Construction of the Value Function and Optimal Rules in Optimal Stopping of One-dimensional Diffusions

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Abstract. A new approach to the solution of optimal stopping problems for one-dimensional diffusions is developed. It arises by imbedding the stochastic problem in a linear programming problem over a space of measures. Optimizing over a smaller class of stopping rules provides a lower bound on the value of the original problem. Then using an auxiliary linear program, the weak duality of a restricted form of the dual linear program provides an upper bound on the value. An explicit formula for the reward earned using a two-point hitting time stopping rule allows one to prove strong duality between these problems and therefore allows one to either optimize over these simpler stopping rules or to solve the restricted dual program. Each optimization problem is parameterized by the initial value of the diffusion and thus one is able to construct the value function by solving the family of optimization problems. This methodology requires little regularity of the terminal reward function. When the reward function is smooth, the optimal stopping locations are shown to satisfy the smooth pasting principle. The procedure is illustrated on a number of examples.

Keywords. optimal stopping, linear programming, duality, non-linear optimization, value function, one-dimensional diffusion

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1 Introduction

This paper develops a new approach to the solution of optimal stopping problems for one-dimensional diffusions. By imbedding the optimal stopping problem in an infinite-dimensional linear program (LP) and examining an auxiliary LP along with its dual, a non-linear optimization problem and a semi-infinite linear program are derived from which the value function is able to be constructed for a wide class of reward functions.

Let x_l and x_r be constants with $-\infty \leq x_l < x_r \leq \infty$. We consider the one-dimensional diffusion X which satisfies the stochastic differential equation

$$dX(t) = \mu(X(t))\, dt + \sigma(X(t))\, dW(t), \qquad X(0) = x \in [x_l, x_r], \tag{1.1}$$

when $X(t) \in [x_l, x_r]$. (In the case $x_l = -\infty$, the interval $[x_l, x_r]$ is to be understood to be $(-\infty, x_r]$, and similarly when $x_r = \infty$.) We assume that all processes are defined on a probability space $(\Omega, \mathcal{F}, P)$ and denote by $\{\mathcal{F}_t\}$ the filtration generated by W. The generator A of X, given by

$$Af(y) = \mu(y)f'(y) + \frac{\sigma^2(y)}{2}\, f''(y),$$

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plays a central role in determining the solution.

We emphasize that throughout this paper, x is reserved to be the initial value of the diffusion. It will never be used as a dummy variable in any expression.

The objective is to select an $\{\mathcal{F}_t\}$-stopping time τ so as to maximize

$$J(\tau; x) := E\left[\int_0^\tau e^{-\alpha s} r(X(s))\, ds + e^{-\alpha\tau} g(X(\tau))\right]. \tag{1.2}$$

In this expression, $\alpha > 0$ denotes a discount rate, r is a running reward function and g represents the reward obtained at the terminal time. The need for a discount factor arises when the time frame for stopping is such that alternative investment possibilities affect the value of the reward earned. This, for example, will be the case for many perpetual options and for many applied problems such as the harvesting of renewable resources. For example, in forest harvesting, r might represent the amenity value or carbon credit of the forest while g would give the value derived from harvesting. We will impose further technical conditions in Sections 3 and 4 that will guarantee finiteness of the discounted reward and existence of optimal stopping rules.

Our definition of $\{\mathcal{F}_t\}$-stopping time follows that of Ethier and Kurtz [6, p. 51] and allows stopping times to take value ∞. Peskir and Shiryaev [13] refer to these random variables as Markov times and reserve the term stopping time to be those Markov times which are finite almost surely. We allow the stopping times to be infinite on a set of positive probability, in which case the decision is not to stop and receive any terminal reward. Clearly this decision should not be, and is not, rewarded when there is no running reward and the terminal reward is positive.

Each of the boundary points x_l and x_r can be classified as a natural, an entrance or an exit boundary point depending on the characteristics of the drift coefficient $\mu(\cdot)$ and diffusion coefficient $\sigma(\cdot)$ ([2, II.10, p. 14-19] or [9, p. 128-131]). When a point is both an exit and an entrance boundary point, the point is called non-singular and the diffusion is not determined uniquely. We therefore assume that the boundary points are singular. When x_l is a natural boundary point and $x > x_l$, the process X will not hit x_l in finite time (a.s.). The point x_l is thus not part of the state space for X. When x_l is an entrance-not-exit boundary point and $x \geq x_l$, $X(t) \in (x_l, x_r]$ (a.s.) for all $t > 0$, so when $x = x_l$, the process immediately enters the interval (x_l, x_r) and never exits at x_l. When x_l is an exit-not-entrance boundary point, there is a positive probability that X hits x_l in finite time after which time X remains at x_l. We therefore interpret $x \in [x_l, x_r]$ to be $x \in (x_l, x_r]$ if x_l is a natural or an exit boundary point and allow $x = x_l$ if x_l is an entrance boundary point. Similar statements apply to x_r. When $x_l = -\infty$ and/or $x_r = \infty$, we require these points to be natural boundaries of the diffusion, which implies the diffusion is non-explosive.

For some diffusions, we impose a restriction on the set of stopping times τ over which we maximize (1.2). When x_l is either a natural or an entrance-not-exit boundary point, each stopping time τ under consideration must satisfy either: (a) there exists some constant M_1^τ with $x_l < M_1^\tau < x$, such that $P(X(t) \geq M_1^\tau, t \leq \tau) = 1$; or (b) $P(X(\tau) \in (x_l, x)) = 0$. Similarly, when x_r is either a natural or an entrance-not-exit boundary point, we require each admissible stopping time τ to satisfy either: (a) there exists some constant M_2^τ with $x < M_2^\tau < x_r$ such that $P(X(t) \leq M_2^\tau, t \leq \tau) = 1$; or (b) $P(X(\tau) \in (x, x_r)) = 0$. Examples of stopping times satisfying these conditions would be the first hitting time of a level $a < x$

and the first hitting time of a level $b > x$. When both x_l and x_r are natural or entrance-not-exit boundary points, both sets of conditions are imposed. These restrictions are not imposed on the set of admissible stopping times when x_l and/or x_r are exit-not-entrance boundary points. Denote the set of admissible stopping times by $\mathcal{A}$.

Typically, one is interested in determining both an optimal stopping time τ^* and the value function

$$V(x) = \sup_{\tau \in \mathcal{A}} J(\tau; x). \tag{1.3}$$

It is helpful to observe that V is a function of the initial position of the diffusion X. This will become important when the problem is imbedded in a family of linear programs parameterized by x.

Optimal stopping of stochastic processes has a long history which has resulted in several solution approaches. Two excellent surveys of the general theory of optimal stopping are [5] and [16]. The book by Shiryaev [15] approaches a non-discounted version of the above problem in which $r \equiv 0$ by seeking the minimal excessive function lying on or above the reward function g. This minimal excessive function is the value function V and an optimal stopping rule is determined by stopping when the process X first hits a point a where $V(a) = g(a)$. The key to this solution technique is identifying the minimal excessive function V along with the set $\{a : V(a) = g(a)\}$. The recent book by Peskir and Shiryaev [13] relates the solution of optimal stopping problems to the solution of free boundary problems and uses the terminology of superharmonic functions in place of excessive functions. The authors consider more general problems that involve processes with jumps and include rewards based on the supremum of the process X as well as running and terminal rewards. For continuous processes, they employ the method of smooth pasting; that is, they seek to determine a (not necessarily connected) open continuation region $\mathcal{C}$, a (not necessarily connected) closed stopping region $\mathcal{S}$ and a function V for which

(i) $\mathcal{C} \cap \mathcal{S} = \emptyset$, $\mathcal{C} \cup \mathcal{S} = [x_l, x_r]$,

(ii) $V \in C^1[x_l, x_r]$, with $V|_{\mathcal{C}} \in C^2(\mathcal{C})$,

(iii) $AV(y) - \alpha V(y) + r(y) = 0$ for all $x \in \mathcal{C}$, and

(iv) $V(y) = g(y)$ for all $x \in \mathcal{S}$.

The moniker "smooth pasting" arises from the fact that one seeks to paste the solution to the differential equation in the region $\mathcal{C}$ from (iii) to the function g on the set $\mathcal{S}$ with the function so defined being continuously differentiable at the boundary points $\mathcal{S} \cap \overline{\mathcal{C}}$. When the process has jumps, the condition of smooth pasting is relaxed to continuous pasting. Optimal stopping of diffusion processes using smooth pasting is also discussed in the text [12] by Øksendal.

The recent paper by Dayanik and Karatzas [4] shows that the excessive functions are characterized as concave functions in a generalized sense. The problem of determining the minimum excessive function which majorizes the reward function is therefore recast as a problem of finding the minimum generalized concave function which majorizes the reward function. Their paper illustrates this approach on a number of optimal stopping problems.

As indicated in the first paragraph, this paper approaches the optimal stopping problem quite differently. The stochastic problem is imbedded in an infinite-dimensional linear program (Section 2). We then optimize over a smaller class of stopping times, specifically the two-point hitting times, and relax the constraints to form an auxiliary linear problem. A dual linear program is derived for which a weak duality relationship exists between the linear programs (Section 3). Strong duality between the problems and sufficiency of a two-point hitting rule is proven in Section 4. The result of strong duality is that the original stopping problem is reformulated as an explicit non-linear optimization problem and as a semi-infinite linear program, both of which can be used to determine the value. This solution technique is then illustrated in Section 5 on a number of examples.

2 LP Problem and Stopping Rule Analysis

2.1 Derivation of the LP

We take the initial position $x \in [x_l, x_r]$ to be arbitrary but fixed in the following discussion.

Let $f \in C^2[x_l, x_r]$ have compact support. Since X satisfies (1.1), an application of Itô's formula yields

$$e^{-\alpha t} f(X(t)) = f(x) + \int_0^t e^{-\alpha s} \left[Af(X(s)) - \alpha f(X(s))\right] ds + \int_0^t e^{-\alpha s} f'(X(s))\, dW(s).$$

Now let τ be any stopping time in $\mathcal{A}$. The optional sampling theorem [6, Theorem 3.6.6] implies that

$$\begin{aligned} e^{-\alpha(t\wedge\tau)} f(X(t\wedge\tau)) - f(x) - \int_0^{t\wedge\tau} e^{-\alpha s} \left[Af(X(s)) - \alpha f(X(s))\right] ds \\ = \int_0^{t\wedge\tau} e^{-\alpha s} f'(X(s))\, dW(s) \end{aligned} \tag{2.1}$$

and so the left-hand side is a martingale. Since f and its derivatives are bounded, taking expectations in (2.1) and letting $t \to \infty$ yields

$$E\left[e^{-\alpha\tau} f(X(\tau)) - \int_0^\tau e^{-\alpha s}[Af(X(s)) - \alpha f(X(s))]\, ds\right] = f(x). \tag{2.2}$$

Notice, in particular, that τ is not assumed to be almost surely finite; on the set $\{\tau = \infty\}$, the discounting drives the first term to 0 and also implies that the integral term is finite. The identity (2.2) holds for all $f \in C_c^2[x_l, x_r]$.

The LP associated with the optimal stopping problem is derived using a discounted occupation measure and discounted stopping distribution. Define the measure μ_τ as

$$\mu_\tau(G) = E\left[e^{-\alpha\tau} I_G(X(\tau))\right], \qquad \forall G \in \mathcal{B}[x_l, x_r]. \tag{2.3}$$

Observe that this measure is well-defined when $\tau = \infty$ even though $X(\tau)$ is not defined since the discounting drives the mass to 0. Furthermore, μ_τ has total mass that is less than or equal to 1. The discounted occupation measure μ_0 is defined by

$$\mu_0(G) = E\left[\int_0^\tau e^{-\alpha s} I_G(X(s))\, ds\right], \qquad \forall G \in \mathcal{B}[x_l, x_r]; \tag{2.4}$$

the total mass of μ_0 is less than or equal to $1/\alpha$.

The identity (2.2) can be expressed in terms of μ_τ and μ_0 as

$$\int f\, d\mu_\tau - \int [Af - \alpha f]\, d\mu_0 = f(x), \qquad \forall f \in C_c^2[x_l, x_r]. \tag{2.5}$$

Notice the dependence of this identity on the initial position x of the diffusion X.

Turning to the objective function (1.2), observe that it can also be expressed in terms of μ_τ and μ_0 as

$$J(\tau; x) = \int r\, d\mu_0 + \int g\, d\mu_\tau. \tag{2.6}$$

We wish to eliminate the running reward from the objective by adjusting the terminal reward function and to do so, we assume the following condition is satisfied.

Condition 2.1 *The reward function r is such that there exists a solution f_r of*

$$Af - \alpha f = r. \tag{2.7}$$

We demonstrate the reformulation for $\tau \in \mathcal{A}$ having associated bounds M_1^τ and M_2^τ; the same reformulation for other $\tau \in \mathcal{A}$ requires a slight adjustment to the argument. Let M denote the pair (M_1^τ, M_2^τ).

Let $\xi_M : [x_l, x_r] \to [0, 1]$ be such that $\xi_M \in C_c^2[x_l, x_r]$ and $\xi_M(y) = 1$ for all $y \in [M_1^\tau, M_2^\tau]$. Define the function $f_{r,M} = f_r \cdot \xi_M$ and note that $f_{r,M} \in C_c^2[x_l, x_r]$. Using $f_{r,M}$ in (2.5), we have

$$\begin{aligned} f_r(x) = f_{r,M}(x) &= \int f_{r,M}\, d\mu_\tau - \int [Af_{r,M} - \alpha f_{r,M}]\, d\mu_0 \\ &= \int f_r\, d\mu_\tau - \int r\, d\mu_0 \end{aligned}$$

and thus the expected reward using the stopping rule τ is

$$J(\tau; x) = \int r\, d\mu_0 + \int g\, d\mu_\tau = \int [f_r + g]\, d\mu_\tau - f_r(x). \tag{2.8}$$

At this point, the question arises as to whether (2.8) depends on the particular choice of solution f_r for (2.7). Consider the homogeneous equation

$$Af - \alpha f = 0. \tag{2.9}$$

It is well-known (see [2, II.10, p. 18,19] or [9, p. 128-131]) that (2.9) has a positive, strictly decreasing solution ϕ and a non-negative, strictly increasing solution ψ as its fundamental solutions. (Both ϕ and ψ depend on α; since we assume α is fixed, we do not use notation that indicates this dependence.) The functions ϕ and ψ are unique up to a multiplicative factor. Furthermore, $\phi(x_l+) = \infty$ and $\psi(x_l+) \geq 0$ when x_l is either a natural or an entrance-not-exit boundary of X and $\phi(x_l-) \in (0, \infty)$ when x_l is an exit-not-entrance boundary point. Similar comments apply to x_r with the roles of ϕ and ψ reversed.

Now consider a stopping time $\tau \in \mathcal{A}$ having an upper bound M_2^τ. Taking $M_1^\tau = x_l$ and letting M denote the pair (x_l, M_2^τ), define $\psi_M = \psi \cdot \xi_M$ and observe that

$$\begin{aligned}\psi(x) = \psi_M(x) &= \int \psi_M \, d\mu_\tau - \int [A\psi_M - \alpha\psi_M] \, d\mu_0 \\ &= \int \psi \, d\mu_\tau. \end{aligned} \tag{2.10}$$

By considering $(f_r + c\psi)\cdot\xi_M$, with c constant, and noting that μ_0 has its support in $[x_l, M_2^\tau]$, we have

$$\begin{aligned}(f_r + c\psi)(x) &= \int (f_r + c\psi) \, d\mu_\tau - \int [A(f_r + c\psi) - \alpha(f_r + c\psi)] \, d\mu_0 \\ &= \int (f_r + c\psi) \, d\mu_\tau - \int r \, d\mu_0 \end{aligned}$$

which, using (2.10), simplifies to

$$\int r \, d\mu_0 = \int f_r \, d\mu_\tau - f_r(x).$$

Slight adjustments to the argument indicates that these results hold for ϕ and for all $\tau \in \mathcal{A}$.

Thus using the function f_r allows the replacement of the running reward of the objective function by suitably adjusting the terminal reward earned at time τ and shifting by the constant $-f_r(x)$. Since the constant shift is the same for each stopping rule $\tau \in \mathcal{A}$, it may be ignored for optimization purposes. To simplify notation, let $g_r = f_r + g$, and define

$$J_r(\tau; x) = J(\tau; x) + f_r(x) = \int g_r \, d\mu_\tau \qquad \text{and} \qquad V_r(x) = V(x) + f_r(x). \tag{2.11}$$

The analysis in the sequel will examine J_r and V_r.

For each stopping time τ and process X satisfying (1.1), the corresponding measures μ_τ and μ_0 satisfy (2.5) and the value $J_r(\tau; x)$ is given by (2.11). Thus the optimal stopping problem is imbedded in the linear program

$$\left\{\begin{array}{ll} \text{Maximize} & \displaystyle\int g_r \, d\mu_\tau \\ \text{Subject to} & \displaystyle\int f \, d\mu_\tau - \int [Af - \alpha f] \, d\mu_0 = f(x), \quad \forall f \in C_c^2[x_l, x_r], \\ & \displaystyle\int 1 d\mu_\tau \leq 1, \\ & \displaystyle\int 1 d\mu_0 \leq 1/\alpha, \\ & \mu_\tau \geq 0, \mu_0 \geq 0. \end{array}\right. \tag{2.12}$$

Denote the value of this linear program by $V_{lp}(x)$. It immediately follows that

$$V_r(x) \leq V_{lp}(x). \tag{2.13}$$

2.2 Analysis of Stopping Rules

The constraints of LP (2.12) can be used to determine the values corresponding to some simple stopping rules that will play a central role in the construction of the value function. In particular, we examine the reward obtained by the first hitting time at levels to the right of, to the left of, or on both sides of x.

