A VARIATIONAL INEQUALITY SUFFICIENT CONDITION FOR OPTIMAL STOPPING WITH APPLICATION TO AN OPTIMAL STOCK SELLING PROBLEM

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Abstract. We give a variational inequality sufficient condition for optimal stopping problems. This result is illustrated by computing solutions to an optimal stock selling problem. The stock selling problem has a model for the stock price which initially has a "hot" growth rate and then "tanks". Solution of the conditions allow computation of both the value function and the optimal stopping times. The implications on the investors behavior of different parameter settings for different utility functions are discussed.

Key words. stock selling times, optimal stopping, variational inequalities

AMS subject classifications. 49J40, 49K15, 49K20, 60G35, 60G40, 62L15, 91B70

1. Introduction. Variational inequalities for solving optimal stopping problems were introduced by Bensoussan and Lions, cf. [1] and [2]. Many variational inequality sufficient conditions have been given. Some examples of these are Bensoussan [1] p. 301, Krylov [7] p. 41 Theorem 7, and Oksendal [9] p. 225.

The sufficiency condition we give is an extremely simple extension of that of [1]. However it points out the apparently previously overlooked requirement that a boundedness condition on the gradient of the value function has in determining the correct solution from the possibly many solutions of the variational inequality. The use of this property is illustrated in our solution of the stock selling problem.

A number of different stock selling problems have been considered. For instance in [9], p. 219 and p. 227, Oksendal considers a stock selling problem whose stock price is given by a geometric Brownian motion with constant coefficients. In [12] Qing Zhang considers a stock selling problem whose stock price is given by a diffusion process whose coefficients are unobservable finite state jump Markov processes. In [3] Beibel and Lerche consider an optimal stock selling problem for a model which is basically the same as the partially observed model we consider below. We discuss comparison of Zhang's results and Beibel and Lerche's results with those of ours in section 9.

The stock problem we consider is motivated by trying to decide when to sell stocks which have rapid growth and then rapidly decline such as the recent behavior of Enron stock. We consider an idealized model of such a situation in which the stock price is given by a geometric Brownian motion which has an initial positive growth rate and after a random time jumps to and stays at a negative growth rate. The investor observes the stock prices but cannot observe the growth rates. Deciding when to sell is based only on past stock prices. Thus this is a partially observed optimal stopping problem. Converting it to a completely observed optimal stopping problem is done similarly to the methods used by Shiryayev [10] p. 200 in his quickest detection of a disorder problem.

We consider the problem with two different utility functions $U(S) = \ln(S)$ and $U(S) = S$. For the utility function $U(S) = \ln(S)$ there are solutions of the variational

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inequality with the appropriate boundedness, and optimal stopping times are calculated. For the utility function $U(S) = S$ it appears to be difficult to find solutions of the variational inequality with the appropriate boundedness for the problem in its original form. We use two changes of probability measures to convert the problem into one for which these boundedness conditions are determined and calculations of the optimal stopping time can be carried out.

2. A Variational Inequality Sufficient Condition for Optimal Stopping. We begin by stating an optimal stopping problem and giving variational inequality sufficient conditions for it. Let $g(z)$ and $h(z)$ denote respectively an n-dimensional vector valued and a $n \times m$-dimensional matrix valued function of the n-dimensional vector z. Let $W(t)$ denote a m-dimensional Wiener process. Assume $g(z)$ and $h(z)$ are regular enough so that solutions of the stochastic differential equation and initial condition

$$dz(t) = g(z(t))\,dt + h(z(t))\,dW(t), \quad z(0) = z, \tag{2.1}$$

exist and are unique. Let $\mathcal{F}_t$ denote the σ-fields

$$\mathcal{F}_t = \sigma[z(r):\ 0 \le r \le t] \tag{2.2}$$

generated by the past of $z(t)$. Let $\beta(z)$ be a continuous scalar function and $U(z)$ a twice continuously differentiable utility function. Let $\mathcal{A}$ denote a class of $\mathcal{F}_t$ stopping times. For each stopping time $\tau \in \mathcal{A}$ consider the expected discounted utility

$$E[e^{\int_0^\tau \beta(z(t))\,dt} U(z(\tau))]. \tag{2.3}$$

The optimal stopping problem is: Find τ in $\mathcal{A}$ which achieves the maximum of (2.3).

The following theorem gives variational inequality sufficient conditions for optimality for this problem.

Theorem 2.1. *Let R be a region in E^n. Assume for each z in R that the solution of (2.1) with initial condition z is contained in R. Let $V(z)$ be a scalar valued function defined on R. Let $V(z)$ be regular enough so that Itô's stochastic differential rule holds for $V(z(t))$. Define the differential operator $A[V](z)$ by*

$$A[V](z) = \beta(z)V(z) + V_z(z)g(z) + \frac{1}{2}\,\mathrm{trace}\,(h(z)h(z)'V_{zz}(z)). \tag{2.4}$$

Let $V(z)$ be a solution of the variational inequality

$$\begin{aligned} &A[V](z) \le 0, \quad V(z) \ge U(z), \\ &(V(z) - U(z))\ A[V](z) = 0, \end{aligned} \tag{2.5}$$

and let the condition

$$E\int_0^\tau e^{2\int_0^t \beta(z(r))\,dr} ||V_z(z(t))h(z(t))||^2\,dt < \infty \tag{2.6}$$

hold for each stopping time τ in $\mathcal{A}$. For $z(t)$ the solution of (2.1) with initial condition z, let

$$\tau(z) = \textit{ first time } z(t) \textit{ hits } \{q:\ V(q) = U(q)\}. \tag{2.7}$$

Let

$$\tau(z) \in \mathcal{A} \text{ for each } z \in R. \tag{2.8}$$

Then

$$V(z) = E[e^{\int_0^{\tau(z)} \beta(z(r))\,dr} U(z(\tau(z)))] = \max_{\tau \in \mathcal{A}} E[e^{\int_0^{\tau} \beta(z(r))\,dr} U(z(\tau))]. \tag{2.9}$$

That is $\tau(z)$ is an optimal stopping time in $\mathcal{A}$ and $V(z)$ is the value function for the optimal stopping problem.

Theorem 2.1 should be known. Its simple proof uses that condition (2.6) implies the expected value of the stochastic integral term in the Itô differential of $e^{\int_0^t \beta(z(r)\,dr} V(z(t))$ is zero. Then standard variational inequality arguments such as in Bensoussan [1] p. 201 give the conclusion. For completeness we give the short proof in an appendix. However the importance of conditions (2.6) and (2.8) holding along with the variational inequality conditions (2.5) does not appear to have been pointed out. Conditions (2.6) and (2.8) make the correct selection of the possibly many solutions of conditions (2.5).

