Thinning and Harvesting in Stochastic Forest Models

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Abstract. This paper analyzes a stochastic forest growth model in which the manager is able to first thin the forest to promote better growth before harvesting. Both Wicksell single thinning-and-harvesting cycle and Faustmann on-going rotation problems are considered. The Wicksell problem is analyzed by first restricting the class of decision times to (thinning,harvesting) pairs that bound the growth away from infinity and imbedding the problem in an infinite-dimensional linear program on a space of triplets of measures. These measures capture the thinning and harvesting decisions along with the behavior of the growth process prior to harvest. An auxiliary linear program then leads to a nonlinear optimization problem for which an optimal value and solution are determined. The values of all the problems are be related through a set of inequalities. The solution of the nonlinear problem determines (random) thinning and harvesting times for the single thinning-and-harvesting cycle which demonstrate the equality of the values of these various problems. Finally for the Wicksell problem, the unrestricted class of thinning-and-harvest times is shown to give the same value as the restricted class. The Faustmann on-going thinning-and-harvesting rotation problem is reduced to a Wicksell problem which then allows for the characterization of the value as the solution to a different nonlinear optimization problem. The effects of the opportunity to thin the forest are illustrated on a mean-reverting stochastic model.

Keywords. stochastic forest models, forest rotation, Wicksell, Faustmann, harvest, thinning, linear programming.

1 Introduction

Consider a stochastic forest model where, without any intervention, the process X is assumed to satisfy

$$dX(t) = \mu(X(t), Y(t))\, dt + \sigma(X(t), Y(t))\, dW(t), \qquad X(0) = x_{new} > 0. \tag{1.1}$$

Here X is a process that captures the volume and quality of the forest stand, x_{new} is the value of a "new" forest, μ denotes the mean growth rate, σ denotes the volatility of the growth rate and W is a standard Brownian motion process which provides the random fluctuations of the forest size. The process Y is an indicator process that identifies whether the forest is new and densely planted ($Y(t) = 1$) or has been thinned ($Y(t) = 2$); we assume $Y(0) = 1$. For our model, we have in mind functions μ and σ such that the process X is always nonnegative

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and represents a growth process. Allowing dependence on the process Y in these coefficients means that the growth dynamics can differ for dense and thinned forests or even that the process X can represent different quantities for the two types of forest. For example, X may represent the volume of fuel wood for a dense forest and the diameter at breast height (dbh) for an average, more valuable tree in the thinned forest. The interventions in the growth of the forest occur when it is thinned and harvested. This paper analyzes both a single thinning-harvesting cycle of Wicksell type and the Faustmann on-going rotation problem for this two-decision model.

For the Wicksell single-cycle problem, the goal of the forest manager is to maximize the expected present-value of the net proceeds of the forest product when thinned and harvested. Let θ denote the time at which the manager thins the forest and let η be the harvest time. The size of the forest after thinning is $X(\theta)$ whereas the size of the stand is $X(\theta-)$ at the time the decision to thin is made; in this model, the thinning decision results in an immediate jump to the process X. Let g_1 and g_2 denote the net profit functions from thinning and harvesting, respectively, and let $\alpha > 0$ denote the discount rate. The single-cycle objective is one of selecting times θ and η, with $\theta \leq \eta$, so as to maximize

$$E\left[e^{-\alpha\theta} g_1(X(\theta-))I_{\{\theta<\infty\}} + e^{-\alpha\eta} g_2(X(\eta))I_{\{\eta<\infty\}}\right]. \tag{1.2}$$

Let $V^w(x_{new}, 1)$ denote the optimal value.

At this point, one must mention the importance of the interpretation of the model for the dense and thinned states. The process X may possibly represent two different quantities in the two states, in which case it may be reasonable for thinning to result in an increase to the process X; $X(\theta) > X(\theta-)$. When the model is such that X represents the same quantity for both dense and thinned forests, it may be necessary to impose an additional condition on the decision times of the model. For example, if X were to represent the volume of lumber on the stand, then a decision to thin would reduce the size of X and the model would therefore restrict the thinning decision to occur only after the process X exceeds the size of the thinned stand. If $X(\theta) = x_{thin}$, in the simplest case, then one would require $\theta > \tau_{x_{thin}}$, where $\tau_{x_{thin}}$ is the first time at which X achieves level x_{thin}. This paper treats the more general model by allowing X to increase in value when thinning occurs; the results can be easily adapted to include additional restrictions on the decision times.

The Faustmann rotation problem replants trees for the cycle to repeat with the new planting resulting in the forest stand returning to value x_{new} so the process X starts from this point following a harvest decision. For $k = 1, 2, 3, \ldots$, let θ_k and η_k denote the random times at which the forest is thinned and harvested, respectively, for the k^{th} time. Then for each k, $X(\theta_k-)$ and $X(\eta_k-)$ represent the values of X when the decisions are made to thin and harvest, respectively, while $X(\theta_k)$ and $X(\eta_k) = x_{new}$ denote the sizes of X immediately following these interventions. The Faustmann on-going rotation problem is one of maximizing

$$E\left[\sum_{k=1}^{\infty} \left(e^{-\alpha\theta_k} g_1(X(\theta_k-))I_{\{\theta_k<\infty\}} + e^{-\alpha\eta_k} g_2(X(\eta_k-))I_{\{\eta_k<\infty\}}\right)\right]. \tag{1.3}$$

Let $V^f(x_{new}, 1)$ denote the optimal value for the Faustmann problem.

The mathematical modeling of forest growth with the aim of determining on-going optimal harvesting decisions began with the paper by Faustmann [10] which considered the case of deterministic growth. Since 1849, a large literature has developed on this topic. The bibliography by Newman [22] provides a partial but extensive up-to-date (2002) list of references on the economics of forest rotation. We concentrate our comments on some papers using stochastic models. Nordstrøm [23] modelled the growth process deterministically but introduced randomness with prices that followed a finite-state Markov chain in discrete time. Miller and Voltaire [20] consider a diffusion process for the tree size and solve the rotation problem. A limitation of this model is that tree sizes will become negative. Clarke and Reed [6] use a geometric Brownian motion for the forest size to ensure positivity and an age-dependent geometric Brownian motion for the price process. Using optimal stopping methods, the paper analyzes both the Wicksell single-period and Faustmann on-going harvest rotation problems. Willassen [26] considers a general stochastic differential equation model for the growth process in continuous time and uses impulse control methods to solve the problem. Buongiorno [4] and others employ Markov decision processes to model the growth process in discrete time; Buongiorno reformulates the problem as a finite-dimensional linear program. Sødal [25] restricts his analysis to decision times that are hitting times of the growth process at fixed levels and uses an intuitive mark-up pricing approach to characterize the value as a nonlinear optimization problem. Additional work on the Wicksell single-period problem for stochastic growth models include papers by Alvarez and Koskela (see, e.g., [2]), among others. Penttinen [24] includes thinning considerations in his models but only in terms of cost, not with the possibility of the growth dynamics improving.

Though our exposition is expressed entirely in terms of forests, their harvest, rotation and thinning, the paper by Miller and Voltaire [20] discusses how the rotation problem is a paradigm that has many additional economic applications.

This paper is organized as follows. Section 2 concentrates on the Wicksell single thinning-and-harvesting cycle problem, beginning by placing a restriction on the class of decision times. The restricted stochastic problem is imbedded in an infinite-dimensional linear program in Section 2.1 having variables in a space of triplets of measures. Section 2.2 develops an auxiliary linear program and corresponding nonlinear optimization problem in such a manner that all of the values for the various problems can be related by inequalities. The solution to the nonlinear problem allows one to identify an optimal pair of thinning and harvest times for the restricted stochastic problem whose value matches the largest bounding value and hence demonstrates that the values of all of these problems are equal. The unrestricted problem is shown to have the same value in Section 2.3 and hence the optimal thinning and harvest times for the restricted problem are seen to be optimal for the unrestricted problem as well. Section 3 examines the Faustmann thinning-and-harvesting rotation problem. The strong Markov property is employed in Section 3.1 to reduce the Faustmann problem to a Wicksell single-cycle problem with a slight modification to the harvest payoff function. As a result, the value is characterized as the solution to a different nonlinear optimization problem. Illustrative examples using a mean-reverting growth process are given in Sections 2.4 and 3.2.

The reformulation of the stochastic problem in terms of an infinite-dimensional linear program has previously been employed by the authors for optimal stopping problems (see [5, 13, 14]), stochastic control problems (see, e.g., [11, 16]) and analysis of uncontrolled stochastic processes (see, e.g., [12]). A benefit of the linear programming formulation is that it enables one to employ numerical methods to approximate the feasible/optimal measures (see, e.g., [11, 12, 16]). For the model of this paper, however, an exact characterization of the solution to the linear program is given.

1.1 Detailed Formulation

We begin with a precise formulation of the growth model with thinning. Assume the coefficients μ and σ are continuous and are such that the process X_y, $y = 1, 2$, is a weak solution of the stochastic differential equation

$$dX_y(t) = \mu(X_y(t), y)\, dt + \sigma(X_y(t), y)\, dW(t), \qquad X_y(0) = 0, \tag{1.4}$$

(see Ethier and Kurtz [9, Section 5.3, p. 291] for details) and that these solutions are unique in distribution. Let A_y denote the generator of the process X_y given by $A_y f(x) = \mu(x, y)f'(x) + (\sigma^2(x, y)/2)f''(x)$. The uniqueness in distribution then implies that the martingale problems for A_y are well-posed and hence that each X_y is a strong Markov process (see [9, Theorem 4.4.2, p. 184]). We wish to take advantage of the strong Markov property to piece together the solutions X_1 and X_2 to form a weak solution (X, Y) of (1.1) at all times other than thinning and harvest times. Let $\{\mathcal{F}_t\}$ denote a common filtration with respect to which X_1 anf X_2 are weak solutions of (1.4) for their respective values of y.

For the Wicksell problem, recall θ and η denote the thinning and harvesting times, respectively; θ and η are required to be $\{\mathcal{F}_t\}$-stopping times with $\theta \leq \eta$. For a given pair (θ, η), define the paired process (X, Y) as follows. The initial values are $X(0) = x_{new}$ and $Y(0) = 1$. For $0 < t < \theta$, $X(t) = x_{new} + X_1(t)$ and $Y(t) = 1$. At time θ, X jumps from $X(\theta-)$ to $X(\theta)$, where $X(\theta)$ has distribution π on a bounded interval $[x_{min}, x_{max}]$, with $x_{min} > 0$, and $Y(\theta) = 2$. The thinned level of the forest $X(\theta)$ is assumed to be independent of the thinning time θ. For $t > \theta$, $X(t) = X(\theta) + X_2(t - \theta)$. Since the Wicksell payoff ends at the harvest time, we may stop the process at η and specify $X(\eta) = X(\eta-)$.

The Faustmann model is quite similar, but with a couple of important changes. The thinning and harvest times are $\{\theta_k\}$ and $\{\eta_k\}$, respectively. Define $\eta_0 = 0$. These decision times must be $\{\mathcal{F}_t\}$-stopping times that satisfy $\eta_{k-1} \leq \theta_k \leq \eta_k$, for each $k \in \mathbb{N}$. Again, for each $k \in \mathbb{N}$, let $\{X_y^{(k)}\}$ be a sequence of independent processes satisfying (1.4), $y = 1, 2$. At each harvest time η_{k-1}, set $(X(\eta_{k-1}), Y(\eta_{k-1})) = (x_{new}, 1)$ and for $\eta_{k-1} \leq t < \theta_k$, set $X(t) = x_{new} + X_1^{(k)}(t - \eta_{k-1})$ and $Y(t) = 1$. At each thinning time θ_k, $X(\theta_k)$ is chosen from $[x_{min}, x_{max}]$ according to π, independently of $\{\theta_k\}$ and $\{\eta_k\}$ and $Y(\theta_k) = 2$. Then for $\theta_k \leq t < \eta_k$, define $X(t) = X(\theta_k) + X_2^{(k)}(t - \theta_k)$ and $Y(t) = 2$. Notice that at the times θ_k, the forest is thinned to a level $X(\theta_k)$ within some range and this is assumed to happen instantaneously, and at the times η_k, the forest is instantaneously harvested and replanted so $X(\eta_k) = x_{new}$, with a corresponding decrease $X(\eta_k) - X(\eta_k-)$ to the process. Observe for each cycle, the process has "poorer growth" dynamics (1.4) with $y = 1$ between η_{k-1} and θ_k and "better growth" dynamics (1.4) with $y = 2$ for times between θ_k and η_k.