Example 2.2 Hitting at Level $b \geq x$
Let $b \geq x$ be fixed and consider the stopping rule $\tau_b = \inf\{t \geq 0 : X(t) = b\}$ which stops the first time the process X hits $\{b\}$. (The possibility that $\tau_b = \infty$ is allowed.) It immediately follows from (2.3) and (2.4) that μ_{τ_b} places all its mass on $\{b\}$ and μ_0 has support in $[x_l, b]$. When x_r is an exit-not-entrance boundary point $\psi(x_r-) < \infty$ and the following argument can simply use ψ, but when x_r is either a natural or an entrance-not-exit boundary point, $\psi(x_r-) = \infty$ and we must make an adjustment. Use the argument for (2.10) with $\xi_b \in C_c^2[x_l, x_r]$ such that $\xi_b(y) = 1$ for $y \in [x_l, b]$. The constraint involving ψ_b is

$$\begin{aligned} \psi(x) = \psi_b(x) &= \int \psi_b \, d\mu_{\tau_b} - \int [A\psi_b - \alpha\psi_b] \, d\mu_0 \\ &= \psi(b)\, \mu_{\tau_b}\{b\} \end{aligned} \tag{2.14}$$

and it immediately follows that $\mu_{\tau_b}\{b\} = \psi(x)/\psi(b)$. Using the fact that (2.8) only depends on the measure μ_{τ_b}, the reward associated with the stopping rule τ_b is

$$J_r(\tau_b; x) = \frac{\psi(x) g_r(b)}{\psi(b)}. \tag{2.15}$$

Example 2.3 Hitting at Level $a \leq x$
Let a be a point in $(x_l, x]$ and consider the stopping rule $\tau_a = \inf\{t \geq 0 : X(t) = a\}$. Then μ_{τ_a} is a point mass at $\{a\}$ and the associated μ_0 has support in $[a, x_r]$. We present the adjustment necessary when x_l is either a natural or an entrance-not-exit boundary point and hence $\phi(x_l+) = \infty$. Let $\zeta_a \in C^2[x_l, x_r]$ be such that $\zeta_a(y) = 0$ for $y \in [x_l, a - \epsilon]$ for some $0 < \epsilon < a \wedge (a - x_l)$, and $\zeta_a(y) = 1$ for $y \in [a, x_r]$. Using the decreasing solution ϕ to (2.9), define $\phi_a = \phi \cdot \zeta_a$. The identity (2.5) is satisfied by ϕ_a. (Note, we use ζ_a to ensure that ϕ_a satisfies the necessary integrability conditions.) Thus

$$\begin{aligned} \phi(x) = \phi_a(x) &= \int \phi_a \, d\mu_{\tau_a} - \int [A\phi_a - \alpha\phi_a] \, d\mu_0 \\ &= \phi(a)\, \mu_{\tau_a}\{a\} \end{aligned} \tag{2.16}$$

so $\mu_{\tau_a}\{a\} = \phi(x)/\phi(a)$ and hence

$$J_r(\tau_a; x) = \frac{\phi(x) g_r(a)}{\phi(a)}. \tag{2.17}$$

Remark 2.4 *In light of the definition (2.3) of μ_τ, the above two examples indicate that the Laplace transform of the first hitting time of the process X at level $\{c\}$ is*

$$E[e^{-\alpha\tau_c}] = \begin{cases} \phi(x)/\phi(c) & \text{for } c \leq x, \\ \psi(x)/\psi(c) & \text{for } c \geq x; \end{cases}$$

see, e.g., [2, II.10,p. 18].

Example 2.5 HITTING $\{a,b\}$
Fix $a \le x \le b$ and consider the stopping rule $\tau_{a,b} = \tau_a \wedge \tau_b = \inf\{t \ge 0 : X(t) \in \{a,b\}\}$, the first hitting time of either level a or level b. Definition (2.3) implies $\mu_{\tau_{a,b}}$ has $\{a,b\}$ for its support and definition (2.4) indicates that the associated occupation measure μ_0 is concentrated on $[a,b]$. Using the two functions ϕ_a and ψ_b in the identity (2.5) and recognizing that, for $c = a, b$, $\phi_a(c) = \phi(c)$ and $\psi_b(c) = \psi(c)$ yields the system

$$\begin{cases} \phi(a)\,\mu_{\tau_{a,b}}(a) + \phi(b)\,\mu_{\tau_{a,b}}(b) = \phi(x) \\ \psi(a)\,\mu_{\tau_{a,b}}(a) + \psi(b)\,\mu_{\tau_{a,b}}(b) = \psi(x). \end{cases}$$

Two cases must be considered. When $a = x = b$, the equation involving ϕ reduces to $\phi(x)\,\mu_{\tau_x}(x) = \phi(x)$ and similarly for ψ with solution $\mu_{\tau_x}(x) = 1$. Now suppose $a < b$. The solution of the system is then

$$\mu_{\tau_{a,b}}(a) = \frac{\phi(x)\,\psi(b) - \phi(b)\,\psi(x)}{\phi(a)\,\psi(b) - \phi(b)\,\psi(a)} \quad \text{and} \quad \mu_{\tau_{a,b}}(b) = \frac{\phi(a)\,\psi(x) - \phi(x)\,\psi(a)}{\phi(a)\,\psi(b) - \phi(b)\,\psi(a)}. \tag{2.18}$$

These masses are non-negative since ϕ is decreasing and ψ is increasing. It therefore follows that the reward associated with the stopping rule $\tau_{a,b}$ is

$$J_r(\tau_{a,b};x) = \begin{cases} g_r(x), & a = b, \\ g_r(a)\cdot\frac{\phi(x)\,\psi(b)-\phi(b)\,\psi(x)}{\phi(a)\,\psi(b)-\phi(b)\,\psi(a)} + g_r(b)\cdot\frac{\phi(a)\,\psi(x)-\phi(x)\,\psi(a)}{\phi(a)\,\psi(b)-\phi(b)\,\psi(a)}, & a < b. \end{cases} \tag{2.19}$$

Examining the expression for $J(\tau_{a,b};x)$ when $a < b$, we see that it simplifies to $g_r(x)$ when either $a = x < b$ or $a < x = b$.

Remark 2.6 *We observe the following limiting results which agree with one's intuition.*

(a) *When x_r is either a natural boundary point or an entrance-not-exit boundary point, $\psi(x_r-) = \infty$. As a result, if we hold the left boundary point a fixed and let $b \to x_r$, the expression for $\mu_{\tau_{a,b}}(a) \to \phi(x)/\phi(a) = \mu_{\tau_a}(a)$ and $\mu_{\tau_{a,b}}(b) \to 0$. Similarly, when x_l is either a natural or exit-not-entrance boundary point, $\phi(x_l+) = \infty$ and holding b fixed and letting $a \to x_l$ yields $\mu_{\tau_{a,b}}(a) \to 0$ and $\mu_{\tau_{a,b}}(b) \to \psi(x)/\psi(b) = \mu_{\tau_b}(b)$. Thus $J_r(\tau_{a,b};x) \to J_r(\tau_a;x)$ as $b \to x_r$ and as $a \to x_l$, $J_r(\tau_{a,b};x) \to J_r(\tau_b;x)$.*

A benefit of these observations is that our optimization of $J_r(\tau_{a,b};x)$ in Section 4 allows for one-sided stopping rules to be seen to be optimal.

(b) *When x_r is an entrance-not-exit boundary point, $\psi(x_r-) < \infty$ and when x_l is an entrance-not-exit boundary point $\phi(x_l+) < \infty$. In either of these cases, letting $b \to x_r$ or $a \to x_l$, the masses (2.18) converge to the masses associated with stopping at x_r or x_l respectively, and $J(\tau_{a,b};x) \to J(\tau_{a,x_r};x)$ as $b \to x_r$ and a similar result holds when $a \to x_l$.*

Remark 2.7 *The expression (2.19) for $J(\tau_{a,b};x)$ exhibits an interesting result when $g_r = c_1\phi + c_2\psi$ for any constants c_1 and c_2. This situation occurs when the running reward $r \equiv 0$ and the terminal reward is $g = c_1\phi + c_2\psi$. For each choice of stopping locations a and b, by explicit computation $J(\tau_{a,b};x) = c_1\phi(x) + c_2\psi(x)$ and thus every two-point stopping rule has the same value. Of course, this observation is merely a special case of (2.10) (suitably generalized to include ϕ).*

Finally observe that for any choice of a and b, with $x_l \le a \le x \le b \le x_r$, the reward obtained using stopping rule $\tau_{a,b}$ is no greater than the optimal reward:

$$J_r(\tau_{a,b}, x) \le V_r(x). \tag{2.20}$$

3 Auxiliary LP, Dual LP and Weak Duality

This section analyzes the LP (2.12) using an auxiliary linear program that involves a relaxation of the constraints, the dual linear program and a further restricted linear program. To begin we place the following additional assumption on the problem to ensure existence of finite values for the linear programs.

Condition 3.1 *For each $x \in (x_l, x_r)$, $\sup_{x_l \le y \le x} \frac{g_r(y)}{\phi(y)} < \infty$ and $\sup_{x \le y \le x_r} \frac{g_r(y)}{\psi(y)} < \infty$.*

For the remainder of this section, fix $x \in [x_l, x_r]$ arbitrarily.

Consider $\tau \in \mathcal{A}$ with corresponding bounds M_1^τ and M_2^τ. Applying the argument leading to (2.8) yields

$$\psi(x) = \int \psi \, d\mu_\tau \qquad \text{and} \qquad \phi(x) = \int \phi \, d\mu_\tau.$$

These identities also hold for all $\tau \in \mathcal{A}$.

Define the auxiliary LP by replacing the infinite number of constraints corresponding to each $f \in C_c^2[x_l, x_r]$ by these two constraints involving ϕ and ψ and eliminating the mass constraint on μ_0.

$$\begin{cases} \text{Maximize} & \displaystyle\int g_r \, d\mu_\tau \\ \text{Subject to} & \displaystyle\int \phi \, d\mu_\tau = \phi(x), \\ & \displaystyle\int \psi \, d\mu_\tau = \psi(x), \\ & \displaystyle\int 1 \, d\mu_\tau \le 1, \\ & \mu_\tau \ge 0. \end{cases} \tag{3.1}$$

Notice this auxiliary LP is expressed solely in terms of the measure μ_τ. We do not assume the optimal value is attained for this auxiliary LP or for any other LP in this section. Sufficient conditions to guarantee the existence of an optimizing measure μ_τ are given in Section 4. Denote the value of this auxiliary LP by $V_{alp}(x)$, where again the LP is parameterized by the initial position x.

Theorem 3.2 *For the initial value $x \in [x_l, x_r]$,*

$$V_{lp}(x) \le V_{alp}(x). \tag{3.2}$$

Proof. Clearly the feasible set of this auxiliary LP (3.1) includes the measure μ_τ for each feasible point in the LP (2.12). $\square$

We now develop a dual LP corresponding to (3.1). Since there are three constraints, there will be three dual variables, which we denote by c_1, c_2 and c_3. The dual LP is

$$\begin{cases} \text{Minimize} & c_1\phi(x) + c_2\psi(x) + c_3 \\ \text{Subject to} & c_1\phi(y) + c_2\psi(y) + c_3 1(y) \geq g_r(y), \ \forall y \in [x_l, x_r], \\ & c_1, c_2 \text{ unrestricted}, \\ & c_3 \geq 0. \end{cases} \tag{3.3}$$

In this linear program, the function 1 represents the constant function taking value 1. Let $V_{dlp}(x)$ denote the value of this dual LP.

Theorem 3.3 *For $x \in [x_l, x_r]$,*

$$V_{dlp}(x) \geq V_{alp}(x). \tag{3.4}$$

Proof. Observe that Condition 3.1 implies the existence of feasible points for the dual LP (3.3) and taking $\mu_\tau = \delta_{\{x\}}$ shows that the feasible set for the auxiliary LP (3.1) is nonempty. A standard weak duality argument therefore establishes the result. □

Finally, we restrict the feasible set of the dual LP (3.3) by setting $c_3 = 0$. This results in a restricted dual LP

$$\begin{cases} \text{Minimize} & c_1\phi(x) + c_2\psi(x) \\ \text{Subject to} & c_1\phi(y) + c_2\psi(y) \geq g_r(y), \ \forall y \in [x_l, x_r], \\ & c_1, c_2 \text{ unrestricted}. \end{cases} \tag{3.5}$$

Denote the value of (3.5) by $V_{rdlp}(x)$. The above discussion implies that the feasible set of the dual LP (3.3) contains the feasible set of restricted dual LP (3.5) so

$$V_{dlp}(x) \leq V_{rdlp}(x). \tag{3.6}$$

Now look carefully at this restricted dual LP. The function $c_1\phi + c_2\psi$ satisfies the differential equation $Af - \alpha f = 0$ and, to be feasible, is required to lie above the reward function. The goal of the LP is to pick the values c_1 and c_2 so as to minimize the objective function.

Combining the set of inequalities in (2.13), (2.20), (3.2), (3.4) and (3.6) yields for every $x_l \leq a \leq x \leq b \leq x_r$,

$$J_r(\tau_{a,b}; x) \leq V_r(x) \leq V_{lp}(x) \leq V_{alp}(x) \leq V_{dlp}(x) \leq V_{rdlp}(x). \tag{3.7}$$

At this point, it will be helpful for our further discussion to state clearly the weak duality result that will be exploited. Let $J_r^*(x) = \sup_{a,b: x_l \leq a \leq x \leq b \leq x_r} J_r(\tau_{a,b}; x)$ denote the optimal value associated by restricting the stopping rules to the set $\{\tau_{a,b} : x_l \leq a \leq x \leq b \leq x_r\}$. Also define $\widehat{J}_r(c_1, c_2; x) = c_1\phi(x) + c_2\psi(x)$ for those c_1 and c_2 that are feasible for (3.5). Then $V_{rdlp}(x) = \inf_{c_1, c_2} \widehat{J}_r(c_1, c_2; x)$.

Theorem 3.4 Weak Duality *Let a and b satisfy $x_l \leq a \leq x \leq b \leq x_r$ and let c_1 and c_2 be feasible for (3.5). Then*

$$J_r(\tau_{a,b}; x) \leq J_r^*(x) \leq V_{rdlp}(x) \leq \widehat{J}_r(c_1, c_2; x). \tag{3.8}$$

4 Optimization of $J_r(\tau_{a,b};x)$ and Strong Duality

We return to the examination of the reward $J_r(\tau_{a,b};x)$ associated with stopping at the first hitting time of $\{a,b\}$ and, in particular, we consider the optimization of this value over all possible choices of a and b with $x_l \leq a \leq x$ and $x \leq b \leq x_r$. Again, let $x \in [x_l, x_r]$ be fixed for this discussion.

We begin by considering the situation in which $g_r \leq 0$ with $x \in (x_l, x_r)$ and with x_l and x_r being either natural boundaries or entrance-not-exit boundaries for the diffusion. This assumption means that $\tau_{x_l,x_r} = \infty$ almost surely and moreover, $\phi(x_l+) = \psi(x_r-) = \infty$. In this setting, an optimal stopping rule is τ_{x_l,x_r} since $J_r(\tau_{x_l,x_r};x) = 0$ and stopping at any finite locations a and b will result in a non-positive expected reward.

For the rest of the optimization discussion, we assume there is some $y \in [x_l, x_r]$ for which $g_r(y) > 0$. We assume $y > x_l$ if x_l is either a natural boundary point or an entrance-not-exit boundary point and similarly $y < x_r$ if x_r is either a natural or entrance-not-exit boundary point.

We now impose conditions which imply that $J_r^*(x)$ is achieved by some points $a^* \in [x_l, x]$ and $b^* \in [x, x_r]$.

Condition 4.1 *Assume g_r satisfies the following:*

(a) *g_r is upper semicontinuous;*

(b) *if x_l is either a natural or an entrance-not-exit boundary point, then $\lim_{y \to x_l} \frac{g_r(y)}{\phi(y)} = 0$; and*

(c) *if x_r is either a natural or an entrance-not-exit boundary point, then $\lim_{y \to x_r} \frac{g_r(y)}{\psi(y)} = 0$.*

Since f_r is continuous, Condition 4.1(a) requires g to be upper semicontinuous.

Theorem 4.2 *Under Conditions 3.1 and 4.1, for each $x \in [x_l, x_r]$, there exist values $a^* = a^*(x) \in [x_l, x]$ and $b^* = b^*(x) \in [x, x_r]$ such that $J_r(\tau_{a^*,b^*};x) = J_r^*(x)$.*

Proof. When $J_r^*(x) = g_r(x)$, the choice of $a^* = x = b^*$ satisfies the claim. So assume that $J_r^*(x) > g_r(x)$. Let $\{(a_n, b_n) : n \in \mathbb{N}\}$ be a sequence with $a_n < x$ and $b_n > x$ for all $n \in \mathbb{N}$ such that $J_r(\tau_{a_n,b_n};x) \to J_r^*(x)$ as $n \to \infty$.

Compactify the interval $[x_l, x_r]$ when x_l and/or x_r are either natural or entrance-not-exit boundary points. It then follows that there exists a subsequence $\{n_k\}$ and values a^* and b^* such that $a_{n_k} \to a^*$ and $b_{n_k} \to b^*$. To simplify notation, assume the original sequence has the properties of this subsequence. The following set of relations then holds.

$$\begin{aligned}
J_r^*(x) &= \lim_{n\to\infty} J_r(\tau_{a_n,b_n};x) \\
&= \lim_{n\to\infty} \left(g_r(a_n) \cdot \frac{\phi(x)\,\psi(b_n) - \phi(b_n)\,\psi(x)}{\phi(a_n)\,\psi(b_n) - \phi(b_n)\,\psi(a_n)} + g_r(b_n) \cdot \frac{\phi(a_n)\,\psi(x) - \phi(x)\,\psi(a_n)}{\phi(a_n)\,\psi(b_n) - \phi(b_n)\,\psi(a_n)} \right) \\
&\leq g_r(a^*) \cdot \frac{\phi(x)\,\psi(b^*) - \phi(b^*)\,\psi(x)}{\phi(a^*)\,\psi(b^*) - \phi(b^*)\,\psi(a^*)} + g_r(b^*) \cdot \frac{\phi(a^*)\,\psi(x) - \phi(x)\,\psi(a^*)}{\phi(a^*)\,\psi(b^*) - \phi(b^*)\,\psi(a^*)} \\
&= J_r(\tau_{a^*,b^*};x) \\
&\leq J_r^*(x);
\end{aligned}$$

the first inequality follows since the continuity of ϕ and ψ imply the convergence of the fractions and g_r is upper semicontinuous. Thus equality holds throughout these relations and τ_{a^*,b^*} is an optimal stopping time.