Remark 2.2. Notice, if

$$\beta(z) \leq 0 \text{ and } ||V_z(z)h(z)||^2 \leq K, \tag{2.10}$$

that

$$E[\int_0^{\tau} e^{2\int_0^t \beta(z(r))\,dr} ||V_z(z(t))h(z(t))||^2\,dt] \leq KE(\tau). \tag{2.11}$$

So if the boundedness condition (2.10) holds, (2.6) holds for stopping times with finite expectations.

Remark 2.2 motivates looking for solutions of (2.5) for which the boundedness condition (2.10) holds.

3. The Stock Price Model. Consider a stock whose price $S(t)$ satisfies the stochastic differential equation and initial condition

$$dS(t) = S(t)(a(t)dt + \sigma dW(t)), \quad S(0) = S. \tag{3.1}$$

In (3.1) S and σ are positive constants, $W(t)$ is a Wiener process and $a(t)$ is a random process which jumps from $a > 0$ to $b < 0$ at a random time J which satisfies

$$Pr[J = 0] = 1 - x, \quad Pr[J > t | J > 0] = e^{-ct}, \tag{3.2}$$

where x and c are constants satisfying $0 \leq x \leq 1$ and $c > 0$.

Let $U(S)$ be a utility function. Let

$$\mathcal{F}_t = \sigma\{S(r) : \ 0 \leq r \leq t\} \tag{3.3}$$

denote the σ-fields generated by the past of the process $S(\cdot)$ up to times t.

Consider finding in a class $\mathcal{A}$ of $\mathcal{F}_t$ stopping times τ, a stopping time which maximizes the expected utility

$$E[U(S(\tau))]. \tag{3.4}$$

4. Reduction to a Completely Observed Problem. It is well known that the solution of the stock price equation (3.1) is given by

$$S(t) = Se^{\int_0^t a(s)ds + \sigma W(t) - \frac{1}{2}\sigma^2 t}. \tag{4.1}$$

This can be checked using Itô's differential rule.

Equation (4.1) implies

$$\ln(S(t)) - \ln(S) + \tfrac{1}{2}\sigma^2 t = \int_0^t a(s)ds + \sigma W(t). \tag{4.2}$$

This implies observing the process $S(t)$ is equivalent to observing the process $y(t)$, where

$$y(t) = \int_0^t a(s)ds + \sigma W(t), \tag{4.3}$$

and that the σ-fields

$$\sigma\{S(r):\ 0 \le r \le t\} \text{ and } \sigma\{y(r):\ 0 \le r \le t\} \tag{4.4}$$

are equal.

Our assumptions about $a(t)$ imply it is a jump Markov process with two states a and b which jumps once from a to b.

Nonlinear filtering results, for instance Davis and Markus [4], Lipster and Shiryayev [8], govern conditional probabilities of states of a jump Markov process given measurements of the type (4.3). In particular, if

$$x(t) = Pr[a(t) = a | y(r),\ 0 \le r \le t], \tag{4.5}$$

then $x(t)$ is a solution of

$$dx(t) = -cx(t)dt + \frac{1}{\sigma}(a-b)(1-x(t))\ x(t)d\nu(t),\ \ x(0) = x. \tag{4.6}$$

In (4.6) $\nu(t)$ is a Wiener process called the innovations process. It satisfies

$$d\nu(t) = \frac{1}{\sigma}(a(t) - (ax(t) + b(1-x(t))))dt + dW(t). \tag{4.7}$$

The relationship (4.7) between the innovations Wiener process $\nu(t)$ and the original Wiener process $W(t)$ implies

$$a(t)dt + \sigma dW(t) = (ax(t) + b(1-x(t)))dt + \sigma d\nu(t). \tag{4.8}$$

Thus we may rewrite the stock price equation (3.1) in terms of the conditional probabilities $x(t)$ and the innovations Wiener process $\nu(t)$ as

$$dS(t) = S(t)[(ax(t) + b(1-x(t)))dt + \sigma d\nu(t)], \quad S(0) = S. \tag{4.9}$$

Since

$$\mathcal{F}(t) = \sigma\{S(r):\ 0 \le r \le t\} = \sigma\{y(r):\ 0 \le r \le t\}, \tag{4.10}$$

and $x(t)$ is defined by (4.5), $x(t)$ is $\mathcal{F}_t$ adapted. Thus the optimization problem given in (3.1)-(3.4) is equivalent to the completely observed optimization problem of choosing the $\mathcal{F}_t$ stopping time τ in our class $\mathcal{A}$ of stopping times to maximize

$$E[U(S(\tau)] \tag{4.11}$$

subject to

$$\begin{aligned} &dS(t) = S(t)[(ax(t) + b(1 - x(t)))dt + \sigma d\nu(t)],\ S(0) = S, \\ &dx(t) = -cx(t)dt + \tfrac{1}{\sigma}(a - b)(1 - x(t))x(t)d\nu(t),\ x(0) = x. \end{aligned} \tag{4.12}$$

Lemma 4.1. *For q in $(0, 1)$, let*

$$T(q) = \text{ first time } x(t) \text{ hits } [0, q]. \tag{4.13}$$

Then

$$E[T(q)] < \infty. \tag{4.14}$$

We give the proof of Lemma 4.1 in the appendix. We remark that (4.14) is false if the interval is replaced by $[q, 1]$.

5. The Logarithmic Utility Problem. Consider the problem (4.11)-(4.12) with utility function

$$U(S) = \ln(S) \tag{5.1}$$

and class of admissible $\mathcal{F}_t$ stopping times τ given by

$$\mathcal{A} = \{\tau :\ E(\tau) < \infty\}. \tag{5.2}$$

For this problem in the notation of Theorem 2.1

$$z = \begin{pmatrix} S \\ x \end{pmatrix},\quad \beta(z) = 0,\quad h(z) = \begin{pmatrix} S\sigma \\ \frac{1}{\sigma}(a - b)x(1 - x) \end{pmatrix}. \tag{5.3}$$

The region R is

$$R = \{(S, x) :\ 0 \le S < \infty,\quad 0 \le x \le 1\}. \tag{5.4}$$

The operator $A[V](S, x)$ is

$$\begin{aligned} A[V](S, x) = {} & S(ax + b(1 - x))V_s(S, x) - cxV_x(S, x) \\ & + \frac{1}{2}S^2\sigma^2 V_{ss}(S, x) + (a - b)(1 - x)xSV_{sx}(S, x) \\ & + \frac{1}{2}\left(\frac{a - b}{\sigma}\right)^2 (1 - x)^2x^2V_{xx}(S, x), \end{aligned} \tag{5.5}$$

and

$$V_z(z)h(z) = S\sigma V_s(S, x) + \frac{a - b}{\sigma}(1 - x)xV_x(S, x), \tag{5.6}$$

where z and (S,x) are related by (5.3).