Let $\mathcal{A}$ denote the set of pairs of admissible decision rules (θ, η) for the Wicksell problem and, with a slight abuse of notation, let $\mathcal{A}$ also denote the sequence of pairs $\{(\theta_k, \eta_k)\}$ for the Faustmann problem.

We place additional restrictions on the coefficients μ and σ through the behavior of the process at the boundaries. Assume ∞ is a natural boundary point (see [3, II.1.6, pp. 14,15] or [15, p. 128-131]) so that the forest does not grow without bounds in a finite time. Also assume that 0 is either a natural, entrance-not-exit or exit-not-entrance boundary point. In the former two cases, the forest will never die out in finite time (since $x_{new} > 0$) whereas the last condition implies that once the forest fails, it never recovers on its own. For models in which 0 is an exit boundary point, the objective function would be non-positive for those decision times θ and η that exceed the hitting time ζ of X at 0.

We note that $\{\mathcal{F}_t\}$ is the filtration associated with the weak solution to (1.1) so it may contain more information than that arising from the observations of the process X. Since the stopping times in $\mathcal{A}$ are $\{\mathcal{F}_t\}$-stopping times, these may in principle be determined using information contained in $\{\mathcal{F}_t\}$ that is not generated by X. Our results nevertheless show that optimal decision rules exist within the subclass of hitting times of the process.

Before addressing the reward structure for the class of problems under consideration, it is important to describe some additional structure to the problem. The change in dynamics occurs only when the thinning and harvesting decisions are made and hence Y changes only at these decision times. As a result, except at the thinning and harvest decision times, the generator of the pair (X, Y) is

$$Af(x,y) = \mu(x,y)\tfrac{\partial f}{\partial x}(x,y) + \tfrac{\sigma^2(x,y)}{2}\,\tfrac{\partial^2 f}{\partial x^2}(x,y)$$

operating on functions $f : \mathbb{R}^+ \times \{1,2\} \to \mathbb{R}$ that are twice-continuously differentiable in x for each y. Due to the discounting at rate α, the eigenvalue problem

$$Af = \alpha f \tag{1.5}$$

plays a central role in the analysis. Under the conditions assumed in this paper, for each $y \in \{1,2\}$, $Af(\cdot,y) = \alpha f(\cdot,y)$ has a nonnegative, strictly increasing solution ψ_y (unique up to a positive multiplicative constant) and furthermore $\psi_y(0+) \geq 0$ and $\lim_{x\to\infty}\psi_y(x) = \infty$ (see [3, II.1.10, pp. 18,19]). The specification of the model and the strictly increasing functions ψ_1 and ψ_2 are used to determine restrictions on the payoff functions g_1 and g_2 allowed in this paper.

Condition 1.1 *The payoff functions g_1 and g_2 are assumed to satisfy the following:*

(a) *g_1 and g_2 are continuous and non-decreasing on $[0,\infty)$;*

(b) *$g_1(x_{new}) < 0$ and $g_2(x_{max}) \leq 0$;*

(c) *there exists some $\overline{x} < \infty$ such that $g_2(\overline{x}) > 0$ and $g_1(\overline{x}) + \frac{\psi_2(x_{min})}{\psi_2(\overline{x})} \cdot g_2(\overline{x}) > 0$; and*

(d) *for $y = 1,2$, $\displaystyle\lim_{x\to\infty}\frac{g_y(x)}{\psi_y(x)} = 0$.*

Condition 1.1(a) implies that a higher value of X yields a larger profit, whereas Condition 1.1(b) indicates there is non-zero cost to immediately thin a replanted forest and no profit from immediately harvesting the thinned forest. Condition 1.1(c) means that a sufficiently large value of X will return a positive profit for both harvesting and for thinning followed by harvesting; should this not be the case, then the optimal times θ and η would be infinite so that no cost is incurred. Finally, Condition 1.1(d) places a restriction on how quickly the reward rate can grow relative to the size of X. This condition will be needed in order to eliminate decisions to thin or harvest at arbitrarily large values from being near-optimal decisions.

2 Wicksell Single Thinning-and-Harvesting Cycle

This section analyzes the single-cycle problem of deciding when to thin and when to harvest a forest so as to maximize (1.2) over $(\theta, \eta) \in \mathcal{A}$ with $\theta \leq \eta$. The examination of this problem, however, begins by restricting the decisions to a smaller collection $\mathcal{A}_1$ of $\{\mathcal{F}_t\}$-stopping times. The restricted problem is imbedded in an infinite-dimensional linear program from which an auxiliary linear program is derived and a finite-dimensional nonlinear optimization problem arises. An optimal pair (θ^*, η^*) of decision times is determined using an optimal solution to the nonlinear problem. Finally, the optimal solution for the restricted problem is shown to be optimal for the unrestricted problem and these results are illustrated by an example.

Let $\mathcal{A}_1$ denote the collection of pairs of $\{\mathcal{F}_t\}$-stopping times (θ, η), with $\theta \leq \eta$, for which there exists some $K < \infty$ such that $P(X(t) \leq K, 0 \leq t \leq \eta) = 1$. For such stopping times, the process X is bounded away from ∞. Note that K may differ for different $(\theta, \eta) \in \mathcal{A}_1$. Let $V_r^w(x_{new}, 1)$ denote the optimal value over the restricted collection of pairs of stopping times.

2.1 Linear Program Imbedding

Select $(\theta, \eta) \in \mathcal{A}_1$ arbitrarily and define the process λ_θ by $\lambda_\theta(t) = I_{[0,t]}(\theta)$. Notice that λ_θ starts at 0 and jumps to 1 at the random time θ after which it remains at 1. Let $\mathcal{D} = C_c^2(\mathbb{R}^+ \times \{1, 2\})$, the space of twice-continuously differentiable functions having compact support. For $f \in \mathcal{D}$, an application of Itô's formula results in

$$
\begin{aligned}
e^{-\alpha t} f(X(t), Y(t)) = f(x_{new}, 1) &+ \int_0^t e^{-\alpha s}[Af - \alpha f](X(s), Y(s))\, ds \\
&+ \int_0^t e^{-\alpha s} \int [f(x, 2) - f(X(s-), Y(s-))]\, \pi(dx)\, d\lambda_\theta(s) \\
&+ \int_0^t e^{-\alpha s} \sigma(X(s), Y(s))\, \tfrac{\partial f}{\partial x}(X(s), Y(s))\, dW(s).
\end{aligned}
$$

The conditions on f imply that the stochastic integral is a martingale so the optional sampling theorem (see [9, Theorem 2.2.13])) establishes that

$$\begin{aligned} e^{-\alpha(t\wedge\eta)} f(X(t\wedge\eta),Y(t\wedge\eta)) - f(x_{new},1) - \int_0^{t\wedge\eta} e^{-\alpha s}[Af-\alpha f](X(s),Y(s))\,ds \\ - \int_0^{t\wedge\eta} e^{-\alpha s}\int [f(x,2)-f(X(s-),Y(s-))]\,\pi(dx)\,d\lambda_\theta(s) \\ = \int_0^{t\wedge\eta} e^{-\alpha s}\sigma(X(s),Y(s))\,\tfrac{\partial f}{\partial x}(X(s),Y(s))\,dW(s) \end{aligned}$$

and hence taking expectations and letting $t\to\infty$ results in Dynkin's formula:

$$\begin{aligned} E\left[e^{-\alpha\eta} f(X(\eta),Y(\eta))I_{\{\eta<\infty\}}\right] - E\left[\int_0^{\eta} e^{-\alpha s}[Af-\alpha f](X(s),Y(s))\,ds\right] \\ - E\left[\int_0^{\eta} e^{-\alpha s}\int [f(x,2)-f(X(s-),Y(s-))]\,\pi(dx)\,d\lambda_\theta(s)\right] \\ = f(x_{new},1). \end{aligned} \tag{2.1}$$

Now define three measures as follows: let ν_η denote the discounted distribution of $(X(\eta),Y(\eta))$; let ν_θ denote the discounted distribution of $(X(\theta-),Y(\theta-))$; and define the expected discounted occupation measure ν_0 so that for each $G_1\in\mathcal{B}(\mathbb{R}^+)$ and $y\in\{1,2\}$

$$\nu_0(G_1\times\{y\}) = E\left[\int_0^{\eta} e^{-\alpha s} I_{G_1\times\{y\}}(X(s),Y(s))\,ds\right].$$

Observe that $Y(\eta)=2$ and $Y(\theta-)=1$ so ν_η and ν_θ can be (and, in the sequel, are) measures on $\mathbb{R}^+$. Notice also that the total masses of ν_η and ν_θ are bounded above by 1 and the total mass of ν_0 is bounded above by $1/\alpha$. Finally observe that the discounting implies that the sets $\{\theta=\infty\}$ and $\{\eta=\infty\}$ contribute no mass to ν_θ and ν_η.

For a function h on $[x_{min},x_{max}]$, define $\langle h,\pi\rangle = \int h(x)\,\pi(dx)$. Using the definitions of the measures ν_θ, ν_η and ν_0 along with the fact that π is a probability measure, (2.1) can be rewritten as

$$\begin{aligned} \int f(x,2)\,\nu_\eta(dx) - \int [Af-\alpha f](x,y)\,\nu_0(dx\times dy) \\ - \int [\langle f(\cdot,2),\pi\rangle - f(x,1)]\,\nu_\theta(dx) = f(x_{new},1) \end{aligned} \tag{2.2}$$

and this identity holds for all $f\in\mathcal{D}$ and $(\theta,\eta)\in\mathcal{A}_1$. (The identity (2.2) also holds for all $(\theta,\eta)\in\mathcal{A}$ but the ensuing argument requires the stopping times to be in $\mathcal{A}_1$.) These measures can be used to evaluate the expected payoff (1.2) resulting in

$$\int g_1(x)\,\nu_\theta(dx) + \int g_2(x)\,\nu_\eta(dx). \tag{2.3}$$

Since for stopping times $(\theta,\eta)\in\mathcal{A}_1$, the measures ν_η, ν_θ and ν_0 satisfy (2.2) and the corresponding value of (1.2) is given by (2.3), the stochastic decision problem on when to thin and harvest over the restricted class of decision rules is imbedded in the linear program

$$\begin{cases} \text{Maximize} & \int g_1\, d\nu_\theta + \int g_2\, d\nu_\eta \\ \text{Subject to} & \int f(x,2)\, \nu_\eta(dx) - \int [Af - \alpha f](x,y)\, \nu_0(dx \times dy) \\ & \qquad - \int [\langle f(\cdot,2), \pi\rangle - f(x,1)]\, \nu_\theta(dx) = f(x_{new},1), \quad \forall f \in \mathcal{D}, \\ & \int 1\, d\nu_\theta \leq 1, \\ & \int 1\, d\nu_\eta \leq 1, \\ & \int 1\, d\nu_0 \leq 1/\alpha, \\ & \nu_\theta, \nu_\eta, \nu_0 \text{ measures.} \end{cases} \tag{2.4}$$

Let $V_{lp}^w(x_{new},1)$ denote the value of (2.4). The above argument immediately implies the following comparison of values.

Theorem 2.1 $V_r^w(x_{new},1) \leq V_{lp}^w(x_{new},1)$.