To be precise, should $a^* = x_l$ with x_l being a natural or an entrance-not-exit boundary point and b^* be an interior point of the interval $[x_l, x_r]$, the limiting expression is $\frac{\psi(x)}{\psi(b^*)} g_r(b^*) = J_r(\tau_{b^*}; x)$ and τ_{b^*} is optimal. A similar comment applies to the case of $b^* = x_r$ with $a^* \in (x_l, x_r)$ yielding τ_{a^*} as an optimal stopping time.

The case in which $a^* = x_l$ and $b^* = x_r$ with both boundary points being either natural or entrance-not-exit does not arise. For if it would, Conditions 4.1(b,c) imply that the coefficients of $g_r(a^*)$ and $g_r(b^*)$ would be 0, corresponding to $\tau_{a^*,b^*} = \infty$ almost surely, and hence $J_r(\tau_{a^*,b^*}; x) = 0$. But there exists some $y \in (x_l, x_r)$ with $g_r(y) > 0$. The stopping time τ_y which stops the process when it first hits $\{y\}$ will have a strictly positive value for $J_r(\tau_y; x)$, contradicting $0 = \lim_{n\to\infty} J_r(\tau_{a_n,b_n}; x) = J_r^*(x)$. □

Remark 4.3 *The above proof only uses upper semicontinuity of g_r at the optimizing points a^* and b^*. One would therefore be able to relax the upper semicontinuity assumption on g_r so that it only is required to hold at the optimizers.*

At this point an observation is very helpful in preparation for the proof of the strong duality theorem. To this point we have been considering a single initial point x and the linear programs related to it. The value function V_r is a function of the initial position and we will prove that the values of the family of LPs parameterized by x give V_r. It is thus beneficial to consider more than a single initial value at at time. For instance, should x be an initial value such that $a^* < x < b^*$, then the optimization of $J(\tau_{a,b}; \tilde{x})$ for every other initial value $\tilde{x} \in (a^*, b^*)$ implies that a^* and b^* are also optimal for $\tilde{x}$. The non-degenerate interval (a^*, b^*) is thus seen to be part of the continuation region $\mathcal{C}$ described in the introduction in which it is optimal to allow the process X to continue without stopping. Let $\mathcal{C} = \cup\{x \in [x_l, x_r] : a^*(x) < x < b^*(x)\}$ be the continuation region. The set $\mathcal{S} = [x_l, x_r] \cap \mathcal{C}^c$ is the stopping region. Let $\overline{\mathcal{C}}$ denote the closure of $\mathcal{C}$ and $\mathcal{S}^\circ$ denote the interior of $\mathcal{S}$.

The following proposition identifies a condition under which x is an element of $\overline{\mathcal{C}}$.

Proposition 4.4 *If $\limsup_{y\to x} g_r(y) < g_r(x)$, then $x \in \overline{\mathcal{C}}$.*

Proof. Suppose x is a point at which $\limsup_{y\to x} g_r(y) < g_r(x)$. Choose δ such that $0 < \delta < g_r(x) - \limsup_{y\to x} g_r(y)$. Let $y_1 < x$ be fixed and consider the stopping rule $\tau_{y_1,x}$ when the initial value is y with $y_1 < y < x$. The value associated with this rule is

$$J_r(\tau_{y_1,x}; y) = \frac{\phi(y)\psi(x) - \phi(x)\psi(y)}{\phi(y_1)\psi(x) - \phi(x)\psi(y_1)}\, g_r(y_1) + \frac{\phi(y_1)\psi(y) - \phi(y)\psi(y_1)}{\phi(y_1)\psi(x) - \phi(x)\psi(y_1)}\, g_r(x).$$

Observe that the coefficient of $g_r(y_1)$ converges to 0 as y converges to x and similarly, the coefficient of $g_r(x)$ converges to 1. Select $y_2 < x$ such that for all $y_2 < y < x$, $g_r(y) < g_r(x) - \delta$,

$$\left| \frac{\phi(y)\psi(x) - \phi(x)\psi(y)}{\phi(y_1)\psi(x) - \phi(x)\psi(y_1)}\, g_r(y_1) \right| < \frac{\delta}{3} \quad \text{and} \quad \frac{\phi(y_1)\psi(y) - \phi(y)\psi(y_1)}{\phi(y_1)\psi(x) - \phi(x)\psi(y_1)}\, g_r(x) > g_r(x) - \frac{\delta}{3}.$$

Then for all $y_2 < y < x$, $J_r(\tau_{y_1,x}; y) > g_r(y)$ and $y \in \mathcal{C}$.

$\square$

Having established the existence of optimizers $a^* = a^*(x)$ and $b^* = b^*(x)$ for the optimal stopping problem restricted to two-point stopping rules, the goal is to prove the optimality of τ_{a^*,b^*} for the general stopping problem. Our approach will be to obtain coefficients c_1^* and c_2^* that are feasible for the restricted dual LP with $\widehat{J}(c_1^*, c_2^*; x) = J(\tau_{a^*,b^*}; x)$ and thus equality will hold throughout (3.7) and (3.8). To achieve this result, we must further restrict the class of reward functions.

Condition 4.5 *For each $x \in (x_l, x_r)$ for which $\limsup_{y\to x} g_r(y) = g_r(x)$,*

(a) $-\infty < \liminf_{y\nearrow x} \frac{g_r(y)-g_r(x)}{y-x} < \infty$ *and* $-\infty < \limsup_{z\searrow x} \frac{g_r(z)-g_r(x)}{z-x} < \infty$; *and*

(b) *if x is a point where $\liminf_{y\nearrow x} \frac{g_r(y)-g_r(x)}{y-x} \neq \limsup_{z\searrow x} \frac{g_r(z)-g_r(x)}{z-x}$, then there exists a sequence $\{y_n < x : n \in N\}$, with $y_n \to x$ and $\frac{g_r(y_n)-g_r(x)}{y_n-x} \to \liminf_{y\nearrow x} \frac{g_r(y)-g_r(x)}{y-x}$ as $n \to \infty$, such that for each n,*

$$\liminf_{y\nearrow y_n} \frac{g_r(y)-g_r(y_n)}{y-y_n} = \limsup_{z\searrow y_n} \frac{g_r(z)-g_r(y_n)}{z-y_n} \tag{4.9}$$

or there exists a sequence $\{z_n > x : n \in \mathbb{N}\}$ with $\lim_{n\to\infty} z_n = x$ and $\lim_{n\to\infty} \frac{g_r(z_n)-g_r(x)}{z_n-x} = \limsup_{z\searrow x} \frac{g_r(z)-g_r(x)}{z-x}$ such that (4.9) is satisfied for each n when z_n replaces y_n.

Observe that Condition 4.5(b) is satisfied, for example, when g_r is C^1 in either a left-neighborhood or a right-neighborhood of x with the left-hand derivative or right-hand derivative of g_r, respectively, existing at x.

Theorem 4.6 Strong Duality *Under Conditions 3.1, 4.1 and 4.5 on g_r, for $x \in [x_l, x_r]$, there exist stopping locations $a^* \in [x_l, x]$ and $b^* \in [x, x_r]$ and coefficients c_1^* and c_2^* such that*

$$J_r(\tau_{a^*,b^*}; x) = J_r^*(x) = V_r(x) = V_{rdlp}(x) = \widehat{J}_r(c_1^*, c_2^*; x).$$

Proof. The existence of a^* and b^* such that $J(\tau_{a^*,b^*}; x) = J_r^*(x)$ follows from Theorem 4.2. Notice in (2.19) that when $a = x$ or $b = x$, $J_r(\tau_{x,b}; x) = g_r(x)$ so

$$J_r^*(x) = \sup_{a,b: x_l \le a \le x \le b \le x_r} J_r(\tau_{a,b}; x) \ge g_r(x). \tag{4.10}$$

It is necessary to consider different cases for the initial value x. Before doing so, however, we examine the value associated with a two-point hitting rule and establish some notation.

Observe that the expression for $J(\tau_{a,b}; x)$ in (2.19) with $a < x < b$ can be rewritten as

$$J_r(\tau_{a,b}; x) = \frac{g_r(a)\psi(b) - g_r(b)\psi(a)}{\phi(a)\,\psi(b) - \phi(b)\,\psi(a)} \cdot \phi(x) + \frac{g_r(b)\phi(a) - g_r(a)\phi(b)}{\phi(a)\,\psi(b) - \phi(b)\,\psi(a)} \cdot \psi(x). \tag{4.11}$$

Define the coefficients c_1 and c_2 by

$$c_1(a, b) = \frac{g_r(a)\psi(b) - g_r(b)\psi(a)}{\phi(a)\,\psi(b) - \phi(b)\,\psi(a)} \quad \text{and} \quad c_2(a, b) = \frac{g_r(b)\phi(a) - g_r(a)\phi(b)}{\phi(a)\,\psi(b) - \phi(b)\,\psi(a)}. \tag{4.12}$$

Now, let $J_{a,b}(y) = c_1(a,b)\phi(y) + c_2(a,b)\psi(y)$ for $y \in [x_l, x_r]$; that is, we define the function $J_{a,b}$ on $[x_l, x_r]$ to have the form of $J_r(\tau_{a,b}; x)$ in (4.11) but do not require the independent variable to lie in the interval (a, b).

Case (a): Suppose $x \in \mathcal{C}$. Then there are points a^* and b^* such that $a^* = a^*(x) < x < b^*(x) = b^*$.

We claim that $J_{a^*,b^*} \geq g_r$. To verify this claim, consider first $a = y < x$ and $b = b^*$. Then $J_r(\tau_{y,b^*}; x) \leq J_r(\tau_{a^*,b^*}; x)$ implies

$$c_1(y, b^*)\phi(x) + c_2(y, b^*)\psi(x) \leq c_1(a^*, b^*)\phi(x) + c_2(a^*, b^*)\psi(x).$$

Using the definitions of c_1 and c_2 in (4.12) and rewriting the expressions as in (2.19), we have

$$\begin{aligned} &\frac{\phi(x)\,\psi(b^*) - \phi(b^*)\,\psi(x)}{\phi(y)\,\psi(b^*) - \phi(b^*)\,\psi(y)} \cdot g_r(y) + \frac{\phi(y)\,\psi(x) - \phi(x)\,\psi(y)}{\phi(y)\,\psi(b^*) - \phi(b^*)\,\psi(y)} \cdot g_r(b^*) \\ &\qquad \leq \frac{\phi(x)\,\psi(b^*) - \phi(b^*)\,\psi(x)}{\phi(a^*)\,\psi(b^*) - \phi(b^*)\,\psi(a^*)} \cdot g_r(a^*) + \frac{\phi(a^*)\,\psi(x) - \phi(x)\,\psi(a^*)}{\phi(a^*)\,\psi(b^*) - \phi(b^*)\,\psi(a^*)} \cdot g_r(b^*)\,. \end{aligned}$$

Isolating $g_r(y)$ on the left-hand-side results in

$$\begin{aligned} g_r(y) &\leq \frac{\phi(y)\,\psi(b^*) - \phi(b^*)\,\psi(y)}{\phi(a^*)\,\psi(b^*) - \phi(b^*)\,\psi(a^*)} \cdot g_r(a^*) + \left(\frac{\phi(y)\,\psi(b^*) - \phi(b^*)\,\psi(y)}{\phi(x)\,\psi(b^*) - \phi(b^*)\,\psi(x)}\right) \\ &\qquad \cdot \left[\frac{\phi(a^*)\,\psi(x) - \phi(x)\,\psi(a^*)}{\phi(a^*)\,\psi(b^*) - \phi(b^*)\,\psi(a^*)} - \frac{\phi(y)\,\psi(x) - \phi(x)\,\psi(y)}{\phi(y)\,\psi(b^*) - \phi(b^*)\,\psi(y)}\right] g_r(b^*) \\ &= \frac{\phi(y)\,\psi(b^*) - \phi(b^*)\,\psi(y)}{\phi(a^*)\,\psi(b^*) - \phi(b^*)\,\psi(a^*)} \cdot g_r(a^*) + \frac{\phi(a^*)\,\psi(y) - \phi(y)\,\psi(a^*)}{\phi(a^*)\,\psi(b^*) - \phi(b^*)\,\psi(a^*)} \cdot g_r(b^*) \\ &= c_1(a^*, b^*)\phi(y) + c_2(a^*, b^*)\psi(y) = J_{a^*,b^*}(y). \end{aligned}$$

Using a similar computation with $a = a^*$ and $b = y > x$ establishes the claim. We therefore see that this choice of $c_1^* = c_1(a^*, b^*)$ and $c_2^* = c_2(a^*, b^*)$ is feasible for the restricted dual linear program (3.5) and (a^*, b^*) is optimal for the problem of maximizing $J_r(\tau_{a,b}; x)$ over a and b, and moreover, by the definition of c_1^* and c_2^*, $J_r(\tau_{a^*,b^*}; x) = \widehat{J}_r(c_1^*, c_2^*; x)$.

Case (b): Suppose $x \in \partial\mathcal{C}$, the boundary of $\mathcal{C}$. There are two cases to consider.

(i): Suppose x is a point at which $\limsup_{y\to x} g_r(y) < g_r(x)$. Since $x \notin \mathcal{C}$, the proof of Proposition 4.4 shows that for y sufficiently close to x, with $y < x$, $y \in \mathcal{C}$ and $b^*(y) = x$. Since $a^*(y) < b^*(y) = x$, the result of Case (a) applies so that fixing $y < x$ sufficiently close to x and defining $c_1^* = c_1(a^*(y), x)$ and $c_2^* = c_2(a^*(y), x)$, the function $c_1^*\phi + c_2^*\psi$ majorizes g_r with equality holding at x.

(ii): Now suppose $x \in \partial\mathcal{C}$ with $\limsup_{y\to x} g_r(y) = g_r(x)$. Then there exists a sequence $\{x_n \in \mathcal{C} : n \in \mathbb{N}\}$ such that $\lim_{n\to\infty} x_n \to x$. Without loss of generality, assume $x_n \searrow x$. By considering a subsequence, if necessary, we may assume that $\frac{g_r(x_n) - g_r(x)}{x_n - x} \to \limsup_{n\to\infty} \frac{g_r(x_n) - g_r(x)}{x_n - x} =: m$. Since each $x_n \in \mathcal{C}$, $a_n^* := a^*(x_n) < b^*(x_n) =: b_n^*$. Observe that $x \leq a_n^* < x_n$ so as $n \to \infty$, $a_n^* \searrow x$. Should b_n^* converge to some value $b^* \in (x, x_r]$, the proof of Case (a) applies. So assume $b_n^* \searrow x$ as $n \to \infty$. Define $c_1^n = c_1(a_n^*, b_n^*)$ and $c_2^n = c_2(a_n^*, b_n^*)$. It then follows that

$c_1^n\phi + c_2^n\psi$ majorizes g_r with equality holding at x_n. We investigate the limit of c_1^n as $n \to \infty$. Observe

$$\begin{aligned}
c_1(a_n^*, b_n^*) &= \frac{g_r(a_n^*)\psi(b_n^*) - g_r(b_n^*)\psi(a_n^*)}{\phi(a_n^*)\psi(b_n^*) - \phi(b_n^*)\psi(a_n^*)} \\
&= \frac{g_r(a_n^*)\psi(b_n^*) - g_r(a_n^*)\psi(a_n^*) + g_r(a_n^*)\psi(a_n^*) - g_r(b_n^*)\psi(a_n^*)}{\phi(a_n^*)\psi(b_n^*) - \phi(a_n^*)\psi(a_n^*) + \phi(a_n^*)\psi(a_n^*) - \phi(b_n^*)\psi(a_n^*)} \\
&= \frac{g_r(a_n^*)\,\frac{\psi(b_n^*)-\psi(a_n^*)}{b_n^*-a_n^*} - \frac{g_r(b_n^*)-g_r(a_n^*)}{b_n^*-a_n^*}\,\psi(a_n^*)}{\phi(a_n^*)\,\frac{\psi(b_n^*)-\psi(a_n^*)}{b_n^*-a_n^*} - \frac{\phi(b_n^*)-\phi(a_n^*)}{b_n^*-a_n^*}\,\psi(a_n^*)}.
\end{aligned}$$

Letting $n \to \infty$, we see that $c_1^n \to \frac{g_r(x)\psi'(x)-m\psi(x)}{\phi(x)\psi'(x)-\phi'(x)\psi(x)} =: c_1^+$. A similar analysis shows that as $n \to \infty$, $c_2^n \to \frac{m\phi(x)-g_r(x)\phi'(x)}{\phi(x)\psi'(x)-\phi'(x)\psi(x)} =: c_2^+$ and thus $c_1^n\phi + c_2^n\psi$ converges to $c_1^+\phi + c_2^+\psi$. Therefore $c_1^+\phi + c_2^+\psi$ majorizes g_r and moreover, $[c_1^+\phi + c_2^+\psi](x) = g_r(x)$.

Case (c): Suppose $x \in \mathcal{S}^\circ$. Then $a^*(x) = x = b^*(x)$. In this case, $c_1^* = c_1(x,x)$ and $c_2^* = c_2(x,x)$ are not defined and we need a different argument.