Let us begin the discussion by asking if there are conditions for which it is optimal to always sell the stock immediately. Since $a(t)$ takes on values a and b and b is negative, $a(t) \le a$. Thus (4.2) implies

$$E[\ln(S(t))] \le \ln(S) + (a - \frac{1}{2}\sigma^2)t. \tag{5.7}$$

If $2a \le \sigma^2$, then $E[\ln(S(t))] \le \ln(S)$, so $\tau = 0$ is the optimal selling time.

Let us assume $2a > \sigma^2$ and find optimal stopping times in the class $\mathcal{A}$ under this condition. We shall see that under this condition for an appropriate $f(x)$

$$V(S,x) = \ln(S) + f(x) \tag{5.8}$$

gives a solution of Theorem 2.1.

Using (5.5)

$$\begin{aligned} &A[\ln(S) + f(x)](S,x) \\ &= ax + b(1-x) - cxf'(x) - \frac{1}{2}\sigma^2 + \frac{1}{2}\left(\frac{a-b}{\sigma}\right)^2 (1-x)^2 x^2 f''(x). \end{aligned} \tag{5.9}$$

Calling the right hand side of (5.9) $B[f](x)$, the variational inequality (2.5) reduces to

$$B[f](x) \le 0, \;\; f(x) \ge 0, \;\text{ and } f(x)B[f](x) = 0. \tag{5.10}$$

We also see that

$$S\sigma V_s(S,x) + \frac{a-b}{\sigma}(1-x)xV_x(S,x) = \sigma + \frac{a-b}{\sigma}(1-x)xf'(x). \tag{5.11}$$

Since $0 \le x \le 1$, this will be bounded if $f'(x)$ is bounded. Thus (2.10) of Remark 2.2 will be satisfied if $f'(x)$ is bounded.

For Itô's differential rule to hold for $\ln(S(t)) + f(x(t))$, $f(x)$ must be at least once continuously differentiable. This continuous differentiability implies if $q \in (0,1)$, and is a boundary point of an interval on which $f(x) = 0$, that $f'(q) = 0$.

Thus we look for a continuously differentiable solution $f(x)$ of (5.10) for which $f'(x)$ is bounded and $f'(q) = 0$ at boundary points q of intervals on which $f(x) = 0$.

Conditions (5.10) imply that if $f(x) \ne 0$, that $B[f](x) = 0$. From (5.9) this equation is

$$ax + b(1-x) - cxf'(x) - \frac{1}{2}\sigma^2 + \frac{1}{2}\left(\frac{a-b}{\sigma}\right)^2 (1-x)^2 x^2 f''(x) = 0. \tag{5.12}$$

Since we are interested in conditions on $f'(x)$ we will set $r(x) = f'(x)$ in (5.12) and solve

$$ax + b(1-x) - cxr(x) - \frac{1}{2}\sigma^2 + \frac{1}{2}\left(\frac{a-b}{\sigma}\right)^2 x^2(1-x)^2 r'(x) = 0 \tag{5.13}$$

for $r(x)$. The general solution of (5.13) is given by

$$\begin{aligned} r(x) &= (1-x)^{-h} x^h e^{\frac{h}{1-x}} \\ &\cdot \left[\int \frac{h}{2c} x^{-(2+h)} (1-x)^{(h-2)} e^{\frac{-h}{1-x}} (\sigma^2 - 2b - 2(a-b)x)dx + C\right] \end{aligned} \tag{5.14}$$

in which C is an arbitrary constant and h is given by

$$h = \frac{2c\sigma^2}{(a-b)^2}. \tag{5.15}$$

Since $h > 0$,

$$\lim_{x \to 1} (1-x)^{-h} x^h e^{h/(1-x)} = +\infty. \tag{5.16}$$

Thus for $r(x)$ to be bounded at $x = 1$ we must have

$$\lim_{x \to 1} \left[\int \frac{h}{2c} x^{-(2+h)} (1-x)^{(h-2)} e^{\frac{-h}{1-x}} (\sigma^2 - 2b - 2(a-b)x)\, dx + C \right] = 0. \tag{5.17}$$

Define $g(x)$ by

$$g(x) = \int_x^1 \frac{-h}{2c} y^{-(2+h)} (1-y)^{(h-2)} e^{-h/(1-y)} (\sigma^2 - 2b - 2(a-b)y)\, dy. \tag{5.18}$$

For (5.17) to hold the quantity in brackets in (5.17) must equal $g(x)$. This follows because both quantities have the same derivative and the same limit at $x = 1$. Thus let

$$r(x) = (1-x)^{-h} x^h e^{h/(1-x)} g(x). \tag{5.19}$$

A calculation using L'Hospital's rule shows that

$$\lim_{x \to 1} r(x) \;=\; \frac{2a - \sigma^2}{2c}. \tag{5.20}$$

LEMMA 5.1. *$r(x)$ has a unique root x^* in $(0,1)$ which satisfies*

$$0 < x^* < \frac{\sigma^2 - 2b}{2(a-b)} \tag{5.21}$$

and $r(x)$ is positive on $(x^*, 1]$.

Proof. Since on the interval $(0,1)$ the function $r(x)$ is given by positive quantities times $g(x)$ its roots will be the same as those of $g(x)$. The integrand of $g(x)$ is given by positive quantities times the linear term

$$2(a-b)y - \sigma^2 + 2b. \tag{5.22}$$

Since $2a > \sigma^2$, (5.22) is positive near $y = 1$. Since $b < 0$ (5.22) is clearly negative near $y = 0$.

Now (5.22) will be positive on

$$\frac{\sigma^2 - 2b}{2a - 2b} < y \le 1. \tag{5.23}$$

Thus

$$g(x) > 0 \quad \text{if} \quad \frac{\sigma^2 - 2b}{2a - 2b} < x < 1. \tag{5.24}$$

The term $y^{-(2+h)}$ with $h > 0$ in the integrand of $g(x)$ and the negativity of (5.22) near $y = 0$ imply

$$\lim_{x\to 0} g(x) = -\infty. \tag{5.25}$$

Thus $g(x)$ and hence $r(x)$ must have a root x^* in $\left(0, \frac{\sigma^2-2b}{2a-2b}\right)$. Since $g(x)$ is monotone increasing on this interval, x^* is unique. This implies $r(x)$ is positive on $(x^*, 1]$. ☐

THEOREM 5.2. *For $f(x)$ defined by*

$$f(x) = \begin{cases} 0 & \text{if } 0 \le x \le x^* \\ \displaystyle\int_{x^*}^{x} r(x)\,dx & \text{if } x^* \le x \le 1, \end{cases} \tag{5.26}$$

the function $V(S, x) = \ln(S) + f(x)$ satisfies the conditions of Theorem 2.1 and

$$T(x^*) = \text{1st time } x(t) \text{ hits } [0, x^*] \tag{5.27}$$

is an optimal stopping time in the class $\mathcal{A}$.

