Pricing Perpetual Russian Options Using Linear Programming

Kurt Helmes

Humboldt University of Berlin, Institute of Operations Research, Germany

Abstract

Let $Y = (Y_t)_{t \geq 0}$ be the price of a stock. The concept of a Russian put option, introduced by Shepp and Shiryaev, refers to a contract when the buyer of the option is guaranteed the larger of two (discounted) values, one being a fixed amount ϱ and the other one being the maximum value of the stock up to the time the option is exercised; it is assumed that the buyer can borrow or lend unlimited amounts of money at a fixed interest rate $r > 0$. Assuming Y to be geometric Brownian motion and no bound on the exercise time, i. e. a perpetual option, Shepp and Shiryaev derived an explicit formula for the fair price of such an option exploiting the equivalence of the pricing problem with optimal stopping problems. In this note we shall compute the price of a Russian option – with and without *average* time constraints – using numerical methods which are based on a linear programming formulation of optimal stopping problems. The LP approach to optimal stopping exploits a characterization of a stopped Markov process through a family of equations which relate the generator of the process with a pair of measures representing the expected occupation of the process and the distribution of the state when the process is stopped. The computational analysis of Russian options leads to bounds on the fair price of such contracts. We illustrate the accuracy of the numerical results by comparing them with the analytical values in the case of no constraints.

1 Introduction

Let $Y = (Y_t)_{t \geq 0}$ denote geometric Brownian motion with drift coefficient μ and diffusion coefficient σ starting at $y_0 > 0$, i. e. Y is the one-dimensional Markov process whose generator G equals, $f \in \mathcal{D} = \text{domain}(G)$, $y \in (0, \infty)$,

$$Gf(y) = \frac{1}{2}\sigma^2 y^2 f''(y) + \mu y f'(y);$$

equivalently, the process can be represented as

$$Y_t = y_0 \exp\left[\sigma W_t - \frac{\sigma^2}{2} t + \mu t\right], \quad t \geq 0,$$

where $W = (W_t)_{t \geq 0}$ is a standard Wiener process on a filtered probability space (Ω, $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}, P$). Let $\mathcal{S} = (\mathcal{S}_t)_{t \geq 0}$ denote the stochastic process

$$\mathcal{S}_t := \varrho \vee \max_{0 \leq u \leq t} Y_u = \max\left\{\varrho, \max_{0 \leq u \leq t} Y_u\right\}, \tag{1}$$

where $\varrho \geq y_0$ is a given number. Let $r > 0$ be fixed and let it represent the interest rate at which investors can either borrow or lend. It is shown in [9], see also [2], [6], [8] and [10], that the pricing of a perpetual Russian option without borrowing or lending restrictions is related to the following family of optimal stopping problems, $\lambda \geq 0$,

$$R(\lambda) = R(\varrho, y_0; \lambda) := \sup_{\tau \in \mathbb{T}} E_{\varrho, y_0} \left[e^{-\lambda\tau} e^{-r\tau} \mathcal{S}_\tau \right], \tag{2}$$

where $\mathbb{T}$ is the set of all finite stopping times adapted to the filtration $\mathcal{F}$. Moreover, see [9],

$$\tilde{X}_t := \frac{\tilde{x}_0 y_0 \vee \max\limits_{0 \leq u \leq t} Y_u}{Y_t}, \qquad t \geq 0, \quad \tilde{x}_0 := \frac{\varrho}{y_0} \geq 1, \tag{3}$$

is a Markov process with respect to the measure $\tilde{P}$, where

$$d\tilde{P}_{|\mathcal{F}_t} = \exp\left[\sigma W_t - \frac{\sigma^2}{2} t\right] dP_{|\mathcal{F}_t}.$$

Actually, $\tilde{X} = (\tilde{X}_t)_{t \geq 0}$ is a diffusion process with instant reflection at the point $\{1\}$ on the state space $\tilde{\mathcal{E}} = [1, \infty)$ whose infinitesimal operator $\tilde{A}$ applied to functions $g \in \mathcal{C}^2((1, \infty))$ equals, $x \in (1, \infty)$,

$$\tilde{A}g(\tilde{x}) = -rxg'(x) + \frac{\sigma^2}{2} x^2 g''(x),$$

and the following condition holds at the boundary point $\{1\}$:

$$g'(1+) = \lim_{\tilde{x} \searrow 1} g'(x) = 0. \tag{4}$$

Let $\tilde{P}_{\tilde{x}_0}$ denote the probability distribution of the process $\tilde{X}$, and let

$$\tilde{R}(\tilde{x}_0; \lambda) := \sup_{\tau} \tilde{E}_{\tilde{x}_0} \left[e^{-\lambda\tau} \tilde{X}_\tau \right], \tag{5}$$

where the supremum is taken over all finite ($\tilde{P}_{\tilde{x}_0}$ – a. s.) Markov times τ. The first main result of [9] is that (2) and (5) are proportional.

