# Computing Optimal Selling Rules for Stocks Using Linear Programming

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Abstract. We consider the model where the price of a stock is described by geometric Brownian motions coupled by a finite state Markov chain. The problem is to find an optimal stopping rule – within the class of policies determined by a *target price* together with a *stop-loss limit* – which maximizes the expected discounted relative price increase. Using a linear programming approach we numerically determine the optimal threshhold values and compute the corresponding mean holding times and the profit- and loss probabilities. For the cases of just one or two hidden states we rely on the analytical results derived by Q. Zhang to illustrate the accuracy of the LP-method.

## 1. Introduction

Let $S = (S_t)_{t\geq 0}$ denote the price of a stock. We assume the stock to be modelled by a finite number of geometric Brownian motions which are driven by a common Wiener process and which are coupled by a finite state Markov chain. Such regime switching models for stock prices have recently become popular because the trajectories of these models represent real data better than less sophisticated models do, while at the same time the typical model to be considered is still simple enough to do explicit computations.

Let $\mathcal{M} = \{1, 2, \ldots, m\}$ denote the state space of the Markov chain $\alpha = (\alpha_t)_{t\geq 0}$ and $Q = (q_{ij})_{1\leq i,j\leq m}$ its generator matrix, i. e. $q_{ij} \geq 0$ for $i \neq j$ and $\sum_{j=1}^{m} q_{ij} = 0$ for each $i \in \mathcal{M}$. We denote the initial distribution of the chain $\alpha$ by $(p_i)_{i\in\mathcal{M}}$, i. e. $p_i = P[\alpha_0 = i]$. Furthermore, for every state $i \in \mathcal{M}$ there is a pair of numbers $(\mu_i, \sigma_i^2)$, $\sigma_i \neq 0$, which specify the drift coefficient and the diffusion coefficient of a geometric Brownian motion driven by a Wiener process $(W_t)_t$. The Wiener process is assumed to be independent of the Markov chain $\alpha$. Thus the stock price satisfies the equation, $t \geq 0$,

$$dS(t) = \mu_{\alpha(t)} S(t)dt + \sigma_{\alpha(t)} S(t)dW_t, \quad S(0) = S_0,$$

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where $S_0 > 0$ denotes its initial price. By the well known solution formula for geometric Brownian motion it follows that $\xi_t := \ln(S_t)$ can be written as $\xi_t = x_0 + X_t$, where $x_0 = \ln(S_0)$, and

$$X(t) = \int_0^t \left(\mu_{\alpha(s)} - \frac{1}{2}\,\sigma^2_{\alpha(s)}\right) ds + \int_0^t \sigma_{\alpha(s)} dW_s.$$

Let $r_i = \mu_i - \frac{1}{2}\,\sigma_i^2$, $i \in \mathcal{M}$. The evolution of the stock price can be equivalently described by switching Brownian motions with drifts $r_i$ and volatilities $\sigma_i^2$, $1 \le i \le m$, i. e.

$$S_t = \exp(\xi_t), \tag{1.1}$$

and

$$d\xi(t) = r_{\alpha(t)}dt + \sigma_{\alpha(t)}dW_t, \quad \xi(0) = x_0 = \ln(S_0). \tag{1.2}$$

Assuming the stock price to be faithfully modelled by (1.1) and (1.2) Q. Zhang [**Z**] analyzes the following fundamental problem in stock trading: Find within a given set $[A_1, A_2] \times [B_1, B_2]$, where $0 < A_1 < A_2 < S_0 < B_1 < B_2 < \infty$, a target price $B \in [B_1, B_2]$ and a stop-loss limit $A \in [A_1, A_2]$ such that when selling the stock at time

$$\tilde{\tau} := \tilde{\tau}_{A,B}(S_0) = \inf\{t \ge 0 \mid S_t \notin (A, B)\}, \tag{1.3}$$

the expected discounted reward

$$E\left[\frac{S(\tilde{\tau}) - S_0}{S_0} \cdot e^{-\varrho\tilde{\tau}}\right] \tag{1.4}$$

is maximized; $\varrho > 0$ is a given discount factor. This optimal stopping problem can be phrased in terms of $(\xi_t)_t$ if we put $a_1 = -\ln(A_2/S_0)$, $b_1 = -\ln(A_1/S_0)$, $a_2 = \ln(B_1/S_0)$ and $b_1 = \ln(B_2/S_0)$. Note, $0 < a_1 \le b_1 < \infty$, and $0 < a_2 \le b_2 < \infty$. Each stopping time $\tilde{\tau}_{A,B}$ determined by $S$, cf. (1.3), is equivalently described by, $a_1 \le z_1 \le b_1$, $a_2 \le z_2 \le b_2$,

$$\tau_{-z_1,z_2}(x_0) = \inf\{t \ge 0 \mid \xi_t \notin (-z_1, z_2)\}. \tag{1.5}$$