Proof. Since $r(x^*) = 0$, $f(x)$ is continuously differentiable and is twice continuously differentiable except at x^*. Thus Itô's differential rule holds for $\ln(S) + f(x)$. Lemma 5.1 implies that $f(x) \ge 0$. Using (5.9)

$$B[f](x) = \begin{cases} ax + b(1-x) - \frac{1}{2}\sigma^2 & \text{if } 0 \le x \le x^* \\ 0 & \text{if } x^* \le x \le 1, \end{cases} \tag{5.28}$$

and since

$$x^* < \frac{\sigma^2 - 2b}{2(a-b)}, \tag{5.29}$$

we have that

$$B[f](x) \le 0. \tag{5.30}$$

Since $f(x) = 0$ if $0 \le x \le x^*$ and $B[f](x) = 0$ if $x^* < x \le 1$, $f(x)B[f](x) = 0$. Thus all the conditions of (5.10) are satisfied. The conditions (5.10) imply the conditions (2.5) for the function (5.8).

Since $r(x)$ is continuous on $[0, 1)$ and has a finite limit at $x = 1$, it is bounded. Thus $f'(x)$ is bounded, which together with (5.11) and Remark2.2 implies condition (2.6) holds for stopping times in $\mathcal{A}$. Condition (2.8) follows from Lemma 4.1. Thus the conditions of Theorem 2.1 are satisfied and $T(x^*)$ given by (5.27) is an optimal stopping time in the class $\mathcal{A}$. ☐

6. The Problem With Utility Function $U(S) = S$. Consider the problem of maximizing

$$E[S(\tau)], \tag{6.1}$$

subject to the equations (4.12) holding, over a class of stopping times $\mathcal{A}$. As mentioned in the introduction we shall use changes of probability measure to convert this problem into one for which we can verify the conditions of Theorem 2.1.

The solution of the stock price equation is given by

$$S(t) = Se^{\int_0^t (ax(r)+b(1-x(r)))\,dr+\sigma\nu(t)-\frac{\sigma^2}{2}t}. \tag{6.2}$$

Again this can be checked by using Itô's differential rule. Let $\tilde{P}$ be a probability measure which is locally absolutely continuous with respect to P through: For each t and $A \in \mathcal{F}_t$

$$\tilde{P}(A) = \int_A e^{\sigma\nu(t)-\frac{\sigma^2}{2}t}\,dP. \tag{6.3}$$

Let $\tilde{E}$ denote taking expectation with respect to $\tilde{P}$. It follows from Theorem 3.4, p. 153 of [6] that for each finite stopping time τ, $\tilde{P}$ is absolutely continuous with respect to P on $\mathcal{F}_\tau$, and for each $A \in \mathcal{F}_\tau$

$$\tilde{P}(A) = \int_A e^{\sigma\nu(\tau)-\frac{\sigma^2}{2}\tau}\,dP. \tag{6.4}$$

Thus since (6.2) also holds with t replaced by a finite stopping time τ, we have

$$E[S(\tau)] = \tilde{E}[Se^{\int_0^\tau ((ax(r)+b(1-x(r)))\,dr}]. \tag{6.5}$$

Now Girsanov's theorem implies that under $\tilde{P}$, $\tilde{\nu}(t)$ defined by

$$\tilde{\nu}(t) = \nu(t) - \sigma t \tag{6.6}$$

is a Wiener process. Thus $x(t)$ is a solution of $x(0) = x$ and

$$dx(t) = (-cx(t) + (a-b)(1-x(t))x(t))\,dt + \frac{a-b}{\sigma}(1-x(t))\,x(t)d\tilde{\nu}(t). \tag{6.7}$$

Thus our original problem is equivalent to the problem: Find a stopping time τ in $\mathcal{A}$ to maximize

$$\tilde{E}[Se^{\int_0^\tau [ax(r)+b(1-x(r)))\,dr}] \tag{6.8}$$

where $x(t)$ is the solution of (6.7).

We shall convert this problem into yet another equivalent problem. Notice that the solution of (6.7) satisfies the integral equation

$$x(t) = xe^{\int_0^t (-c+(a-b)(1-x(r))-\frac{1}{2}(\frac{a-b}{\sigma})^2(1-x(r))^2)\,dr+\int_0^t \frac{a-b}{\sigma}(1-x(r))\,d\tilde{\nu}(r)}. \tag{6.9}$$

Again this may be checked using Itô's differential rule. Formula (6.9) may be rearranged to give

$$\begin{aligned} &e^{\int_0^t (ax(r)+b(1-x(r)))\,dr} \\ &= \frac{xe^{(a-c)t}}{x(t)}\,e^{\int_0^t \frac{a-b}{\sigma}(1-x(r))\,d\tilde{\nu}(r)-\int_0^t \frac{1}{2}(\frac{a-b}{\sigma})^2(1-x(r))^2\,dr}. \end{aligned} \tag{6.10}$$

This suggests making the change of measures defined through, for each t and $A \in \mathcal{F}_t$

$$\tilde{\tilde{P}}(A) = \int_A e^{\int_0^t \frac{a-b}{\sigma}(1-x(r))\,d\tilde{\nu}(r)-\int_0^t \frac{1}{2}(\frac{a-b}{\sigma})^2(1-x(r))^2\,dr}\,d\tilde{P}. \tag{6.11}$$

Then setting $y(t) = \frac{1}{x(t)}$, and again using Theorem 3.4 of [6], denoting expectation with respect to $\tilde{\tilde{P}}$ by $\tilde{\tilde{E}}$, (6.10) with t replaced by a finite stopping time τ implies

$$E[S(\tau)] = \tilde{E}[Se^{\int_0^\tau (ax(r)+b(1-x(r)))\,dr}] = \tilde{\tilde{E}}[Sxe^{(a-c)\tau}\, y(\tau)]. \tag{6.12}$$

Girsanov's Theorem implies that under $\tilde{\tilde{P}}$

$$\tilde{\tilde{\nu}}(t) = \tilde{\nu}(t) - \int_0^t \frac{a-b}{\sigma}(1-x(r))\,dr \tag{6.13}$$

is a Wiener process.

A calculation using Itô's rule, (6.7) and (6.13) gives that $y(t)$ is the solution of

$$dy(t) = ((c-a+b)y(t) + a - b)\,dt + \frac{a-b}{\sigma}(y(t)-1)\,d\tilde{\tilde{\nu}}(t),\ y(0) = \frac{1}{x}. \tag{6.14}$$

Thus we arrive at the problem choose $\tau \in \mathcal{A}$ to maximize

$$\tilde{\tilde{E}}[Sxe^{(a-c)\tau} y(\tau)] \tag{6.15}$$

where $y(t)$ is the solution of (6.14).

