. we define $B = [0, b]$ in Theorem 1, two linear programs whose optimal values sandwhich the expected pay-off when using τ_b, $0 \leq x \leq b$,

$$\varphi(b) := \Psi(0, b) = E_x \left[R(X_{\tau_b}) + \int_0^{\tau_b} \ell(X_s) ds \right] .$$

To formulate these programs we let $a(x) = \sum_{i=0}^{N_a} \alpha_i x^i$, $b(x) = \sum_{i=0}^{N_b} \beta_i x^i$, $\ell(x) = \sum_{i=0}^{N_\ell} \gamma_i x^i$ and $R(x) = \sum_{i=0}^{N_R} \delta_i x^i$, where N_a, N_b, N_ℓ and N_R are integers, and $\alpha_0, \ldots, \alpha_{N_a}$, $\beta_0, \ldots, \delta_1, \ldots, \delta_{N_R} \in \mathbb{R}$. For any integer $M \geq N :=$

$\max\{N_a, N_b, N_\ell, N_R\}$ we define for $\boldsymbol{\mu} = (\mu_0, \mu_1, \ldots, \mu_M) \in \mathbb{R}^{M+1}$ the i-th iterated differences of $\boldsymbol{\mu}$, $(i,n) \in \mathfrak{M} := \{(i,n) \mid 0 \le i \le M,\ 0 \le n \le M-i\}$,

$$(-1)^i \Delta^i \boldsymbol{\mu}(n) = \sum_{k=0}^{i} (-1)^k \binom{i}{k} \mu_{k+n}, \tag{7}$$

and call

$$\mathcal{H}^M := \left\{ \boldsymbol{\mu} \in \mathbb{R}^{M+1} \mid \Delta^i \boldsymbol{\mu}(n) \ge 0,\ (i,n) \in \mathfrak{M} \right\} \subset \mathbb{R}^{M+1}$$

the Hausdorff polytope of order M, cf. Helmes [5] and, for the generalization of this concept to higher dimensions, Röhl [16]. For vectors $\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_{N_a}, 0, \ldots, 0) \in \mathbb{R}^{M+1}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$ and $\boldsymbol{\delta}$ similarly defined, we denote their scalar product with vectors $\boldsymbol{\mu} \in \mathcal{H}^M$ by

$$\langle \boldsymbol{\mu}, \boldsymbol{\alpha} \rangle := \sum_{k=0}^{M} \mu_k \alpha_k;$$

for further use we shall also introduce the following abbreviations, $k \in \mathcal{K} := \{k \in \mathbb{N} \mid 0 \le k \le M - N\}$, $\eta_k := \eta_k(x, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\mu}^{(1)}, \boldsymbol{\mu}^{(0)})$ and

$$\eta_k := \frac{k(k-1)}{2} \sum_{i=0}^{N_a} \alpha_i \mu_{i+k-2}^{(0)} + k \sum_{i=0}^{N_b} \beta_i \mu_{i+k-1}^{(1)} + x^k - \mu_k^{(1)}.$$

Next we define two linear programming problems, $\mathcal{P}_{\min}$ and $\mathcal{P}_{\max}$:

$$\underline{\varphi}(b) := \min_{\boldsymbol{\mu}^{(0)}, \boldsymbol{\mu}^{(1)}} \left\{ \langle \boldsymbol{\mu}^{(1)}, \boldsymbol{\delta} \rangle + \langle \boldsymbol{\mu}^{(0)}, \boldsymbol{\gamma} \rangle \;\middle|\; \begin{array}{l} \boldsymbol{\mu}^{(0)}, \boldsymbol{\mu}^{(1)} \in \mathcal{H}^M,\ \mu_0^{(1)} = 1, \\ \eta_k = 0,\ k \in \mathcal{K} \end{array} \right\}$$

and

$$\bar{\varphi}(b) := \max_{\boldsymbol{\mu}^{(0)}, \boldsymbol{\mu}^{(1)}} \left\{ \langle \boldsymbol{\mu}^{(1)}, \boldsymbol{\delta} \rangle + \langle \boldsymbol{\mu}^{(0)}, \boldsymbol{\gamma} \rangle \;\middle|\; \begin{array}{l} \boldsymbol{\mu}^{(0)}, \boldsymbol{\mu}^{(1)} \in \mathcal{H}^M,\ \mu_0^{(1)} = 1, \\ \eta_k = 0,\ k \in \mathcal{K} \end{array} \right\}.$$