Thus, due to the pay-off function (1.4), the problem can be formulated as follows: Find a pair $(z_1, z_2) \in \mathcal{I} := [a_1, b_1] \times [a_2, b_2]$ which maximizes

$$\sum_{i=1}^{m} p_i v_{0,i}(z_1, z_2), \tag{1.6}$$

where, for general initial value $x$, $-z_1 \le x \le z_2$, $x = \xi(0)$, $j = \alpha(0)$,

$$v_{x,j}(z_1, z_2) := E_{x,j}\Big[\Phi\left(\xi_{\tau_{-z_1,z_2}(x)}\right) \exp\big(-\varrho\tau_{-z_1,z_2}(x)\big)\Big],$$

and

$$\Phi(x) = e^x - 1.$$

Note, the objective function (1.6) clearly reveals the fact that the market-trend indicator process $\alpha$ is assumed to be unobservable, and admissible selling strategies $\tau$, see (1.5), are to be based only an $S_\tau$, $Q$ and $p$. In [**Z**] it is shown that for fixed numbers $z_1$ and $z_2$ the function $(x, i) \mapsto v_{x,i}(z_1, z_2) =: v_{x,i}$, $x \in [-z_1, z_2]$, $i \in \mathcal{M}$, satisfies the system of ODEs,

$$\frac{1}{2}\,\sigma_i^2\,\frac{dv_{x,i}}{dx^2} + r_i\,\frac{dv_{x,i}}{dx} - \varrho v_{x,i} + Qv_{x,\cdot}(i) = 0, \tag{1.7}$$

with boundary conditions

$$v_{-z_1,i} = \Phi(-z_1) \quad \text{and} \quad v_{z_2,i} = \Phi(z_2); \tag{1.8}$$

for any vector $f \in \mathbb{R}^{|\mathcal{M}|}$, $Qf(\cdot)(i) = \sum_{j \neq i} q_{ij}(f(j) - f(i))$. Moreover, explicit formulas for the solutions of (1.7) and (1.8) are given when $|\mathcal{M}| = 1$ and $|\mathcal{M}| = 2$. The solutions turn out to be linear combinations of exponential functions whose parameters depend in a convoluted way on $z_1$ and $z_2$, see [**Z**] for details. Similar results are given by Zhang for the expected holding time

$$(x, i) \mapsto T_{x,i}(z_1, z_2) := E_{x,i}\left[\tau_{-z_1,z_2}(x)\right], \tag{1.9}$$

and the profit- and loss probabilities

$$(x, i) \mapsto P^{(r)}_{x,i}(z_1, z_2) := P_{x,i}\Big[\xi\big(\tau_{-z_1,z_2}(x)\big) \geq z_2\Big], \tag{1.10}$$

and

$$(x, i) \mapsto P^{(\ell)}_{x,i}(z_1, z_2) := P_{x,i}\Big[\xi\big(\tau_{-z_1,z_2}(x)\big) \leq -z_1\Big]. \tag{1.11}$$

Using a non linear optimization solver one can then compute the optimal threshholds $(z_1^*, z_2^*)$ for the cases of one or two hidden states.

In this paper we shall present a numerical method based on linear programming for analyzing, in particular, the optimal stopping problem described above for the case of more than two hidden states and for more general switching diffusions. Different numerical methods for optimal stopping using linear and non linear programming techniques have recently been proposed by Cho [**C1**], Cho and Stockbridge [**C2**], Helmes [**H4**] and Röhl [**R**]. The linear programming approach to optimal stopping and, more generally, to stochastic control is an extension of work by Manne [**M**] who initiated the formulation of stochastic control problems as linear programs over a space of stationary distributions for the long-term average control of finite-state Markov chains. The generalization of the LP-formulation for continuous time, general state and control spaces, and different objective functions has been established by Stockbridge [**S**], Kurtz and Stockbridge [**K1**], [**K2**] and Bhatt and Borkar [**B**]. Their results show the equivalence of such control problems to infinite dimensional linear programs whose variables are measures. Different numerical methods are determined by the way these infinite dimensional LPs are approximated by finite dimensional ones. A particular approximation employing power functions has been proposed by Helmes et al. [**H2**], see also Helmes [**H4**] and Röhl [**R**]. An approximation based on finite dimensional semi-definite programs has recently been proposed by [**L**]. A novel feature to be presented in this paper is the use of explicit formulas for the corner points of 1-dimensional Hausdorff polytopes. Such polytopes are the fundamental sets for the approximating LPs. The formulas for the corner points speed up computing time and enhance the numerical accuracy of the finite dimensional LPs.