7. Solution of the Third Equivalent Problem. Since S and x are positive constants we consider the problem of chosing $\tau \in \mathcal{A}$ to maximize

$$\tilde{\tilde{E}}[e^{(a-c)\tau}\, y(\tau)] \tag{7.1}$$

where $y(t)$ is the solution of

$$dy(t) = ((c-a+b)y(t) + a - b)\,dt + (\frac{a-b}{\sigma})(y(t)-1)\,d\tilde{\tilde{\nu}}(t),\ y(0) = \frac{1}{x}. \tag{7.2}$$

Define

$$T(q) = \text{ first time } y(t) \in [q, \infty). \tag{7.3}$$

In order to utilize Theorem 2.1 we shall first consider maximizing (7.1) over the set of stopping times $\mathcal{A}$ defined by

$$\mathcal{A} = \{\tau :\ \tau \leq T(q) \text{ for some } q\} \tag{7.4}$$

and then show an optimal stopping time in $\mathcal{A}$ is also optimal in the class $\mathcal{B}$ of all finite stopping times.

Notice that if $a \geq c$ and q_n is a sequence approaching infinity that

$$\tilde{\tilde{E}}[e^{(a-c)T(q_n)} y(T(q_n))] \geq q_n. \tag{7.5}$$

This leads to the very counterintuitive conclusion that no matter how negative the rate of decline b is, if $a \geq c$ arbitrarily large expected returns can be obtained by using the stopping times $T(q_n)$.

Let us assume $a < c$ and solve the problem given in terms of (7.1), (7.2) and (7.4). The quantity $A[V](y)$ for this problem is given by

$$\begin{aligned} A[V](y) = {} & (a-c)V(y) + ((c-a+b)y + (a-b))V'(y) \\ & + \frac{1}{2}\left(\frac{a-b}{\sigma}\right)^2 (y-1)^2 V''(y). \end{aligned} \tag{7.6}$$

The general solution of $A[V](y)=0$ is given by

$$(7.7)\ \ V(y)=C_1\,W_{k,m}(\frac{h}{y-1})e^{\frac{h}{2(y-1)}}(y-1)^k+C_2\,M_{k,m}(\frac{h}{y-1})e^{\frac{h}{2(y-1)}}(y-1)^k,$$

where

$$(7.8)\quad k=1+\frac{\sigma^2(a-b-c)}{(a-b)^2},\quad h=\frac{2c\sigma^2}{(a-b)^2},\quad m=\left[\left(k-\frac{1}{2}\right)^2+h\frac{(c-a)}{c}\right]^{\frac{1}{2}},$$

and $W_{k,m}(\cdot)$ and $M_{k,m}(\cdot)$ are the classical Whittaker functions.

LEMMA 7.1. *The condition $c>a$ implies the conditions (a) and (b) hold where*

$$(a)\quad m+k-\frac{3}{2}>0 \qquad\qquad (b)\quad \frac{h}{(m-k+\frac{1}{2})(m+k-\frac{1}{2})}>1.$$

The proof of Lemma 7.1 is given in the appendix.

Let

$$(7.9)\qquad g(y)=W_{k,m}(\frac{h}{y-1)})e^{\frac{h}{2(y-1)}}(y-1)^k$$

and let us see if we can construct a solution of the conditions of Theorem 2.1 from the function $C_1g(y)$.

The $W_{k,m}(\cdot)$ function, Spain [11] p. 122, has integral representation given by

$$(7.10)\qquad W_{k,m}\left(\frac{h}{y-1}\right)=\left[\int_0^\infty e^{-x}x^{m-k-\frac{1}{2}}\left(1+\frac{x(y-1)}{h}\right)^{m+k-\frac{1}{2}}dx\right]\cdot\left[\frac{1}{\Gamma(m-k+\frac{1}{2})}\left(\frac{h}{y-1}\right)^k e^{\frac{-h}{2(y-1)}}\right].$$

Thus $g(y)$ given by (7.9) has the representation

$$(7.11)\qquad g(y)=\frac{h^k}{\Gamma(m-k+\frac{1}{2})}\int_0^\infty e^{-x}x^{m-k-\frac{1}{2}}(1+\frac{x(y-1)}{h})^{m+k-\frac{1}{2}}dx.$$

Differentiating (7.11) under the integral sign gives

$$(7.12)\quad g'(y)=\frac{h^{k-1}(m+k-\frac{1}{2})}{\Gamma(m-k+\frac{1}{2})}\int_0^\infty e^{-x}x^{m-k+\frac{1}{2}}(1+\frac{x(y-1)}{h})^{m+k-\frac{3}{2}}\,dx$$

and

$$(7.13)\qquad g''(y)=\left[\int_0^\infty e^{-x}x^{m-k+\frac{3}{2}}\left(1+\frac{x(y-1)}{h}\right)^{m+k-\frac{5}{2}}dx\right]\cdot\left[\frac{(m+k-\frac{1}{2})(m+k-\frac{3}{2})h^{k-2}}{\Gamma(m-k+\frac{1}{2})}\right].$$

It follows from (a) of Lemma 7.1 that for $1\le y<\infty$ (7.12) and (7.13) are greater than 0. This implies $g(y)$ is increasing and strictly convex on $1\le y<\infty$.

In order for

$$V(y) = \begin{cases} C_1 g(y) & \text{if } 1 \le y < y^* \\ y & \text{if } y^* \le y < \infty \end{cases} \tag{7.14}$$

to satisfy the conditions of Theorem 2.1 it must be at least once continuously differentiable. For this to happen C_1 and y^* must be a solution of

$$C_1 g(y^*) = y^*, \quad C_1 g'(y^*) = 1. \tag{7.15}$$

If y^* is a solution of

$$\frac{g(y)}{g'(y)} = y, \tag{7.16}$$

(7.15) will hold with $C_1 = \dfrac{1}{g'(y^*)}$.

Lemma 7.2. *There is a solution $y^* > 1$ of (7.16).*

Proof.

$$\frac{g(y)}{g'(y)} = \frac{h}{m+k-\frac{1}{2}} \frac{\displaystyle\int_0^\infty e^{-x} x^{m-k-\frac{1}{2}} (1 + \frac{x(y-1)}{h})^{m+k-\frac{1}{2}}\, dx}{\displaystyle\int_0^\infty e^{-x} x^{m-k+\frac{1}{2}} (1 + \frac{x(y-1)}{h})^{m+k-\frac{3}{2}}\, dx}. \tag{7.17}$$

Thus

$$\frac{g(1)}{g'(1)} = \frac{h}{m+k-\frac{1}{2}} \frac{\displaystyle\int_0^\infty e^{-x} x^{m-k-\frac{1}{2}}\, dx}{\displaystyle\int_0^\infty e^{-x} x^{m-k+\frac{1}{2}}\, dx}. \tag{7.18}$$