(i): To begin, consider the situation in which $\liminf_{y\nearrow x} \frac{g_r(y)-g_r(x)}{y-x} = \liminf_{z\searrow x} \frac{g_r(z)-g_r(x)}{z-x} = m$. We seek constants c_1 and c_2 such that the function $c_1\phi + c_2\psi$ majorizes g_r and equality holds at the initial value x. Consider the system of equations

$$\begin{aligned}
c_1\phi(x) + c_2\psi(x) &= g_r(x) \\
c_1\phi'(x) + c_2\psi'(x) &= m
\end{aligned} \tag{4.13}$$

which arises by requiring the first derivative of $c_1\phi + c_2\psi$ to equal m as well as $c_1\phi + c_2\psi$ to equal g_r at x. The solution to this system is

$$c_1 = \frac{g_r(x)\psi'(x) - m\psi(x)}{\phi(x)\psi'(x) - \phi'(x)\psi(x)} \quad \text{and} \quad c_2 = \frac{m\phi(x) - g_r(x)\phi'(x)}{\phi(x)\psi'(x) - \phi'(x)\psi(x)}. \tag{4.14}$$

We claim the function $c_1\phi + c_2\psi$ majorizes g_r.

To see this, let $\epsilon > 0$ be chosen arbitrarily and let $y < x$ be arbitrary and $z > x$ be chosen as will be specified later. The optimality of τ_x implies that $J_r(\tau_{y,z};x) \leq J_r(\tau_x;x) = g_r(x)$. Writing $J_r(\tau_{y,z};x)$ as in (2.19) and isolating $g_r(y)$ leads to the inequality

$$\begin{aligned}
g_r(y) &\leq \frac{\phi(y)\psi(z) - \phi(z)\psi(y)}{\phi(x)\psi(z) - \phi(z)\psi(x)} \cdot g_r(x) - \frac{\phi(y)\psi(x) - \phi(x)\psi(y)}{\phi(x)\psi(z) - \phi(z)\psi(x)} \cdot g_r(z) \\
&= \frac{g_r(x)\psi(z) - g_r(z)\psi(x)}{\phi(x)\psi(z) - \phi(z)\psi(x)} \cdot \phi(y) + \frac{g_r(z)\phi(x) - g_r(x)\phi(z)}{\phi(x)\psi(z) - \phi(z)\psi(x)} \cdot \psi(y).
\end{aligned} \tag{4.15}$$

Now as in Case (b,ii) examine the coefficient of $\phi(y)$. We have

$$\begin{aligned}
\frac{g_r(x)\psi(z) - g_r(z)\psi(x)}{\phi(x)\psi(z) - \phi(z)\psi(x)} &= \frac{g_r(x)\psi(z) - g_r(x)\psi(x) + g_r(x)\psi(x) - g_r(z)\psi(x)}{\phi(x)\psi(z) - \phi(x)\psi(x) + \phi(x)\psi(x) - \phi(z)\psi(x)} \\
&= \frac{g_r(x)\left(\frac{\psi(z)-\psi(x)}{z-x}\right) - \left(\frac{g_r(z)-g_r(x)}{z-x}\right)\psi(x)}{\phi(x)\left(\frac{\psi(z)-\psi(x)}{z-x}\right) - \left(\frac{\phi(z)-\phi(x)}{z-x}\right)\psi(x)}
\end{aligned}$$

and thus letting $\{z_n > z : n \in \mathbb{N}\}$ be a sequence with $z_n \to x$ such that $\frac{g_r(z_n)-g_r(x)}{z_n-x} \to m$, it follows that

$$\lim_{n\to\infty} \frac{g_r(x)\psi(z_n) - g_r(z_n)\psi(x)}{\phi(x)\psi(z_n) - \phi(z_n)\psi(x)} = \frac{g_r(x)\psi'(x) - m\psi(x)}{\phi(x)\psi'(x) - \phi'(x)\psi(x)}.$$

Arguing similarly with the coefficient of $\psi(y)$ yields

$$\lim_{n\to\infty} \frac{g_r(z_n)\phi(x) - g_r(x)\phi(z_n)}{\phi(x)\psi(z_n) - \phi(z_n)\psi(x)} = \frac{m\phi(x) - g_r(x)\phi'(x)}{\phi(x)\psi'(x) - \phi'(x)\psi(x)}.$$

Recalling that y is fixed, let $N \in \mathbb{N}$ be such that for all $n \geq N$

$$\left| \frac{g_r(x)\psi(z_n) - g_r(z_n)\psi(x)}{\phi(x)\psi(z_n) - \phi(z_n)\psi(x)} - \frac{g_r(x)\psi'(x) - m\psi(x)}{\phi(x)\psi'(x) - \phi'(x)\psi(x)} \right| < \frac{\epsilon}{\phi(y) + \psi(y)} \tag{4.16}$$

and

$$\left| \frac{g_r(z_n)\phi(x) - g_r(x)\phi(z_n)}{\phi(x)\psi(z_n) - \phi(z_n)\psi(x)} - \frac{m\phi(x) - g_r(x)\phi'(x)}{\phi(x)\psi'(x) - \phi'(x)\psi(x)} \right| < \frac{\epsilon}{\phi(y) + \psi(y)}. \tag{4.17}$$

Using the estimates in (4.16) and (4.17) in (4.15) yields

$$\begin{aligned} g_r(y) &\leq \frac{g_r(x)\psi'(x) - m\psi(x)}{\phi(x)\psi'(x) - \phi'(x)\psi(x)} \cdot \phi(y) + \frac{m\phi(x) - g_r(x)\phi'(x)}{\phi(x)\psi'(x) - \phi'(x)\psi(x)} \cdot \psi(y) + \epsilon \\ &= c_1\phi(y) + c_2\psi(y) + \epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, the claim holds for all $y < x$.

A similar argument with $z > x$ chosen arbitrarily and $y_n < x$ chosen in a similar approximating sequence close enough to x establishes the relation for $z > x$. Thus when $\liminf_{y \nearrow x} \frac{g_r(y)-g_r(x)}{y-x} = \liminf_{z \searrow x} \frac{g_r(z)-g_r(x)}{z-x} = m$ at x and $a^* = x = b^*$, we see that defining c_1 and c_2 as in (4.14) produces the function $c_1\phi + c_2\psi$ which majorizes g_r with $[c_1\phi + c_2\psi](x) = g_r(x)$ and hence $\tau_x = 0$ is optimal.

(ii): Suppose g_r satisfies Condition 4.5(b) at x with a "$>$" inequality and for simplicity assume that there is a left-approximating sequence $\{y_n < x : n \in \mathbb{N}\}$ and let $m_n = \liminf_{y \nearrow y_n} \frac{g_r(y)-g_r(y_n)}{y-y_n}$. A similar argument will apply when a right approximating sequence $\{z_n > x : n \in \mathbb{N}\}$ exists.

Let $m = \liminf_{y \nearrow x} \frac{g_r(y)-g_r(x)}{y-x}$ and define the coefficients c_1^- and c_2^- to be the solutions of the system (4.13). We claim that the function $c_1^-\phi + c_2^-\psi$ majorizes g_r and satisfies $[c_1^-\phi + c_2^-\psi](x) = g_r(x)$. The latter condition follows immediately from the first equation in the system so we only need to show that $c_1^-\phi + c_2^-\psi$ majorizes g_r.

Since $x \in \mathcal{S}^\circ$, there is some δ such that for all $\tilde{x} \in (x-\delta, x)$, $a^*(\tilde{x}) = \tilde{x} = b^*(\tilde{x})$. For $y_n \in (x-\delta, x)$, define c_1^n and c_2^n as in (4.14) with m_n replacing m. Case (b,i) then implies that $c_1^n\phi + c_2^n\psi$ majorizes g_r and $[c_1^n\phi + c_2^n\psi](y_n) = g_r(y_n)$. Since $\frac{g_r(y_n)-g_r(x)}{y_n-x} \to m$ as $n \to \infty$, it follows that $g_r(y_n) \to g_r(x)$. Letting $n \to \infty$, the continuity of the derivatives of ϕ and ψ and the existence of the finite limit m of $\frac{g_r(y_n)-g_r(x)}{y_n-x}$ implies $c_1^n \to c_1^-$ and $c_2^n \to c_2^-$ and hence $c_1^n\phi + c_2^n\psi \to c_1^-\phi + c_2^-\psi$. Therefore $c_1^-\phi + c_2^-\psi$ majorizes g_r.

Observe that when g_r satisfies has both approximating sequences in Conditions 4.5(b), one is able to make this argument on both sides to obtain pairs of coefficients (c_1^-, c_2^-) and (c_1^+, c_2^+) such that both $c_1^-\phi + c_2^-\psi$ and $c_1^+\phi + c_2^+\psi$ majorize g_r and agree with g_r at x. Using convex combinations shows that the whole family of coefficients $(\lambda c_1^- + (1-\lambda)c_1^+, \lambda c_2^- + (1-\lambda)c_2^+)$, where $0 \leq \lambda \leq 1$, also provide majorizing functions.

(iii): Suppose g_r satisfies Condition 4.5(b) at x with a "$<$" inequality. The same proof as in Case (b,ii) applies to establish that $c_1^-\phi + c_2^-\psi$ majorizes g_r, with equality at x. However, $c_1^-\phi + c_2^-\psi$ is smooth at x so $[c_1^-\phi + c_2^-\psi]'(x) \leq \liminf_{y\nearrow x} \frac{g_r(y)-g_r(x)}{y-x} < \limsup_{z\searrow x} \frac{g_r(z)-g_r(x)}{z-x}$. Thus $[c_1^-\phi + c_2^-\psi](z) < g_r(z)$ for some $z > x$ sufficiently close to x, a contradiction. Hence $x \in \mathcal{S}^\circ$ implies that Condition 4.5(b) can only be satisfied with a "$>$" inequality. $\square$

The proof of Case (b,iii) indicates a condition which implies $x \in \mathcal{C}$. We formalize this result in the following proposition.

Proposition 4.7 *Suppose g_r satisfies Conditions 3.1, 4.1 and 4.5. If x is a point at which*

$$\liminf_{y\nearrow x} \frac{g_r(y) - g_r(x)}{y - x} < \limsup_{z\searrow x} \frac{g_r(z) - g_r(x)}{z - x},$$

then $x \in \mathcal{C}$.

An implication of Theorem 4.6 is that the optimal stopping problem has been reformulated as two different optimization problems. One may solve the non-linear maximization problem $J(\tau_{a,b}; x)$ over the values of $a \leq x$ and $b \geq x$. One may also solve the restricted dual linear program (3.5) over coefficients c_1 and c_2. Only Conditions 3.1, 4.1 and 4.5 are imposed on g_r so little regularity is required.

We emphasize the constructive nature of this approach. For each initial position x, the optimizing values a^* and b^* determine an interval $[a^*, b^*]$ which may be degenerate. Consider an x for which $a^* < x < b^*$ so that the interval $[a^*, b^*]$ is not degenerate. Then for each $x \in [a^*, b^*]$, the corresponding optimizing values are also given by a^* and b^*. So for each x in the interval, the coefficients c_1^* and c_2^* given by (4.12) are constant. Moreover, on $[a^*, b^*]$, $c_1^*\phi + c_2^*\psi$ is the minimal harmonic function which majorizes g_r. Thus a single optimization determines the value function over the interval;

$$V_r(x) = c_1^*\phi(x) + c_2^*\psi(x), \qquad x \in [a^*, b^*].$$

For the degenerate interval $[a^*, b^*] = \{x\}$, the proof of Theorem 4.6 shows how to find coefficients c_1^* and c_2^* such that $c_1^*\phi + c_2^*\psi$ majorizes g_r with $c_1^*\phi(x) + c_2^*\psi(x) = g_r(x)$. Thus the value function can be constructed by solving the family of non-linear optimization problems or by solving the family of restricted dual LPs (3.5) or by some combination of these approaches.

4.1 Smooth Pasting

Suppose now that g_r has some additional smoothness. Specifically, suppose g_r is C^1 in a neighborhood of the optimizing values a^* and b^*. For each a and b with $a \leq x \leq b$ and $a < b$, define functions c_1 and c_2 by (4.12). Since we are interested in optimizing with respect to a and b, we simplify notation by letting $h(a, b) = J(\tau_{a,b}; x) = c_1(a, b)\phi(x) + c_2(a, b)\psi(x)$. Using the smoothness of g_r, taking partial derivatives with respect to a and b and simplifying the expressions yields

$$\frac{\partial h}{\partial a}(a,b) = \left[g_r'(a) - \left(\frac{g_r(a)\psi(b)-g_r(b)\psi(a)}{\phi(a)\psi(b)-\phi(b)\psi(a)}\,\phi'(a) + \frac{g_r(b)\phi(a)-g_r(a)\phi(b)}{\phi(a)\psi(b)-\phi(b)\psi(a)}\,\psi'(a)\right)\right] \cdot \left[\frac{\phi(x)\psi(b)-\phi(b)\psi(x)}{\phi(a)\psi(b)-\phi(b)\psi(a)}\right] \tag{4.18}$$

$$= [g_r'(a) - (c_1(a,b)\phi + c_2(a,b)\psi)'(a)] \cdot \left[\frac{\phi(x)\psi(b)-\phi(b)\psi(x)}{\phi(a)\psi(b)-\phi(b)\psi(a)}\right] \tag{4.19}$$

and

$$\frac{\partial h}{\partial b}(a,b) = \left[g_r'(b) - \left(\frac{g_r(a)\psi(b)-g_r(b)\psi(a)}{\phi(a)\psi(b)-\phi(b)\psi(a)}\,\phi'(b) + \frac{g_r(b)\phi(a)-g_r(a)\phi(b)}{\phi(a)\psi(b)-\phi(b)\psi(a)}\,\psi'(b)\right)\right] \cdot \left[\frac{\phi(a)\psi(x)-\phi(x)\psi(a)}{\phi(a)\psi(b)-\phi(b)\psi(a)}\right] \tag{4.20}$$

$$= [g_r'(b) - (c_1(a,b)\phi + c_2(a,b)\psi)'(b)] \cdot \left[\frac{\phi(a)\psi(x)-\phi(x)\psi(a)}{\phi(a)\psi(b)-\phi(b)\psi(a)}\right]. \tag{4.21}$$

Consider the expression on the right-hand side of (4.18). When $x = b$, the second factor is 0 indicating that there is no change in h as one moves the stopping location a. This is intuitively clear since $x = b$ implies the process is stopped immediately. For $a \leq x < b$, the second factor is strictly positive and less than or equal to 1. A similar analysis of the right-hand side expression in (4.20) shows that the second factor is 0 when $x = a$ and is strictly positive and bounded by 1 for $a < x \leq b$.

From these observations, we see that setting $h_a = 0$ and $h_b = 0$ requires either $a = x$ or from (4.19)

$$g_r'(a) = (c_1(a,b)\phi + c_2(a,b)\psi)'(a) \tag{4.22}$$

and $b = x$ or from (4.21)

$$g_r'(b) = (c_1(a,b)\phi + c_2(a,b)\psi)'(b). \tag{4.23}$$

Thus when $a^* \neq x$ and $b^* \neq x$, the optimization over a and b imposes equality of the first derivatives of the functions g_r and $c_1(a,b)\phi + c_2(a,b)\psi$ at the optimizers. At the beginning of Section 4, it is shown that

$$g_r(a) = (c_1(a,b)\phi + c_2(a,b)\psi)(a) \qquad \text{and} \qquad g_r(b) = (c_1(a,b)\phi + c_2(a,b)\psi)(b) \tag{4.24}$$

for every choice of a and b. This means that at an optimal pair (a^*, b^*) of stopping locations, either $a^* = x$ and the process stops immediately or a^* satisfies the smooth pasting condition and similarly either $b^* = x$ or the smooth pasting condition is satisfied at b^*.

In the case of natural or entrance-not-exit boundary points, we point out that a^* could be x_l in which case $J_r(\tau_{x_l,b}; x)$ has expression (2.15) and the smooth pasting condition is only required at b^* and similarly should $b^* = x_r$, the expression for $J_r(\tau_{a,x_r}; x)$ is (2.17) and smooth pasting is only required at a^*.

The above argument assumes that the partial derivatives $\frac{\partial h}{\partial a}$ and $\frac{\partial h}{\partial b}$ actually equal 0 for some a and b with $x_l \leq a \leq x$ and $x \leq b \leq x_r$. The optimal value could also occur with $a = x_l$ or $b = x_r$ without either the smooth pasting condition holding or a^* or b^* being x. The endpoints must also be considered when determining the optimizing values of a and b.

5 Examples

This section illustrates how to construct the value function V using the non-linear optimization method, the restricted dual LP and a combination of these approaches. We consider a geometric Brownian motion process for the first five examples. This choice of diffusion implies that $x_l = 0$ and $x_r = \infty$ and that both boundaries are natural. We conclude this section with examples of optimal stopping problems for other types of diffusions. Additional examples involving other types of processes may be found in [8], though for those examples the initial position is assumed to be small. For later reference, we begin by determining the important results concerning geometric Brownian motion.

Let $\alpha > 0$ denote the discount rate and let $\mu < \alpha$ and $\sigma > 0$ be constants. We assume the process X satisfies the stochastic differential equation

$$dX(t) = \mu X(t)\,dt + \sigma X(t)\,dW(t), \qquad X(0) = x > 0. \tag{5.1}$$

The generator A of the process X is $Af(y) = \frac{\sigma^2}{2} y^2 f''(y) + \mu y f'(y)$ so the solutions of the differential equation (2.9) are $\phi(y) = y^{\gamma_1}$ and $\psi(y) = y^{\gamma_2}$, in which

$$\gamma_1 := \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2\alpha}{\sigma^2}} < 0 < \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2\alpha}{\sigma^2}} =: \gamma_2. \tag{5.2}$$

Consider a general solution $f = c_1\phi + c_2\psi$ in which $c_1, c_2 > 0$. Evaluating the derivative we have

$$f'(y) = \gamma_1 c_1 y^{(\gamma_1 - 1)} + \gamma_2 c_2 y^{(\gamma_2 - 1)}.$$

Setting $f' = 0$ and solving for y yields

$$y_c = \left(\frac{-c_1\gamma_1}{c_2\gamma_2}\right)^{1/(\gamma_2-\gamma_1)}.$$

Observe $f''(y) > 0$ for $y > 0$ since $\alpha > \mu$ implies $\gamma_2 > 1$. Thus f' is strictly decreasing for $y < y_c$ and strictly increasing for $y > y_c$ and hence y_c is a minimizer of f.