Proposition 1.

$$R(\lambda) = y_0 \tilde{R}(\tilde{x}_0; \lambda), \tag{6}$$

and (6) is finite whenever $\lambda > 0$.

Next, following Shepp and Shiryaev [9], we consider the jump-diffusion process $X = (X_t)_{t \geq 0}$ on $\mathcal{E} = [1, \infty) \cup \{0\}$ with instant reflection at $\{1\}$ and cemetery state $\{0\}$ whose infinitesimal operator A on twice-continuously differentiable functions g vanishing at zero equals, $x \in (1, \infty)$,

$$Ag(x) = \tilde{A}g(x) - \lambda g(x), \tag{7}$$

and the boundary condition (4) holds.

We put $x_0 = \tilde{x}_0$ and consider

$$V^*(x_0) := \sup_\tau E_{x_0}[X_\tau], \tag{8}$$

where the supremum is over all finite Markov times τ, and E_{x_0} denotes expectation with respect to the distribution of the process X.

The second main result of [9] is the equality of (5) and (8) and an explicit formula for V^*.

Proposition 2. Let $\lambda > 0$, $x_0 \geq 1$, then

$$\tilde{R}(x_0; \lambda) = V^*(x_0);$$

$$V^*(x_0) = \begin{cases} \dfrac{A^*}{z_2 - z_1}\left[(z_2 - 1)\left(\dfrac{x_0}{A^*}\right)^{z_1} + (1 - z_1)\left(\dfrac{x_0}{A^*}\right)^{z_2}\right] & , \quad 1 \leq x_0 < A^*, \\ x_0 & , \quad x_0 \geq A^*, \end{cases} \tag{9}$$

where

$$z_{1|2} := \frac{\sigma^2 - 2r}{2\sigma^2} \mp \sqrt{\left(\frac{\sigma^2 - 2r}{2\sigma^2}\right)^2 + \frac{2\lambda}{\sigma^2}},$$

and

$$A^* = \left(\frac{z_2}{z_1} \cdot \frac{z_1 - 1}{z_2 - 1}\right)^{\frac{1}{z_2 - z_1}}. \tag{10}$$

Moreover, the optimal time τ^* to exercise a Russian option is when the process $\tilde{X}$, see (3), passes the threshhold A^*, i. e.

$$\tau^* = \inf\left\{t \geq 0 \;\middle|\; \tilde{X}_t \geq A^*\right\}.$$

In this paper we shall take the first part of Proposition 2 as a starting point for the computational analysis of perpetual Russian options. Extending the applicability of two numerical methods, see [4], from analyzing stopped diffusions to analyzing stopped jump-diffusion processes we find the approximate value for and (numerical) error bounds on the fair price of a perpetual Russian put option, with and without *average* time constraints.

The methods are based on a linear programming approach to exit time problems and to stopping time problems of Markov processes. The first method exploits the fact that the values of quantities of interest of exit time problems can be sandwiched between the values of appropriately defined (finite-dimensional) linear programs, cf. [5]. Applying line search techniques to these bounds while choosing varying threshholds as exit boundaries we obtain accurate estimates of $V^*(x_0)$ and determine a range for A^*. Whenever the line search (numerically) converges we get “exact” values for the price and the exercise policy of a Russian option.

The second method is a modification of a numerical approach to general stopping problems recently proposed by Röhl [7], see also [1], and it replaces Röhl's iteration technique by a pair of optimization problems, one being linear the other one being non-linear, and a second LP-problem which constitutes the verification step of the procedure. The second method has the advantage over the first one that additional (average) time constraints can be easily incorporated.

In Section 2 we formulate the approximating finite dimensional linear programs and give a detailed computational analysis of the pricing of a Russian option *without* time constraints. In Section 3 we compute the price of a perpetual Russian option when an *average* time constraint is imposed.

2 Approximating LP-problems

According to the theory and the results described in [4] and given the generator A defined by (7) and (4) and the objective specified by (8) we consider the following two LP-problems, $b \in (1, C)$, $C > 1$, $M \in \mathbb{N}$,

$$\min_{p,\vec{\mu}}\{bp\} \qquad \text{and} \qquad \max_{p,\vec{\mu}}\{bp\} \tag{11}$$

subject to, $p \in \mathbb{R}$, $\vec{\mu} \in \mathbb{R}^{\frac{(M+1)(M+2)}{2}}$,

$$0 \le p \le 1, \quad \text{and} \quad (-1)^i \vec{\mu}(i,j) \ge 0, \quad {\scriptstyle 0 \le i \le M, \quad 0 \le j \le M-i,} \tag{12}$$

$$\vec{\mu}(i,j) = \vec{\mu}(i-1, j+1) - \vec{\mu}(i-1, j), \quad {\scriptstyle 1 \le i \le M, \quad 0 \le j \le M-i,} \tag{13}$$

$$p + \lambda \cdot \vec{\mu}(0,0) = 1 \tag{14}$$

$$\begin{aligned} p - \left(\frac{x_0 - 1}{b-1}\right)^m \; &- \; \left(-m \cdot r + m(m-1)\frac{\sigma^2}{2} - \lambda\right) \vec{\mu}(0,m) \\ &- \; \left(\frac{-m \cdot r + m(m-1)\sigma^2}{(b-1)}\right) \vec{\mu}(0, m-1) \\ &- \; \left(\frac{m(m-1)\sigma^2}{2(b-1)^2}\right) \vec{\mu}(0, m-2) \; = \; 0, \qquad {\scriptstyle m = 2, \ldots, M.} \end{aligned} \tag{15}$$