Since the Hausdorff polytope includes the set of all moment sequences up to order M, we obtain the inequalities

$$\min_{0 \le b \le 1} \underline{\varphi}(b) \le \min_{0 \le b \le 1} \varphi(b) \le \min_{0 \le b \le 1} \bar{\varphi}(b).$$

Furthermore, if there is a b^* such that

$$\min_{0 \le b \le 1} \underline{\varphi}(b) = \underline{\varphi}(b^*) = \bar{\varphi}(b^*) = \min_{0 \le b \le 1} \bar{\varphi}(b) \tag{8}$$

then τ_{b^*} is the optimal stopping time in the class of all stopping rules $\{\tau_b\}_{0 \le b \le 1}$. If

$$\varepsilon^* := \min_{0 \le b \le 1} \bar{\varphi}(b) - \min_{0 \le b \le 1} \underline{\varphi}(b) > 0, \tag{9}$$

and $\bar{b}^*$, $\underline{b}^*$ resp., is a solution of $\mathcal{P}_{\max}$, $\mathcal{P}_{\min}$ resp., then $\tau_{\bar{b}^*}$ and $\tau_{\underline{b}^*}$ are ε^*-optimal stopping times within the class $\{\tau_b\}_b$.

These observations underly the following numerical procedure, where the tacit assumption about $\underline{\varphi}(b)$ and $\bar{\varphi}(b)$ is that unimodality of φ might ensure unimodality of $\underline{\varphi}$ and $\bar{\varphi}$, at least for large values of M.

Method I.

> *Assume that $\varphi(b)$ is a unimodal function of b. Apply a line search technique, e. g. the Golden Section rule, etc. to $\underline{\varphi}(b)$ and $\bar{\varphi}(b)$. If equality (8), inequality (9) resp., holds then the line search will determine an optimal, ε^*-optimal resp., stopping rule within the class of all stopping times $\{\tau_b\}_{0\le b\le 1}$.*

We shall illustrate Method I by analyzing Shiryaev's quickest detection problem.

Example (The detection problem for a Wiener process, part I).

The detection problem, sometimes called the disruption problem, for Brownian motion is to detect the onset, assumed to be conditionally exponentially distributed and independent of the noise, of a drift value r. The decision is to choose a random variable τ, the time at which an "alarm signal" is given, such that a linear combination of the *probability of false alarm* and the *average delay of detecting the occurence of disruption* is minimized. The disruption problem for a Wiener process is equivalent to an optimal stopping time problem of a diffusion. This problem has been solved by Shiryaev using a variational inequality approach, see [19] for more details. The stopping problem has the form

$$\inf_{\tau, E_x[\tau]<\infty} \left\{ (1 - X_\tau) + c \int_0^\tau X_s \, ds \right\}, \tag{10}$$

where $c > 0$ is a given number, and $(X_t)_t$ satisfies the stochastic differential equation, x, r, σ, λ positive parameters, $0 < x < 1$,

$$dX_t = \lambda(1 - X_t)dt + \frac{r}{\sigma} X_t(1 - X_t)d\bar{W}_t, \quad X(0) = x, \tag{11}$$

where $(\bar{W}_t)_t$, a Brownian motion, is the innovation process determined by the original noise process and accumulated estimates up to time t. The process $(X_t)_{t\ge 0}$ represents the conditional probability of the events $\{\theta \ge t\}$, θ the time of disruption, given the observations $\mathfrak{F}_t^Y = \sigma(Y_s, 0 \le s \le t)$, where the data are described by

$$Y_t = \begin{cases} \sigma W_t, & 0 \le t \le \theta \\ r(t-\theta) + \sigma W_t, & t \ge \theta, \end{cases}$$

and $(W_t)_t$ is a Wiener process.