For detailed discussions and explanations of the basic ideas of the LP-approach to the analysis of exit and stopping time problems and to some stochastic control problems we refer to the literature cited above and to [**H1**] and [**H3**].

This paper is organized as follows. In Section 2 we describe the finite dimensional approximating linear programs for the stock trading problem. In Section 3 we present some numerical results for the cases $|\mathcal{M}| = 1$ and $|\mathcal{M}| = 2$ and compare our results with numerical results based on the analytical formulas derived in [**Z**].

In Section 4 we consider two numerical examples for $|\mathcal{M}| = 3$ and 4, and draw some conclusions.

## 2. Linear and non linear optimization models associated with the stock trading problem

It follows from the general results to which we referred above, see [**K1**], that the optimal selling problem described in the Introduction is equivalent to the following infinite dimensional optimization problem. The objective is to choose $0 \le \vec{p}^{(s)} \in \mathbb{R}^m$, $1 \le s \le 2$, $z_s \in [a_s, b_s]$, $\mu(dx,i)$, $i \in \mathcal{M}$, non negative measures on $[-z_1, z_2]$, such that

$$\sup_{z_1,z_2,\vec{p}^{(1)},\vec{p}^{(2)},\mu} \left\{ \Phi(-z_1) \sum_{i=1}^{m} \vec{p}^{(1)}(i) + \Phi(z_2) \sum_{i=1}^{m} \vec{p}^{(2)}(i) \right\},$$

subject to, $\forall\, j \in \mathcal{M}$, $\forall\, f \in \mathcal{C}^2$,

$$\begin{aligned} &\vec{p}^{(1)}(j) f(-z_1) + \vec{p}^{(2)}(j) f(z_2) \\ &\quad = \; p_j f(0) + \int_{-z_1}^{z_2} \left( \frac{\sigma_j^2}{2} f''(x) + r_j f'(x) - \varrho f(x) \right) \mu(dx, j) \\ &\qquad + \sum_{i=1}^{m} q_{ij} \int_{-z_1}^{z_2} f(x) \mu(dx, i), \end{aligned} \tag{2.1}$$

is attained. Note, if $z_1$ and $z_2$ are given the non linear problem becomes a linear one. The resulting infinite dimensional linear programs with variables $\mu$ and $\vec{p} = \left(\vec{p}^{(1)}, \vec{p}^{(2)}\right)$ can be approximated by finite dimensional LPs as follows: Choose $N \in \mathbb{N}$ and consider, cf. [**R**], the corner points of the Hausdorff polytope of order $N$, i. e. the vectors $\vec{e}_k \in \mathbb{R}^{N+1}$, $0 \le k \le N$, where $0 \le j \le N$,

$$\vec{e}_k(j) = \begin{cases} \dfrac{\binom{N-j}{k-j}}{\binom{N}{k}} & , \quad j \le k \\ \quad 0 & , \quad \text{else.} \end{cases}$$

Next, choose variables $\kappa_k^{(s)}(i) \ge 0$, $s \in \{1, 2\}$, $0 \le k \le N$, $i \in \mathcal{M}$, and define the quantities, $0 \le n \le N$, $i \in \mathcal{M}$, $z_1$ and $z_2$ being fixed,

$$\vec{\mu}^{(1)}(n,i) \;:=\; (-z_1)^n \sum_{k=0}^{N} \kappa_k^{(1)}(i) \vec{e}_k(n),$$

$$\vec{\mu}^{(2)}(n,i) \;:=\; z_2^n \sum_{k=0}^{N} \kappa_k^{(2)}(i) \vec{e}_k(n),$$

$$\vec{\mu}(n,i) \;:=\; \vec{\mu}^{(1)}(n,i) + \vec{\mu}^{(2)}(n,i), \tag{2.2}$$

and

$$\vec{\mu}_\tau(n,i) \;:=\; (-z_1)^n \vec{p}^{(1)}(i) + z_2^n \vec{p}^{(2)}(i). \tag{2.3}$$