We utilize this structure of f in some of the following examples.

Example 5.1 Perpetual Put Option
For this example, X represents the price of a risky asset in a Black-Scholes market. Let $K > 0$ denote the option's strike price. The goal is to select a stopping time τ so as to maximize

$$E\left[e^{-\alpha\tau}(K - X(\tau))^+\right]. \tag{5.3}$$

For this optimization problem to give the risk-neutral price of the option, α is the interest rate on the non-risky asset and the expectation is taken with respect to the risk-neutral measure with the result that the mean rate of return of the risky asset is $\mu = \alpha$. As a result, $\gamma_1 = -\frac{2\alpha}{\sigma^2}$ and $\gamma_2 = 1$.

This optimal stopping problem has no running reward r; a reward g is only earned when the option is exercised at the stopping time. The reward function is $g(y) = (K - y)^+$. Our goal is, for each $x \in (0, \infty)$, to maximize (2.19) over stopping locations a and b with $a \leq x \leq b$.

We note that since $r = 0$, $J_r = J$ so we drop the subscript for both J and $\widehat{J}$ in the discussion of this example and for the other examples that lack a running reward.

Solution 1: Maximizing $J(\tau_{a,b}; x)$

Initial Analysis of Stopping Location a to the Left of x. Consider the case when $x > K$. For any stopping location a with $a > K$ and hence any b with $b > K$ as well, $J(\tau_{a,b}; x) = 0$ by simple evaluation of (2.19) since $g(a) = 0 = g(b)$. But for each $a \in (0, K)$, $J(\tau_{a,b}; x) > 0$ so the optimal choice a^* must be less than K.

Analysis of Stopping Location b to the Right of x. Now consider the case $x > 0$ and let $a < x \wedge K$. First consider the case when $x < b < K$. The function $J_{a,b}$ having expression (4.11) but which takes the independent variable in $[x_l, x_r]$ is strictly convex and satisfies $J_{a,b}(a) = K - a$ and $J_{a,b}(b) = K - b$ and hence it follows that $J_{a,b}(x) < K - x = g_r(x)$.

Now consider any $b > K \vee x$ and the stopping time $\tau_{a,b}$. The second summand in (2.19) is 0 and the first summand only depends on b in its mass. In fact, this mass is an increasing function of b and hence the maximal value occurs when $b = \infty$. This means that an optimal stopping rule has the form τ_a which exercises the option when the stock price first hits some value a. The goal now reduces to maximizing $J(\tau_a; x)$ given by (2.17) over values of $a \in [0, K]$.

Fix $x > 0$ and let $h(a) = \frac{g(a)}{\phi(a)} \cdot \phi(x) = (Ka^{-\gamma_1} - a^{1-\gamma_1})x^{\gamma_1}$ for $0 < a < K$. Setting $h'(a) = 0$ results in $a^* = \frac{-K\gamma_1}{1-\gamma_1}$. A simple analysis shows that a^* is a maximizer. This means that for any $x \geq a^*$, an optimal stopping rule is given by τ_{a^*}. The relationship between the functions g and $\frac{g(a^*)}{\phi(a^*)}\,\phi$ is displayed in Figure 1(a).

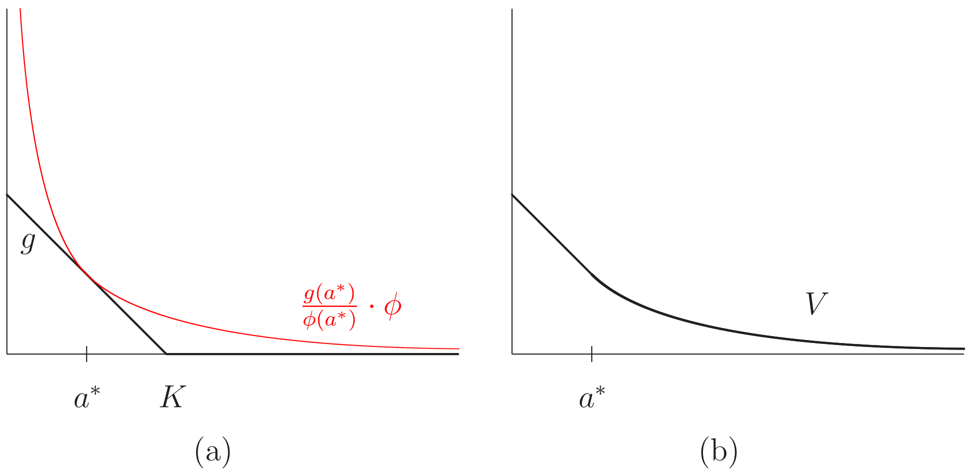


Figure 1: Relation between g and $(g(a^*)/\phi(a^*))\phi$ and the value function V

Analysis when $x < a^*$. Now consider $0 < x < a^*$. Suppose there were an optimal stopping location $a_1 < x < a^*$. Since for $0 < y < K$, g is continuously differentiable, a_1 would satisfy the smooth pasting principle. However, a^* is the unique value at which $\frac{g(a)}{\phi(a)}\phi'(a) = -1 = g'(a)$. Thus for no $a_1 < x < a^*$ is a_1 an optimal stopping location. It then follows that the optimal stopping rule is to stop immediately.

The value function V and optimal stopping time τ^* are given by

$$V(x) = \begin{cases} K - x, & \text{for } 0 \leq x \leq a^*, \\ (K - a^*)\left(\frac{x}{a^*}\right)^{\gamma_1}, & \text{for } x \geq a^*. \end{cases} \quad \text{and} \quad \tau^* = \begin{cases} \tau_x = 0, & \text{for } x \leq a^*, \\ \tau_{a^*}, & \text{for } x \geq a^*. \end{cases}$$

The value function V is displayed in Figure 1(b).

Solution 2: Minimizing $\widehat{J}(c_1, c_2; g_r)$.

We demonstrate how to use the restricted dual LP (3.5) to obtain the value function. Let $x > 0$ be fixed. First observe that the majorizing condition of $c_1\phi + c_2\psi$ over g along with $\phi(0+) = \infty$ and the strict positivity of g near 0 implies that c_1 must be positive. Also since $\psi(\infty) = \infty$ and $\phi(\infty) = 0$, c_2 must also be non-negative. Since $\psi(x) > 0$, the objective function would be minimized if $c_2 = 0$. In this case, the majorizing condition reduces to $c_1\phi \geq g$ yielding $c_1^* = \sup_{0<a\leq x} \frac{g(a)}{\phi(a)}$. Observe that $a^* = \left(\frac{-\gamma_1}{1-\gamma_1}\right) K$ is the optimizer. Thus for any initial value $x \geq a^*$, the pair $(c_1^*, 0)$ provides the optimal solution for the restricted dual LP.

For $x < a^*$, c_2 must be positive. In trying to minimize $[c_1\phi + c_2\psi](x)$ subject to the majorizing condition, we must have $[c_1\phi + c_2\psi](x) \geq g(x)$. The question then arises as to whether it is possible to have $[c_1\phi + c_2\psi](x) = g(x)$. Since g is C^1 on $(0, a^*)$, the majorizing requirement implies $[c_1\phi + c_2\psi]'(x) = g'(x)$ when $[c_1\phi + c_2\psi](x) = g(x)$. This then sets up the system (4.13) of linear equations that determine the optimal choice of c_1^* and c_2^* given by (4.14). Since $[c_1^*\phi + c_2^*\psi]'' > 0$, $[c_1^*\phi + c_2^*\psi]'$ is strictly increasing and it follows that $c_1^*\phi + c_2^*\psi$ majorizes g.

Example 5.2 PERPETUAL "UP-AND-OUT" BARRIER PUT OPTION

Our second example is a modification of the perpetual put option in Example 5.1 in which we consider an "up-and-out" barrier for which the option becomes worthless at the time when the stock price first hits a barrier level. This example is solved by Dayanik and Karatzas [4] using a characterization of the excessive functions as generalized concave functions combined with a change in variable argument. We determine the solution using the approaches of this paper.

Again, let the stock price process satisfy (5.1) with $\mu = \alpha$, where α denotes the risk-neutral interest rate. Let $b_0 > K$ denote the barrier such that the option expires at time $\tau_{b_0} = \inf\{t \geq 0 : X(t) \geq b_0\}$. We observe that $\tau_{b_0} = 0$ for initial values $X(0) = x \geq b_0$. The option price is determined by optimizing

$$J(\tau \wedge \tau_{b_0}; x) = E\left[e^{-\alpha(\tau\wedge\tau_{b_0})}(K - X(\tau \wedge \tau_{b_0}))^+\right] \tag{5.4}$$

over all stopping times $\tau \in \mathcal{A}$. We note that the dynamics (5.1) of the stock price process X are determined under the risk-neutral measure and that for initial positions $x \geq b_0$ the definition of τ_{b_0} implies $J(\tau \wedge \tau_{b_0}; x) = 0$ for every $\tau \in \mathcal{A}$.

As in Example 5.1, $\gamma_1 = -\frac{2\alpha}{\sigma^2}$ and $\gamma_2 = 1$ so the decreasing solution of $Af - \alpha f = 0$ is $\phi(y) = y^{\gamma_1}$ and the increasing solution is $\psi(y) = y$. Let $\tau_{a,b}$ denote the two-point stopping rule with $a \leq x \leq b$ and $a < b$. We observe that when $b \leq b_0$, $\tau_{a,b} \wedge \tau_{b_0} = \tau_{a,b}$ and that $\tau_{a,b} \wedge \tau_{b_0} = \tau_{a,b_0}$ when $b \geq b_0$. We therefore optimize the value

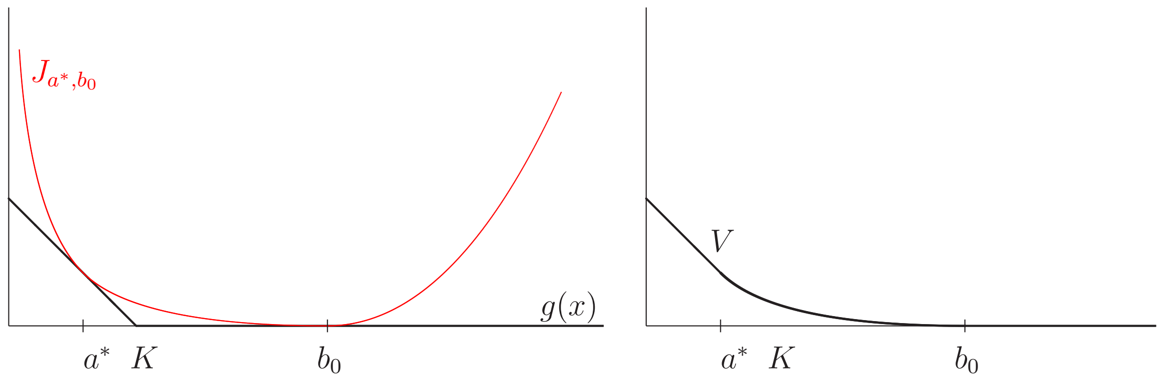
$$J(\tau_{a,b}; x) = (K - a)^+ \cdot \frac{bx^{\gamma_1} - xb^{\gamma_1}}{ba^{\gamma_1} - ab^{\gamma_1}} + (K - b)^+ \cdot \frac{xa^{\gamma_1} - ax^{\gamma_1}}{ba^{\gamma_1} - ab^{\gamma_1}} \tag{5.5}$$

over the possible values of a and b with $a \le x \le b$.

Consider the case of $a < x < b < K$ in which both terms of (5.5) are positive and define the function $J_{a,b}$ on $[x_l, x_r]$ by (5.5) with the independent variable replacing x. We have already established that $J_{a,b}(a) = (K-a)^+$ and $J_{a,b}(b) = (K-b)^+$. Observe $J_{a,b}$ is strictly convex, whereas the function $(K-y)^+$ is linear over $[0, K]$. It therefore follows that $J(\tau_{a,b}; x) = J_{a,b}(x) < (K-x)^+$ so $\tau_{a,b}$ is not optimal. Thus either the optimal $b = x$ and it is optimal to stop immediately or the optimal choice of b is greater than K. In the latter case, the second term of (5.5) is 0 and the first term is strictly increasing in b. Since the option becomes worthless at the time the process first hits the barrier, the optimal choice of b is $b^* = b_0$. The same analysis as in Example 5.1 indicates that $a^* < K$. Since the second term in (5.5) is 0, the optimization problem reduces to maximizing the first term of $J(\tau_{a,b_0}; x)$ over the values of a. A brief examination shows that $J(\tau_{0,b_0}; x) = 0 = J(\tau_{K,b_0;x})$ with $J(\tau_{a,b_0}; x) > 0$ for all $a \in (0, K)$. Setting the derivative with respect to a equal to 0 results in the transcendental equation

$$-b_0\gamma_1 K a^{\gamma_1 - 1} + b_0 (\gamma_1 - 1) a^{\gamma_1} + K b_0^{\gamma_1} = 0 \tag{5.6}$$

whose solution is the optimal value a^*, provided $a^* \le x$. Figure 2(a) displays the relation between g and J_{a^*,b_0}, the function having form $J(\tau_{a^*,b_0}; x)$ without restriction on the independent variable.



(a) g and $J(\tau_{a^*,b_0}; x)$ (b) The Value Function V

Figure 2: Relation between g and $J(\tau_{a^*,b_0}; x)$ and the value function V

For $x < a^*$, the same arguments in Example 5.1 yields the optimal stopping location $a = x$ so that stopping occurs immediately. Thus the optimal stopping rule is

$$\tau^* = \begin{cases} \tau_{a^*,b_0}, & \text{for } a^* \le x \le b_0, \\ 0, & \text{otherwise,} \end{cases}$$

and the value function is

$$V(x) = \begin{cases} K - x, & \text{for } 0 \le x \le a^*, \\ \left(\dfrac{(K-a^*)}{a^*((a^*)^{\gamma_1 - 1} - b_0^{\gamma_1 - 1})}\right) x^{\gamma_1} - \left(\dfrac{(K-a^*) b_0^{\gamma_1 - 1}}{a^*((a^*)^{\gamma_1 - 1} - b_0^{\gamma_1 - 1})}\right) x, & \text{for } a^* \le x \le b_0, \\ 0, & \text{for } x \ge b_0. \end{cases}$$

The value function V is displayed in Figure 2(b). Equation (5.6) agrees with equation (6.7) of [4] for determining the optimal stopping location a^*, and the formulas for V are the same. Our expression for V exhibits the function as a linear combination of the fundamental solutions ϕ and ψ to $Af - rf = 0$ on $[a^*, b_0]$ and equal to g otherwise.

Example 5.3 Capped Call Option on a Dividend-Paying Asset
Consider now a process X satisfying (5.1) with $\mu = \alpha - \delta$ in which $\alpha > 0$ denotes the interest rate on the risk-free asset and $\delta > 0$ gives the dividend rate for the risky asset. Let $K > 0$ denote the strike price of the call option and let $L > K$ denote the cap. The option pays $g(y) = (y \wedge L - K)^+$ when exercised at a time that the stock price value is y. The goal is to determine an optimal exercise time $\tau \in \mathcal{A}$ so as to maximize

$$E\left[e^{-\alpha\tau} g(X(\tau))\right].$$

The generator of the dividend-paying asset is $Af(y) = (\alpha - \delta)yf'(y) + (\sigma^2/2)y^2 f''(y)$. Define

$$\gamma_1 := \frac{1}{2} + \frac{\delta-\alpha}{\sigma^2} - \sqrt{\left(\frac{1}{2} + \frac{\delta-\alpha}{\sigma^2}\right)^2 + \frac{2\alpha}{\sigma^2}} < 0 < \frac{1}{2} + \frac{\delta-\alpha}{\sigma^2} + \sqrt{\left(\frac{1}{2} + \frac{\delta-\alpha}{\sigma^2}\right)^2 + \frac{2\alpha}{\sigma^2}} =: \gamma_2.$$

It then follows that the decreasing and increasing solutions to the equation $Af - \alpha f = 0$ are $\phi(y) = y^{\gamma_1}$ and $\psi(y) = y^{\gamma_2}$, respectively. Note that $\gamma_2 \geq 1$ with equality only when $\delta = 0$.