The random variable θ is assumed to be distributed according to

$$P[\theta = 0] = x \quad \text{and} \quad P[\theta \geq t \mid \theta > 0] = e^{-\lambda t}.$$

Since the Markov process X satisfies Equation (11), the generator A of this process equals, $f \in \mathfrak{C}^2(\mathbb{R})$, cf. (1),

$$Af(x) = \lambda(1-x)f'(x) + \frac{r^2}{2\sigma^2}\,x^2(1-x)^2 f''(x).$$

The following (infinite dimensional) LP-problem, cf. Section 2, solves (10):

$$v^*(x) = \min_{\mu_\tau,\mu} \left\{ \int_0^1 (1-\xi)\mu_\tau(d\xi) + c\int_0^1 \xi\mu(d\xi) \;\middle|\; \begin{array}{l} \mu_\tau - \delta_x - A^*\mu = 0, \\ \langle \mu_\tau, \mathbb{1}\rangle = 1, \\ \mu_\tau, \mu \geq 0 \end{array} \right\}.$$

Below we compare the numerical results for the optimal value (as a function of x) using Method I with the exact values based on Shiryaev's formula, viz.

$$v^*(x) = \begin{cases} (1-A^*) - \displaystyle\int_\xi^{A^*} \psi^*(z)\,dz, & x \in [0, A^*) \\ 1 - X, & x \in [A^*, 1] \end{cases} \tag{12}$$

where, $\varLambda = 2\sigma^2\lambda/r^2$, $C = 2\sigma^2 c/r^2$, $H(y) = \ln(y/(1+y)) - 1/y$,

$$\psi^*(z) = -C\int_0^z \exp\Big[-\varLambda[H(z) - H(y)]\Big]\frac{dy}{y(1-y)^2},$$

and A^* is the root of the equation

$$\psi^*(A^*) = -1. \tag{13}$$

Figures 1 and 2 show the graphs of the functions $A^*(c)$ and $v^*(x)$ as defined by (12) and (13) for a particular set of parameters.

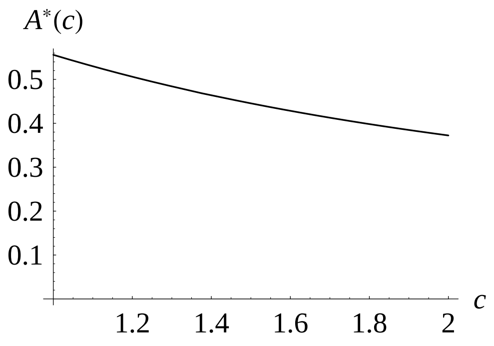

Fig. 1. The optimal stopping point A^* as a function of c ($r = \sigma = \lambda = 1$).

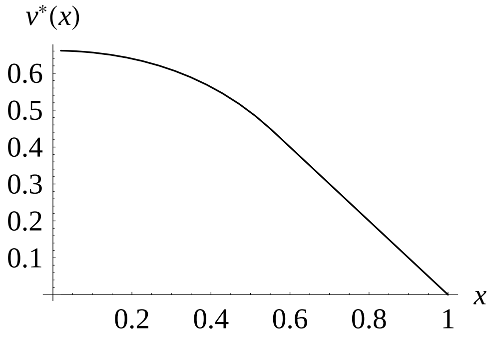

Fig. 2. The value function v^* on $[0.02, 1]$ $(r = \sigma = \lambda = c = 1)$.