We let C be a number large enough so that the optimal stopping point A^*, cf. (8) and (10), is included in the interval $(1, C)$, and we denote the optimal value of the minimization problem, cf. (11), maximization problem resp., by $\underline{\varphi}(b)$, $\bar{\varphi}(b)$ resp. By construction, the following inequality holds:

$$\max_{1 \le b \le C} \underline{\varphi}(b) \; \le \; V^*(x_0) \; \le \; \max_{1 \le b \le C} \bar{\varphi}(b).$$

Furthermore, if there is a b^* such that

$$\max_{1 \le b \le C} \underline{\varphi}(b) \; = \; \underline{\varphi}(b^*) \; = \; \bar{\varphi}(b^*) \; = \; \max_{1 \le b \le C} \bar{\varphi}(b^*) \tag{16}$$

then τ_{b^*} is the optimal stopping time for (8) in the class of all stopping rules $\{\tau_b\}_{1\le b\le C}$ where

$$\tau_b \;=\; \inf\left\{t \ge 0 \mid \tilde{X}_t \ge b\right\}.$$

If

$$\varepsilon^* \;:=\; \max_{1\le b\le C} \bar{\varphi}(b) \;-\; \min_{1\le b\le C} \underline{\varphi}(b) \;>\; 0, \tag{17}$$

and $\underline{b}^*$, $\bar{b}^*$ resp., is a solution of the minimization problem (11), the maximization problem resp., then $\tau_{\underline{b}^*}$ and $\tau_{\bar{b}^*}$ are ε^*-optimal stopping times within the class $\{\tau_b\}_b$. Thus the following procedure, Method I, can be used to find the price of a Russian option:

Method I.

> *Apply a line search technique, e. g. the Golden Section rule or the Fibonacci rule, etc. to $\underline{\varphi}(b)$ and $\bar{\varphi}(b)$. If equality (16), inequality (17) resp., holds then the line search determines the value of A^*, an ε^*-optimal stopping point resp., and*
>
> $$y_0 \cdot \underline{\varphi}(b^*) \;=\; y_0 \cdot \bar{\varphi}(b^*)$$
>
> *is the price of the Russian option, cf. (6) and Proposition 2.*

The following tables illustrate typical results which we obtained. Table 1, Table 2 resp., reports the numerical values for A^* and V^* as a function of the parameter λ, the initial position x_0 resp., using Method I. In each case we have applied the Golden Section rule to the appropriate min- and max-LP-problem terminating the line search after 30 iterations. Note the excellent agreement of the numerical results with the exact values which were obtained employing Mathematica to evaluate (9) and (10).

Table 1:
The optimal stopping point A^* and the optimal value V^* as a function of $\lambda \in [0.01, 0.1]$ compared with the results when using Method I ($r = 0.07$, $\sigma = 0.4$, $x_0 = 1$)

λ value	objective of LPs	optimal stopping point	exact value for A^*	optimal value	exact value for V^*
0.01	min	5.750417	5.819119	3.156561	3.157538
	max	5.911762		3.157812	
0.02	min	3.896830	3.896875	2.309536	2.309543
	max	3.897405		2.309566	
0.03	min	3.100739	3.102079	1.959601	1.959601
	max	3.101702		1.959604	
0.04	min	2.656646	2.656733	1.762739	1.762739
	max	2.656608		1.762740	
0.05	min	2.369417	2.369419	1.635011	1.635011
	max	2.369413		1.635011	
0.06	min	2.167847	2.167929	1.544895	1.544895
	max	2.167933		1.544895	
0.07	min	2.018520	2.01852	1.477677	1.477677
	max	2.018520		1.477677	
0.08	min	1.903190	1.903191	1.425503	1.425503
	max	1.903200		1.425503	
0.09	min	1.811419	1.811419	1.383773	1.383773
	max	1.811419		1.383773	
0.10	min	1.736629	1.736629	1.349603	1.349603
	max	1.736629		1.349603	