An integration by parts gives

$$\int_0^\infty e^{-x} x^{m-k+\frac{1}{2}}\, dx = \left(m-k+\frac{1}{2}\right) \int_0^\infty e^{-x} x^{m-k-\frac{1}{2}}\, dx, \tag{7.19}$$

so by (7.18) and (b) of Lemma 7.1

$$\frac{g(1)}{g'(1)} = \frac{h}{(m-k+\frac{1}{2})(m+k-\frac{1}{2})} > 1. \tag{7.20}$$

Dividing both numerator and denominator of (7.17) by $(y-1)^{(m+k-\frac{1}{2})}$ gives

$$\frac{g(y)}{g'(y)} = \frac{h(y-1)}{m+k-\frac{1}{2}} \frac{\int_0^\infty e^{-x} x^{m-k-\frac{1}{2}} \, (\frac{1}{(y-1)} + \frac{x}{h})^{m+k-\frac{1}{2}}\, dx}{\int_0^\infty e^{-x} x^{m-k+\frac{1}{2}} \, (\frac{1}{(y-1)} + \frac{x}{h})^{m+k-\frac{3}{2}}\, dx}. \tag{7.21}$$

Now

$$\lim_{y\to\infty} \frac{\int_0^\infty e^{-x} x^{m-k-\frac{1}{2}} (\frac{1}{y-1} + \frac{x}{h})^{m+k-\frac{1}{2}}\, dx}{\int_0^\infty e^{-x} x^{m-k+\frac{1}{2}} (\frac{1}{y-1} + \frac{x}{h})^{m+k-\frac{3}{2}}\, dx} = \frac{1}{h}. \tag{7.22}$$

Thus asymptotically as $y \to \infty$

$$\frac{g(y)}{g'(y)} \sim \frac{y-1}{m+k-\frac{1}{2}}\,. \tag{7.23}$$

Since $m+k-\frac{1}{2} > 1$, $\dfrac{g(y)}{g'(y)}$ asymptotically has slope less than one. Since its value at one is bigger than one, and it has asymptotic slope less than one, $\dfrac{g(y)}{g'(y)}$ must cross the line given by y at some $y^* > 1$. □

Theorem 7.3. *For y^* the solution of (7.16), the function*

$$V(y) = \begin{cases} \frac{1}{g'(y^*)} g(y) & \text{if } 1 \le y \le y^* \\ y & \text{if } y^* \le y < \infty \end{cases} \tag{7.24}$$

satisfies the conditions of Theorem 2.1 for the problem given by (7.1), (7.2) and (7.4). The stopping time $T(y^*)$ is an optimal stopping time in the class $\mathcal{A}$.

Proof. Since $g'(y) > 0$ and $g(y)$ is strictly convex, the conditions

$$\frac{1}{g'(y^*)}\, g(y^*) = y^*, \quad \frac{1}{g'(y^*)}\, g'(y^*) = 1 \tag{7.25}$$

imply y is a line of support of $\frac{1}{g'(y^*)}\, g(y)$ at y^*. Thus

$$\frac{1}{g'(y^*)}\, g(y) \ge y \tag{7.26}$$

which implies

$$V(y) \ge y. \tag{7.27}$$

Since $A[\frac{1}{g'(y^*)}\, g(\cdot)](y) = 0$, for $V(y)$ given by (7.24),

$$A[V](y) = \begin{cases} 0 & \text{if } 1 \le y < y^* \\ by + (a-b) & \text{if } y^* < y < \infty. \end{cases} \tag{7.28}$$

Evaluating $A[\frac{1}{g'(y^*)}\, g(\cdot)](y) = 0$ at y^*, using (7.6) and the conditions (7.25), gives

$$(a-c)y^* + (cy^* - (a-b)(y^*-1)) + \frac{1}{2}(\frac{a-b}{\sigma})^2 (y^*-1)^2\, \frac{g''(y^*)}{g'(y^*)} = 0. \tag{7.29}$$

Since $g(y)$ is increasing and strictly convex the last term in (7.29) is positive. Thus (7.29) implies

$$by^* + a - b < 0. \tag{7.30}$$

Since b is negative

$$by + (a-b) < 0, \quad \text{if } \ y^* \le y < \infty, \tag{7.31}$$

or, from (7.28), that $A[V](y) \le 0$ for $V(y)$ given by (7.24). That $(V(y) - y)A[V](y) = 0$ follows immediately from (7.24) and (7.28).

To verify the condition (2.6) we see

$$V'(y) = \begin{cases} \dfrac{1}{g'(y^*)}\, g'(y) & \text{if } 1 \le y \le y^* \\ 1 & \text{if } y^* < y < \infty. \end{cases} \tag{7.32}$$

From (7.12) and (a) of Lemma 7.1, $g'(1) > 0$. Since $g'(y)$ is increasing (7.32) implies

$$0 \le V'(y) \le 1. \tag{7.33}$$

The quantity in (2.6) for this problem is

$$\tilde{\tilde{E}}[\int_0^\tau e^{2(a-c)t}(V^{'}(y(t))^2(\frac{a-b}{\sigma})^2(y(t)-1))^2\, dt. \tag{7.34}$$

Since $\tau \le T(q)$ for some q and (7.33) holds, (7.34) is bounded by

$$\int_0^\infty e^{2(a-c)t}(\frac{a-b}{\sigma})^2(q-1)^2\, dt = \frac{(q-1)^2}{2(c-a)}(\frac{a-b}{\sigma})^2. \tag{7.35}$$

Thus condition (2.6) holds. Condition (2.8) follows from Lemma 4.1 and the defintion of $T(y^*)$. Thus the conditions of Theorem 2.1 are satisfied and $T(y^*)$ is an optimal stopping time in the class $\mathcal{A}$. ▯

THEOREM 7.4. *The stopping time $T(y^*)$ is optimal in the class $\mathcal{B}$ of all finite stopping times.*

Proof. If τ is a finite stopping time and q_n is a sequence approaching ∞, satisfying $q_n > 1/x$, the definition of $T(q_n)$ implies

$$y(\tau \wedge T(q_n)) = \begin{cases} y(\tau) \text{ if } \tau < T(q_n) \\ q_n \text{ if } \tau \ge T(q_n). \end{cases} \tag{7.36}$$

It can be shown that $\lim_{n\to\infty} T(q_n) = +\infty$ with probability one.