The quantities $\vec{\mu}(n,i)$ include the moments of all occupation measures $\mu(dx,i)$ on $(-z_1, z_2)$ of the process $(X_t, \alpha_t)_t$ killed at rate $\varrho$. We finalize the construction of the approximating finite dimensional LPs by expanding the pay-off function

$\Phi(\xi) = e^{\xi} - 1$ up to the power $N$ and by restricting the test functions $f(x)$, cf. (2.1), to be monomials $x^n$, $0 \le n \le N$. The following result is a consequence of the general theory of the LP-aproach.

THEOREM 2.1. *The optimal value $V_N^*$ of the finite dimensional linear program*

$$\max_{\vec{p},\vec{\kappa}} \left\{ \sum_{n=1}^{N} \sum_{i=1}^{m} \frac{\vec{\mu}_\tau(n,i)}{n!} \right\} =: V_N^*(z_1, z_2) =: V_N^*$$

*subject to*

(i) $\vec{\kappa} = \big(\kappa_k^{(s)}(i)\big) \ge 0$, $s \in \{1,2\}$, $0 \le k \le N$, $1 \le i \le m$,

(ii) $\forall\, 0 \le n \le N$, $j \in \mathcal{M}$

$$\begin{aligned}\vec{\mu}_\tau(n,j) &= p_j 0^n + \frac{n(n-1)\sigma_j^2}{2}\,\vec{\mu}(n-2,j) + n r_j \vec{\mu}(n-1,j) - \varrho\vec{\mu}(n,j) \\ &\quad + \sum_{i=1}^{m} q_{ij}\vec{\mu}(n,i),\end{aligned}$$

*where $\vec{\mu}_\tau$ and $\vec{\mu}$ are defined by (2.2) and (2.3), tends, for $N \to \infty$, to the value of the stock trading problem if the target price equals $z_2$ and the stop-loss limit is $-z_1$.*

REMARK 2.2. If the pay-off function were a polynomial of order less than $N$ then $V_N^*$ would be a corresponding upper bound. The finite expansion of the exponential function interferes with this property to hold. For all practical purposes, however, $V_N^*$ will be an upper bound whenever $N$ is reasonably large.

Next, we use a non linear optimization technique to find the best stopping points $-z_1$ and $z_2$. For the examples considered in Section 3 and 4 it turns out that $z_1 = b_1$, and the 2-dimensional optimization problem with variables $(z_1, z_2)$ becomes a 1-dimensional problem about $z_2$ for which line search techniques can be applied. Once optimal threshholds $z_1^* = z_1^*(\varrho, N)$ and $z_2^* = z_2^*(\varrho, N)$ are determined based on the finite dimensional LPs, cf. Theorem 2.1, we find approximate values of the mean holding time $T^*(\varrho)$ and the ratio of the profit-to-loss probabilities $R^*(\varrho)$ of the trading policy by solving the associated exit time problem for the jump-diffusion process $(X_t)_t$, cf. [**H2**]. The corresponding finite dimensional LPs, see below, provide upper as well as lower bounds for the quantities of interest. To this end, we again consider variables $\vec{p} = \left(p_i^{(s)}\right)_{s,i}$, $1 \le s \le 2$, $i \in \mathcal{M}$, and $\vec{\nu} = \big(\nu(n,i)\big)_{n,i}$, $0 \le n \le N$, $i \in \mathcal{M}$, such that, this time,

$$p_i^{(s)} \ge 0 \quad \text{and} \quad \sum_{s,i} p_i^{(s)} = 1, \tag{2.4}$$

$$\begin{aligned}\nu(n,i) &= \nu^{(1)}(n,i) + \nu^{(2)}(n,i), \quad \text{where} \\ \nu^{(1)}(n,i) &= (-z_1^*)^n \sum_{k=0}^{N} \alpha_k^{(1)}(i)\vec{e}_k(n), \quad \alpha_k^{(1)}(i) \ge 0,\ 0 \le k \le N,\ i \in \mathcal{M}, \\ \nu^{(2)}(n,i) &= (z_2^*)^n \sum_{k=0}^{N} \alpha_k^{(2)}(i)\vec{e}_k(n), \quad \alpha_k^{(2)}(i) \ge 0,\end{aligned} \tag{2.5}$$