Table 2:
The optimal stopping point A^* and the optimal value V^* as a function of the starting point x_0 compared with the results when using Method I ($r = 0.07$, $\sigma = 0.4$, $\lambda = 0.1$)

initial position x_0	objective of LPs	optimal stopping point	optimal value	exact value for V^*
1.1	min	1.736625	1.357751	1.357751
	max	1.736629	1.357751	
1.2	min	1.736625	1.381242	1.381242
	max	1.736629	1.381242	
1.3	min	1.736625	1.418994	1.418994
	max	1.736629	1.418994	
1.4	min	1.736625	1.470263	1.470263
	max	1.736629	1.470263	
1.5	min	1.736625	1.534537	1.534537
	max	1.736629	1.534537	
1.6	min	1.736625	1.611467	1.611467
	max	1.736629	1.611467	
1.7	min	1.736629	1.700822	1.700822
	max	1.736629	1.700822	
1.8	min	1.800000	1.799998	1.800000
	max	1.800000	1.799998	
1.9	min	1.900000	1.899999	1.900000
	max	1.900000	1.899999	

For the same set of parameters Figures 1 and 2 show the graphs of the functions $A^*(\lambda)$ and $V^*(x_0)$ as defined by (10) and (9).

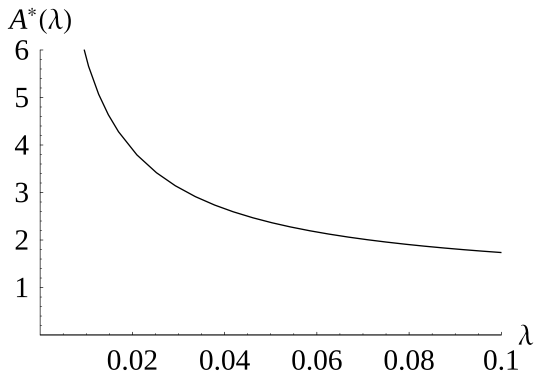

Fig. 1: The optimal stopping point A^* as a function of λ ($r = 0.07$, $\sigma = 0.4$, $x_0 = 1$).

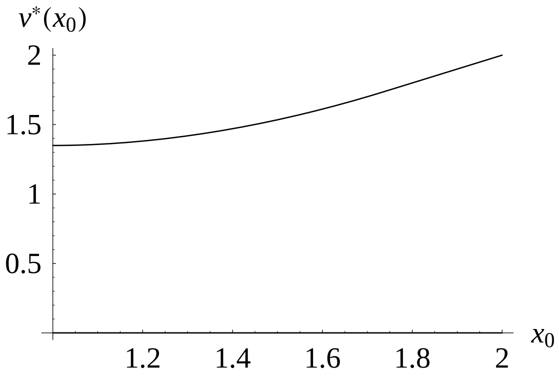

Fig. 2: The value function V^* with $x_0 \in [0, 2.2]$ ($r = 0.07$, $\sigma = 0.4$, $\lambda = 0.1$).

While Method I can only be applied should an optimal stopping rule be an element of $\{\tau_b\}_{b\in\mathbb{R}}$, as in the case of pricing a perpetual Russian option, the following method is applicable in general, cf. [4]. This time we associate with the optimal stopping problem (8) *one* finite dimensional linear *maximization* problem whose optimal value $\bar{v}_M(x_0)$ is an upper bound on $V^*(x_0)$, viz.

$$\max_{\vec{\mu}^{(\tau)},\vec{\mu}} \left\{(C-1)\vec{\mu}^{(\tau)}(0,1)+\vec{\mu}^{(\tau)}(0,0)\right\} =: \bar{v}_M(x_0) \tag{18}$$

subject to, $\vec{\mu}^{(\tau)},\ \vec{\mu} \in \mathbb{R}^{\frac{(M+1)(M+2)}{2}}$,

$$(-1)^i\vec{\mu}^{(\tau)}(i,j) \geq 0 \quad \text{and} \quad (-1)^i\vec{\mu}(i,j) \geq 0, \quad {\scriptstyle 0\leq i\leq M,\quad 0\leq j\leq M-i,} \tag{19}$$

$$\vec{\mu}^{(\tau)}(i,j) = \vec{\mu}^{(\tau)}(i-1,j+1)-\vec{\mu}^{(\tau)}(i-1,j), \quad {\scriptstyle 1\leq i\leq M,\quad 0\leq j\leq M-i,} \tag{20}$$

$$\vec{\mu}(i,j) = \vec{\mu}(i-1,j+1)-\vec{\mu}(i-1,j), \quad {\scriptstyle 1\leq i\leq M,\quad 0\leq j\leq M-i,} \tag{21}$$

$$\vec{\mu}^{(\tau)}(0,0)+\lambda\cdot\vec{\mu}(0,0) = 1, \tag{22}$$

$$\begin{aligned}\vec{\mu}^{(\tau)}(0,m) - \left(\frac{x_0-1}{C-1}\right)^m &- \left(-m\cdot r+m(m-1)\frac{\sigma^2}{2}-\lambda\right)\vec{\mu}(0,m)\\ &- \left(\frac{-m\cdot r+m(m-1)\sigma^2}{C-1}\right)\vec{\mu}(0,m-1)\\ &- \left(\frac{m(m-1)\sigma^2}{2(C-1)^2}\right)\vec{\mu}(0,m-2) \;=\; 0, \quad {\scriptstyle m=2,\ldots,M.}\end{aligned} \tag{23}$$