For the same parameters Table 1 reports the numerical results for $A^*(c)$ and v^* as a function of c using Method I, while Table 2 reports the results for $v^*(x)$ as a function of the initial position x. In each case we have applied the Golden Section rule to the min and max LP-problems using moments up to order 30 and terminated the line search after 40 iterations. Note the excellent agreement of the numerical results with the exact values which were obtained employing Mathematica to evaluate formulas (12) and (13). Table 3 illustrates the limitations of the method if more extreme parameter settings, e. g. $r/\sigma \gg 1$, for instance $r/\sigma = 10$, are analyzed. In Table 3 we display the computed quantities $v^*(0.3)$ and A^* (for $\lambda = 1$, $r = 10$ and $\sigma = 1$) as a function of the number of moments used.

Table 1. The optimal stopping point A^* and the optimal value v^* as a function of c using Method I ($r = \sigma = \lambda = 1$, $x = 0.3$ and $M = 30$)

c value	objective of LPs	optimal stopping point	exact value for A^*	optimal value	exact value for v^*
1.0	min	0.556066	0.556066	0.609534	0.609534
	max	0.556066		0.609534	
1.2	min	0.506103	0.506093	0.637820	0.637820
	max	0.506091		0.637820	
1.4	min	0.463688	0.463688	0.658360	0.658360
	max	0.463687		0.658360	
1.6	min	0.427376	0.427384	0.673251	0.673254
	max	0.427376		0.673251	
1.8	min	0.396020	0.396014	0.683900	0.683910
	max	0.396015		0.683900	
2.0	min	0.368711	0.368709	0.691282	0.691308
	max	0.368709		0.691282	

Table 2. The optimal stopping point A^* and the optimal value v^* as a function of the initial position using Method I ($r = \sigma = \lambda = c = 1$ and $M = 30$)

initial position	objective of LPs	optimal stopping point	optimal value	exact value for v^*
0.1	min	0.556064	0.656103	0.656103
	max	0.556075	0.656103	
0.2	min	0.556066	0.639540	0.639540
	max	0.556067	0.639540	
0.3	min	0.556066	0.609534	0.609534
	max	0.556066	0.609534	
0.4	min	0.556065	0.562906	0.562906
	max	0.556064	0.562906	
0.5	min	0.556065	0.494628	0.494628
	max	0.556066	0.494628	
0.6	min	0.600000	0.400000	0.400000
	max	0.600000	0.400000	
0.7	min	0.700000	0.300000	0.300000
	max	0.700000	0.300000	
0.8	min	0.800000	0.200000	0.200000
	max	0.800000	0.200000	
0.9	min	0.900000	0.100000	0.100000
	max	0.900000	0.100000	

Table 3. The computed values v^* and A^* as functions of M ($\lambda = \sigma = 1$, $r = 10$)

number of moments used M	objective of LPs	optimal stopping point	optimal value
30	min	0.975893	0.125080
	max	0.977355	0.134995
40	min	0.979529	0.126333
	max	0.976450	0.132733
50	min	0.973717	0.126144
	max	0.979016	0.131450
60	min	0.979407	0.126074
	max	0.978503	0.130838
70	min	0.976796	0.126813
	max	0.977235	0.130920
80	min	0.978065	0.126408
	max	0.982095	0.130700
90	min	0.979725	0.126207
	max	0.978065	0.130427
100	min	0.975573	0.126339
	max	0.979408	0.130085

This parameter setting provides an example where $\underline{\varphi}(b)$ and $\bar{\varphi}(b)$ are not unimodal functions. Using numerical integration we found the optimal stopping point to be $A^* \doteq 0.977968$ and the optimal value $v^*(0.3) \doteq 0.129128$.