Hence

$$\lim_{n\to\infty} e^{(a-c)\tau\wedge T(q_n)} y(\tau \wedge T(q_n)) = e^{(a-c)\tau} y(\tau). \tag{7.37}$$

Theorem 7.3 implies

$$\tilde{\tilde{E}}[e^{(a-c)\tau\wedge T(q_n)} y(\tau \wedge T(q_n))] \le \tilde{\tilde{E}}[e^{(a-c)T(y^*)} y(T(y^*))]. \tag{7.38}$$

Fatou's Lemma and (7.37) imply

$$\tilde{\tilde{E}}[e^{(a-c)\tau} y(\tau)] \le \liminf_{n\to\infty} \tilde{\tilde{E}}[e^{(a-c)\tau\wedge T(q_n)} y(\tau \wedge T(q_n))]. \tag{7.39}$$

Thus (7.38) and (7.39) imply

$$\tilde{\tilde{E}}[e^{(a-c)\tau} y(\tau)] \le \tilde{\tilde{E}}[e^{(a-c)T(y^*)} y(T(y^*))] \tag{7.40}$$

which gives the optimality of $T(y^*)$ in the class $\mathcal{B}$ of all finite stopping times. ▯

8. Computations. In Table 8.2 we give results of computing optimal selling regions for a variety of conditions. To explain the coefficients used in Table 8.2, notice that equation (4.1) implies if the growth rate $a(t)$ has the constant value a the expected stock price is

$$E[S(t)] = Se^{at} \tag{8.1}$$

and similarly if $a(t)$ has the constant value b

$$E[S(t)] = Se^{bt}. \tag{8.2}$$

The value $a = .1$ gives $e^{.1} \approx 1.105$, or a growth rate of approximately 10 % per year. The value $b = -.92$ gives $e^{-.92} \approx .398$, or a rate of decline of approximately 60 % per year. Table 8.2 was computed in terms of values of a, b, c, σ but is expressed in terms of approximate percents of growth and decline to aid intuitive understanding of the results. The relationships between the values of a and b used in the computation of Table 8.2 and the approximate percents of growth and decline per year are listed in Table 8.1.

Table 8.1
Correspondence of parameter values and approximate percents of growth and decline per year

a	.1	.18	b	-.92	-1.6
% growth	10	20	% decline	60	80

Except in the last case in Table 8.2 the mean of the jump time was taken to be two years or $c = \frac{1}{2}$. In literature using actual stock data, values for the stock variance σ near .3 or .4 are common. We use those and also the value $\sigma = .5$ to illustrate a case with large variance.

Table 8.2
Optimal selling regions for both criteria and different parameter values

Problem Coefficients				$U(S) = \ln(S)$	$U(S) = S$
growth	decline	mean years	σ	selling region	selling region
10%	60%	2	.3	$0 \le x \le .911$	$0 \le x \le .766$
10%	60%	2	.4	$0 \le x \le .978$	$0 \le x \le .831$
10%	60%	2	.5	$0 \le x \le 1.00$	$0 \le x \le .857$
10%	80%	2	.3	$0 \le x \le .927$	$0 \le x \le .760$
10%	80%	2	.4	$0 \le x \le .986$	$0 \le x \le .856$
10%	80%	2	.5	$0 \le x \le 1.00$	$0 \le x \le .891$
20%	60%	2	.3	$0 \le x \le .678$	$0 \le x \le .441$
20%	60%	2	.4	$0 \le x \le .849$	$0 \le x \le .601$
20%	60%	2	.5	$0 \le x \le .939$	$0 \le x \le .685$
20%	80%	2	.3	$0 \le x \le .651$	$0 \le x \le .368$
20%	80%	2	.4	$0 \le x \le .913$	$0 \le x \le .587$
20%	80%	2	.5	$0 \le x \le .953$	$0 \le x \le .709$
20%	80%	6	.4	$0 \le x \le .659$	For the sequence of selling regions $0 \le x \le a_n$ where $a_n \downarrow 0$, the problem is unbounded

This table shows the optimal stock selling regions for $U(S) = \ln(S)$ and $U(S) = S$ are quite different. The optimal selling regions for $\ln(S)$ are uniformly longer than those of S. In some cases these differences are large. In the 10 % growth, $\sigma = .5$ cases

$$2a = .2 \quad < \quad .25 = \sigma^2,$$

so selling immediately always happens for the $\ln(S)$ criteria, while for the S criteria selling does not occur immediately if x is large enough. If the growth rate is approximately 20 % for an exponentially distributed length of time with mean six years

$$a = .18 > \frac{1}{6} = c.$$

This implies there is a sequence of values of x approaching zero, so that stopping at these values gives unbounded expected return for the criteria S. For this case the criteria $\ln(S)$ is finite and its optimal selling region is $0 \le x \le .659$.

9. Comparison with Reference [12] by Q. Zhang and Reference [3] by Beibel and Lerche. In [12] Q. Zhang considers a stock selling problem in which the stock price is given by a diffusion process with unobserved drift and diffusion coefficients given by jump Markov processes. This model is more general than that of the current paper whose drift coefficient is a one jump Markov process and whose diffusion coefficient is constant. However the classes of stopping times optimized over are different. In [12] the stopping times are first times the stock price leaves a given interval. In the present paper they are stopping times adapted to the past measurements of the stock price. These include the first times the stock price leaves an interval so there should be a larger optimal expected return in the present case. There is an advantage to using the stopping times of [12] in that they are easier to implement.

The difference between the two problems is especially clear in the case in which the initial probability x of being in the increasing state a is less than one. (Some people assert this is “always” the case by the time a small investor invests in a stock.) The stopping times of [12] do not depend on x. If x is small enough to be in the optimal selling region for the current problem, the investor should not buy the stock. If he is using stopping times of [12], he holds the stock until its price leaves an interval.

In [3] Beibel and Lerche consider an optimal stock selling problem with utility function $U(S) = S$ which is roughly the same as our second equivalent problem. Their methods applied to this problem would be roughly the following. Let $x(t)$ be the solution of (6.7) with initial condition $x(0) = 1$. Let $0 < x < 1$ and

$$T(x) = \text{ first time } x(t) = x. \tag{9.1}$$

Then using elegant probabilistic arguments they show

$$E[e^{\int_0^{T(x)} a(x(r))+b(1-x(r))\,dr}] \tag{9.2}$$

has a a maximum attained at some x^* and for the problem with $x(t)$ given by (6.7) with initial condition γ for which $x^* \le \gamma \le 1$ that

$$T(x^*) = \text{ first time } x(t) = x^* \tag{9.3}$$

maximizes the criteria (6.8).

However they do not carry out any computations. It appears that computing (9.2) would encounter similar difficulties to those we encountered in solving the variational inequalities for this case.

Appendix. In this appendix we provide the proofs of Theorem 2.1, Lemma 4.1 and Lemma 7.1.