Conditions (19) and (20) constitute the finite dimensional analogue of the Hausdorff conditions, cf. [5]. The Hausdorff conditions are necessary and sufficient for an infinite sequence to be equal to the sequence of moments of a distribution whose support is contained in $[0,1]$. Hence,

$$V^*(x_0) \;\leq\; \bar{v}_M(x_0); \tag{24}$$

for the corresponding infinite dimensional LP-problem, i. e. $M=\infty$, equality holds in (24). Next, whenever $\vec{\mu}^{(\tau)}$ is a solution of (18) – (23) then the transformation $\mathcal{T}_M$, to be defined below, cf. [3] and [4] or [7], yields valuable information which helps to improve the upper bound $\bar{v}_M(x_0)$:

$$\vec{\mu}^{(\tau)} \overset{\mathcal{T}_M}{\longmapsto} \left(q_{\frac{k}{M}}\right)_{0\leq k\leq M} \qquad \text{and} \qquad q_{\frac{k}{M}} := (-1)^{M-k}\binom{M}{k}\vec{\mu}^{(\tau)}(M-k,k).$$

Along with the optimization problem (18) – (23) we shall consider a non-linear optimization problem which is almost identical with the linear one, execpt for an additional constraint and a transformation of variables:

$$\max_{\vec{\mu},\vec{p},\vec{b}} \left\{(C-1)\vec{\mu}^{(\tau)}(0,1)+\vec{\mu}^{(\tau)}(0,0)\right\} =: \hat{v}_M(x_0) \tag{25}$$

subject to $\vec{b}$, $\vec{p} \in \mathbb{R}^{N_p}$, $N_p \in \mathbb{N}$, $\vec{p} \geq 0$, $\sum_{i=1}^{N_p} p_i = 1$, $1 < b_1 \leq b_2 \leq \cdots \leq b_{N_p} < C$, (19) – (23) and

$$\vec{\mu}^{(\tau)}(0,m) = \sum_{j=1}^{N_p} p_j b_j^m, \quad 0 \leq m \leq M. \tag{26}$$

In light of Proposition 2 we shall choose $N_p = 1$ for the case of a Russian option.

While in general nothing can be said about the relative size of $V^*(x_0)$ and $\hat{v}_M(x_0)$, it happens in many applications that both numbers are equal (close) up to numerical accuracy, see below. The real benefit of the non-linear optimization problem (25), (26) and (19) – (23) is that a solution suggests refinements of (18) – (23) which yield better upper bounds on $V^*(x_0)$ than $\bar{v}_M(x_0)$. To see how this is done let $\hat{p}$ and $\hat{b}$, assuming $N_p = 1$ for simplicity, be elements of an optimal solution of the non-linear problem. Then we cover the interval $[1, C]$ by the following three subintervals, $0 < \varepsilon \ll 1$,

$$[1, C] = \left[1,\, \hat{b} - \varepsilon\right] \cup \left[\hat{b} - \varepsilon,\, \hat{b} + \varepsilon\right] \cup \left[\hat{b} + \varepsilon,\, C\right],$$

and express the variables $\vec{\mu}^{(\tau)}$ as follows:

$$(C-1)^m \vec{\mu}^{(\tau)}(0,m) = \vec{\nu}^{(1)}(m) + \vec{\nu}^{(2)}(m) + \vec{\nu}^{(3)}(m), \quad 0 \leq m \leq M, \tag{27}$$

where

$$\begin{aligned}
\vec{\nu}^{(1)}(m) &= \left(\hat{b} - \varepsilon - 1\right)^m \vec{\mu}^{(1)}(0,m), \\
\vec{\nu}^{(2)}(m) &= \sum_{k=0}^{m} \binom{m}{k} (2\varepsilon)^k \left(\hat{b} - \varepsilon - 1\right)^{m-k} \vec{\mu}^{(2)}(0,m), \\
\vec{\nu}^{(3)}(m) &= \sum_{k=0}^{m} \binom{m}{k} \left(C - \hat{b} - \varepsilon\right)^k \left(\hat{b} + \varepsilon\right)^{m-k} \vec{\mu}^{(3)}(0,m),
\end{aligned} \tag{28}$$

and $\left(\vec{\mu}^{(k)}(i,j)\right)$, $1 \leq k \leq 3$, satisfy the constraints (19) and (20).

The value of the refined problem will be denoted by $\bar{v}_M^*(x_0)$. The following inequalities hold by construction:

$$V^*(x_0) \leq \bar{v}_M^*(x_0) \leq \bar{v}_M(x_0).$$

Tables 3 and 4 display the results of our numerical analysis of the pricing of a perpetual Russian option using a three step procedure based on the optimization problem described above: *Step 1:* Solve the linear problem (18) – (23). *Step 2:* Solve the non-linear problem (25), (26) and (19) – (23). Use the solution of the linear problem together with the transformation $\mathcal{T}_M$ to initialize the non-linear problem. *Step 3:* Use the solution of the non-linear problem to determine a refinement of the linear one; choose ε to be small, e. g. equal to 10^{-2} or 10^{-3}. Take $\bar{v}_M^*(x_0)$ as an upper bound for $V^*(x_0)$.