4 Method II

According to Theorem 2, choosing as variables the first $M+1$ moments of measures defined on I we can associate with each optimal stopping problem (3) *one* finite dimensional linear *minimization* problem $\underline{\mathcal{P}}_M$ whose optimal value $\underline{v}_M(x)$ is a lower bound on $v^*(x)$. Let $a(x)$, $b(x)$, $\ell(x)$, $R(x)$, $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, $\boldsymbol{\delta}$, $\boldsymbol{\mu}$, $\mathfrak{M}$, $\mathcal{K}$ and $(\eta_k)_{k\in\mathcal{K}}$ be defined as in Section 3, then we have the following inequality:

$$
\begin{aligned}
v^*(x) &= \inf_{\mu_\tau,\mu\geq 0} \{\langle \mu_\tau, R\rangle + \langle \mu, \ell\rangle \mid \langle \mu_\tau, \mathbb{1}\rangle = 1,\ \ \mu_\tau - \delta_x - A^*\mu = 0\} \\
&\geq \min_{\boldsymbol{\mu}^{(\tau)},\boldsymbol{\mu}} \left\{ \langle \boldsymbol{\mu}^{(\tau)}, \delta\rangle + \langle \boldsymbol{\mu}, \boldsymbol{\gamma}\rangle \;\middle|\; \begin{array}{l} \Delta^i \boldsymbol{\mu}^{(\tau)}(n),\ \Delta^i \boldsymbol{\mu}(n) \geq 0, \\ (i,n) \in \mathfrak{M}, \\ \mu_0^{(\tau)} = 1,\ \eta_k = 0,\ k \in \mathcal{K} \end{array} \right\} \qquad (14) \\
&=: \underline{v}_M(x).
\end{aligned}
$$

Note that for a general stopping problem the method of moments for the LP-approach does not determine a linear optimization problem which bounds $v^*(x)$ from above. However, as will be illustrated below, the transformation T_M, to be defined next, cf. Feller [4] and Röhl [16], when applied to a solution $\underline{\boldsymbol{\mu}}^{(\tau)}$ yields valuable information which allows to improve the lower bound $\underline{v}_M(x)$:

$$
\underline{\boldsymbol{\mu}}^{(\tau)} \overset{T_M}{\longmapsto} \left(q_{\frac{k}{M}}\right)_{0\leq k\leq M} \qquad \text{and} \qquad q_{\frac{k}{M}} := \binom{M}{k} (-1)^{M-k} \Delta^{M-k} \underline{\boldsymbol{\mu}}^{(\tau)}(k).
$$

Along with the optimization problem $\underline{\mathcal{P}}_M$, see (14), we shall consider a non-linear optimization problem $\hat{\mathcal{P}}_M$. The problem $\hat{\mathcal{P}}_M$ differs from $\underline{\mathcal{P}}_M$ in that convex combinations of moments (up to M) of a (fixed) finite set of Dirac measures are substituted for the variables $\left(\mu_k^{(\tau)}\right)_{0\leq k\leq M}$. For instance, deciding on N_p Dirac measures at (variable) points $0 \leq b_1, b_2, \ldots, b_{N_p} \leq 1$ and a (variable) non-negative vector $\boldsymbol{p} \in \mathbb{R}^{N_p}$, $\sum_{j=1}^{N_p} p_j = 1$, we put

$$
\boldsymbol{\mu}_m^{(\tau)} = \sum_{j=1}^{N_p} p_j b_j^m, \qquad 0 \leq m \leq M. \tag{15}
$$

The non-linear problem $\hat{\mathcal{P}}_M$ is defined as

$$
\hat{v}_M(x) := \min_{\boldsymbol{\mu},\boldsymbol{p},\boldsymbol{b}} \left\{ \langle \boldsymbol{\mu}^{(\tau)}, \boldsymbol{\delta}\rangle + \langle \boldsymbol{\mu}, \boldsymbol{\gamma}\rangle \;\middle|\; \begin{array}{l} \boldsymbol{\mu} \in \mathcal{H}^M,\ \boldsymbol{\mu}^{(\tau)} \text{ satisfies (15)} \\ \text{and } \eta_k = 0,\ k \in \mathcal{K} \end{array} \right\}
$$