2.2 Auxiliary Linear Program and Nonlinear Optimization

The goal now is to simplify the linear program into an auxiliary linear program in a manner that becomes more tractable and for which the values can be easily related. Choose any $(\theta,\eta) \in \mathcal{A}_1$ and let K denote the bound corresponding to this pair. Recall, for each $y \in \{1,2\}$, ψ_y is a strictly increasing solution to $Af(\cdot,y) = \alpha f(\cdot,y)$. Let $f(x,y) = a_1\psi_1(x)I_{\{1\}}(y) + a_2\psi_2(x)I_{\{2\}}(y)$ for some $a_1, a_2 \in \mathbb{R}$. This function does not have compact support so cannot be immediately used in (2.2). Let $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ be a mollifying function satisfying $\xi(x) = x$ for $x \leq K$ and $\xi \in C_c^2(\mathbb{R}^+)$. Considering the function $\tilde{f}(x,y) = \xi(x)\cdot f(x,y)$, we see that $\tilde{f} \in \mathcal{D}$ so can be used in (2.2). Moreover, for $x \leq K$, $\frac{\partial \tilde{f}}{\partial x} = \frac{\partial f}{\partial x}$ and $\frac{\partial^2 \tilde{f}}{\partial x^2} = \frac{\partial^2 f}{\partial x^2}$. Since $X(t) \leq K$ a.s. for $t \leq \eta$, it immediately follows that $[A\tilde{f} - \alpha\tilde{f}](X(t),Y(t)) = 0$ for all $t \leq \eta$ and therefore

$$\int a_2\psi_2(x)\, \nu_\eta(dx) - \int [a_2\langle\psi_2,\pi\rangle - a_1\psi_1(x)]\, \nu_\theta(dx) = a_1\psi_1(x_{new}). \tag{2.5}$$

Thus when restricting the pairs of decision times (θ,η) to the subcollection $\mathcal{A}_1$, the identity (2.2) extends to the function $f = a_1\psi_1 I_{\{1\}} + a_2\psi_s I_{\{2\}}$, for any $a_1, a_2 \in \mathbb{R}$.

We now define an auxiliary linear program by replacing the infinite collection of constraints in (2.4) by two constraints derived from (2.5) in which the first constraint selects $a_1 = a_2 = 1$, while the second constraint takes $a_1 = 0$ and $a_2 = 1$. Further relax the conditions by removing the total mass constraints on ν_θ, ν_η and ν_0, thus eliminating ν_0 from the linear program. The auxiliary linear program is

$$\begin{cases} \text{Maximize} & \int g_1(x)\, \nu_\theta(dx) + \int g_2(x)\, \nu_\eta(dx) \\ \text{Subject to} & \int [\psi_1(x) - \langle\psi_2,\pi\rangle]\, \nu_\theta(dx) + \int \psi_2(x)\, \nu_\eta(dx) = \psi_1(x_{new}), \\ & \int \langle\psi_2,\pi\rangle\, \nu_\theta(dx) - \int \psi_2(x)\, \nu_\eta(dx) = 0, \\ & \nu_\eta, \nu_\theta \text{ measures.} \end{cases} \tag{2.6}$$

Inherent in the formulation (2.6) is the feasibility requirement on the measures that the function ψ_1 is integrable with respect to ν_θ and ψ_2 is integrable with respect to ν_η. Denoting the value of the auxiliary linear program by $V_{aux}^w(x_{new},1)$, again the next theorem follows from the above discussion.

Theorem 2.2 $V_{lp}^{w}(x_{new}, 1) \le V_{aux}^{w}(x_{new}, 1)$.

We now turn to the analysis of (2.6) which is a special case of a general linear programming problem (A.1) in Appendix A. A remark is necessary, however, in order that Proposition A.1 can be applied. One assumption on the model (A.1) is that the integrands in the constraints be positive. The only integrand in (2.6) for which this is questionable is $\psi_1(x) - \langle \psi_2, \pi \rangle$. Recall, the solutions ψ_1 and ψ_2 are unique up to a multiplicative constant so one may multiply ψ_1 by a sufficiently large constant in order to achieve this positivity; we assume the solution ψ_1 satisfies $\psi_1(x_{min}) > \langle \psi_2, \pi \rangle$. Note this condition is satisfied when $\psi_1(x_{min}) > \psi_2(x_{max})$.

Theorem 2.3 *Assume g_1 and g_2 satisfy Condition 1.1. Let $F = \{(u, v) : u \ge x_{new}, v \ge x_{max}\}$. Then the value of the auxiliary linear program has the following bound:*

$$V_{aux}^{w}(x_{new}, 1) \le \psi_1(x_{new}) \cdot \sup_{u,v \in F} \frac{g_1(u)\psi_2(v) + \langle \psi_2, \pi \rangle g_2(v)}{\psi_1(u)\psi_2(v)}. \tag{2.7}$$

Moreover, optimizers (u_w^*, v_w^*) of the right-hand side of (2.7) exist in F and the hitting times

$$\theta^* = \inf\{t > 0 : X(t-) = u_w^*\} \quad \textit{and} \quad \eta^* = \inf\{t \ge \theta^* : X(t-) = v_w^*\} \tag{2.8}$$

are optimal thinning and harvesting times, respectively, for the restricted *Wicksell single-cycle problem. The optimal value for the restricted problem is*

$$V_r^{w}(x_{new}, 1) = \frac{g_1(u_w^*)\psi_2(v_w^*) + \langle \psi_2, \pi \rangle g_2(v_w^*)}{\psi_1(u_w^*)\psi_2(v_w^*)} \cdot \psi_1(x_{new}). \tag{2.9}$$

Proof. Begin by normalizing the first constraint of (2.6) by dividing both sides by $\psi_1(x_{new})$. The resulting formulation then matches the general linear program (A.1) so Proposition A.1 can be applied, establishing the bound (2.7). Now rewrite the bounding ratio in (2.7) as

$$\frac{\psi_1(x_{new})}{\psi_1(u)} \cdot \left[g_1(u) + \frac{\langle \psi_2, \pi \rangle}{\psi_2(v)} \cdot g_2(v) \right]$$

from which one can see that optimizing over v only involves optimizing the ratio $\frac{g_2(v)}{\psi_2(v)}$. By Condition 1.1(c), there exists some $\overline{x}$ for which $g_2(\overline{x}) > 0$. The function ψ_2 is strictly positive on $(0, \infty)$ so the ratio $\frac{g_2(\overline{x})}{\psi_2(\overline{x})} > 0$. By continuity of g_2 and ψ_2 and Condition 1.1(d), the maximum value is achieved at some location $v_w^* < \infty$.

Now turning to the optimization over u, we seek to maximize the ratio $\frac{g_1(u) + \frac{\langle \psi_2, \pi \rangle g_2(v_w^*)}{\psi_2(v_w^*)}}{\psi_1(u)}$. Condition 1.1(c) implies this ratio will be strictly positive for some sufficiently large value of u. Since $\lim_{u \to \infty} \psi_1(u) = \infty$, the increase of g_1 in the numerator by the constant $\frac{\langle \psi_2, \pi \rangle g_2(v_w^*)}{\psi_2(v_w^*)}$ does not affect the limiting value as $u \to \infty$ and hence Condition 1.1(d) establishes the existence of a maximizer $u_w^* < \infty$.

Finally, using a pair (u_w^*, v_w^*) of maximizers of the ratio in (2.7), define the decision times θ^* and η^* as in (2.8). The measure ν_{θ^*} will concentrate all of its mass on $\{u_w^*\}$ and similarly

ν_{η^*} is a point mass on $\{v_w^*\}$. The two constraints of the auxiliary linear program (2.6) form the system

$$\begin{cases} [\psi_1(u_w^*) - \langle\psi_2,\pi\rangle]\,\nu_{\theta^*}(\{u_w^*\}) + \psi_2(v_w^*)\,\nu_{\eta^*}(\{v_w^*\}) = \psi_1(x_{new}), \\ \langle\psi_2,\pi\rangle\,\nu_{\theta^*}(\{u_w^*\}) - \psi_2(v_w^*)\,\nu_{\eta^*}(\{v_w^*\}) = 0 \end{cases}$$

from which one readily determines that $\nu_{\theta^*}(\{u_w^*\}) = \frac{\psi_1(x_{new})}{\psi_1(u_w^*)}$ and $\nu_{\eta^*}(\{v_w^*\}) = \frac{\psi_1(x_{new})}{\psi_1(u_w^*)} \cdot \frac{\langle\psi_2,\pi\rangle}{\psi_2(v_w^*)}$. Hence this choice of thinning and harvesting times achieves the upper bound (2.9). $\square$

Remark 2.4 *Let $\theta_u = \inf\{t > 0 : (X(t-), Y(t-)) = (u,1)\}$ denote the first hitting time of u by X under dynamics given by $Y(t) = 1$ when $X(0) = x_{new}$. It is well-known that*

$$E\left[e^{-\alpha\theta_u}\right] = \frac{\psi_1(x_{new})}{\psi_1(u)} \tag{2.10}$$

(see [3, II.1.10, p. 18]). Now let $\eta_v = \inf\{t > 0 : (X(t), Y(t)) = (v,2)\}$ be the first hitting time by X of v under dynamics given by $Y(t) = 2$. Conditioning on $X(\theta_u) = x$, we have $E\left[e^{-\alpha(\eta_v-\theta_u)}|X(\theta_u) = x\right] = \psi_2(x)/\psi_2(v)$ and since $X(\theta_u)$ has distribution π on $[x_{min}, x_{max}]$, it follows that $E\left[e^{-\alpha(\eta_v-\theta_u)}\right] = \langle\psi_2,\pi\rangle/\psi_2(v)$ and hence that $E\left[e^{-\alpha\eta_v}\right] = \psi_1(x_{new})/\psi_1(u)\cdot\langle\psi_2,\pi\rangle/\psi_2(v)$. The ratio in (2.7) can be expressed as

$$\begin{aligned} &\frac{\psi_1(x_{new})}{\psi_1(u)} \cdot g_1(u) + \frac{\psi_1(x_{new})}{\psi_1(u)} \cdot \frac{\langle\psi_2,\pi\rangle}{\psi_2(v)} \cdot g_2(v) \\ &\qquad = E\left[e^{-\alpha\theta_u} g_1(X(\theta_u-)) I_{\{\theta_u<\infty\}} + e^{-\alpha\eta_v} g_2(X(\eta_v)) I_{\{\eta_v<\infty\}}\right]. \end{aligned} \tag{2.11}$$

Thus the nonlinear optimization problem (2.7) maximizes (1.2) over the hitting times of the paired process (X,Y) at thinning and harvesting levels $(u,1)$ and $(v,2)$, respectively.

Remark 2.5 *Let $d_1(u) = E\left[e^{-\alpha\theta_u}\right] = \psi_1(x_{new})/\psi_1(u)$ denote the expected discount factor arising from the rule to thin the forest when (X,Y) reaches level $(u,1)$ and let $d_2(v) = E\left[e^{-\alpha(\eta_v-\theta_u)}\right] = \langle\psi_2,\pi\rangle/\psi_2(v)$ be the expected discount factor associated with the rule to harvest when (X,Y) hits $(v,2)$, starting at time θ_u in location $X(\theta_u)$. The bound (2.7) then takes the form $d_1(u)[g_1(u) + d_2(v)g_2(v)]$. The first order optimality conditions are therefore*

$$\begin{cases} d_1'(u)[g_1(u) + d_2(v)g_2(v)] + d_1(u)g_1'(u) = 0 \\ d_2'(v)g_2(v) + d_2(v)g_2'(v) = 0 \end{cases}$$

which can be rewritten as

$$\begin{cases} \dfrac{u\frac{\partial}{\partial u}[g_1(u) + d_2(v)g_2(v)]}{[g_1(u) + d_2(v)g_2(v)]} = -\dfrac{ud_1'(u)}{d_1(u)} \\ \dfrac{vg_2'(v)}{g_2(v)} = -\dfrac{vd_2'(v)}{d_2(v)}. \end{cases}$$

The second equation shows that the optimal harvesting level v_w^* occurs where the elasticity of the harvest payoff function g_2 equals the negative of the elasticity of the harvesting discount factor d_2. The optimal thinning level u_w^*, however, occurs where the partial *elasticity of the combined payoffs for thinning g_1 and discounted harvesting $d_2(v)g_2(v)$ equals the negative of the elasticity of the thinning discount factor d_1.*

2.3 Optimality for the Unrestricted Problem

The results of Section 2.2 determine an optimal pair of thinning and harvesting times $(\theta^*, \eta^*) \in \mathcal{A}_1$ for the restricted problem. This subsection demonstrates that this pair of decision times is also optimal for the unrestricted problem.