and, $\forall\, 0 \le n \le N,\ j \in \mathcal{M}$

$$
\begin{aligned}
&(-z_1^*)^n p_j^{(1)} + (z_2^*)^n p_j^{(2)} \\
&\quad = \quad 0^n p_j + \frac{n(n-1)}{2}\,\sigma_j^2 \nu(n-2,j) + n r_j \nu(n-1,j) + \sum_{i=1}^m q_{ij}\nu(n,i).
\end{aligned} \tag{2.6}
$$

We define the set of feasible points $\mathcal{S} := \mathcal{S}(m, N, z_1^*, z_2^*) \subset \mathbb{R}^{2m}\times\mathbb{R}^{m(N+1)}$ as follows:

$$\mathcal{S} := \big\{(\vec{p},\vec{\nu}) \;\big|\; \vec{p} \text{ and } \vec{\nu} \text{ satisfy } (2.4)-(2.6)\big\}.$$

The values of the following finite dimensional linear programs provide, for given $z_1^*$ and $z_2^*$, upper and lower bounds for exit probabilities and mean holding times of the corresponding exit problems:

$$\bar{T}_N := \max_{(\vec{p},\vec{\nu})\in\mathcal{S}} \left\{\sum_{i=1}^m \nu(0,i)\right\} \quad \text{and} \quad \underline{T}_N := \min_{(\vec{p},\vec{\nu})\in\mathcal{S}} \left\{\sum_{i=1}^m \nu(0,i)\right\};$$

for $1 \le s \le 2$,

$$\bar{P}_N^{(s)} := \max_{(\vec{p},\vec{\nu})\in\mathcal{S}} \left\{\sum_{i=1}^m p_i^{(s)}\right\} \quad \text{and} \quad \underline{P}_N^{(s)} := \min_{(\vec{p},\vec{\nu})\in\mathcal{S}} \left\{\sum_{i=1}^m p_i^{(s)}\right\},$$

and

$$\bar{R}_N := \frac{\bar{P}_N^{(2)}}{\underline{P}_N^{(1)}} \quad \text{and} \quad \underline{R}_N := \frac{\underline{P}_N^{(2)}}{\bar{P}_N^{(1)}}.$$

The essential properties of this particular construction are summarized in the following theorem.

THEOREM 2.3. *Let $z_1^*$ and $z_2^*$ be given. For every $N \ge 1$ the following inequalities hold:*

$$\text{(a)} \qquad \underline{T}_N \;\le\; \sum_{i=1}^m p_i E_{0,i}\big[\tau_{-z_1^*,z_2^*}(0)\big] \;=:\; T(z_1^*,z_2^*) \;\le\; \bar{T}_N;$$

$$\text{(b)} \qquad \underline{R}_N \;\le\; \frac{\sum_{i=1}^m p_i P_{0,i}^{(r)}(z_1^*,z_2^*)}{\sum_{i=1}^m p_i P_{0,i}^{(\ell)}(z_1^*,z_2^*)} \;=:\; R(z_1^*,z_2^*) \;\le\; \bar{R}_N.$$

REMARK 2.4. Below, we assume that $z_1^*$ and $z_2^*$ are approximate values for the optimal stop-loss limit and the optimal target price (depending on $\varrho$) and that we haven choosen $N$ large enough so that up to numerical accuracy

$$\underline{T}_N \doteq \bar{T}_N \qquad \text{and} \qquad \underline{R}_N \doteq \bar{R}_N.$$

We then take $T(z_1^*,z_2^*)$, $R(z_1^*,z_2^*)$ resp., as an approximate value for $T^*(\varrho)$, $R^*(\varrho)$ resp.

## 3. Numerical results for one or two hidden states

To test the accuracy of the LP approximations we have repeated the computations by Q. Zhang for the case $|\mathcal{M}| = 1$. Using the LPs described in Theorems 2.1 and 2.2 we have duplicated all numbers to be in found in Tables 2 − 4 of [**Z**]. In addition we have analyzed the case of one hidden state for $r = r_1 = 0.26$, $\sigma = 0.51$ and $\mathcal{I} = [0.01, 0.36] \times [0.01, 2.4]$, i. e. the loss is limited to $\sim 30\,\%$ while the maximum profit is allowed to be $\sim 1100\,\%$. The choice of the parameters is motivated by the two-state model to be considered below. It turns out that in all these cases the LP results show the stop-loss limit to be either equal to $-b_1$ or close to $-b_1$; this again agrees with Zhang's results.