In general, nothing can be said about the relative size of $v^*(x)$ and $\hat{v}_M(x)$. In many applications, however, $\hat{v}_M(x)$ equals $v^*(x)$ up to numerical accuracy. A benefit of computing $\hat{v}_M(x)$ is that an optimal solution of $\hat{\mathcal{P}}_M$ yields a refinement of $\underline{\mathcal{P}}_M$ which yields a better lower bound than $\underline{v}_M(x)$. Again, to simplify the exposition, let us assume that an optimal solution of $\hat{\mathcal{P}}_M$ assigns weights to only two points, $\hat{b}_1$ and $\hat{b}_2$; the general case is a straight forward extension of this special one. Then we cover the unit interval by five subintervals, $0 \leq \varepsilon_1, \varepsilon_2 \ll 1$,

$$\begin{aligned}[0,1] &= [0, \hat{b}_1 - \varepsilon_1] \cup [\hat{b}_1 - \varepsilon_1, \hat{b}_1 + \varepsilon_1] \cup [\hat{b}_1 + \varepsilon_1, \hat{b}_2 - \varepsilon_2] \\ &\quad \cup [\hat{b}_2 - \varepsilon_2, \hat{b}_2 + \varepsilon_2] \cup [\hat{b}_2 + \varepsilon_2, 1] \\ &= \bigcup_{j=1}^{5} I_j.\end{aligned}$$

For any such covering of $[0,1]$ the infinite-dimensional linear program (6) can be written as

$$\inf_{\mu^{(1)},\ldots,\mu^{(5},\mu \geq 0} \left(\sum_{i=1}^{5} \langle \mu^{(j)}, R \rangle + \langle \mu, \ell \rangle \right)$$

subject to

$$\sum_{i=1}^{5} \langle \mu^{(j)}, \mathbb{1} \rangle = 1, \qquad \text{support}\left(\mu^{(j)}\right) \subset I_j, \qquad 0 \leq j \leq 5,$$

and

$$\sum_{i=1}^{5} \mu^{(i)} - \delta_k - A^* \mu = 0.$$

Switching from measures $\mu^{(j)}$ to finite sequences $\boldsymbol{\mu}^{(j)} \in \mathbb{R}^{M+1}$ such that each vector $\boldsymbol{\mu}^{(j)}$ satisfies the analogue of the "Hausdorff conditions" (7) for measures defined on a general interval $[a,b]$, $0 \leq a < b \leq 1$, i.e.

$$\boldsymbol{\mu}_n^{(j)} = \sum_{k=0}^{n} \binom{n}{k} (b-a)^k a^{n-k} \boldsymbol{\nu}_k^{(j)},$$

where each vector $\boldsymbol{\nu}^{(j)}$ satisfies (7), we obtain a refinement of $\underline{\mathcal{P}}_M$. The value of the refined problem will be denoted by $\underline{v}_M^*(x)$. By construction the following inequalities hold:

$$\underline{v}_M(x) \leq \underline{v}_M^*(x) \leq v^*(x).$$

A refinement of $\underline{\mathcal{P}}_M$ typically yields a much improved lower bound. The following procedure, Method II, formalizes the ideas described above.

Method II.

Step 1. *Solve* $\underline{\mathcal{P}}_M$.

Step 2. *Solve* $\hat{\mathcal{P}}_M$.
Use the solution of $\underline{\mathcal{P}}_M$ *and the transformtion* T_M *to specify an initial value for a non-linear solver.*

Step 3. *Use the solution of* $\hat{\mathcal{P}}_M$ *to determine a refinement of* $\underline{\mathcal{P}}_M$*; choose* ε_i *"small", e. g.* 10^{-4} *or* 10^{-5}.
If $\underline{v}^*_M(x) \approx \hat{v}_M(x)$ *take* $\underline{v}^*_M(x)$ *as an estimate (lower bound) for* $v^*(x)$.