Using arguments similar to those used for the stopping location b to the right of x in Exercise 5.1 but for the stopping location a to the left of x, it follows that the optimal choice for the left stopping boundary is $a^* = 0$ and the value associated with the exercise rule which says to stop when first hitting level b is

$$J(\tau_b; x) = \frac{g(b)}{\psi(b)} \cdot \psi(x).$$

We wish to maximize the function

$$h(y) = \begin{cases} \dfrac{(y-K)^+}{y^{\gamma_2}}, & \text{for } K \leq y \leq L, \\ \dfrac{(L-K)^+}{y^{\gamma_2}}, & \text{for } y \geq L. \end{cases}$$

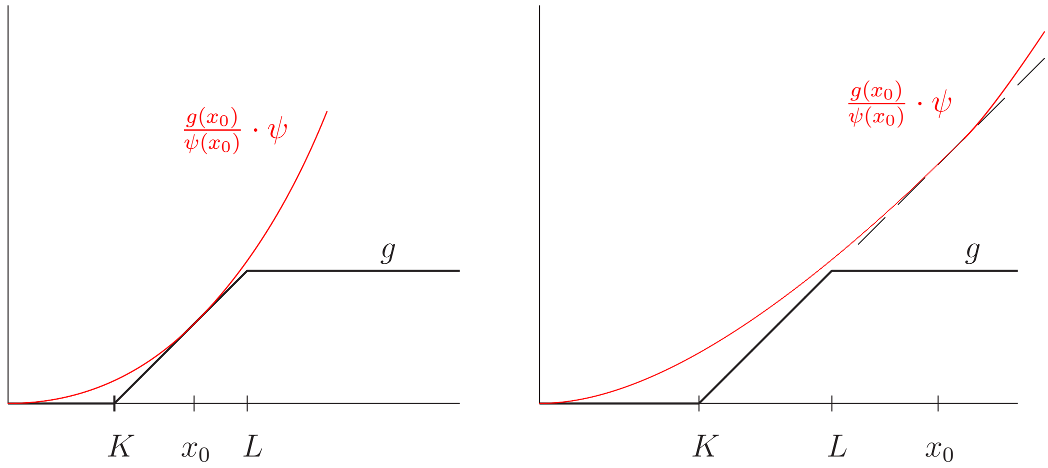
Clearly, the maximum of the second expression occurs when $y = L$. The maximum of the first expression occurs at the point $x_0 = \frac{K\gamma_2}{\gamma_2 - 1}$ provided $x_0 \leq L$. When $x > x_0 \wedge L$, one can determine coefficients c_1^- and c_2^- as in Case (c,ii) of the proof of Theorem 4.6 and hence it is optimal to stop immediately when $x \geq x_0 \wedge L$. The value function is therefore given by

$$V(x) = \begin{cases} \dfrac{x_0 \wedge L - K}{x_0^{\gamma_2}} \cdot x^{\gamma_2}, & \text{for } 0 \leq x \leq x_0 \wedge L, \\ \\ x \wedge L - K, & \text{for } x \geq x_0 \wedge L. \end{cases}$$

The relation between the $\frac{g(x_0)}{\psi(x_0)} \cdot \psi$ and g is displayed in Figure 3. Notice in Figure 3(a) that x_0 is the location where the functions $\frac{g(x_0)}{\psi(x_0)} \cdot \psi$ and g are tangent. Thus x_0 is the stopping location for the optimal exercise rule. In Figure 3(b), however, the function $\frac{g(x_0)}{\psi(x_0)} \cdot \psi$ is strictly greater than g so it is not possible to realize the larger function by any stopping location b. Observe the smaller function $\frac{g(L)}{\psi(L)} \cdot \psi$ majorizes g as required and gives the value corresponding to the stopping rule τ_L. The optimal exercise time is

$$\tau^* = \begin{cases} \tau_{x_0 \wedge L}, & \text{for } x \leq x_0 \wedge L, \\ 0, & \text{for } x \geq x_0 \wedge L. \end{cases}$$

The value function V and the optimal exercise rule agree with the results of Example 6.3 of [4]. This problem was first studied by Broadie and Detemple [3].



(a) $x_0 \leq L$ (b) $x_0 > L$

Figure 3: Maximizer x_0 of $(y-K)^+/\psi(y)$

Example 5.4 Forest Harvest with Carbon Credits
Let X satisfy (5.1) with $\mu, \sigma > 0$. The process X now represents the quantity of lumber in a stand of forest. When the stand is harvested, it earns a net profit of $g(y) = k_1 y^\beta - k_2$, in which $k_1, k_2 > 0$ and $\beta > 0$. Until harvest, the owner is paid a carbon credit that is proportional to the same power of the size of the forest, so $r(y) = Ry^\beta$. The owner's objective is to select a stopping time τ so as to maximize

$$E\left[\int_0^\tau e^{-\alpha t} R X^\beta(t)\, dt + e^{-\alpha\tau}(k_1 X^\beta(\tau) - k_2)\right]. \tag{5.7}$$

We make the following assumptions about the relation between the parameters. In order to have a finite maximum in (5.7), we impose the condition that $\beta < \gamma_2$, where γ_2 is defined in (5.2); otherwise, the owner can receive arbitrarily large discounted rewards by choosing to stop when X hits sufficiently large values. The assumption $0 < \beta < \gamma_2$ also implies that $(\sigma^2/2)\beta(\beta-1) + \mu\beta - \alpha < 0$. In addition, we assume $k_1[\alpha - \beta\mu - (\sigma^2/2)\beta(\beta-1)] > R$, which will imply the existence of a finite optimal stopping time.

Applying the differential operator $Af - \alpha f$ to the function $f_r(y) = \frac{R}{(\sigma^2/2)\beta(\beta-1)+\mu\beta-\alpha} y^\beta$ results in $Af_r - \alpha f_r = Ry^\beta$. Let $k_3 = k_1 + \frac{R}{(\sigma^2/2)\beta(\beta-1)+\mu\beta-\alpha}$. Thus the function $g_r(y) =$

$f_r(y) + g(y) = k_3 y^\beta - k_2$. We note that the assumptions on the parameters imply $k_3 > 0$ so that the owner has an incentive to harvest the lumber at some point. It then follows that g_r is strictly increasing, $g_r(0) = -k_2$ and $g_r(y_0) = 0$ for $y_0 = \left(\frac{k_2}{k_3}\right)^{1/\beta}$. For $x < y_0$, choosing any value a such that $0 < a < x$ has value $J(\tau_{a,b}; a) = g_r(a) < 0$. Selecting $a = 0$ and $b \geq y_0$ and hence $c_1(a, b) = 0$, however, yields

$$J(\tau_{a,b}; x) = J(\tau_b; x) = \frac{g_r(b)}{\psi(b)} \cdot x^{\gamma_2},$$

with the result that its value is non-negative for all $x \geq 0$. Thus for x sufficiently small, the optimal stopping time will be τ_b for some $b > x \vee y_0$. We now seek the optimal value of b.

Define $h(b) = g_r(b)/\psi(b) = k_3 b^{\beta-\gamma_2} - k_2 b^{-\gamma_2}$. Setting the derivative of h equal to 0 yields

$$0 = k_3(\beta - \gamma_2) b^{\beta-\gamma_2-1} + k_2\gamma_2 b^{-\gamma_2-1} = [k_3(\beta - \gamma_2) b^\beta + k_2\gamma_2] b^{-\gamma_2-1}$$

and hence a unique maximum occurs at $b^* = \left(\frac{k_2\gamma_2}{k_3(\gamma_2-\beta)}\right)^{1/\beta}$.

Now consider the situation for $x \in (b^*, \infty)$. Since for $x > b^*$, $g_r \in C^1(b^*, \infty)$. If there were two distinct points a^* and b^* for which the stopping time τ_{a^*,b^*} would be optimal, the points would need to satisfy the smooth pasting conditions (4.22) and (4.23). These conditions would imply that $g_r'(y) = c_1\phi'(y) + c_2\psi'(y)$ for at least two values of y. Differentiating $h_1(y) = \frac{c_1\phi'(y)+c_2\psi'(y)}{g_r'(y)} = \frac{c_1\gamma_1}{k_3\beta} y^{\gamma_1-\beta} + \frac{c_2\gamma_2}{k_3\beta} y^{\gamma_2-\beta}$ yields

$$h_1'(y) = \frac{c_1\gamma_1(\gamma_1 - \beta)}{k_3\beta} y^{\gamma_1-\beta-1} + \frac{c_2\gamma_2(\gamma_2 - \beta)}{k_3\beta} y^{\gamma_2-\beta-1} > 0.$$

The ratio $\frac{c_1\phi'+c_2\psi'}{g_r'}$ is therefore strictly increasing and b^* is the only value which satisfies the smooth pasting principle. Thus, for $x > b^*$, there cannot be two distinct optimal points a^* and b^*. The only optimal stopping rule is $\tau^* = 0$.

The value function is therefore

$$V_r(x) = \begin{cases} \left(\dfrac{g_r(b^*)}{(b^*)^{\gamma_2}}\right) x^{\gamma_2}, & \text{for } x \leq b^*, \\[2ex] k_3 x - k_2, & \text{for } x \geq b^*. \end{cases}$$

The value function for the original stopping problem is

$$V(x) = \begin{cases} \left(\dfrac{g_r(b^*)}{(b^*)^{\gamma_2}} - \dfrac{R}{(\sigma^2/2)\beta(\beta-1) + \mu\beta - \alpha}\right) x^{\gamma_2}, & \text{for } x \leq b^*, \\[2ex] k_1 x^\beta - k_2, & \text{for } x \geq b^*. \end{cases}$$

Example 5.5 Discontinuous Reward
In order that the algebra be tractable, we consider the specific model in which $\mu = \frac{\sigma^2}{2}$ and $\alpha = 2\sigma^2$. With this choice of parameters, $\phi(y) = y^{-2}$ and $\psi(y) = y^2$.

Let X denote the quantity of some product. Let g_1 and g_2 be positive constants with $g_1 < g_2$. Consider the reward function $g(y) = g_1 I_{[0,x_1)}(y) + g_2 I_{[x_1,\infty)}(y)$. The discontinuity of the reward function represents the possibility that the reward changes dramatically once the quantity achieves a certain threshold. The goal of the decision maker is to determine a stopping time $\tau \in \mathcal{A}$ so as to maximize $E\left[e^{-\alpha\tau} g(X(\tau))\right]$.

Consider any initial point $x \geq x_1$. In this case, it is simpler to solve the restricted dual linear program. Observe that g satisfies Condition 4.5(b). Define $c_1 = \frac{g_2}{2} x^2$ and $c_2 = \frac{g_2}{2} x^{-2}$. which are the solutions of (4.13). Since $g'(y) = 0$ for $y > x_1$, the function $c_1\phi(y) + c_2\psi(y) = \frac{g_2}{2}\left(\frac{x^2}{y^2} + \frac{y^2}{x^2}\right)$ has minimum value g_2 at x and hence majorizes g. Therefore the optimal stopping time is $\tau = 0$ corresponding to selecting $a^* = x = b^*$.

Consider $x \in (0, x_1)$. Selecting $a = x = b$ results in $\tau_{a,b} = 0$ and a value $J(\tau_{a,b}; x) = g(x) = g_1$. Now consider other stopping locations. Clearly $a < x < x_1$ so $g(a) = g_1$. When b is chosen so that $x < b < x_1$, defining the strictly convex function $J_{a,b}$ on $[x_l, x_r]$, along with $J_{a,b}(a) = g_1 = J_{a,b}(b)$, indicates that $J(\tau_{a,b}; x) = J_{a,b}(x) < g(x)$ which implies that $\tau_{a,b}$ is suboptimal. When $b \geq x_1$, the possibility exists for the value to exceed g_1. We now examine this case more carefully.

Since $g(y) = g_2$ for all $y \geq x_1$ and choosing $b > x_1$ requires a longer time for the process to hit b than for it to hit x_1 and hence more discounting to occur, it is clear that the optimal choice for stopping to the right of x is $b^* = x_1$.

Consider now $h(a) := J(\tau_{a,x_1}; x)$ given by (2.19). For the particular geometric Brownian motion under consideration, the expression simplifies to

$$h(a) = \frac{g_2 x^2 x_1^2 + g_1\left[(x_1/x)^2 - x^2\right] a^2 - g_2 (x_1/x)^2 a^4}{x_1^4 - a^4}.$$

Setting $h'(a) = 0$ results in the equation

$$\frac{2\left(x_1^4/x^2 - x^2\right)}{x_1^4 - a^4} \cdot a\left[g_1 x_1^4 - 2 g_2 x_1^2 a^2 + g_1 a^4\right] = 0$$

having solutions $a = 0$ and

$$a = \sqrt{\frac{g_2}{g_1} - \sqrt{\left(\frac{g_2}{g_1}\right)^2 - 1}} \cdot x_1. \tag{5.8}$$

Observe that for x sufficiently close to 0, $J(\tau_{0,b}; x) = \left(\frac{g_2}{x_1^2}\right) x^2 < g_1$ and thus $a = 0$ is not an optimal choice. Let a_* be given by (5.8). Then for $a_* \leq x \leq x_1$, τ_{a_*,x_1} is an optimal stopping rule. Define the function

$$J_{a_*,x_1}(y) = \frac{g_2 x_1^2 - g_1 a_*^2}{x_1^4 - a_*^4}\, y^2 + \frac{g_1 a_*^2 - g_2 a_*^4}{x_1^4 - a_*^4}\,(x_1/y)^2, \qquad y \in [0, \infty).$$

Thus J_{a_*,x_1} is the same function as $J(\tau_{a_*,x_1}; x)$ but without the restriction on the independent variable that $a_* \leq x \leq x_1$. Figure 4 displays the relationship between g and J_{a_*,x_1}.

For $x \leq a_*$, any choice of $a < x$ results in $J(\tau_{a,x_1}; x) < g(x)$ so the optimal stopping rule is to stop immediately.

From the above analysis, we have determined the value function V to be

$$V(x) = \begin{cases} g_1, & \text{for } 0 \leq x \leq a_*, \\ c_1(a_*, x_1)x^{-2} + c_2(a_*, x_1)x^2, & \text{for } a_* \leq x \leq x_1, \\ g_2, & \text{for } x \geq x_1. \end{cases}$$

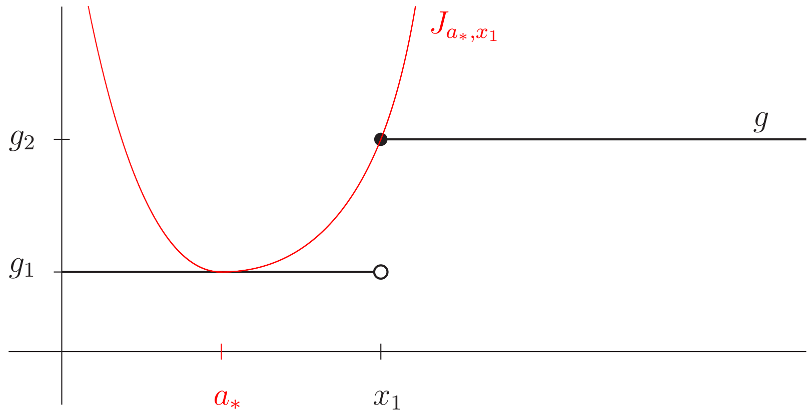


Figure 4: Relation between g and J_{a_*,x_1}

This example is similar to an example of Salminen [14] in which he considers a drifted Brownian motion on $(-\infty, \infty)$ having piecewise constant reward function. A significant distinction is that in our example, the reward function is upper semicontinuous, contrasting with the lower semicontinuous reward function of [14], and resulting in the existence of an optimal stopping time.

We now investigate several examples in which the diffusion is not a geometric Brownian motion process. We begin by revisiting the "up-and-out" barrier put option of Example 5.2.

Example 5.6 Constant-Elasticity-of-Variance Model
This example is analyzed in Example 6.2 of [4] using the generalized concavity approach.

Let $\beta \in (0, 1)$ be given. The stock price process is given by the constant-elasticity-of-variance model:

$$dX(t) = \alpha X(t)\,dt + \sigma X^{1-\beta}(t)\,dW(t), \quad X(0) = x,$$

in which $\alpha > 0$ denotes the interest rate on the non-risky asset, b_0 denotes the "up-and-out" barrier and the dynamics of X are given relative to the risk-neutral measure. The objective is to maximize over all admissible τ

$$E\left[e^{-\alpha\tau}(K - X(\tau))^+\right]$$

in which $K < b_0$ denotes the strike price of the option.

Again for $x \geq b_0$, the barrier is immediately reached or exceeded so the option is worthless. Thus for every admissible stopping time τ, $J(\tau; x) = 0$ and hence $V(x) = 0$ for $x \geq b_0$.

Now consider x with $0 \leq x \leq b_0$. The process X lives on $[0, \infty)$ and has generator $Af(y) = \frac{\sigma^2}{2} y^{2(1-\beta)} f''(y) + \alpha y\, f'(y)$ defined for $f \in C^2[0, \infty)$. The decreasing and increasing solutions, respectively, of $Af - \alpha f = 0$ are

$$\phi(y) = y \int_y^\infty \frac{1}{z^2}\, e^{-\frac{\alpha}{\beta\sigma^2} z^{2\beta}}\,dz, \qquad \text{and} \qquad \psi(y) = y.$$

Straightforward calculations show that $\phi(0+) = 1$ and $\phi''(x) > 0$ for each $x > 0$. Note in this example, 0 is an exit-not-entrance boundary point and so is part of the state space for X.