Table 1 reports approximate values of the optimal selling point, the expected pay-off, the expected holding time and the ratio of the profit-to-loss probabilities for some values of $\varrho$. The numbers reveal what is intuitively clear: For larger discount factors the target prices are set lower, the expected holding times as well as the pay-offs decrease, while the ratios of the probabilities increase.

Table 1. Approximate values for the optimal stopping point $z_2^*$, the value function, and upper bounds for the expected holding time and the ratio of the profit-to-loss probabilities; 1-state model, $r_1 = 0.26$, $\sigma_1 = 0.51$; $N = 50$.

| $\varrho$ | $z_2^*$ | $E\Big[\big(e^{X_{\tau^*}} - 1\big)\, e^{-\varrho\tau^*}\Big]$ | $E[\tau^*]$ | $\frac{\text{profit-prob.}}{\text{loss-prob.}}$ |
|---|---|---|---|---|
| 0.6 | 1.312 | 0.279 | 2.036 | 1.136 |
| 0.8 | 0.895 | 0.196 | 1.311 | 1.265 |
| 1 | 0.712 | 0.157 | 1.012 | 1.399 |
| 2 | 0.381 | 0.091 | 0.508 | 1.998 |
| 3 | 0.276 | 0.071 | 0.359 | 2.485 |
| 4 | 0.227 | 0.062 | 0.293 | 2.889 |
| 5 | 0.197 | 0.056 | 0.252 | 3.237 |

Next, based on the estimates for $r_i$, $\sigma_i$ and $q_{ij}$, $1 \le i, j \le 2$, for Microsoft stocks that were given in [**Z**], i.e. $r_1 = 1.5$, $r_2 = -1.61$, $\sigma_1 = 0.44$, $\sigma_2 = 0.63$, $q_{12} = 6.04$ and $q_{21} = 8.9$, we have analyzed the case with two hidden states choosing $\mathcal{I}$ as above and assuming $\alpha(0)$ to be uniformly distributed. In Table 2 we report the LP-results for the 2-state model for the same quantities of interest as above, again as a function of $\varrho$, and compare the values with those numbers (in parantheses) which are obtained by applying Mathematica's ODE-solver to (1.7)−(1.11) when $z_1 = -b_1$ and $z_2 = z_2^*$. Note the excellent agreement of the numbers if $\varrho \ge 1$. For small values of $\varrho$ neither the ODE solver nor the LP-results can be fully trusted. While the LP-numbers for the mean holding time show at least the right qualitative behaviour the ODE-results for the mean holding time are garbage, see values for $\varrho = 0.6$ and $\varrho = 0.8$.

The difference between some of our values and the numbers reported in [**Z**] on p. 79 are due to a trivial typo in Zhang's original code (private communication by Q. Zhang). Finally, Table 3 reports the expected relative increase (no discounting) of the stock value and the relative increase should the investor be lucky, i.e. the process $(X_t)_t$ exits from the interval $(-z_1^*, z_2^*)$ at $z_2^*$. For the particular parameter constellation the odds are slightly favourable to experience a 66 % increase while holding the stock on average for around 5 months, see Tables 2 and 3, row $\varrho = 2$; the average relative increase is 21 %.

Table 2. LP-results of the optimal threshhold, the optimal value, the mean holding time and the ratio of profit-to-loss probabilities as a function of $\varrho$. The numbers in parantheses are based on Mathematica's ODE solver and expressions (1.7) – (1.11); 2-dim. case, $q_{12} = 6.04$, $q_{21} = 8.9$, $\sigma_1 = 0.44$, $\sigma_2 = 0.61$, $r_1 = 1.5$, $r_2 = -1.61$, uniform initial distribution; $N = 100$.