Proof of Theorem 2.1. Itô's differential rule implies that

$$\begin{aligned}
\text{(A1)} \qquad de^{\int_0^t \beta(z(r))dr} V(z(t)) = {} & \beta(z(t)) e^{\int_0^t \beta(z(r))dr} V(z(t))dt \\
& + e^{\int_0^t \beta(z(r))dr} \Big[V_z(z(t)) g(z(t)) \\
& + \tfrac{1}{2} \operatorname{trace} \big(h(z(t)) h'(z(t)) V_{zz}(z(t)) \big) \Big] dt \\
& + e^{\int_0^t \beta(z(r))dr} V_z(z(t)) h(z(t)) dW(t)
\end{aligned}$$

or expressed in integrated form and using (2.4)

$$\begin{aligned}
\text{(A2)} \qquad e^{\int_0^t \beta(z(r))dr} V(z(t)) - V(z) = {} & \int_0^t e^{\int_0^s \beta(z(r))dr} A[V](z(s)) ds \\
& + \int_0^t e^{\int_0^s \beta(z(r))dr} V_z(z(s)) h(z(s)) dW(s).
\end{aligned}$$

Since (A2) holds sample function wise it also holds with t replaced by a finite stopping time τ.

It follows from Remark 1, p. 29 of Gihman and Skorohod (1972), that condition (2.6) implies for each stopping time τ in $\mathcal{A}$ that

$$\text{(A3)} \qquad E[\int_0^\tau e^{\int_0^s \beta(z(r))\, dr}\, V_z(z(s)) h(z(s))\, dW(s)] = 0.$$

Thus for each τ in $\mathcal{A}$

$$\text{(A4)} \qquad E[e^{\int_0^\tau \beta(z(r))\, dr}\, V(z(\tau))] = V(z) + E[\int_0^\tau e^{\int_0^s \beta(z(r))\, dr}\, A[V](z(s))\, ds].$$

Now from (2.5),

$$V(z) \geq U(z) \text{ and } A[V](z) \leq 0,$$

so for each τ in $\mathcal{A}$

$$\text{(A5)} \qquad E[e^{\int_0^\tau \beta(z(r))\, dr} U(z(\tau))] \leq V(z).$$

From (2.5) for

$$\tau(z) = \text{ first time } z(t) \text{ hits } \{y : \ U(y) = V(y)\}$$

it follows that

$$\text{(A6)} \qquad A[V](z(s)) = 0 \text{ on } 0 \leq s < \tau(z)$$

and

$$\text{(A7)} \qquad U(z(\tau(z))) = V(z(\tau(z))).$$

Thus (A4) implies

$$\text{(A8)} \qquad E[e^{\int_0^{\tau(z)} \beta(z(r))\, dr}\, U(z(\tau(z)))] = V(z).$$

Since condition (2.8) requires $\tau(z)$ to belong to $\mathcal{A}$, (A5) and (A8) imply $\tau(z)$ is an optimal stopping time in the class $\mathcal{A}$. $\square$

Proof of Lemma 4.1. We may assume $x > q$, for otherwise $T(q) = 0$ and $E[T(q)] = 0$. For $x(t)$ given by (4.8), from Theorem 2, p. 149 of [5], appropriately modified, it follows that for $t > 0$

$$P[x(t) < 1] = 1. \tag{A9}$$

A solution $K(x)$ of the differential equation

$$-cxK'(x) + \frac{1}{2}r^2(1-x)^2x^2K''(x) + 1 = 0 \tag{A10}$$

on $[0,1]$ satisfying $K(q) = 0$ and $K'(x)$ bounded on $[q,1]$ is given by

$$\begin{aligned} K(x) = \int_q^x dz \Bigg[& \frac{2}{r^2}\, e^{\frac{2c}{r^2(1-z)}}\, z^{\frac{2c}{r^2}}\, (1-z)^{-\frac{2c}{r^2}} \\ & \cdot \int_z^1 (1-y)^{\frac{-2(r^2-c)}{r^2}}\, y^{\frac{-2(r^2+c)}{r^2}}\, e^{\frac{-2c}{r^2(1-y)}}\, dy \Bigg]. \end{aligned} \tag{A11}$$

To see that $K'(x)$ is bounded, a L'Hospital's rule argument implies that

$$\lim_{x\to 1} K'(x) \; = \; \frac{r^4}{2c}, \tag{A12}$$

and from this and the form of $K'(x)$ boundedness follows. Notice that $K(x) \geq 0$ for $x \geq q$.

Set

$$T(q) = \inf\{t : x(t) = q\} \tag{A13}$$

and for a fixed time T greater than 0

$$\tau_T = \min(T, T(q)). \tag{A14}$$

From (A9) and (A13), it follows for $s < \tau_T$ that $x(s)$ is contained in $[q,1]$ over which $K(x)$ is defined. Itô's formula implies that

$$\begin{aligned} & K(x(\tau_T)) - K(x) \\ & \quad = \frac{a-b}{\sigma}\int_0^{\tau_T} (1-x(s))x(s)K'(x(s))\, dW(s) \\ & \quad\quad + \int_0^{\tau_T} \left[-cx(s)K'(x(s)) + \frac{1}{2}(\frac{a-b}{\sigma})^2(1-x(s))^2x(s)^2K''(x(s))\right] ds. \end{aligned} \tag{A15}$$

For $K(x)$ the solution of (A10) with $r = \frac{a-b}{\sigma}$, (A15) implies

$$K(x(\tau_T)) - K(x) = -\tau_T + \frac{a-b}{\sigma}\int_0^{\tau_T} (1-x(s))x(s)K'(x(s))\, dW(s). \tag{A16}$$

Since the integrand in the stochastic integral is bounded and τ_T is bounded the expected value of the stochastic integral is zero giving

$$E[\tau_T] = K(x) - E[K(x(\tau_T))]. \tag{A17}$$

Now τ_T increases monotonically to $T(q)$ as T approaches infinity. This and the positivity of $K(x(\tau_T))$ imply

$$E[T(q)] \leq K(x). \tag{A18}$$ □

Proof of Lemma 7.1. To show (a): Since $m \geq 0$, if $k - \frac{3}{2} > 0$ there is nothing to prove. Thus consider $k - \frac{3}{2} \leq 0$. A calculation using (7.8) shows

$$m^2 - (\frac{3}{2} - k)^2 = \frac{2\sigma^2(-b)}{(a-b)^2} > 0. \tag{A19}$$

Thus $m^2 > (\frac{3}{2} - k)^2$, and since both m and $\frac{3}{2} - k$ are ≥ 0,

$$m > \frac{3}{2} - k \quad \text{or} \quad m + k - \frac{3}{2} > 0, \tag{A20}$$

giving (a).

To show (b), using (7.8)

$$\begin{aligned} \frac{h}{(m - k + \frac{1}{2})(m + k - \frac{1}{2})} &= \frac{h}{m^2 - (k - \frac{1}{2})^2} \\ &= \frac{\frac{2c\sigma^2}{(a-b)^2}}{(k - \frac{1}{2})^2 + \frac{2\sigma^2(c-a)}{(a-b)^2} - (k - \frac{1}{2})^2} \\ &= \frac{c}{c-a} \ > \ 1. \end{aligned} \tag{A21}$$ □

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