Let $0 \le a < x < b \le b_0$ be given and consider the two-sided stopping rule $\tau_{a,b}$. The value (2.19) associated with $\tau_{a,b}$ is

$$J(\tau_{a,b}; x) = \frac{\phi(x)\psi(b) - \phi(b)\psi(x)}{\phi(a)\psi(b) - \phi(b)\psi(a)}(K-a)^+ + \frac{\phi(a)\psi(x) - \phi(x)\psi(a)}{\phi(a)\psi(b) - \phi(b)\psi(a)}(K-b)^+. \tag{5.9}$$

The same arguments as in Example 5.2 establish (i) that the optimal b is $b^* = b_0$ for every a and (ii) that the optimal choice for a is in $[0, K)$. Notice that $K < b_0 = b^*$ implies that the second term in (5.9) is 0 and hence we only need to optimize the first summand in a. Observe also that the restriction $a \in [0, K)$ implies $(K-a)^+ = K-a$ and this is continuously differentiable on $[0, K)$. There are two possibilities for the optimal a: either $a \in (0, K)$ or $a = 0$. The former case holds when the derivative (with respect to a) of $J(\tau_{a,b_0}; x)$ equals 0. This requirement is satisfied when the following condition holds:

$$h(a) := \phi(b_0)a - b_0\phi(a) - (K-a)[b_0\phi'(a) - \phi(b_0)] = 0. \tag{5.10}$$

Observe that $h(K) < 0$. The fact that $\phi'' > 0$ implies that $\lim_{a \searrow 0} \phi'(a)$ exists. Denote this limit by $\phi'(0+)$ and observe that $\phi'(0+)$ may equal $-\infty$. The condition (5.10) is satisfied by some (unique) $a \in [0, K)$ when

$$\phi'(0+) \le \frac{\phi(b_0)}{b_0} - \frac{1}{K}\,. \tag{5.11}$$

We remark that when $\beta \le \frac{1}{2}$, one can show that $\phi'(0+) = -\infty$ which guarantees the existence of an optimizing $a \in (0, K)$; the value of $\phi'(0+)$ is unknown for $\frac{1}{2} < \beta < 1$. Should (5.11) fail, then the optimizer is $a^* = 0$. Letting a^* denote the optimizer for (5.9), the optimal stopping rule is

$$\tau^* = \begin{cases} \tau_{a^*,b_0}, & \text{for } a^* \le x \le b_0, \\ 0, & \text{otherwise} \end{cases}$$

and the value function is

$$V(x) = \begin{cases} K - x, & \text{for } 0 \le x \le a^*, \\ \frac{K-a^*}{\phi(a^*)b_0 - \phi(b_0)a^*}\left[b_0 \cdot \phi(x) - \phi(b_0)x\right], & \text{for } a^* \le x \le b_0, \\ 0, & \text{for } x \ge b_0. \end{cases}$$

Example 5.7 Optimal Stopping Problem for a Mean Reverting Process
This example comes from Example 6.10 of [4]. The diffusion process X satisfies the stochastic differential equation

$$dX(t) = \mu X(t)(R - X(t))\,dt + \sigma X(t)\,dW(t), \qquad X(0) = x > 0, \tag{5.12}$$

in which μ, R and σ are positive constants. Notice that $x_l = 0$ and $x_r = \infty$ and both boundaries are natural. The generator of X is $Af(y) = \frac{\sigma^2}{2}y^2 f''(y) + \mu y(R-y)f'(y)$ for $f \in C^2(0, \infty)$.

Let $\alpha > 0$ denote the discount rate and set $K > 0$. The goal is to maximize

$$E\left[e^{-\alpha\tau}(X(\tau) - K)^+\right] \tag{5.13}$$

over all $\tau \in \mathcal{A}$. We remark that this problem differs from pricing a perpetual call option having strike K since the expectation is taken with respect to the 'real-world' probability measure, not the 'risk-neutral' measure.

We begin by considering the restricted dual LP. Let ϕ and ψ denote the decreasing and increasing solutions of $Af - \alpha f = 0$, respectively. Since 0 is a natural boundary, $\phi(0+) = \infty$. As a result of $g(y) = (y - K)^+$ being non-negative, c_1 must also be non-negative. The goal of the restricted dual LP is to minimize $c_1\phi(x) + c_2\psi(x)$ so taking $c_1 = 0$, if possible, would be best. We therefore optimize

$$J(\tau_b; x) = \frac{(b - K)^+}{\psi(b)} \cdot \psi(x)$$

over $b > 0$.

Let M denote the Kummer M-function

$$M(a, b, z) = 1 + \frac{az}{b} + \frac{(a)_2}{(b)_2}\frac{z^2}{2!} + \cdots + \frac{(a)_n}{(b)_n}\frac{z^n}{n!} + \cdots ,$$

where for $c = a, b$, $(c)_n := c(c+1)(c+2) \cdot \ldots \cdot (c+n-1)$, $(c)_0 := 1$ (see [1] for details). $M(a, b, z)$ is the increasing solution of the ordinary differential equation

$$zf''(z) + (b - z)f'(z) - af(z) = 0.$$

Let $\gamma_1 = \left(\frac{1}{2} - \frac{\mu R}{\sigma^2}\right) - \sqrt{\left(\frac{1}{2} - \frac{\mu R}{\sigma^2}\right)^2 + \frac{2\alpha}{\sigma^2}}$ and $\gamma_2 = \left(\frac{1}{2} - \frac{\mu R}{\sigma^2}\right) + \sqrt{\left(\frac{1}{2} - \frac{\mu R}{\sigma^2}\right)^2 + \frac{2\alpha}{\sigma^2}}$, noting that $\gamma_1 < 0 < \gamma_2$, and let $c = \frac{2\mu}{\sigma^2}$. The increasing solution ψ of $Af - \alpha f = 0$ is

$$\psi(y) = (cy)^{\gamma_2}\, M(\gamma_2, 2\gamma_2 + cR, cy), \qquad y \geq 0.$$

Using the integral representation of the confluent hypergeometric function M, ψ can also be expressed as

$$\psi(y) = \frac{\Gamma(2\gamma_2 + cR)}{\Gamma(\gamma_2 + cR)\Gamma(\gamma_2)}\,(cb)^{\gamma_2} \int_0^1 e^{cbt} t^{\gamma_2 - 1}(1 - t)^{-\gamma_1}\, dt, \qquad g \geq 0.$$

Thus we need to maximize the ratio

$$h(b) = \frac{(b - K)^+}{b^{\gamma_2} M(\gamma_2, 2\gamma_2 + c, cb)} = \frac{(b - K)^+}{b^{\gamma_2} \int_0^1 e^{cyt} t^{\gamma_2 - 1}(1 - t)^{-\gamma_1}\, dt} \tag{5.14}$$

over $b > 0$. A brief examination of (5.14) indicates that $h(b) = 0$ for $b \leq K$ and the denominator dominates the numerator for large b so $\lim_{b\to\infty} h(b) = 0$. Furthermore $h(b) > 0$ for $b > K$. Using the integral representation and seeking the critical point(s) of h by setting the derivative equal to zero results (after some algebra) in the equation (for $b > K$)

$$\gamma_2 K - \left[(\gamma_2 - 1)b + cb(b - K)\frac{\int_0^1 e^{cbt} t^{\gamma_2}(1 - t)^{-\gamma_1}\, dt}{\int_0^1 e^{cbt} t^{\gamma_2 - 1}(1 - t)^{-\gamma_1}\, dt}\right] = 0. \tag{5.15}$$

Examining the lefthand-side of (5.15) we see that the expression is initially positive, but is strictly decreasing in b (for $b > K$). Hence there is a unique value b^* for which equality holds in (5.15) and it follows that b^* is a maximizer. We have therefore identified the optimal value for each $x \le b^*$.

When $x > b^*$, the fact that $g'(b) = 1$ for $b > x > b^* > K$ allows the same smooth pasting argument as in solution 1 of Example 5.1 to be applied to conclude that it is optimal to stop immediately. Hence the value function is

$$V(x) = \begin{cases} \dfrac{(b^* - K)^+}{(cb^*)^\gamma M(\gamma, 2\gamma + cR, cb^*)} \cdot (cx)^\gamma M(\gamma, 2\gamma + cR, cx), & \text{for } 0 \le x \le b^*, \\ \\ (x - K)^+, & \text{for } x \ge b^*, \end{cases}$$

and the optimal stopping rule is

$$\tau^* = \begin{cases} \tau_{b^*}, & \text{for } 0 < x \le b^*, \\ 0, & \text{for } x > b^*. \end{cases}$$

Example 5.8 Brownian Motion with Piecewise Linear Reward
Let $\mu \equiv 0$ and $\sigma(x) \equiv 1$ so that X is a Brownian motion process whose generator is $Af = \frac{1}{2}f''$ for $f \in C^2(-\infty, \infty)$. The boundary points $\pm\infty$ are both natural boundaries. Let $x_0 > 0$ and $c > 0$ be constants such that $1 - cx_0 > 0$. Define the terminal reward function

$$g(x) = \begin{cases} 1, & \text{for } x \le 0, \\ 1 - cx, & \text{for } 0 \le x \le x_0, \\ 1 - cx_0, & \text{for } x \ge x_0. \end{cases}$$

The objective is to maximize $E\left[e^{-\alpha\tau} g(X(\tau))\right]$ over admissible stopping times τ. This example has been considered by Øksendal and Reikvam [11] and also appears as Example 6.11 of [4].

The decreasing and increasing solutions to $Af - \alpha f = 0$ are $\phi(y) = e^{-\sqrt{2\alpha}y}$ and $\psi(y) = e^{\sqrt{2\alpha}y}$, respectively. Let x be fixed and consider a and b such that $a \le x \le b$ with $a < b$. The value (2.19) corresponding to the stopping rule $\tau_{a,b}$ is

$$J(\tau_{a,b}; x) = \left(\frac{e^{\sqrt{2\alpha}(b-x)} - e^{-\sqrt{2\alpha}(b-x)}}{e^{\sqrt{2\alpha}(b-a)} - e^{-\sqrt{2\alpha}(b-a)}} \right) g(a) + \left(\frac{e^{\sqrt{2\alpha}(x-a)} - e^{-\sqrt{2\alpha}(x-a)}}{e^{\sqrt{2\alpha}(b-a)} - e^{-\sqrt{2\alpha}(b-a)}} \right) g(b). \tag{5.16}$$

Consider first the case in which $x < 0$. Taking $c_1 = \frac{1}{2}e^{\sqrt{2\alpha}x}$ and $c_2 = \frac{1}{2}e^{-\sqrt{2\alpha}x}$, the function $[c_1\phi + c_2\psi](y) = \cosh(y - x)$ has minimum value 1 at x and therefore majorizes g. Thus, it is optimal to stop the process immediately.

We now consider $x \ge 0$. Since g is piecewise linear, there are two cases to analyze. The optimal left endpoint a^* must satisfy the smooth pasting conditions when $0 < a^* < x_0$ or should $a^* = 0$, the only conditions would be that $c_1\phi(0) + c_2\psi(0) = 1$ and $c_1\phi'(0) + c_2\psi'(0) \ge g'(0+)$.

We begin by examining the second case; that is, we seek a linear combination $c_1\phi + c_2\psi$ of ϕ and ψ for which $(c_1\phi + c_2\psi)(0) = 1$ and $(c_1\phi + c_2\psi)'(0) \ge -c = g'(0+)$. More specifically, we are considering the case in which the left stopping location is $a = 0$ and we wish to optimize over the right stopping location b. The smooth pasting condition must be satisfied at the optimal choice b^*. This situation is illustrated in Figure 5(a).

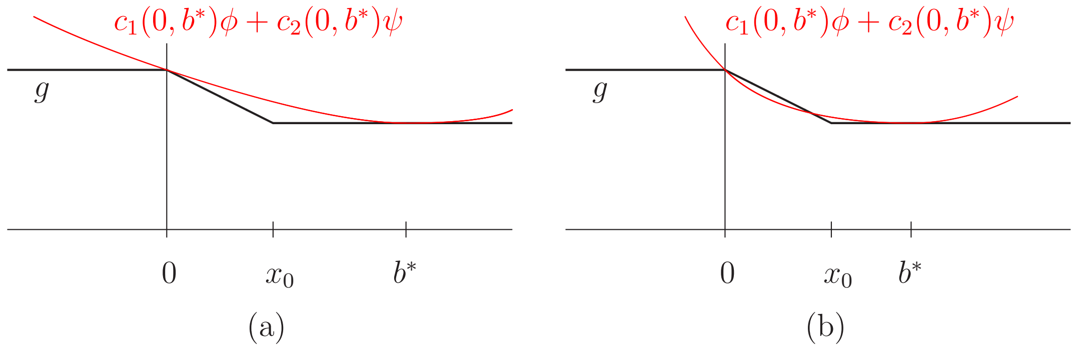


Figure 5: Possibilities for $c_1(0,b^*)\phi + c_2(0,b^*)\psi$ when $c_1(0,b^*)\phi(0) + c_2(0,b^*)\psi(0) = 1$.

The smooth pasting conditions require

$$\begin{cases} 1 - cx_0 = c_1(0,b)e^{-\sqrt{2\alpha}b} + c_2(0,b)e^{\sqrt{2\alpha}b} \\ \qquad 0 = -\sqrt{2\alpha}c_1(0,b)e^{-\sqrt{2\alpha}b} + \sqrt{2\alpha}c_2(0,b)e^{\sqrt{2\alpha}b}. \end{cases}$$

The second equation implies $c_1(0,b) = c_2(0,b)e^{2\sqrt{2\alpha}b}$ which, when used in the first equation, yields

$$1 - cx_0 = 2c_2(0,b)e^{\sqrt{2\alpha}b}.$$

Using (4.12), one can determine that the value of b satisfying this equation is

$$b^* = \frac{1}{\sqrt{2\alpha}} \ln\left(\frac{1}{1-cx_0}\left(1 + \sqrt{1-(1-cx_0)^2}\right)\right) \tag{5.17}$$

and the corresponding coefficients c_1 and c_2 are

$$c_1(0,b^*) = \frac{1}{2} + \frac{1}{2}\sqrt{1-(1-cx_0)^2} \quad \text{and } c_2(0,b^*) = \frac{1}{2} - \frac{1}{2}\sqrt{1-(1-cx_0)^2}.$$

The condition $(c_1\phi + c_2\psi)'(0) \geq -c$ is satisfied when the parameters satisfy the condition

$$\frac{4\alpha x_0}{1 + 2\alpha x_0^2} \leq c, \tag{5.18}$$

which is the condition assumed in [11].

For $x \geq b^*$, the function $[c_1\phi + c_2\psi](y) = (1 - cx_0)\cosh(y - x)$ majorizes g and equals g at x. Therefore τ_x is an optimal stopping time.

Thus when the parameters satisfy (5.18), the value function is

$$V(x) = \begin{cases} 1, & \text{for } x \leq 0, \\ (\frac{1}{2} + \frac{1}{2}\sqrt{1-(1-cx_0)^2})e^{-\sqrt{2\alpha}x} + (\frac{1}{2} - \frac{1}{2}\sqrt{1-(1-cx_0)^2})e^{\sqrt{2\alpha}x}, & \text{for } 0 \leq x \leq b^*, \\ 1 - cx_0, & \text{for } x \geq b^*. \end{cases}$$

Now consider the case in which the parameters α, c and x_0 do not satisfy (5.18). Figure 5(b) displays the function $c_1(0,b)\phi(x) + c_2(0,b)\psi(x)$ when b satisfies the smooth pasting conditions. Clearly this function does not majorize g so $a = 0$ is not optimal. Thus $a^* > 0$

and the smooth pasting conditions must be satisfied at both a^* and b^*. These conditions are:

$$\begin{cases} 1-ca = c_1(a,b)e^{-\sqrt{2\alpha}a} + c_2(a,b)e^{\sqrt{2\alpha}a} \\ 1-cx_0 = c_1(a,b)e^{-\sqrt{2\alpha}b} + c_2(a,b)e^{\sqrt{2\alpha}b} \\ -c = -\sqrt{2\alpha}c_1(a,b)e^{-\sqrt{2\alpha}a} + \sqrt{2\alpha}c_2(a,b)e^{\sqrt{2\alpha}a} \\ 0 = -\sqrt{2\alpha}c_1(a,b)e^{-\sqrt{2\alpha}b} + \sqrt{2\alpha}c_2(a,b)e^{\sqrt{2\alpha}b}. \end{cases}$$

Observe that the second and fourth equations are the same as in the previous case, so we immediately have

$$c_1(a,b) = \frac{1-cx_0}{2}\, e^{\sqrt{2\alpha}b} \qquad \text{and} \qquad c_2(a,b) = \frac{1-cx_0}{2}\, e^{-\sqrt{2\alpha}b}.$$

Using these values in the third equation establishes

$$b-a = \frac{1}{\sqrt{2\alpha}} \sinh^{-1}\left(\frac{c\sqrt{2\alpha}}{1-cx_0}\right).$$

The first equation can be written as

$$\frac{1-ca}{1-cx_0} = \cosh(\sqrt{2\alpha}(b-a))$$

which solving for a results in

$$a^* = \frac{1}{c}\left[1-(1-cx_0)\cosh\left(\sinh^{-1}\left(\frac{\sqrt{c2\alpha}}{1-cx_0}\right)\right)\right]$$

and hence

$$b^* = \frac{1}{c}\left[1-(1-cx_0)\cosh\left(\sinh^{-1}\left(\frac{\sqrt{c2\alpha}}{1-cx_0}\right)\right)\right] + \frac{1}{\sqrt{2\alpha}} \sinh^{-1}\left(\frac{c\sqrt{2\alpha}}{1-cx_0}\right).$$

Thus for $a^* \le x \le b^*$, the value function V is determined. For $x \ge b^*$, the argument of the previous case indicates that optimal stopping must occur immediately and hence

$$V(x) = \begin{cases} 1, & \text{for } x \le 0, \\ 1-cx, & \text{for } 0 \le x \le a^*, \\ (1-cx_0)\cosh\left(\sqrt{2\alpha}(b^*-x)\right), & \text{for } a^* \le x \le b^*, \\ 1-cx_0, & \text{for } x \ge b^*, \end{cases}$$

and the optimal stopping rule is

$$\tau^* = \begin{cases} \tau_{a^*,b^*}, & \text{for } a^* \le x \le b^*, \\ 0, & \text{otherwise.} \end{cases}$$

Example 5.9 Drifted Brownian Motion with Running Cost
A version of this optimal stopping problem arose in the study of an optimal stochastic control problem by Karatzas and Ocone [10] and was examined in Example 6.9 of [4].

Let θ be a positive constant. The process X satisfies the stochastic differential equation

$$dX(t) = -\theta\,dt + dW(t), \qquad X(0) = x,$$

in which W is a standard Brownian motion and $x \in (-\infty, \infty)$. The objective is to select an admissible stopping time τ so as to minimize

$$E\left[\int_0^\tau e^{-\alpha t} X^2(t)\,dt + \delta e^{-\alpha\tau} X^2(\tau)\right] \tag{5.19}$$

in which $\alpha > 0$ denotes the discount rate and δ is a positive constant. The generator of X is $Af(y) = -\theta f'(y) + \frac{1}{2} f''(y)$, defined for all $f \in C^2(-\infty, \infty)$.

Our formulation of this optimal stopping problem differs from that in Example 6.9 of [4] in that we allow initial positions $x < 0$ and do not have 0 as an absorbing barrier. The more restricted problem can be solved by considering $\tau \wedge \tau_0$, where τ_0 denotes the first hitting time of $\{0\}$ and only taking $x \geq 0$.