| $\varrho$ | $z_2^*$ | optimal value | mean holding time | profit-loss prob. ratio |
|---|---|---|---|---|
| 0.6 | 2.390 | 0.362 (0.350) | 2.191 (large negative number) | 0.517 (0.495) |
| 0.8 | 1.380 | 0.218 (0.217) | 1.148 ($-0.170$) | 0.620 (0.620) |
| 1 | 1.038 | 0.162 (0.162) | 0.848 (0.848) | 0.725 (0.725) |
| 2 | 0.504 | 0.077 (0.077) | 0.411 (0.411) | 1.158 (1.158) |
| 3 | 0.358 | 0.055 (0.055) | 0.297 (0.297) | 1.474 (1.474) |
| 4 | 0.278 | 0.045 (0.045) | 0.235 (0.235) | 1.769 (1.769) |
| 5 | 0.231 | 0.039 (0.039) | 0.199 (0.199) | 2.026 (2.026) |

Table 3. LP-values of the expected relative increase (no discounting) of the stock value as a function of $z_2^*(\varrho)$ together with the probability of winning, $P^*(r)$, and related quantities; the parameters are the same as in Table 2.

| $\varrho$ | 0.6 | 0.8 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|
| $z_2^*$ | 2.390 | 1.380 | 1.038 | 0.504 | 0.358 | 0.278 | 0.231 |
| $\Phi(z_2^*)$ | 9.913 | 2.975 | 1.824 | 0.655 | 0.431 | 0.321 | 0.259 |
| $P^*(r)$ | 0.341 | 0.385 | 0.420 | 0.537 | 0.596 | 0.639 | 0.669 |
| $\Phi(z_2^*)P^*(r)$ | 3.380 | 1.144 | 0.766 | 0.352 | 0.256 | 0.205 | 0.174 |
| $E\big[\Phi(x_{\tau^*})\big]$ | 3.180 | 0.958 | 0.591 | 0.212 | 0.134 | 0.096 | 0.074 |

## 4. Some numerical results for three and four hidden states

An advantage of the LP-approach over other methods is that one can easily handle more than two hidden states. If, for instance, AMPL or a similar program is used as an interface to implement the finite dimensional LP then only trivial modifications of the data file will be necessary; instead of 2-vectors $(r_i)$, $(\sigma_i)$ and

$(p_i)$, and a $2 \times 2$ matrix $Q$ only corresponding higher dimensional analogues have to be specified. Table 4 reports the values of the quantities which were considered in Section 3 when, besides the up-state 1 and down-state 2, an additional state 3 models the possibility that the stock price fluctuates around its present value for a while. Specifically, for the same set $\mathcal{I}$ as above we haven chosen $\vec{r} = (r_1, r_2, r_3) = (1.51,\ -1.61,\ 0)$, $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3) = (0.44,\ 0.63,\ 0.4)$, $\vec{P} = (p_1, p_2, p_3) = \left(\frac{1}{4},\ \frac{1}{4},\ \frac{1}{2}\right)$ and

$$Q = \begin{pmatrix} -14.08 & 6.04 & 8.04 \\ 8.9 & -17.8 & 8.9 \\ 1.5 & 1 & -2.5 \end{pmatrix}.$$

Table 4. An excerpt of the analogue of Table 2 for an example with 3 hidden states. The numbers for the mean holding time and the ratio of probabilities are the upper bounds which, up to 5 decimals, are identical to the lower bounds. The parameters are specified in Section 4; $N = 50$.

| $\varrho$ | $\hat{z}_2$ | optimal value | mean holding time | profit-loss prob. ratio |
|---|---|---|---|---|
| 0.6 | 0.6150 | 0.0890 | 1.030280 | 0.78880 |
| 1 | 0.4070 | 0.0630 | 0.675408 | 1.15020 |
| 2 | 0.2550 | 0.0458 | 0.422430 | 1.67476 |
| 5 | 0.1516 | 0.0360 | 0.252380 | 2.66432 |

Finally, Table 5 illustrates the consequences of model misspecification. Should the stock price be governed by the 3-state model considered in this Section but should the optimal trading policy of the 2-state model described in Section 3 be implemented then the investor would hold the stock on average for too long and his expectations about making a profit would be too high.