In Table 3 we compare the values $\bar{v}_M$, $\hat{v}_M$, $\bar{v}_M^*$ and V^* as functions of $\lambda \in \{0.01, \ldots, 0.1\}$ for the parameters $r = 0.07$, $\sigma = 0.4$ and $x_0 = 1$. For the individual verification steps, i. e. Step 3, we take appropriate coverings; for instance, for $\lambda = 0.01$ we use

$$[1,\ 5.817] \cup [5.817,\ 5.823] \cup [5.823,\ 6]. \tag{29}$$

This covering was suggested by the result of the corresponding non-linear problem and a first run of problem (18) – (23), (27) and (28).

Table 3:
The values $\bar{v}_M$, $\hat{v}_M$, $\bar{v}_M^*$ and V^* as functions of λ; $x_0 = 1$, $r = 0.07$, $\sigma = 0.4$ and $M = 20$

λ	$\bar{v}_M$	$\hat{v}_M$	$\bar{v}_M^*$	V^*
0.01	3.16007	3.15858	3.15858	3.157538
0.02	2.33217	2.31014	2.30955	2.309543
0.03	1.99198	1.95962	1.95971	1.959601
0.04	1.80326	1.76275	1.76274	1.762739
0.05	1.68103	1.63501	1.63502	1.635011
0.06	1.59552	1.54490	1.54490	1.544895
0.07	1.53278	1.47768	1.47768	1.477677
0.08	1.48229	1.42550	1.42551	1.425503
0.09	1.44513	1.38377	1.38378	1.383773
0.10	1.41380	1.34960	1.34961	1.349603

Applying transformation T_M to the solution of the refined problem and using the covering (29) we obtain the numbers in Table 4.

Table 4:
The values $q^{(i)}_{k/M}$, $0 \le k \le M = 25$, for an optimal solution of the refined problem; $x_0 = 1$, $r = 0.07$, $\sigma = 0.4$

k	$q^{(1)}_{k/M}$	$q^{(2)}_{k/M}$	$q^{(3)}_{k/M}$
0	0	0	0
1	0	0	0
2	0	0	0
3	0	0	0
4	0	0	0
5	0	0	0
6	0	0	0
7	0	0	0
8	0	0	0
9	0	0	0
10	0	0	0
11	0	0	0
12	0	0	0
13	0	0	0
14	0	0	0
15	0	0	0
16	0	0	0
17	0	0	0
18	0	0.542589	0
19	0	0	0
20	0	0	0
21	0	0	0
22	0	0	0
23	0	0	0
24	0	0	0
25	0	0	0

They show that the solution of the refined LP "concentrates its mass" on the interval [5.817, 5.823]. The index of the non-zero entry in Table 4, 18, suggests using

$$5.817 + \frac{18}{25} \cdot 0.006 = 5.82132$$

as an (approximate) optimal threshhold for the stopping problem. If we combine the results of both methods, i. e. we take $\underline{\varphi}(\underline{b}) = 3.156561$ and $\bar{v}_M^* = 3.15858$, then we obtain the following range for the price of a perpetual Russian option, $x_0 = 1$, $r = 0.07$, $\sigma = 0.4$ and $\lambda = 0.01$,

$$[3.1565,\ 3.1586];$$

the midpoint of this interval is in excellent agreement with the true value, see Table 3.

3 Pricing problems with constraints

In this section we solve the stopping problem (8) under the additional constraint that all stopping times τ satisfy the condition, $T > 0$,

$$E[\tau] \le T. \tag{30}$$

In our LP-models the variable $\vec{\mu}(0,0)$ is approximately equal to $E[\tau]$. Thus, to take care of (30), we add to all optimization models which are part of Method II the extra line

$$\vec{\mu}(0,0) \le T. \tag{31}$$

This way, whenever we are using Method II, we can easily handle average time constraints. As far as Method I goes the following modification yields good results as long as the constraint (31) is binding. We replace the maximization problem, see (11), by

$$\max_{p,w,\vec{\mu}} \{bp - \kappa w\} =: \bar{\Psi}(b;T), \tag{32}$$

where w is a non-negative real variable and the parameter κ is some "large" number; to the constraints (12) – (15) we add the following two inequalities:

$$\vec{\mu}(0,0) - w \le T \quad \text{and} \quad w - \vec{\mu}(0,0) \le T. \tag{33}$$

We then apply a line search technique to $\bar{\Psi}(b;T)$. While we have a counterpart for $\bar{\varphi}$ we do not have one for $\underline{\varphi}$.