The first task is to rephrase the minimization problem as the negative of the maximization of the negative of the running and terminal cost functions. Secondly, by considering $f_r(y) = \frac{1}{\alpha} y^2 - \frac{2\theta}{\alpha^2} y + \frac{\alpha+2\theta^2}{\alpha^3}$, $Af(y) - \alpha f(y) = -y^2$ and the objective function (5.19) becomes

$$E\left[e^{-\alpha\tau}\left(\frac{1-\alpha\delta}{\alpha} X^2(\tau) - \frac{2\theta}{\alpha^2} X(\tau) + \frac{\alpha + 2\theta^2}{\alpha^3}\right)\right];$$

we have omitted from this objective function both the constant correction $-f_r(x)$ and the negative of the entire expression. These must be taken into account when determining the value function for the original minimization problem. The adjusted terminal reward function is $g_r(y) = \frac{1-\alpha\delta}{\alpha} y^2 - \frac{2\theta}{\alpha^2} y + \frac{\alpha+2\theta^2}{\alpha^3}$.

Let $\gamma_1 = \theta - \sqrt{\theta^2 + 2\alpha}$ and $\gamma_2 = \theta + \sqrt{\theta^2 + 2\alpha}$ and note that $\gamma_1 < 0 < \gamma_2$. The decreasing and increasing solutions to $Af - \alpha f = 0$ are $\phi(y) = e^{\gamma_1 y}$ and $\psi(y) = e^{\gamma_2 y}$, respectively.

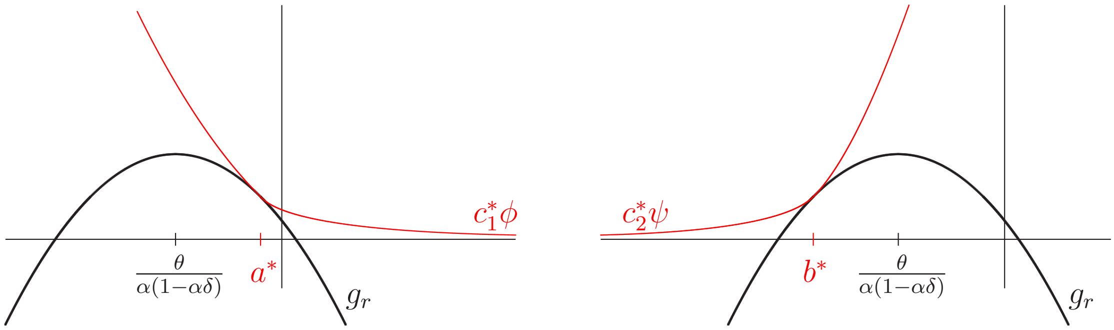
There are several cases to analyze depending on the value of $1 - \alpha\delta$.

Case i: $1-\alpha\delta < 0$**.** In this case, g_r is a quadratic function with a negative leading coefficient. This case is illustrated in Figure 6. The maximum value of g_r occurs at $\frac{\theta}{\alpha(1-\alpha\delta)} < 0$ and $g_r(0) > 0$. Observe that $y^2/e^{\gamma_1 y} \to 0$ as $y \to -\infty$ and similarly $y^2/e^{\gamma_2 y} \to 0$ as $y \to \infty$, which implies that c_1 and c_2 must be non-negative in order for $c_1\phi + c_2\psi$ to majorize g_r. Since the restricted dual LP seeks to minimize $[c_1\phi + c_2\psi](x)$, it would be best to set either $c_1 = 0$ or $c_2 = 0$, if possible.

Consider the second case. Maximizing the ratio $\frac{g_r(y)}{\phi(y)} = \frac{1-\alpha\delta}{\alpha} y^2 e^{-\gamma_1 y} - \frac{2\theta}{\alpha^2} y e^{-\gamma_1 y} + \frac{\alpha+2\theta^2}{\alpha^3} e^{-\gamma_1 y}$ determines the optimal coefficient c_1^*. Straightforward calculations show that $\left(\frac{g_r(y)}{\phi(y)}\right)'$ is positive when $y = \frac{\theta}{\alpha(1-\alpha\delta)}$ and negative when $y = 0$ and monotone decreasing on $\left(\frac{\theta}{\alpha(1-\alpha\delta)}, 0\right)$ so there is a unique value a^* where the derivative is 0. The value a^* is the solution of

$$-\alpha^2\gamma_1(1-\alpha\delta)\,y^2 + 2(\alpha^2(1-\alpha\delta) + \alpha\gamma_1\theta)\,y - (2\alpha\theta + \gamma_1(\alpha + 2\theta^2)) = 0 \tag{5.20}$$

that is greater than $\frac{\theta}{\alpha(1-\alpha\delta)}$. Let $c_1^* = \frac{g_r(a^*)}{\phi(a^*)}$. Then for $x \geq a^*$, the optimal stopping rule is τ_{a^*} and $V_r(x) = c_1^*\phi(x)$.



(a) $x > a^*$ (b) $x < b^*$

Figure 6: g_r, $c_1^*\phi$ and $c_2^*\psi$ when $1 - \alpha\delta < 0$

A similar analysis applies to find b^* and $c_2^* = \frac{g_r(b^*)}{\psi(b^*)}$ such that for $x \le b^*$ the optimal stopping time is τ_{b^*} and the value function is $V_r(x) = c_2^*\psi(x)$. In fact, b^* is the root of equation (5.20), in which γ_1 is replaced by γ_2, with the root being less than $\frac{\theta}{\alpha(1-\alpha\delta)}$.

Figure 6 illustrates the relation between g_r and the optimal multiples of ϕ and ψ.

For $b^* \le x \le a^*$, using the stopping rule $\tau_{a,b}$ with $a < x < b$, the concavity of g_r along with the convexity of $c_1(a,b)\phi + c_2(a,b)\psi$ and the equality of g_r and $c_1(a,b)\phi + c_2(a,b)\psi$ at $y = a$ and $y = b$ results in $J_r(\tau_{a,b}; x) < g_r(x)$. Hence the optimal stopping time is $\tau^* = 0$ with resulting value function $V_r(x) = g_r(x)$.

Summarizing, the optimal stopping rule τ^* and corresponding value function V_r are

$$\tau^* = \begin{cases} \tau_{b^*}, & \text{for } x \le b^*, \\ 0, & \text{for } b^* \le x \le a^*, \\ \tau_{a^*}, & \text{for } x \ge a^*, \end{cases} \qquad \text{and} \qquad V_r(x) = \begin{cases} c_2^*\psi(x), & \text{for } x \le b^*, \\ g_r(x), & \text{for } b^* \le x \le a^*, \\ c_1^*\phi(x), & \text{for } x \ge a^*. \end{cases}$$

The value function V for the original optimal stopping problem which seeks to minimize (5.19) is

$$V(x) = \begin{cases} \frac{1}{\alpha}x^2 - \frac{2\theta}{\alpha^2}x + \frac{\alpha+2\theta^2}{\alpha^3} - c_2^* e^{\gamma_2 x}, & \text{for } x \le b^*, \\ \delta x^2, & \text{for } b^* \le x \le a^*, \\ \frac{1}{\alpha}x^2 - \frac{2\theta}{\alpha^2}x + \frac{\alpha+2\theta^2}{\alpha^3} - c_1^* e^{\gamma_1 x}, & \text{for } x \ge a^*, \end{cases}$$

Case ii: $1 - \alpha\delta = 0$. In this case, g_r is a linear function with negative slope and the analysis of the previous case with x large applies to find the root $a^* = \frac{\alpha+2\theta^2}{2\alpha^2\theta} + \frac{1}{\alpha\gamma_1}$ of (5.20), recognizing that the quadratic term has coefficient 0. The optimal stopping rule is to stop immediately when $x \le a^*$ and to use the hitting time τ_{a^*} when $x \ge a^*$. The resulting value function V for the original optimal stopping problem is

$$V(x) = \begin{cases} \delta x^2, & \text{for } x \le a^*, \\ \frac{1}{\alpha}x^2 - \frac{2\theta}{\alpha^2}x + \frac{\alpha+2\theta^2}{\alpha^3} - c_1^* e^{\gamma_1 x}, & \text{for } x \ge a^*, \end{cases}$$

Case iii: $1 - \alpha\delta > 0$. In this case, g_r is a quadratic function with positive leading coefficient. Since g_r is continuously differentiable, we seek two points a^* and b^* which satisfy the smooth pasting conditions. Using coefficients $c_1(a,b)$ and $c_2(a,b)$ given by (4.12) implies that $c_1(a,b)\phi(a) + c_2(a,b)\psi(a) = g_r(a)$ and similarly for b, for every choice of a and b. Thus the smooth pasting conditions are imposed on the derivatives and require

$$\begin{cases} \gamma_1 c_1(a,b)e^{\gamma_1 a} + \gamma_2 c_2(a,b)e^{\gamma_2 a} = g_r'(a) \\ \gamma_1 c_1(a,b)e^{\gamma_1 b} + \gamma_2 c_2(a,b)e^{\gamma_2 b} = g_r'(b), \end{cases}$$

or

$$\begin{cases} \left(\gamma_1 e^{\gamma_1 a+\gamma_2 b} - \gamma_2 e^{\gamma_2 a+\gamma_1 b}\right) g_r(a) + (\gamma_1 - \gamma_2)\, e^{(\gamma_1+\gamma_2)a} g_r(b) \\ \qquad\qquad = \left(\frac{2(1-\alpha\delta)}{\alpha} a - \frac{2\theta}{\alpha^2}\right)\left(e^{\gamma_1 a+\gamma_2 b} - e^{\gamma_1 b+\gamma_2 a}\right) \\ \\ (\gamma_1 - \gamma_2)\, e^{(\gamma_1+\gamma_2)b} g_r(a) + \left(\gamma_2 e^{\gamma_1 a+\gamma_2 b} - \gamma_1 e^{\gamma_2 a+\gamma_1 b}\right) g_r(b) \\ \qquad\qquad = \left(\frac{2(1-\alpha\delta)}{\alpha} b - \frac{2\theta}{\alpha^2}\right)\left(e^{\gamma_1 a+\gamma_2 b} - e^{\gamma_1 b+\gamma_2 a}\right). \end{cases} \tag{5.21}$$

Let $a^* < b^*$ denote the solutions of (5.21) and let $c_1^* = c_1(a^*, b^*)$ and $c_2^* = c_2(a^*, b^*)$. Figure 7 illustrates the relation between g_r and $c_1^*\phi + c_2^*\psi$.

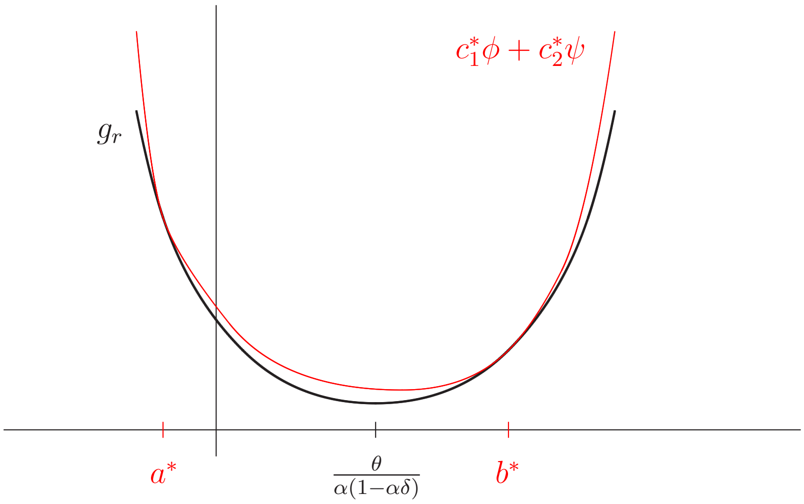


Figure 7: Relation between g_r and $c_1^*\phi + c_2^*\psi$

The optimal stopping time and value function V_r are

$$\tau^* = \begin{cases} \tau_{a^*,b^*}, & \text{for } a^* \le x \le b^*, \\ 0, & \text{otherwise,} \end{cases} \qquad \text{and} \qquad V_r(x) = \begin{cases} c_1^*\phi(x) + c_2^*\psi(x), & \text{for } a^* \le x \le b^*, \\ g_r(x), & \text{otherwise.} \end{cases}$$

The value function V for the original optimal stopping problem is

$$V(x) = \begin{cases} \frac{1}{\alpha} x^2 - \frac{2\theta}{\alpha^2} x + \frac{\alpha+2\theta^2}{\alpha^3} - c_1^* e^{\gamma_1 x} - c_2^* e^{\gamma_2 x}, & \text{for } a^* \le x \le b^*, \\ \\ \delta x^2, & \text{otherwise.} \end{cases}$$

Example 5.10 Cantor Set Indicator as Terminal Reward
Again consider a Brownian motion process X having generator $Af = \frac{1}{2}f''$ and solutions $\phi(y) = e^{-\sqrt{2\alpha}y}$ and $\psi(y) = e^{\sqrt{2\alpha}y}$ of $Af - \alpha f = 0$. Let $\mathbb{C}_1$ denote the Cantor set in $[0,1]$ obtained by removing successive middle thirds and define the set $\mathbb{C}$ to be the union of all

integer translations of $\mathbb{C}_1$. Let $g = I_{\mathbb{C}}$ be the terminal reward function. Since $\mathbb{C}$ is perfect, g is upper semicontinuous. The objective is to optimize $E\left[e^{-\alpha\tau} g(X(\tau))\right]$ over all admissible stopping times τ.

To determine the value function V and optimal stopping rules for this problem, two cases must be analyzed.

Case i: Suppose the initial value x is an element of $\mathbb{C}$. Define c_1^* and c_2^* to be the solution of the system (4.13) with $g_r(x) = 1$ and replacing $g_r'(x)$ by 0. It then follows that $c_1^*\phi + c_2^*\psi$ has minimum value 1 at x and majorizes g_r. The stopping rule corresponding to this choice of c_1^* and c_2^* is $\tau^* = \tau_x = 0$ and it is optimal to stop the process immediately.

We observe that when $x \in \mathbb{C}$, even though $\limsup_{y \nearrow x} \frac{g(y)-g(x)}{y-x} = \infty$ and $\liminf_{z \searrow x} \frac{g(z)-g(x)}{z-x} = -\infty$, g satisfies Condition 4.5(a) with the two values being equal.

Case ii: Suppose $x \notin \mathbb{C}$. Then x is an element of some open interval which does not intersect $\mathbb{C}$. Let

$$a_x = \max\{c \in \mathbb{C} : c < x\} \qquad \text{and} \qquad b_x = \min\{c \in \mathbb{C} : c > x\}. \tag{5.22}$$

We claim that τ_{a_x,b_x} is an optimal stopping rule. Observe that the expression (4.11) for $J(\tau_{a_x,b_x}; \tilde{x})$ is strictly positive for all $\tilde{x} \in (a_x, b_x)$ whereas picking $a_x \leq a \leq x \leq b \leq b_x$ with either $a \neq a_x$ or $b \neq b_x$ results in $J(\tau_{a,b}; \tilde{x}) < J(\tau_{a_x,b_x}; \tilde{x})$. So $a^*(x) \leq a_x$ and $b^*(x) \geq b_x$. Clearly the optimal stopping boundaries must be elements of $\mathbb{C}$. Should one choose $a \in \mathbb{C}$ with $a < a_x$, then $\tau_{a,b} > \tau_{a_x,b}$ resulting in more discounting. Hence a_x is an optimal left hitting boundary. A similar argument applies for b_x.

The value function and optimal stopping times for this problem are

$$V(x) = \begin{cases} 1, & \text{for } x \in \mathbb{C}, \\ \dfrac{e^{\sqrt{2\alpha}(b_x - x)} - e^{-\sqrt{2\alpha}(b_x - x)}}{e^{\sqrt{2\alpha}(b_x - a_x)} - e^{-\sqrt{2\alpha}(b_x - a_x)}} + \dfrac{e^{\sqrt{2\alpha}(x - a_x)} - e^{-\sqrt{2\alpha}(x - a_x)}}{e^{\sqrt{2\alpha}(b_x - a_x)} - e^{-\sqrt{2\alpha}(b_x - a_x)}}, & \text{for } x \notin \mathbb{C}, \end{cases}$$

and

$$\tau^* = \begin{cases} \tau_x, & \text{for } x \in \mathbb{C}, \\ \tau_{a_x,b_x}, & \text{for } x \notin \mathbb{C}. \end{cases}$$

6 Concluding Remarks

This paper establishes two alternative optimization approaches to the solution of optimal stopping problems for one-dimensional diffusions. One method recasts the problem as a non-linear maximization over two-point stopping locations while the other determines a semi-infinite linear program over the coefficients of the harmonic functions. The combination of an explicit formula for the expected reward obtained using a two-point hitting rule and duality analysis proves that the class of such two-point hitting times contains an optimal stopping rule.

The method is local in nature in that the optimization problems are parameterized by the initial position x of the diffusion. Therefore, in principle, it is necessary to solve the entire family of optimization problems in order to determine the value function. In practice, however, the structure of the two-point stopping rules determines the value function over intervals of initial values. Strong duality between the optimization problems allows one to choose whichever problem is easier to analyze for a given initial value. As demonstrated by some of these examples, one is also able to use knowledge of the stochastic process and its hitting times to determine the optimal stopping rules.

The restricted dual linear program (3.5) is quite similar to the approach of Shiryaev [15] in that it seeks a minimal harmonic function which majorizes the terminal reward function. Shiryaev's approach seeks a minimal super-harmonic function for all values of x since this function is the value function V. Our approach only determines the value function piecewise so is able to utilize the fundamental solutions of the differential equation $Af - \alpha f = 0$ to characterize all harmonic functions. When the pieces are connected, the resulting value function is, of course, super-harmonic.

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