Table 5. Relevant quantities for sub-optimal decisions for the 3-state case; the paramters, except that $\hat{z}$ is replaced by $z_2^*(\varrho)$, are as in Table 4.

| $\varrho$ | 0.6 | 0.8 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|
| $z_2^*(\varrho)$ | 2.390 | 1.380 | 1.038 | 0.504 | 0.358 | 0.278 | 0.231 |
| mean holding time | 4.537 | 2.435 | 1.785 | 0.840 | 0.593 | 0.460 | 0.383 |
| profit-loss prob. ratio | 0.318 | 0.432 | 0.525 | 0.930 | 1.245 | 1.552 | 1.828 |

Based on our analysis we draw several conclusions. Investors, by choosing a stop-loss limit and a target price, face the following gambling situation. For a typical regime switching model higher target prices imply less favourable odds of making a profit and increase the average holding duration. Thus it seems reasonable for most investors to follow a policy of setting, relative to the stop-loss limit, a "small" target price to experience, if possible, on average modest but repeated gains while the capital is only tied up over a relatively short period of time. Such a policy would still have to take transaction costs and taxes into account. The discount factor $\varrho$ in our model and the associated objective function (1.4) is one way to rationalize the setting of the threshhold values $-z_1^*$ and $z_2^*$. For instance, if $\varrho$ is chosen in a small neighbourhood of 0.6, e. g. $(0.56, 0.64)$, then the objective (1.4) can be interpreted as the average quotient of risky gains by investing in a stock and guaranteed interest payments when buying a riskless asset paying compound interest at an annual rate of $\sim 1\,\% - \sim 6\,\%$. But in general, the factor $\varrho$ is kind of an artifact which changes a typically unbounded (if $B_2 = \infty$, $A_1 = 0$) problem into a bounded one. Since for typical switching models the stop-loss limit will always be set at its lowest feasible point an alternative approach would be to take the target price as the fundamental control factor and, for given model parameters, compute exit probabilities, mean holding times and related quantities on which the final choice will be based. The linear programming technique is well suited for both approaches if jump-diffusions are used to model the stock price movements. It provides accurate results not only for models with one or two hidden states but, in principle, for any number of states. Using the explicit formulas for the corner points of the Hausdorff polytopes, instead of relying on the characterization by iterated differences, enhances the accuracy of the LP-technique and makes the code run faster. An additional advantage of the LP-technique is that constraints like, "do not on average hold the stock for more than 6 months", could be easily incorporated into the approximating LPs; typically this requires but a few exta lines in a code which is based on AMPL. Preliminary studies of such type of problems have been made but work continues on larger models.

At the end we briefly analyze just one example with 4 hidden states and compute the corresponding exit probabilities. The example is similar to the one discussed by Zhang, see Introduction in [**Z**]. We assume the Markov chain $\alpha(t) = \big(\alpha_I(t), \alpha_{II}(t)\big)$, where $\alpha_I(t) \in \{\text{UP, DOWN}\}$ and $\alpha_{II}(t) \in \{\text{up, down}\}$, to represent, by its first component, the state of the primary market trend, and its second component to be an indicator of the state of the secondary market movement. Thus the state space of $\alpha$ equals $\big\{(\text{UP, up}), (\text{DOWN, up}), (\text{UP, down}), (\text{DOWN, down}\big\}$. For our numerical example we assume its generator matrix to be

$$Q = \begin{pmatrix} -14.08 & 6.04 & 8.04 & 0 \\ 8.9 & -16.94 & 0 & 8.04 \\ 5.5 & 0 & -11.54 & 6.04 \\ 0 & 5.5 & 8.9 & -14.4 \end{pmatrix}.$$

Furthermore, we take $\vec{r} = (1.5, 0.31, -1, -1.8)$, $\vec{\sigma} = (0.44, 0.5, 0.63, 0.3)$ and $\vec{p} = (0.35, 0.35, 0.15, 0.15)$. If we set $z_1^* = 0.36$ and $z_2^* = 0.24$, then chances are roughly $50 - 50$ (the profit-to-loss ratio equals 1.00354) that the mean holding time is $\sim 3$ months ($T^*(\varrho) = 0.252$); the exit probabilities are given by Table 6. While the parameters of the 2-state model are estimates based on real data the parameters of the 3- and 4-state models are chosen for illustrative purposes.

A detailed analysis of the example shows that critical parameters which need to be carefully estimated are the volatilities associated with different hidden states.

TABLE 6. The exit probabilities for an example with 4 states. The parameters are given as above.

| state | $P^{\ell}(i)$ | $P^{r}(i)$ |
|---|---|---|
| 1 | 0.0196539 | 0.254447 |
| 2 | 0.0514260 | 0.113069 |
| 3 | 0.2668960 | 0.128278 |
| 4 | 0.1611420 | 0.00508889 |

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