Theorem 2.6 *Assume g_1 and g_2 satisfy Condition 1.1. Then $V^w(x_{new}, 1) = V_r^w(x_{new}, 1)$ and hence the thinning and harvest times (θ^*, η^*) of Theorem 2.3 is an optimal pair for the unrestricted stochastic forestry problem.*

Proof. Choose $(\theta, \eta) \in \mathcal{A}$ arbitrarily. Select a sequence $K_n \nearrow \infty$ and define $\tau_{K_n} = \inf\{t > 0 : X(t) = K_n\}$. Since ∞ is a natural boundary point, it follows that $\tau_{K_n} \nearrow \infty$ as $n \to \infty$. Observe that $\{(\theta \wedge \tau_{K_n}, \eta \wedge \tau_{K_N})\}$ is a sequence of decision rules within the restricted collection $\mathcal{A}_1$ and hence

$$\begin{aligned} V_r^w(x_{new}, 1) \geq{}& E\left[e^{-\alpha(\theta\wedge\tau_{K_n})} g_1(X(\theta \wedge \tau_{K_n})) I_{\{\theta\wedge\tau_{K_n})<\infty\}} \right. \\ &\quad \left. + e^{-\alpha(\eta\wedge\tau_{K_n})} g_2(X(\eta \wedge \tau_{K_n})) I_{\{\eta\wedge\tau_{K_n})<\infty\}}\right] \\ ={}& E\left[e^{-\alpha(\theta\wedge\tau_{K_n})} g_1(X(\theta \wedge \tau_{K_n})) I_{\{\theta\wedge\tau_{K_n})<\infty\}} I_{\{\theta<\infty\}}\right] \\ &\quad + E\left[e^{-\alpha(\theta\wedge\tau_{K_n})} g_1(X(\theta \wedge \tau_{K_n})) I_{\{\theta\wedge\tau_{K_n})<\infty\}} I_{\{\theta=\infty\}}\right] \\ &\quad + E\left[e^{-\alpha(\eta\wedge\tau_{K_n})} g_2(X(\eta \wedge \tau_{K_n})) I_{\{\eta\wedge\tau_{K_n}<\infty\}} I_{\{\eta<\infty\}}\right] \\ &\quad + E\left[e^{-\alpha(\eta\wedge\tau_{K_n})} g_2(X(\eta \wedge \tau_{K_n})) I_{\{\eta\wedge\tau_{K_n}<\infty\}} I_{\{\eta=\infty\}}\right]. \end{aligned} \tag{2.12}$$

Consider the first expectation on the right-hand side of (2.12). Observe that on the set $\{\theta < \infty\}$, $\theta(\omega) \wedge \tau_{K_n}(\omega) = \theta(\omega)$ for n sufficiently large. Thus

$$\lim_{n\to\infty} e^{-\alpha(\theta\wedge\tau_{K_n})} g_1(X(\theta \wedge \tau_{K_n})) I_{\{\theta\wedge\tau_{K_n}<\infty\}} I_{\{\theta<\infty\}} = e^{-\alpha\theta} g_1(X(\theta)) I_{\{\theta<\infty\}}$$

and hence Fatou's lemma implies that

$$E\left[e^{-\alpha\theta} g_1(X(\theta)) I_{\{\theta<\infty\}}\right] \leq \lim_{n\to\infty} E\left[e^{-\alpha(\theta\wedge\tau_{K_n})} g_1(X(\theta \wedge \tau_{K_n})) I_{\{\theta\wedge\tau_{K_n}<\infty\}} I_{\{\theta<\infty\}}\right].$$

Analyzing the second term, notice that on the set $\{\theta = \infty\}$, $\theta \wedge \tau_{K_n} = \tau_{K_n}$ so

$$\begin{aligned} E\left[e^{-\alpha(\theta\wedge\tau_{K_n})} g_1(X(\theta \wedge \tau_{K_n})) I_{\{\theta\wedge\tau_{K_n}\}} I_{\{\theta=\infty\}}\right] &= E\left[e^{-\alpha\tau_{K_n}} g_1(X(\tau_{K_n})) I_{\{\tau_{K_n}<\infty\}} I_{\{\theta=\infty\}}\right] \\ &\leq g_1(K_n) E\left[e^{-\alpha\tau_{K_n}} I_{\{\tau_{K_n}<\infty\}}\right] \\ &= g_1(K_n) \cdot \frac{\psi_1(x_{new})}{\psi_1(K_n)} \end{aligned}$$

and the right-hand side converges to 0 by Condition 1.1(d). The same analyses applies to the third and fourth expectations in (2.12) with the result that

$$V_r^w(x_{new}, 1) \geq E\left[e^{-\alpha\theta} g_1(X(\theta)) I_{\{\theta<\infty\}} + e^{-\alpha\eta} g_2(X(\eta)) I_{\{\eta<\infty\}}\right].$$

Taking the supremum over $(\theta, \eta) \in \mathcal{A}$ implies $V_r^w(x_{new}, 1) \geq V^w(x_{new}, 1)$. The reverse inequality follows immediately from the fact that $\mathcal{A}_1 \subset \mathcal{A}$. □

2.4 Example

For simplicity, we illustrate the single cycle thinning-and-harvesting problem by looking at the case in which X is a mean reverting process for both dense and thinned forests. The key feature of this model are particular mean levels for dense and thinned forests to which the process is attracted. In our formulation, the mean levels are $1/\gamma_y$, with $\gamma_y > 0$. When the process is smaller than $1/\gamma_y$, the drift will be positive, but when the process exceeds this level, the drift will be negative. These levels therefore act as natural targets for the sizes of the dense and thinned forest or for the size of an individual tree in a dense or thinned forest, depending on what the process X models for each type of forest. Specifically, the process X satisfies (1.1), where for $y = 1, 2$, $\mu(x, y) = \bar{\mu}(1 - \gamma_y x)$ and $\sigma(x, y) = \bar{\sigma}\sqrt{x}$, with $\bar{\mu}$ and $\bar{\sigma}$ being fixed positive constants. This model is an example of a process on the state space $(0, \infty)$ for which the left end-point 0 is an entrance-not-exit boundary point. For simplicity, we take the distribution π of the size of the thinned forest stand $X(\theta)$ to be a unit point mass at x_{thin} which means $X(\theta) = x_{thin}$ following the decision to thin.

The differential operators for the mean-reverting model are $A^{(y)}f(x) = \bar{\mu}(1-\gamma_y x)f'(x) + (\bar{\sigma}^2/2)xf''(x)$, $y = 1, 2$. The increasing solutions of the eigenfunction equation (1.5) are given by

$$\psi_y(x) = K_M\left(\frac{\alpha}{\gamma_y\bar{\mu}}, \frac{2\bar{\mu}}{\bar{\sigma}^2}, \frac{2\gamma_y\bar{\mu}}{\bar{\sigma}^2}x\right), \tag{2.13}$$

in which $K_M(a, b, z)$ denotes the Kummer M-function

$$K_M(a, b, z) = 1 + \frac{az}{b} + \frac{(a)_2}{(b)_2}\frac{z^2}{2!} + \cdots + \frac{(a)_n}{(b)_n}\frac{z^n}{n!} + \cdots,$$

where for $c = a, b$, $(c)_n := c(c+1)(c+2) \cdot \ldots \cdot (c+n-1)$, $(c)_0 := 1$ (see [1] for details). For instance, $K_M(a, b, z)$ is a solution of the ordinary differential equation

$$zf''(z) + (b - z)f'(z) - af(z) = 0;$$

an alternative notation for the Kummer M-function is $K_M(a, b, z) = {}_1F_1(a, b; z)$ in [1].

For illustrative purposes we choose the thinning and harvesting reward functions g_1 and g_2 as follows:

$$g_y(x) = x \cdot \delta_y \cdot \tfrac{1+\tanh(\varrho_y(x-\mathfrak{z}_y))}{2} - c_y, \qquad y = 1, 2, \tag{2.14}$$

where for $y = 1$ and 2, δ_y, ϱ_y, $\mathfrak{z}_y$ and c_y are nonnegative constants. Nice interpretations can be given to the parameters: δ_y represents the long-term growth rate of timber values (for large values of x) when $\varrho > 0$; the quantity c_y may include a common shift in value that applies to all timber sizes (e.g., property tax or transportation costs per tree [7, Table 1]); ϱ_y is a scaling factor and $\mathfrak{z}_y$ is close to the inflection point where profits increase the most. This form for the harvesting payoff function g_2 gives a reasonable family of functions that provide good approximations to the "Tree Value Conversion Standards" (see Mendel et. al., [19]) which estimate the harvesting payoffs based on the size (diameter at breast height - dbh) and grade of the tree. An example of the fit for sugar maples is provided in Appendix B; appropriate choices of parameters also provide good fits of g_2 for the TVCS of other tree varieties as well. (The tree value conversion standards were revisited in the publication [8] by DeBald and Dale; we base our illustration on the original paper [19].) The form (2.14) of payoff function can be used for g_1 to obtain good approximations of the "Fuel Value" (see Morrow [21]). Notice that this class of functions includes affine functions when $\varrho = 0$.

The paper [21] by Morrow lists the TVCS values corresponding to hardwood trees having dbh in the range from 10 inches to 28 inches and comments that trees with dbh in the range of 4 to 10 inches are worth nothing except for fuelwood, but are in the most need of thinning. He further adds that trees with dbh in the range of 10-14 inches have marginal value for timber and the rate of value increase is high, especially for thinned trees. The situation is reversed when the trees have dbh in the range 24 to 28 inches with high values but low growth rates. The function class (2.14) for g_2 captures this type of value change while appropriate choices of γ_y in the mean-reverting stochastic growth model will provide the observed type of growth. Morrow also indicates that trees with dbh of 20 inches "do not just occur; they are the result of thinning young stands, good sites, or a combination of both."

Table 1 illustrates the optimal thinning and harvesting levels u_w^* and v_w^*, respectively, for a particular choice of parameters along with the optimal expected discounted payoff $V^w(x_{new}, 1)$ for thinning and harvesting. The particular choice of $\varrho_1 = 0$ and $\mathfrak{z}_1 = 0$ indicates that a good fit to the fuelwood values is given by an affine function. First observe that the value of $g_1(u_w^*)$ is slightly negative indicating that it is optimal to spend money to thin the forest to receive the benefit of larger and more valuable trees when one harvests. Thus thinning is important in order to develop more valuable trees rather than as a source of income. Notice also the effect of increasing the thinned size of the forest is to thin earlier but this has no impact on the decision to harvest. Intuitively, when x_{thin} is small, the decision to thin would remove larger trees so as to allow smaller, good quality trees to grow to full size and be harvested. When x_{thin} is large, however, the results indicate that the smaller competing trees would be thinned leaving larger trees for later harvest.

x_{thin}	u_w^*	v_w^*	$V^w(x_{new}, 1)$
10.	29.3	61.7	3.257
12.5	28.0	61.7	3.493
15.	26.5	61.7	3.770
17.5	24.9	61.7	4.097
20.	23.1	61.7	4.487
22.5	21.0	61.7	4.957
25.	18.8	61.7	5.526

Table 1: Optimal thinning levels u_w^* and harvesting levels v_w^* (in cm) for the Wicksell-Model as well as values of $V^w(x_{new}, 1)$ for various x_{thin} values (in cm) using two mean reverting processes; $\mu_1 = \mu_2 = 1$, $\sigma_1^2 = \sigma_2^2 = 0.03$; $1/\gamma_1 = 100$, $1/\gamma_2 = 120$, $\alpha = 0.03$, $x_{new} = 0.5$, $\delta_1 = 0.7345$, $\varrho_1 = 0$, $\mathfrak{z}_1 = 0$, $c_1 = 9.1748$, $\delta_2 = 1.8254$, $\varrho_2 = 0.04502$, $\mathfrak{z}_2 = 56.6523$, $c_2 = 4.3862$.