Table 5 displays the values of $\bar{v}_M^*$, checked against the numbers $\hat{v}_M$ and $\bar{\Psi}(b^*;T)$, and the optimal stopping points $b^* = b^*(x_0)$, together with the values of $\vec{\mu}(0,0)$ and the dual variables corresponding to the constraint (31) as a function of the initial position x_0 when $T = 1$. Please note that the constraint (31) is no longer binding should the initial position get close to the point where immediate stopping is best.

Table 5:
The values b^* and $\bar{v}^*_M$ as functions of x_0 when $E[\tau]=1$; $\lambda=0.1$, $r=0.07$, $\sigma=0.4$ and $M=20$

initial position	b^*	$\bar{v}^*_M$	$\mu(0,0)$	dual variable
1.0	1.44807	1.30326	1	0.089348
1.1	1.46201	1.31581	1	0.082935
1.2	1.50025	1.35022	1	0.0662675
1.3	1.5567	1.40111	1	0.0470164
1.4	1.62606	1.46356	1	0.0264184
1.5	1.70437	1.53398	1	0.00705969
1.6	1.73663	1.61147	0.720713	0
1.7	1.73663	1.70082	0.206181	0
1.8	1.8	1.8	0	0
1.9	1.9	1.9	0	0

Figure 3 shows the graphs of $T \mapsto \Psi(b^*;T)$ and $T \mapsto b^*(T)$, when $T \in \{0.01, 0.02, \ldots, 0.99, 1\}$ and $\lambda = 0.1$, $r = 0.07$, $\sigma = 0.4$ and $x_0 = 1$. The values of $\bar{\Psi}(b^*;T)$ have been checked against the values $\hat{v}_M$ and $\bar{v}^*_M$.

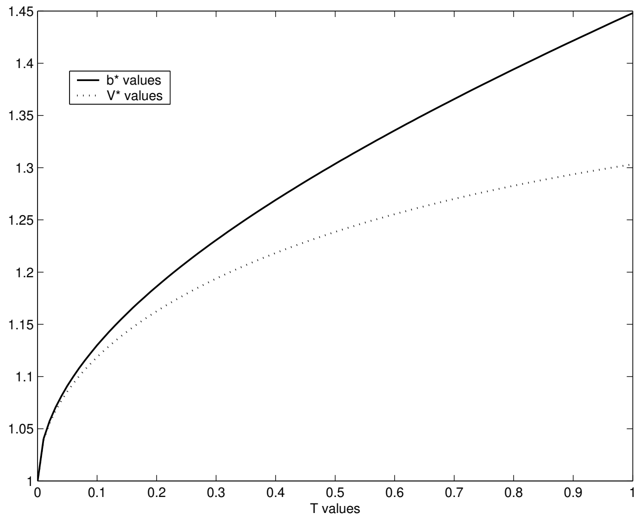

Fig. 3: The graphs of $T \mapsto \Psi(b^*;T)$ and $T \mapsto b^*(T)$, $T \in \{0.01, 0.02, \ldots, 0.99, 1\}$; $x_0 = 1$, $\lambda = 0.1$, $r = 0.07$, $\sigma = 0.4$ and $M = 20$.

4 Conclusions

We have formulated finite dimensional linear programming problems, derived from a LP-approach to optimal stopping, and have computed the price of a discounted Russian option. For various parameter constellations we have compared the numerical results with the analytical ones. Our computational analysis has shown that the linear programming technique is an easy to use and accurate method to compute the price of a Russian option for a large range of parameter values, cf. results in Section 2 and some additional results collected in Appendix A. Appendix B contains an example of an AMPL-model file used for some of the computations. The fact that the file is so short illustrates how useful interfaces like AMPL and GAMS are.

Such programs are easy to modify should changes be warranted. For instance, it only requires an extra line to be added to the program in Appendix B to compute the price of a Russian option with an average time constraint.

Appendix A

Table 6:
The optimal stopping point A^* and the optimal value V^* as a function of $\lambda \in [0.001, 0.01]$ compared with the results when using Method I; $r = 0.07$, $\sigma = 0.4$, $x_0 = 1$

λ	objective of LPs	optimal stopping point	exact stopping point	optimal value	exact value
0.001	min	20.504658	21.33600	10.130699	10.20230
	max	21.448930		10.407792	
0.002	min	14.291994	14.55350	7.080385	7.09311
	max	14.587538		7.109600	
0.003	min	11.533998	11.59840	5.743571	5.74944
	max	11.983427		5.760774	
0.004	min	9.639487	9.85697	4.958397	4.96243
	max	10.035447		4.967894	
0.005	min	8.629283	8.67978	4.432446	4.43305
	max	8.760390		4.435024	
0.006	min	7.775992	7.81822	4.046460	4.04716
	max	7.827776		4.047705	
0.007	min	7.084095	7.15399	3.750360	3.75064
	max	7.140455		3.750778	
0.008	min	6.641438	6.62270	3.514061	3.51412
	max	6.631472		3.514352	
0.009	min	6.179400	6.18591	3.320052	3.32013
	max	6.174981		3.320405	
0.010	min	5.844771	5.81912	3.157457	3.15754
	max	5.849002		3.157612	