Remark. Whenever the solution of $\hat{\mathcal{P}}_M$ involves but one Dirac measure δ_{b^*} a further heuristic is to combine methods I and II and to compare the values $\underline{v}^*_M(x)$ and $\min_{0\le b\le 1}\bar{\varphi}(b)$. Should the numbers be close then these numbers determine a (reasonable) range for the optimal value of a general stopping problem.

Example (The quickest detection problem for a Wiener process, part II).

Tables 4 − 7 display the results of our analysis of the quickest detection problem using Method II for the parameters $r = \sigma = \lambda = 1$; $c = 1$ and $x = 0.3$ if fixed. In Table 4 we compare the values $\underline{v}_M(x)$, $\hat{v}_M(x)$ and $\underline{v}^*_M(x)$, $M = 25$, with $v^*(x)$ as functions of x, and in Table 5 we compare the values of $A^*(c)$, as a function of c, with the values for A^* derived from the optimization problems $\underline{\mathcal{P}}_M$, $\hat{\mathcal{P}}_M$ and a refinement of $\underline{\mathcal{P}}_M$. For the verification step (Step 3 of Method II) we used the covering

$$[0.0556] \cup [0.556, 0.55612] \cup [0.55612, 1] \tag{16}$$

which is suggested by the numbers in Table 6.

Table 4. The values $\underline{v}_M(x)$, $\hat{v}_M(x)$, $\underline{v}^*_M(x)$ and $v^*(x)$ as functions of x; $r = c = \sigma = \lambda = 1$, $M = 25$

initial position	$\underline{v}_M(x)$	$\hat{v}_M(x)$	$\underline{v}^*_M(x)$	exact value $v^*(x)$
0.1	0.63958	0.65610	0.656101	0.656103
0.2	0.62301	0.63954	0.639538	0.639540
0.3	0.59301	0.60953	0.609533	0.609534
0.4	0.54643	0.56291	0.562904	0.562906
0.5	0.47995	0.49463	0.494630	0.494628
0.6	0.39497	0.4	0.4	0.4
0.7	0.29941	0.3	0.3	0.3
0.8	0.19997	0.2	0.2	0.2
0.9	0.09999	0.1	0.1	0.1

Table 5. Approximating values of $A^*(c)$ based on $\underline{\mathcal{P}}_M$, $\hat{\mathcal{P}}_M$ and a refinement of $\underline{\mathcal{P}}_M$ $(r = \sigma = \lambda = 1,\ x = 0.3,\ M = 25)$

c value	estimate of A^* based on Step 1	estimate of A^* based on Step 2	estimate of A^* based on Step 3	exact value $A^*(c)$
1	0.56	0.55607194	0.5561	0.556066
1.2	0.52	0.50609462	0.5065	0.506093
1.4	0.47	0.46368731	0.4640	0.463688
1.6	0.44	0.42737578	0.4270	0.427384
1.8	0.40	0.39601437	0.3960	0.396014
2	0.40	0.36870895	0.3687	0.368709

In Table 6 we illustrate how the transformation T_M is used. The numbers shown, $(q_{k/M})_{0\le k\le M}$, are the image of $\boldsymbol{\mu}^{(\tau)}$, an optimal solution of $\underline{\mathcal{P}}_M$, under T_M; $x = 0.3$ in this case. From these numbers we can infer that the non-linear solver should be initialized at a point nearby $14/25 = 0.56$; $14/25$ is our first estimate of the optimal stopping point.

The same idea is applied in Step 3 of Method II. Table 7 illustrates this part of the procedure. The solution of $\hat{\mathcal{P}}_M$, $b^* = 0.55607194$, is an estimate of the optimal stopping time and b^* suggests the covering (16). Applying transformation T_M to each solution vector $\boldsymbol{\mu}^{(i)}$ associated with covering (16) we obtain Table 7. It shows that the solution of the refined LP concentrates "all" its mass on $[0.556, 0.55612]$; the spurious mass $q_0^{(3)} = 0.0242408$ actually dissapears for larger values of M. So we use $0.556 + 14/25 \cdot 0.00012 = 0.550672$ and $0.556 + 16/25 \cdot 0.00012 = 0.550738$ to specify a range for the optimal stopping point.