These values in Table 1 should be compared with those when thinning is not applied and the process X only evolves according to the mean-reverting regime with parameters $\mu = 1$, $\sigma^2 = 0.03$ and $1/\gamma_1 = 100$, and the reward function is given by g_2. The choice of g_2 for this comparison provides an *optimistic* value since the trees in a dense forest will not grow to the same size or grade as for a thinned forest, but there will be some trees of sufficient size so that the use of the fuelwood payoff function g_1 would not be appropriate. The optimal harvesting level of the corresponding Wicksell problem equals 58.8 and the optimal value is 4.47. Moreover, the expected harvesting time is ~ 87 years. For the case with thinning, when $x_{thin} = 20$ cm or $x_{thin} = 22.5$ cm in Table 1 (and the associated selection of parameters and payoff functions), for example, the two phases – before thinning and after thinning – have an average length of 22-25 years and 61-64 years respectively. Hence, thinning might slightly increase the average optimal cash flow while the average time up to harvesting is about the same. (Appendix C derives an integral formula for the mean hitting times of a mean-reverting process; this formula has been used to determine these mean thinning and harvest times.)

The optimal values u_w^*, v_w^* and thinning level x_{thin} serve as recommendations to the forester who would be given the task of determining how to implement thinning and harvesting. Additional comments about the model and implementation are given in Section 4 of concluding remarks.

3 Faustmann Infinite-Cycle Rotation Problem

In contrast with the Wicksell problem whose goal is to maximize the expected discounted reward for thinning and harvesting a forest one time, the Faustmann problem allows the forest to be replanted after each harvest and therefore rewards are earned over an infinite number of thinning and harvesting cycles.

3.1 Reduction to a Wicksell Single-Cycle Problem

This section analyzes the Faustmann problem by using the strong Markov property to relate it to a Wicksell problem for an adjusted payoff function.

We assume the dynamics of the forest growth process X follow (1.1), that thinning occurs at times $\{\theta_k\}$ with the thinned state $X(\theta_k)$ having distriubtion π and that harvesting and replanting happens at times η_k at which point the process X reinitializes at x_{new}. The Faustmann objective is to maximize (1.3) over all $\{(\theta_k, \eta_k)\} \in \mathcal{A}$. Note, we assume $\theta_k \leq \eta_k \leq \theta_{k+1}$ for every k. Recall $V^f(x_{new}, 1)$ denotes the optimal value for the Faustmann problem.

To facilitate understanding of the argument for this section, we use a subscript on the expectation operator, e.g. $E_{x_{new}}[\cdot]$, to indicate the initial position of the process X. This notation is important when using the strong Markov property.

Theorem 3.1 *Assume g_1 and g_2 satisfy Condition 1.1. Then the optimal value for the Faustmann infinite thinning-and-harvesting cycle problem is characterized by the nonlinear optimization problem*

$$V^f(x_{new}, 1) = \sup_{u,v \in F} \frac{g_1(u)\psi_2(v) + \langle \psi_2, \pi \rangle g_2(v)}{\psi_1(u)\psi_2(v) - \psi_1(x_{new})\langle \psi_2, \pi \rangle} \cdot \psi_1(x_{new}). \tag{3.1}$$

Moreover, an optimal pair (u_f^*, v_f^*) exists and the thinning and harvest times are given by the successive hitting times of X to the levels u_f^* and v_f^*, namely, setting $\eta_0^* = 0$, define, for

$k = 1, 2, 3, \ldots,$

$$\theta_k^* = \inf\{t > \eta_{k-1}^* : X(t-) = u_f^*\} \quad \text{and} \quad \eta_k^* = \inf\{t > \theta_k^* : X(t-) = v_f^*\}. \tag{3.2}$$

The optimal value is therefore

$$V^f(x_{new}, 1) = \frac{g_1(u_f^*)\psi_2(v_f^*) + \langle \psi_2, \pi \rangle g_2(v_f^*)}{\psi_1(u_f^*)\psi_2(v_f^*) - \psi_1(x_{new})\langle \psi_2, \pi \rangle} \cdot \psi_1(x_{new}). \tag{3.3}$$

Proof. In preparation for the analysis, notice that for $k \geq 2$, $\theta_k \geq \eta_1$; define $\tilde{\theta}_{k-1} = \theta_k - \eta_1$. Also recall that on the set $\{\eta_1 < \infty\}$, $X(\eta_1) = x_{new}$ and observe that the summands of (1.3) are 0 for $k \geq 2$ on the set $\{\eta_1 = \infty\}$. Then using the strong Markov property in the second last equality below, we have

$$\begin{aligned}
&E_{x_{new}}\left[\sum_{k=1}^{\infty} e^{-\alpha\theta_k} g_1(X(\theta_k-)) I_{\{\theta_k<\infty\}}\right] \\
&= E_{x_{new}}\left[e^{-\alpha\theta_1} g_1(X(\theta_1-)) I_{\{\theta_1<\infty\}}\right] + E_{x_{new}}\left[\sum_{k=2}^{\infty} e^{-\alpha\theta_k} g_1(X(\theta_k-)) I_{\{\theta_k<\infty\}}\right] \\
&= E_{x_{new}}\left[e^{-\alpha\theta_1} g_1(X(\theta_1-)) I_{\{\theta_1<\infty\}}\right] \\
&\quad + E_{x_{new}}\left[E_{x_{new}}\left[e^{-\alpha\eta_1} I_{\{\eta_1<\infty\}} \sum_{k=2}^{\infty} e^{-\alpha(\theta_k-\eta_1)} g_1(X(\eta_1 + (\theta_k - \eta_1)-)) I_{\{\eta_1+(\theta_k-\eta_1)<\infty\}} \middle| \mathcal{F}_{\eta_1}\right]\right] \\
&= E_{x_{new}}\left[e^{-\alpha\theta_1} g_1(X(\theta_1-)) I_{\{\theta_1<\infty\}}\right] \\
&\quad + E_{x_{new}}\left[e^{-\alpha\eta_1} I_{\{\eta_1<\infty\}} E_{x_{new}}\left[\sum_{k=1}^{\infty} e^{-\alpha\tilde{\theta}_k} g_1(X(\eta_1 + \tilde{\theta}_k-)) I_{\{\eta_1+\tilde{\theta}_k<\infty\}} \middle| \mathcal{F}_{\eta_1}\right]\right] \\
&= E_{x_{new}}\left[e^{-\alpha\theta_1} g_1(X(\theta_1-)) I_{\{\theta_1<\infty\}}\right] \\
&\quad + E_{x_{new}}\left[e^{-\alpha\eta_1} I_{\{\eta_1<\infty\}} E_{X(\eta_1)}\left[\sum_{k=1}^{\infty} e^{-\alpha\tilde{\theta}_k} g_1(X(\tilde{\theta}_k-)) I_{\{\tilde{\theta}_k<\infty\}}\right]\right] \\
&= E_{x_{new}}\left[e^{-\alpha\theta_1} g_1(X(\theta_1-)) I_{\{\theta_1<\infty\}}\right] \\
&\quad + E_{x_{new}}\left[e^{-\alpha\eta_1} I_{\{\eta_1<\infty\}} E_{x_{new}}\left[\sum_{k=1}^{\infty} e^{-\alpha\tilde{\theta}_k} g_1(X(\tilde{\theta}_k-)) I_{\{\tilde{\theta}_k<\infty\}}\right]\right].
\end{aligned}$$

Define $\tilde{\eta}_{k-1} = \eta_k - \eta_1$, for $k \geq 2$. The same analysis can be applied to the rewards obtained from harvesting and thus

$$\begin{aligned}
&E_{x_{new}}\left[\sum_{k=1}^{\infty} \left(e^{-\alpha\theta_k} g_1(X(\theta_k-)) I_{\{\theta_k<\infty\}} + e^{-\alpha\eta_k} g_2(X(\eta_k-)) I_{\{\eta_k<\infty\}}\right)\right] \\
&= E_{x_{new}}\Bigg[e^{-\alpha\theta_1} g_1(X(\theta_1-)) I_{\{\theta_1<\infty\}} + e^{-\alpha\eta_1} g_2(X(\eta_1-)) I_{\{\eta_1<\infty\}} \qquad (3.4) \\
&\quad + e^{-\alpha\eta_1} I_{\{\eta_1<\infty\}} E_{x_{new}}\left[\sum_{k=1}^{\infty} \left(e^{-\alpha\tilde{\theta}_k} g_1(X(\tilde{\theta}_k-)) I_{\{\tilde{\theta}_k<\infty\}} + e^{-\alpha\tilde{\eta}_k} g_2(X(\tilde{\eta}_k-)) I_{\{\tilde{\eta}_k<\infty\}}\right)\right]\Bigg].
\end{aligned}$$

Denote the infinite sequences by $(\tilde{\theta}, \tilde{\eta}) = \{(\tilde{\theta}_k, \tilde{\eta}_k)\}$ and notice $(\tilde{\theta}, \tilde{\eta}) \in \mathcal{A}$. Observe

$$\begin{aligned}
&V^f(x_{new}, 1) \\
&\quad = \sup_{(\tilde{\theta},\tilde{\eta})\in\mathcal{A}} E_{x_{new}}\left[\sum_{k=1}^{\infty}\left(e^{-\alpha\tilde{\theta}_k} g_1(X(\tilde{\theta}_k-)) I_{\{\tilde{\theta}_k<\infty\}} + e^{-\alpha\tilde{\eta}_k} g_2(X(\tilde{\eta}_k-)) I_{\{\tilde{\eta}_k<\infty\}}\right)\right].
\end{aligned}$$

Taking the supremum of the right-hand side of (3.4) over $(\tilde{\theta}, \tilde{\eta}) \in \mathcal{A}$ implies

$$\begin{aligned}
&E_{x_{new}}\left[\sum_{k=1}^{\infty}\left(e^{-\alpha\theta_k} g_1(X(\theta_k-)) I_{\{\theta_k<\infty\}} + e^{-\alpha\eta_k} g_2(X(\eta_k-)) I_{\{\eta_k<\infty\}}\right)\right] \\
&\quad \leq E_{x_{new}}\left[e^{-\alpha\theta_1} g_1(X(\theta_1-)) I_{\{\theta_1<\infty\}} + e^{-\alpha\eta_1} g_2(X(\eta_1-)) I_{\{\eta_1<\infty\}} + e^{-\alpha\eta_1} I_{\{\eta_1<\infty\}} V^f(x_{new}, 1)\right] \\
&\quad = E_{x_{new}}\left[e^{-\alpha\theta_1} g_1(X(\theta_1-)) I_{\{\theta_1<\infty\}} + e^{-\alpha\eta_1} [g_2(X(\eta_1-)) + V^f(x_{new}, 1)] I_{\{\eta_1<\infty\}}\right] \\
&\quad = E_{x_{new}}\left[e^{-\alpha\theta_1} g_1(X(\theta_1-)) I_{\{\theta_1<\infty\}} + e^{-\alpha\eta_1} \tilde{g}_2(X(\eta_1-)) I_{\{\eta_1<\infty\}}\right]
\end{aligned}$$

where $\tilde{g}_2(x) = g_2(x) + V^f(x_{new}, 1)$. Thus the Faustmann infinite-cycle rotation problem is bounded above by a Wicksell single-cycle problem using the thinning payoff function g_1 and a shifted harvesting payoff function $\tilde{g}_2$. In a similar manner, begin by taking the supremum of the left-hand side of (3.4) over $(\theta, \eta) \in \mathcal{A}$, then the supremum of the right-hand side over $(\tilde{\theta}, \tilde{\eta}) \in \mathcal{A}$ and finally the supremum of the right-hand side over (θ_1, η_1) in the Wicksell admissible set to obtain the opposite inequality and hence the value $V^f(x_{new}, 1)$ equals the value $\tilde{V}^w(x_{new}, 1)$ of this Wicksell problem. Theorem 2.3 establishes existence of optimizers u_f^* and v_f^* for which the value of this Wicksell problem (and hence the Faustmann problem) is

$$V^f(x_{new}, 1) = \tilde{V}^w(x_{new}, 1) = \frac{g_1(u_f^*)\psi_2(v_f^*) + \langle\psi_2, \pi\rangle \tilde{g}_2(v_f^*)}{\psi_1(u_f^*)\psi_2(v_f^*)} \cdot \psi_1(x_{new}). \tag{3.5}$$

Substituting $\tilde{g}_2(v_f^*) = g_2(v_f^*) + V^f(x_{new}, 1)$ into (3.5) and solving for $V^f(x_{new}, 1)$ yields

$$\begin{aligned}
V^f(x_{new}, 1) &= \frac{\psi_1(x_{new})[g_1(u_f^*)\psi_2(v_f^*) + \psi_2(x_{thin}) g_2(v_f^*)]}{\psi_1(u_f^*)\psi_2(v_f^*)} \cdot \left[1 - \frac{\psi_1(x_{new})\langle\psi_2, \pi\rangle}{\psi_1(u_f^*)\psi_2(v_f^*)}\right]^{-1} \\
&= \frac{g_1(u_f^*)\psi_2(v_f^*) + \psi_2(x_{thin}) g_2(v_f^*)}{\psi_1(u_f^*)\psi_2(v_f^*) - \psi_1(x_{new})\langle\psi_2, \pi\rangle} \cdot \psi_1(x_{new}). \qquad (3.6)
\end{aligned}$$

Now define $(\theta, \eta) \in \mathcal{A}$ to be the successive hitting times of u_f^* and v_f^* defined in (3.2). Then $X(\theta_k^*-) = u_f^*$ and $X(\eta_k^*-) = v_f^*$ and the evaluation of (1.3) then involves determining the value of the infinite series comprised of the Laplace transform of the successive hitting times of the levels u_f^* or v_f^*. The evaluation of these series is given in Appendix D with the result that the expected discounted reward obtained using this policy achieves the bound (3.6).