Table 7:
The optimal stopping point A^* and the optimal value V^* as a function of $r \in [0.01, 0.11]$ compared with the results when using Method I; $\lambda = 0.05$, $\sigma = 0.4$, $x_0 = 1$

r	objective of LPs	optimal stopping point	exact stopping point	optimal value	exact value
0.01	min	3.265851	3.26447	1.838161	1.83816
	max	3.264860		1.838176	
0.02	min	3.026813	3.02688	1.792408	1.79241
	max	3.032295		1.792411	
0.03	min	2.837939	2.84043	1.752841	1.75284
	max	2.839670		1.752843	
0.04	min	2.689689	2.68966	1.718148	1.71815
	max	2.689669		1.718149	
0.05	min	2.564593	2.56485	1.687385	1.68738
	max	2.564711		1.687385	
0.06	min	2.459568	2.45958	1.659850	1.65985
	max	2.459396		1.659850	
0.07	min	2.369418	2.36942	1.635011	1.63501
	max	2.369479		1.635011	
0.08	min	2.290938	2.29121	1.612452	1.61245
	max	2.291212		1.612452	
0.09	min	2.222671	2.22263	1.591841	1.59184
	max	2.222642		1.591841	
0.10	min	2.161918	2.16194	1.572915	1.57291
	max	2.161941		1.572915	
0.11	min	2.107788	2.10779	1.555455	1.55545
	max	2.107602		1.555455	

Appendix B

```
        # This is the LP-model (18)-(23)

        # Options
option cplex_options 'feasibility=1.0e-8 advance=0';

        # Parameters:
param M default 25;
param lam default 0.01;
param r default 0.07;
param sig default 0.4;
param x0 default 1;
param C default 6;

        # Sets:
set ORD := 0 .. M;

        # Variables
var mu {ORD, ORD} ;
var mu1 {ORD, ORD} ;

        # Objective function
maximize goal:
    (C-1)*mu1[0, 1]  +  mu1 [0, 0];

        # Constraints
subject to Hausdorff_diff_mu {i in 1 .. M , j in 0 .. M  - i}:
   mu[i, j] = mu[i - 1, j + 1] - mu[i - 1, j];
subject to Hausdorff_diffineq_mu {i in ORDNUNG, j in 0 .. M  - i}:
   (-1)^i * mu[i, j] >= 0;

subject to Hausdorff_diff_mu1 {i in 1 .. M , j in 0 .. M  - i}:
   mu1[i, j] = mu1[i - 1, j + 1] - mu1[i - 1, j];
subject to Hausdorff_diffineq_mu1 {i in ORDNUNG, j in 0 .. M  - i}:
   (-1)^i * mu1[i, j] >= 0;

subject to dynamic_0 :
        mu1[0,0]  + lam*mu[0,0]  = 1;
subject to dynamic {m in 2 .. M}:
   mu1[0, m]*(C-1)^m  - (x0-1)^(m)
   -(-r * m + sig^2*m*(m-1)/2 -lam )* mu[0, m]*(C-1)^m
   -(-r * m + 2*sig^2*m*(m-1)/2 )* mu[0, m-1]*(C-1)^(m-1)
   -( sig^2*m*(m-1)/2 )* mu[0, m-2]*(C-1)^(m-2) = 0;
```

References

[1] Cho M. J., Stockbridge R. H. Linear Programming Formulation for Optimal Stopping Problems. To appear in SIAM J. Control Optim.

[2] Duffie D., Harrison J. M. (1993) Arbitrage Pricing of Russian Options and Perpetual Look-Back Options. Annals Appl. Probability **3**, 641–573

[3] Feller W. (1965) An Introduction to Probability Theory and its Applications. Vol. 2, Wiley, New York

[4] Helmes K. Numerical Methods for Optimal Stopping Using Linear and Non-Linear Programming. Submitted for publication

[5] Helmes K., Röhl S., Stockbridge R. H. (2001) Computing Moments of the Exit Time Distribution for Markov Processes by Linear Programming. Operations Research **49**, 516–530

[6] Kallianpur G., Karandikar R. L. (2000) Introduction to Option Pricing Theory, Birkhäuser, Boston

[7] Röhl S. (2001) Ein linearer Programmierungsansatz zur Lösung von Stopp- und Steuerungsproblemen. Ph. D. Dissertation, Humboldt-Universität zu Berlin, Berlin, Germany

[8] Shepp L., Shiryaev A. N. (1993) The Russian Option: Reduced Regret. Ann. Appl. Probab. **3**, 631–640

[9] Shepp L., Shiryaev A. N. (1993) A New Look at Pricing of the "Russion Option". Theory Probab. Appl. **39**, 103–119

[10] Shiryaev A. N. (2001) Essentials of Stochastic Finance: Facts, Models, Theory. World Scientific, Singapore