Since the detection problem can be analyzed by Method I as well as Method II we can combine the values $\underline{v}_M^*(x)$ and $\bar{\varphi}(b^*)$, see Table 2 for the latter value, to get an estimate of $v^*(x)$ and to obtain a (numerical) error bound, viz.

$$0.609533 + 10^{-6}.$$

Table 6. The values $q_{k/M}$, $0 \le k \le M = 25$; $r = c = \sigma = \lambda = 1$, $x = 0.3$ for an optimal solution of $\underline{\mathcal{P}}_M$

k	$q_{k/M}$	k	$q_{k/M}$	k	$q_{k/M}$	k	$q_{k/M}$	k	$q_{k/M}$
0	0	6	0	12	0	18	0	24	0
1	0	7	0	13	0.299145	19	0	25	0
2	0	8	0	14	0.512821	20	0		
3	0	9	0	15	0.188034	21	0		
4	0	10	0	16	0	22	0		
5	0	11	0	17	0	23	0		

Table 7. The values $q_{k/M}$, $0 \leq k \leq M = 25$, for an optimal solution of $\underline{\mathcal{P}}_M$; $r = c = \sigma = \lambda = 1$, $x = 0.3$

k	$q^{(1)}_{k/M}$	$q^{(2)}_{k/M}$	$q^{(3)}_{k/M}$
0	0	0	0.0242408
1	0	0	0
2	0	0	0
3	0	0	0
4	0	0	0
5	0	0	0
6	0	0	0
7	0	0	0
8	0	0	0
9	0	0	0
10	0	0	0
11	0	0	0
12	0	0	0
13	0	0	0
14	0	0.333593	0
15	0	0.48927	0
16	0	0.152897	0
17	0	0	0
18	0	0	0
19	0	0	0
20	0	0	0
21	0	0	0
22	0	0	0
23	0	0	0
24	0	0	0
25	0	0	0

5 Concluding Remarks

We have described numerical procedures for analyzing optimal stopping problems of Markov processes. Both methods are based on a linear programming approach to such kind of decision problems. We have illustrated these methods by numerically analyzing Shiryaev's quickest detection problem for a Wiener process. This example was chosen for its importance and for the fact that the numerical results can be compared with analytical ones.

We conclude by commenting on some of our computational experiences. We used AMPL as a convenient interface and employed the CPLEX solver. When using Method I to analyze the quickest detection problem we ran LP problems with M up to 140 accepting unscaled infeasibilities. If $M = 50$ an individual LP-run typically requires ~ 500 iterations, most of them in phase I of the Simplex algorithm. To be able to check computations within loops, etc. we set AMPL- and CPLEX-options in such way that previously computed bases were not used. Large values of M and N naturally increase the run time of Method I. For different parameter settings we made 'ad hoc' decisions to strike a compromise between accuracy and run time.

When using Method II we used smaller values for M, $M \leq 40$; for larger values we very often experienced the program to exit because of detected (numerical) infeasibilities. For the larger linear and non-linear problems of Method II the solvers, CPLEX for the linear problems and MINOS for the non-linear ones, typically required $1000-1500$ iterations, half of them during phase I of the Simplex algorithm. For the detection problem, as far as refinements of $\underline{\mathcal{P}}_M$ are concerned, we had to use but one Dirac measure δ_b (cf. Method II); depending on the size of M and the parameters given we used $\varepsilon = 10^{-4}$ or 10^{-5}.

In light of the excellent agreement between the numerical and analytical results for a large set of different parameters we consider the LP-techniques a convenient and easy to use tool for analyzing the detection problem as well as similarly structured ones, e. g. pricing perpetual Russian options, etc.

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