The final point to be addressed is the demonstration that (u_f^*, v_f^*) is an optimizing pair for the nonlinear optimization problem (3.1). To see this, the equality of $V^f(x_{new}, 1)$ and $\tilde{V}^w(x_{new}, 1)$ in (3.5) implies that

$$V^f(x_{new}, 1) = \sup_{u,v\in F} \frac{g_1(u)\psi_2(v) + \langle\psi_2, \pi\rangle \tilde{g}_2(v)}{\psi_1(u)\psi_2(v)} \cdot \psi_1(x_{new})$$

and hence for each $u, v \in F$,

$$V^f(x_{new}, 1) \geq \frac{g_1(u)\psi_2(v) + \langle \psi_2, \pi \rangle \tilde{g}_2(v)}{\psi_1(u)\psi_2(v)} \cdot \psi_1(x_{new})$$
$$= \frac{g_1(u)\psi_2(v) + \langle \psi_2, \pi \rangle g_2(v)}{\psi_1(u)\psi_2(v)} \cdot \psi_1(x_{new}) + \frac{\psi_1(x_{new})\langle \psi_2, \pi \rangle}{\psi_1(u)\psi_2(v)} \cdot V^f(x_{new}, 1).$$

Solving for $V^f(x_{new}, 1)$ yields for each $u, v \in F$,

$$V^f(x_{new}, 1) \geq \frac{g_1(u)\psi_2(v) + \langle \psi_2, \pi \rangle g_2(v)}{\psi_1(u_2^{\psi}(v) - \psi_1(x_{new})\langle \psi_2, \pi \rangle} \cdot \psi_1(x_{new})$$

with equality for (u_f^*, v_f^*), establishing (3.1). $\square$

Remark 3.2 *A similar remark to that of Remark 2.5 holds for the Faustmann problem. Specifically, let*

$$d_1(u, v) = \frac{\psi_1(x_{new})\psi_2(v)}{\psi_1(u)\psi_2(v) - \psi_1(x_{new})\langle \psi_2, \pi \rangle} \text{ and } d_2(u, v) = \frac{\psi_1(x_{new})\langle \psi_2, \pi \rangle}{\psi_1(u)\psi_2(v) - \psi_1(x_{new})\langle \psi_2, \pi \rangle}$$

denote the sums of the expected thinning discount factors and the expected harvesting discount factors, respectively, when thinning occurs at level u and harvesting occurs at level v. Observe the nonlinear function (3.1) to be maximized takes the form $d_1(u, v)g_1(u) + d_2(u, v)g_2(v)$. Also note that $d_1(u, v) = \frac{\psi_2(v)}{\langle \psi_2, \pi \rangle} \cdot d_2(u, v)$. Letting $g(u, v) = \frac{\psi_2(v)}{\langle \psi_2, \pi \rangle} \cdot g_1(u) + g_2(v)$, (3.1) can be written as $d_2(u, v)g(u, v)$. The first order optimality conditions for (u_f^*, v_f^*) yield

$$\begin{cases} \dfrac{u_f^* \frac{\partial g}{\partial u}(u_f^*, v_f^*)}{g(u_f^*, v_f^*)} = -\dfrac{u_f^* \frac{\partial d_2}{\partial u}(u_f^*, v_f^*)}{d_2(u_f^*, v_f^*)} \\[2ex] \dfrac{v_f^* \frac{\partial g}{\partial v}(u_f^*, v_f^*)}{g(u_f^*, v_f^*)} = -\dfrac{v_f^* \frac{\partial d_2}{\partial v}(u_f^*, v_f^*)}{d_2(u_f^*, v_f^*)} \end{cases}$$

and since

$$\frac{\frac{\partial d_1}{\partial u}(u, v)}{d_1(u, v)} = \frac{\frac{\partial d_2}{\partial u}(u, v)}{d_2(u, v)}$$

it follows that at a pair (u_f^*, v_f^*) of optimizers, the partial elasticities of the payoff function g equal the negative of the partial elasticities of the discount factors:

$$\begin{cases} \mathbb{EL}_u[g](u_f^*, v_f^*) = -\mathbb{EL}_u[d_1](u_f^*, v_f^*) \\ \mathbb{EL}_v[g](u_f^*, v_f^*) = -\mathbb{EL}_v[d_2](u_f^*, v_f^*) \end{cases}$$

Notice that the payoff function g is valued in currency at the time of harvesting, not its present-value.

3.2 Example

The illustration of the Faustmann solution uses the same model formulation as for the Wicksell problem. Recall the growth process X satisfies (1.1) with $\mu(x, y) = \bar{\mu}(1 - \gamma_y x)$ and $\sigma(x, y) = \bar{\sigma}\sqrt{x}$, where $\bar{\mu}$, $\bar{\sigma}$ and γ_y are fixed positive constants, while each thinning decision produces a value of $X(\theta_k) = x_{thin}$ and harvesting restarts the forest value at $X(\eta_k) = x_{new}$. Also recall that the differential operators for the mean-reverting model are $A^{(y)} f(x) = \bar{\mu}(1 - \gamma_y x) f'(x) + (\bar{\sigma}^2/2) x f''(x)$, $y = 1, 2$, the increasing solutions of the eigenfunction equation (1.5) are given by (2.13) and the payoff functions are as in (2.14).

Table 2 displays the optimal thinning and harvesting levels u_f^* and v_f^* along with the optimal payoff for the Faustmann problem. Comparing these values with Table 1, one observes the effect of optimizing over infinitely many cycles reduces both the optimal thinning and harvesting levels while at the same time increases the payoff received. Moreover, these optimal thinning and harvesting levels are in agreement with the comments by Morrow [21] on the need to thin a stand of hardwoods when dbh is in the 4 to 10 inch range and the valuation of the trees is high when dbh is 24 to 28 inches ($\sim$ 60 cm to $\sim$ 70 cm).

x_{thin}	u_f^*	v_f^*	$V^f(x_{new}, 1)$
10.	28.5	60.7	3.405
12.5	27.0	60.7	3.677
15.	25.3	60.6	4.002
17.5	23.5	60.4	4.396
20.	21.4	60.3	4.879
22.5	19.0	60.1	5.481
25.	16.1	59.9	6.249

Table 2: Optimal thinning levels u_f^* and harvesting levels v_f^* (in cm) for the Faustmann-Model as well as values of $V^f(x_{new}, 1)$ for various x_{thin} values (in cm) using two mean reverting processes; $\mu_1 = \mu_2 = 1$, $\sigma_1^2 = \sigma_2^2 = 0.03$; $1/\gamma_1 = 100$, $1/\gamma_2 = 120$, $\alpha = 0.03$, $x_{new} = 0.5$, $\delta_1 = 0.7345$, $\varrho_1 = 0$, $\mathfrak{z}_1 = 0$, $c_1 = 9.1748$, $\delta_2 = 1.8254$, $\varrho_2 = 0.04502$, $\mathfrak{z}_2 = 56.6523$, $c_2 = 4.3862$.

4 Concluding Remarks

This paper has examined a variation of the Wicksell single cycle and Faustmann on-going harvest rotation problems that includes thinning of the forest to promote better growth dynamics. The model adopts a deterministic pricing function. This choice can be understood to be derived from aggregate data over a long-term horizon and thus the problem under consideration treats rotation management as a long-term investment. One might use this problem to determine an approximate age or size of tree at which harvesting would occur. Then when one approaches the harvesting and marketing of the forest, one would take a short-term view and use a stochastic pricing model to decide on the precise harvesting time.

The analysis in this paper replaces the stochastic model by an infinite-dimensional linear program over a space of (deterministic) measures. This approach proved to be quite tractable for the problems under study in this paper. An interesting feature of this analysis is the reduction of the Faustmann infinite-cycle problem to a Wicksell single-cycle problem through the use of the strong Markov property for the growth process.

The results for the thinning-and-harvest problems immediately reduce to the known results for the harvest only problems. When there is no change in dynamics allowed, there is only one strictly increasing solution to the eigenvalue problem $Af = \alpha f$ and there would be no dependence on a Y process. For the Wicksell problem, one eliminates the g_2 term from the optimization problem (2.7) since this would provide a second harvest opportunity. Theorem 2.3 then reduces to the known solution (see, e.g., Sødal [25]). The reduction for the Faustmann problem is even more immediate. When harvesting is the only decision to make, there would be a single payoff function g, only one reinitializing point x_{new}, one decision level u^* and as above only one increasing function ψ. The expression (3.3) in Theorem 3.1 then simplifies to the known result.

This methodology can be easily adapted to include additional features to the model. For instance, the owner of the forest stand may receive a running payment stream that depends on the size of the forest; such payments might represent the amenity value of the forest or carbon credit payments that are received as long as the forest is allowed to grow (see Helmes and Stockbridge [14] for a single-cycle example using this methodology). The dynamics may also include sudden destruction due to fire or pests. Assuming the occurrences of such destruction are modelled by a Poisson process with some probability distribution on the size of the forest following the occurrence, the methods of this paper apply with only a minor modification to the generator A and an adjustment to the discount factor α.

The imbedding of the stochastic problem in an infinite-dimensional linear program can often be sharpened to show equivalence between the formulations. Kurtz and Stockbridge [18] establishes this equivalence for absolutely continuous stochastic control problems, Cho and Stockbridge [5] proves this for optimal stopping problems in which the processes exclude singular behavior (such as the thinning decisions of this paper) and Helmes and Stockbridge [13] extends this result to processes having singular behavior. These results are proven with less regularity of the payoff functions, typically only requiring semi-continuity. The linear programming formulation then allows one to employ numerical techniques to approximate the optimal solutions. A variety of such numerical approaches are possible, including approximating the diffusion process by a continuous-time Markov chain, characterizing the measures using their moments and using finite-elements to determine densities for approximating measures. The model and analysis of this paper do not require the equivalence since the equivalence of the values is derived from the exact analysis.

Appendix A Optimization of Linear Program (2.6)

In this appendix, we prove that an optimal choice of measures for (2.6) will place point masses on locations which maximize (2.7). To do so, however, we phrase the problem more generally as one of finding finite measures ν_1 and ν_2 so as to solve the linear programming problem

$$\begin{cases} \text{Maximize} & \displaystyle\int g_1(x)\,\nu_1(dx) + \int g_2(y)\,\nu_2(dy) \\ \text{Subject to} & \displaystyle\int F_1(x)\,\nu_1(dx) + \int F_2(y)\,\nu_2(dy) = 1, \\ & \displaystyle\int G_1(x)\,\nu_1(dx) - \int G_2(y)\,\nu_2(dy) = 0, \end{cases} \tag{A.1}$$

in which F_i are non-negative, measurable, G_i are positive, measurable and g_i are measurable functions on measurable spaces $(E_i, \mathcal{F}_i)$, $i = 1, 2$. To be feasible, the measures ν_1 and ν_2 must be such that for $i = 1, 2$, F_i and G_i are integrable with respect to ν_i.

Proposition A.1 *Define the functions $H(x, y) = F_1(x)G_2(y) + G_1(x)F_2(y)$ and $h(x, y) = g_1(x)G_2(y) + G_1(x)g_2(y)$ and assume H is strictly positive. Then an upper bound on the optimal value of (A.1) is given by*

$$\int g_1(x)\,\nu_1(dx) + \int g_2(y)\,\nu_2(dy) \le \sup_{(x,y)} \frac{h(x, y)}{H(x, y)}$$

and if the function h/H has a global maximum, then this bound is achieved.

Proof. Let (ν_1, ν_2) be a feasible pair of measures and note that at least one (hence both) measures have positive mass. Let $\nu = \nu_1 \times \nu_2$ be the product measure on $(E_1 \times E_2, \mathcal{F}_1 \times \mathcal{F}_2)$. Observe that

$$\begin{aligned} \int H\,d\nu &= \int F_1\,d\nu_1 \int G_2\,d\nu_2 + \int G_1\,d\nu_1 \int F_2\,d\nu_2 \\ &= \left(\int F_1\,d\nu_1 + \int F_2\,d\nu_2\right) \int G_2\,d\nu_2 \end{aligned}$$

where the second constraint has been used and hence

$$\int \frac{H}{\int G_2\,d\nu_2}\,d\nu = 1.$$

Thus a probability measure $\tilde{\nu}$ can be defined by taking $H/(\int G_2\,d\nu_2)$ to be the density with respect to ν.

Now observe that

$$\begin{aligned} \int h\,d\nu &= \int g_1\,d\nu_1 \int G_2\,d\nu_2 + \int G_1\,d\nu_1 \int g_2\,d\nu_2 \\ &= \left(\int g_1\,d\nu_1 + \int g_2\,d\nu_2\right) \int G_2\,d\nu_2. \end{aligned}$$

So as a result,

$$\int g_1\,d\nu_1 + \int g_2\,d\nu_2 = \int \frac{h}{\int G_2\,d\nu_2}\,d\nu = \int \frac{h}{H} \cdot \frac{H}{\int G_2\,d\nu_2}\,d\nu = \int \frac{h}{H}\,d\tilde{\nu}.$$

For any probability measure $\tilde{\nu}$, the value of the objective is bounded above by $\sup_{x,y} \frac{h(x,y)}{H(x,y)}$. Moreover, when the ratio h/H achieves its maximum, say at (x^*, y^*), an optimal pair of measures (ν_1^*, ν_2^*) can be determined by taking $\tilde{\nu} = \delta_{\{(x^*,y^*)\}}$. This then implies that the corresponding product measure ν^* also places a point mass on $\{(x^*, y^*)\}$ having mass $\frac{\int G_2\, d\nu_2}{H(x^*,y^*)}$ and moreover that the measure ν_1^* is a point mass on $\{x^*\}$ and similarly ν_2^* is a point mass on $\{y^*\}$. Now utilizing the two constraints of the linear program, we are able to determine the masses of ν_1^* and ν_2^* from the system of equations

$$\begin{aligned} F_1(x^*)\, \nu_1^*\{x^*\} + F_2(y^*)\, \nu_2^*\{y^*\} &= 1 \\ G_1(x^*)\, \nu_1^*\{x^*\} - G_2(y^*)\, \nu_2^*\{y^*\} &= 0 \end{aligned}$$

yielding

$$\nu_1^*\{x^*\} = \frac{G_2(y^*)}{F_1(x^*)G_2(y^*) + G_1(x^*)F_2(y^*)}, \qquad \nu_2^*\{y^*\} = \frac{G_1(x^*)}{F_1(x^*)G_2(y^*) + G_1(x^*)F_2(y^*)}.$$

□

Acknowledgement. The authors would like to thank Hans Volkmer for the simple proof of the optimization result in Proposition A.1.

Appendix B Payoff Function g_2

The form of the harvesting payoff function $g_2(x) = x\delta_2 \cdot \frac{1+\tanh(\varrho_2(x-\mathfrak{z}_2))}{2} - c_2$ is chosen since the parameters δ_2, ϱ_2, $\mathfrak{z}_2$ and c_2 provide enough flexibility to accurately capture pricing data. For example, Morrow [21] reports pricing data (based on 1972 prices) over their growth cycle for a number of different varieties of trees. Table 3 displays the data for the growth of a sugar maple; dbh stands for the diameter at breast height. Tree Value Conversion Standards (TVCS) provide a measure of a tree's worth, based on the comparative value of the quantity and quality of expected yield of one-inch lumber, taking into account conversion costs such as harvesting, transporting, and milling. It is a standard by which trees of different sizes can be compared, but it excludes most price effects of inflation and the marketplace. A fit of this pricing data using the parameters $\delta_2 = 1.8254$, $\varrho_2 = 0.04502$, $\mathfrak{z}_2 = 56.6523$ and $c_2 = 4.3862$ is displayed in Figure 1. We can see this family of pricing functions provides an excellent fit to the data; very good fits are also provided with an appropriate change in the parameters for the prices corresponding to other varieties of trees.

Appendix C Mean Hitting Times

To aid in the selection of model parameters, we determine formulae for the mean thinning times and mean harvest times of various stochastic models that can be used for the growth process. Observe the aim is to determine the mean hitting time for thinning the forest stand and these cycles always begin with X at x_{new} and have $Y(t) = 1$ for $\eta_{k-1} \leq t < \theta_k$. Similarly, when we consider the mean harvest time from the time θ_k of thinning, Y is constant at 2. To simplify the notation in this appendix, we therefore assume $\mu(x, y) = \mu(x)$ and

dbh (in cm)	Tree Value Conversion Standards (TVCS)
25	0
30	1
35	3
40	8
45	17
50	29
60	58
70	94

Table 3: Typical Change in Size and Value with Growth of a Sugar Maple

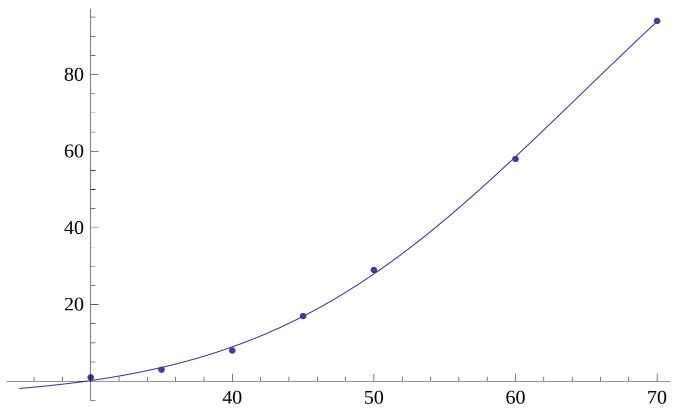

Figure 1: Fit of g_2 to (dbh,TVCS) data in Table 3.

$\sigma(x, y) = \sigma(x)$ and the generator A of the process X is $Af(x) = \mu(x)f'(x) + (\sigma^2(x)/2)f''(x)$, defined for every $f \in C^2(0, \infty)$. To cover both the thinning and harvesting situations, assume $X(0) = x_0$, where x_0 is either x_{new} or some fixed $x \in [x_{min}, x_{max}]$.

It is possible, of course, to estimate these means using Monte Carlo simulation. One can also view (2.10) as an expression for the Laplace transform of the thinning time θ_u (in which α is the variable) and employ numerical differentiation to approximate the mean.

We state and prove a general theorem that applies to many models and follow this by specifying the results to the mean-reverting model used to illustrate the solution approach in this paper. We begin by defining some notation.

First, define the function s by $s(x) = e^{-\int^x [2\mu(y)/\sigma^2(y)]\,dy}$ using the indefinite integral. Also define the scale function S by the indefinite integral $S(x) = \int^x s(y)\,dy$, the speed density $m(x) = 1/[\sigma^2(x)s(x)]$ and the speed measure M on $(0, \infty)$ by $M[a, b] = \int_a^b m(y)\,dy$ for $0 < a < b < \infty$. Finally, define the measure S on $(0, \infty)$ by $S[a, b] = S(b) - S(a)$ for $0 < a < b < \infty$. (This notation follows Karlin and Taylor [17, pp. 194, 227].)

Proposition C.1 *Let X satisfy (1.1) with μ and σ not depending on y. Let $b > x_0$ be a fixed level and set $\tau_b = \inf\{t \geq 0 : X(t-) = b\}$ to be the time when the process X first reaches level b. Suppose the model coefficients μ and σ are such that, for each $x > 0$,* $\lim_{a \searrow 0} S[a, x] = \infty$

and $\lim_{a\searrow 0} M[a,x] < \infty$. *Then the expected time until harvest at level* b *is given by*

$$E[\tau_b] = \int_{x_0}^{b}\int_{0}^{y} e^{-\int_z^y \frac{2\mu(u)}{\sigma^2(u)}\,du} \cdot \frac{2}{\sigma^2(z)}\,dz\,dy. \tag{C.2}$$

Proof. Select a such that $0 < a < x_0 < b$, where the last inequality is given by hypothesis. Define $\tau_a = \inf\{t > 0 : X(t-) = a\}$, define τ_b similarly and let $\tau_{a,b} = \tau_a \wedge \tau_b = \inf\{t \geq 0 : X(t-) = a \text{ or } X(t-) = b\}$. Karlin and Taylor [17, pp.192-197] show that

$$E[\tau_{a,b}] = 2\left\{\frac{S[a,x_0]}{S[a,b]}\int_a^b\int_a^y m(z)\,dz\,dS(y) - \int_a^{x_0}\int_a^y m(z)\,dz\,dS(y)\right\}. \tag{C.3}$$

Now observe that $\tau_{a,b} \leq \tau_b$ and since 0 is assumed to be either a natural or an entrance-not-exit boundary point, $\tau_{a,b} \nearrow \tau_b$ as $a \searrow 0$. The monotone convergence theorem implies that we need to analyze the limit of (C.3) as a decreases to 0. The key observation is that (C.3) can be rewritten in the form

$$2\left\{\int_{x_0}^b\int_a^y m(z)s(y)\,dz\,d(y) - \frac{S[x_0,b]}{S[a,b]}\int_a^b\int_a^y m(z)s(y)\,dz\,d(y)\right\}.$$

The first term is continuous in a so taking the limit as a goes to 0 yields (C.2). We therefore need to show the limit of the second term is 0. The numerator of the fraction is a fixed constant so can be ignored. Observe that the remainder of the term can be expressed as

$$\int_a^b\left(\int_a^y m(z)\,dz\right)\frac{s(y)}{S[a,b]}\,dy \tag{C.4}$$

and, in particular, notice that for each choice of a, the fraction forms a probability density on the interval $[a,b] \subset [0,b]$. The condition $\lim_{a\searrow 0} S[a,x] = \infty$ implies that this family of probability measures converges weakly to a point mass at 0. The fact that (C.4) converges to 0 is therefore a consequence of the finiteness of $\lim_{a\searrow 0} M[a,y]$ for each y. □

Simple estimates verify that the conditions $\lim_{a\searrow 0} S[a,x] = \infty$ and $\lim_{a\searrow 0} M[a,x] < \infty$ are satisfied by the mean-reverting model used in this paper. We therefore identify the mean hitting time for this model.

Example C.2 MEAN REVERTING MODEL
For the mean-reverting growth model, the coefficients are $\mu(x) = \mu(1-\gamma x)$ and $\sigma(x) = \sigma\sqrt{x}$ for some constants $\gamma, \mu, \sigma > 0$. Let $\kappa = \frac{2\mu}{\sigma^2}$ and $\varrho = \gamma\kappa$, and assume $b < 1/\gamma$; since $\mu > 0$ implies $\kappa > 0$, no additional constraint needs to be imposed on the relation between μ and σ. The mean harvest time for the mean-reverting model is

$$E[\tau_b] = \frac{2}{\sigma^2}\int_{x_0}^b\int_0^y e^{\varrho(y-z)}y^{-\kappa}z^{\kappa-1}\,dz\,dy.$$

Finally, we report the mean hitting times for several other stochastic models that have been used in the literature to model the growth process.

Example C.3 Geometric Brownian Motion Model
Recall, the geometric Brownian motion growth model has $\mu(x) = \mu x$ and $\sigma(x) = \sigma x$ for some constants $\mu, \sigma > 0$. Let $\kappa = \frac{2\mu}{\sigma^2}$. The conditions on S and M are satisfied provided $\kappa > 1$ (that is, $2\mu - \sigma^2 > 0$) resulting in the mean harvest time being

$$E[\tau_b] = \frac{2}{2\mu - \sigma^2} \ln(b/x_0).$$

Example C.4 Logistic I Model
For the logistic I growth model, the coefficients are $\mu(x) = \mu x(1 - \gamma x)$ and $\sigma(x) = \sigma x$, with $\gamma, \mu, \sigma > 0$. Again, let $\kappa = \frac{2\mu}{\sigma^2}$ and $\varrho = \gamma\kappa$. Then when $b < 1/\gamma$ and $2\mu - \sigma^2 > 0$, the mean harvest time for the logistic I model is

$$E[\tau_b] = \frac{2}{\sigma^2} \int_{x_0}^{b} \int_{0}^{y} e^{\varrho(y-z)} y^{-\kappa} z^{\kappa-2} \, dz \, dy.$$

Example C.5 Logistic II Model
The coefficients of the logistic II growth model are $\mu(x) = \mu x(1-\gamma x)$ and $\sigma(x) = \sigma x(1-\gamma x)$ with $\gamma, \mu, \sigma > 0$. Again, let $\kappa = \frac{2\mu}{\sigma^2}$ with $\kappa > 1$. Then when $b < 1/\gamma$, the mean harvest time at level b for the logistic II model is

$$E[\tau_b] = \frac{2}{\sigma^2} \int_{x_0}^{b} \int_{0}^{y} \left(\frac{y}{1-\gamma y}\right)^{-\kappa} \left(\frac{z}{1-\gamma z}\right)^{\kappa} \frac{1}{z^2(1-\gamma z)^2} \, dz \, dy.$$

Appendix D Expected Discount Factors

Consider the thinning-and-harvesting rule whereby a dense forest is thinned whenever it reaches level u and is harvested whenever a thinned forest achieves size v. Recall, the initial state of the forest is $(X(0), Y(0)) = (x_{new}, 1)$ indicating that the stand is dense and new. Define the "zeroeth" harvesting time $\eta_0 = 0$. Now define the successive thinning and harvest times (for $k = 1, 2, 3, \ldots$) by $\theta_k = \inf\{t \geq \eta_{k-1} : X(t-) = u\}$ and $\eta_k = \inf\{t \geq \theta_k : X(t-) = v\}$. We seek to determine the expressions for the expected discount factors

$$E\left[\sum_{k=1}^{\infty} e^{-\alpha\theta_k}\right] \quad \text{and} \quad E\left[\sum_{k=1}^{\infty} e^{-\alpha\eta_k}\right].$$

To determine $E\left[e^{-\alpha\eta_k}\right]$ define $\tilde{\eta}_k = \eta_k - \theta_k$ and observe that $\tilde{\eta}_k$ gives the first hitting time of level v of the process X under the dynamics with $Y(t) = 1$ starting at time θ_k in location $X(\theta_k)$ having distribution π on $[x_{min}, x_{max}]$ independent of θ_k. Recall from Remark 2.4, the Laplace transform of $\tilde{\eta}_k$ is given by $\langle\psi_2, \pi\rangle/\psi_2(v)$. Using the strong Markov property and subscripts on the expectation operator to indicate the initial value of X, we have

$$\begin{aligned}
E_{x_{new}}\left[e^{-\alpha\eta_k}\right] &= E_{x_{new}}\left[e^{-\alpha[\theta_k + (\eta_k - \theta_k)]}\right] \\
&= E_{x_{new}}\left[e^{-\alpha\theta_k} E_{x_{new}}\left[e^{-\alpha(\eta_k - \theta_k)} \middle| \mathcal{F}_{\theta_k}\right]\right] \\
&= E_{x_{new}}\left[e^{-\alpha\theta_k} E_{X(\theta_k)}\left[e^{-\alpha\tilde{\eta}_k}\right]\right] \\
&= E_{x_{new}}\left[e^{-\alpha\theta_k}\right] E\left[E_{X(\theta_k)}\left[e^{-\alpha\tilde{\eta}_k}\right]\right] \\
&= E_{x_{new}}\left[e^{-\alpha\theta_k}\right] \cdot \frac{\langle\psi_2, \pi\rangle}{\psi_2(v)}.
\end{aligned} \tag{D.1}$$

Similarly, let $\tilde{\theta}_k = \theta_k - \eta_{k-1}$, note $\tilde{\theta}_k$ is the first hitting time of X (with $Y(t) = 1$) started at time η_{k-1} in location x_{new} and $E[e^{-\alpha\tilde{\theta}_k}] = \psi_1(x_{new})/\psi_1(u)$. Applying the strong Markov property, we have

$$\begin{aligned} E_{x_{new}}\left[e^{-\alpha\theta_k}\right] &= E_{x_{new}}\left[e^{-\alpha[\eta_{k-1}+(\theta_k-\eta_{k-1})]}\right] \\ &= E_{x_{new}}\left[e^{-\alpha\eta_{k-1}} E_{x_{new}}\left[e^{-\alpha(\theta_k-\eta_{k-1})}\middle| \mathcal{F}_{\eta_{k-1}}\right]\right] \\ &= E_{x_{new}}\left[e^{-\alpha\eta_{k-1}} E_{X(\eta_{k-1})}\left[e^{-\alpha\tilde{\theta}_k}\right]\right] \\ &= E_{x_{new}}\left[e^{-\alpha\eta_{k-1}}\right] \cdot \frac{\psi_1(x_{new})}{\psi_1(u)} . \end{aligned} \tag{D.2}$$

Iterating (D.1) and (D.2) determines the summands of each series and hence yields

$$\begin{aligned} E\left[\sum_{k=1}^{\infty} e^{-\alpha\eta_k}\right] = \sum_{k=1}^{\infty} \left(\frac{\psi_1(x_{new})}{\psi_1(u)} \cdot \frac{\langle\psi_2,\pi\rangle}{\psi_2(v)}\right)^k &= \frac{1}{1-\frac{\psi_1(x_{new})\langle\psi_2,\pi\rangle}{\psi_1(u)\psi_2(v)}} \cdot \frac{\psi_1(x_{new})\langle\psi_2,\pi\rangle}{\psi_1(u)\psi_2(v)} \\ &= \frac{\psi_1(x_{new})\langle\psi_2,\pi\rangle}{\psi_1(u)\psi_2(v) - \psi_1(x_{new})\langle\psi_2,\pi\rangle} \end{aligned}$$

and

$$\begin{aligned} E\left[\sum_{k=1}^{\infty} e^{-\alpha\theta_k}\right] = \sum_{k=1}^{\infty} \left(\frac{\psi_1(x_{new})}{\psi_1(u)}\right)^k \cdot \left(\frac{\langle\psi_2,\pi\rangle}{\psi_2(v)}\right)^{k-1} &= \frac{\psi_2(v)}{\langle\psi_2,\pi\rangle} \sum_{k=1}^{\infty} \left(\frac{\psi_1(x_{new})}{\psi_1(u)} \cdot \frac{\langle\psi_2,\pi\rangle}{\psi_2(v)}\right)^k \\ &= \frac{\psi_1(x_{new})\psi_2(v)}{\psi_1(u)\psi_2(v) - \psi_1(x_{new})\langle\psi_2,\pi\rangle} . \end{aligned}$$

References

[1] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions,* Dover, New York, (1965).

[2] L.H.R. Alvarez and E. Koskela, Wicksellian theory of forest rotation under interest rate variability, *J. Econ. Dyn. Control,* **29** (2005), 529–545.

[3] A.N. Borodin and P. Salminen, *Handbook of Brownian Motion - Facts and Formulae,* 2nd. ed., Birkhäuser, Basel, (2002).

[4] J. Buongiorno, Generalization of Faustmann's formula for stochatic forest growth and prices with Markov decision process models, *Forest Sci.,* **47** (2001), 466-474.

[5] MJ Cho and RH Stockbridge, Linear programming formulation for optimal stopping problems, *SIAM J. Control Optim.,* **40** (2002), 1965–1982.

[6] H.R. Clarke and W.J. Reed, The tree-cutting problem in a stochastic environment, *J. Econ. Dyn. Control,* **13** (1989), 569–595.

[7] K. Davies, Towards more accurate growth simulations and NPV appraisals: Using INFORM to project tree grade and market value increases, *The Compiler,* **14** (1996), 18–23.

[8] P.S. DeBald and M.E. Dale, Tree value conversion standards revisited, *USDA Forest Service Research Paper NE-645,* 1991.

[9] S.N. Ethier and T.G. Kurtz, *Markov Processes: Characterization and Convergence*, Wiley, New York (1986).

[10] M. Faustmann, Calculation of the value which forest land and immature stands possess for forestry, *Allgemeine Forst-und Jagdzeitung,* **15** (1849), 441–455.

[11] K Helmes, S Röhl and RH Stockbridge, Computing moments of the exit time distribution for Markov processes by linear programming, *Oper. Res.,* **49** (2001), 516–530.

[12] K Helmes and RH Stockbridge, Extension of Dale's moment conditions with application to the Wright-Fisher model, *Stochastic Models,* **19** (2003), 255–267.

[13] K Helmes and RH Stockbridge, Linear Programming Approach to the Optimal Stopping of Singular Stochastic Processes, *Stochastics,* **79** (2007), 309-335.

[14] K. Helmes and R.H. Stockbridge, Construction of the value function and stopping rules in optimal stopping of one-dimensional diffusions, to appear in *Adv. Applied Probab.,* **42** (2010).

[15] K. Itô and H.P. McKean, Jr., *Diffusion Process and their Sample Paths,* Springer-Verlag, New York, 1974.

[16] P Kaczmarek, ST Kent, GA Rus, RH Stockbridge, BA Wade, Numerical solution of a long-term average control problem for singular stochastic processes, *Math. Meth. Oper. Res.,* **66** (2007), 451–473.

[17] S. Karlin and H.M. Taylor, *A Second Course in Stochastic Processes,* Academic Press, New York, 1981.

[18] T.G. Kurtz and R.H. Stockbridge, Existence of Markov controls and characterization of optimal Markov controls, *SIAM J. Control Optim.,* **36** (1998), 609–653, Erratum: *SIAM J. Control Optim.,* **37** (1999), 1310–1311.

[19] J.J. Mendel, P.S. DeBald and M.E. Dale, Tree value conversions standards for hardwood sawtimber, *USDA Forest Service Research Paper NE-337,* 1976.

[20] R.A. Miller and K. Voltaire, A stochastic analysis of the tree paradigm, *J. Econ. Dyn. Control,* **6** (1983), 371–386.

[21] R.R. Morrow, Tree value: A basis for woodland management, *Cornell University Dept. Nat. Res. Cons. Circular,* **19** (1981), 101-104.

[22] D.H. Newman, Forestry's golden rule and the development of the optimal forest rotation literature, *J. Forest Econ.,* **8** (2002), 5-27.

[23] C.J. Nordstrøm, A stochastic model for the growth period decision in forestry, *Swedish J. Econ.,* **77** (1975), 329–337.

[24] M.J. Penttinen, Impact of stochastic price and growth processes on optimal rotation age, *Eur. J. Forest Res.,* **125** (2006), 335-343.

[25] S. Sødal, The stochastic rotation problem: A comment, *J. Econ. Dyn. Control,* **26** (2002), 509-515.

[26] Y. Willassen, The stochastic rotation problem: A generalization of Faustmann's formula to stochastic forest growth, *J. Econ. Dyn. Control,* **22** (1998), 573–596.