The Beneš-Problem and Related Problems Revisited

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***Abstract*— We show how the Beneš-Problem, i. e. the problem of how to choose a nonanticapting control process u whose absolute value is bounded by 1 such that the second moment at time T of the controlled diffusion process X with drift process u is as small as possible, can be solved by analyzing a special entry-and-exit problem. A characterization of the optimal strategy of general entry-and-exit problems can be phrased in terms of a finite-dimensional nonlinear optimization problem. This nonlinear optimization problem can be solved explicitly for the case of switching controls of Brownian motion with a quadratic cost function of the state. The explicit solution is an essential ingredient of a new proof of the Beneš-Problem as well as related problems.**

I. INTRODUCTION

Over the years, the Beneš-Problem has stimulated the development of different solution techniques for stochastic control problems, see [1], [2], [4], [5], [9], [10], [12], [17]. In this note we shall present one more proof which is based on a characterization of the optimal strategy of a special entry-and-exit problem. Furthermore, all relevant properties of the value function are derived in "frequency space", i. e. we study the behavior of the Laplace transform of the second moment function $t \mapsto E_x[X_t^2]$, x fixed, $dX(t) = -sign(X(t))dt + dW(t)$, $X(0) = x$, without exploiting the explicit formula for $E_x[X^2(t)]$ given in [13]. The technique can also be used when analyzing problems related to the Beneš-Problem, for example, when the objective function includes a particular additive penalty term involving u, see, for instance, [3], [15], or when the control set is asymmetric, i. e. $-a \le u \le b$, $a, b > 0$.

In Section 2 we describe a particular class of entry-and-exit problems and state a characterization theorem of the optimal strategies of such switching problems in terms of a finite-dimensional nonlinear optimization problem. In Section 3 we derive a formula for the aforementioned Laplace transform. The proof of the optimality of the $-\mathrm{sgn}(X_t)$ control process is spelled out in Section 4.

II. SWITCHING CONTROLS

Switching controls, cf. [7], are a special case of entry-and-exit problems. Entry-and-exit problems have a long history in many areas of applications, e. g. in economics, engineering, production, inventory and queueing theory, etc; for details and further references of applications within each area see, for instance, [6], [14], [16] and [18]. In [11] we derive a characterization of the optimal strategy of general entry-and-exit problems in terms of a nonlinear optimizatin problem. We exploit this general result for the case of switching controls. The simplest case of a switching control problem is the following one: Let $\alpha > 0$ be a discount factor and let $c > 0$ denote the cost of switching the drift value of a Brownian motion from μ to $-\mu$, $\mu > 0$, and vice versa. The objective of the decision maker is to choose Markov times $\left(\tau_j^{(0)}\right)_{j\ge 1}$ and $\left(\tau_j^{(1)}\right)_{j\ge 1}$, $0 = \tau_0^{(1)} < \tau_1^{(0)} < \tau_1^{(1)} < \tau_2^{(0)} < \cdots < \tau_j^{(1)} < \tau_j^{(0)} < \cdots$, where the sequences have no points of accumulation, such that the total cost, $x_0 \in \mathbb{R}$, $y_0 = 1$,

$$M^{(sw)}(x_0, y_0) \; := \; E\left[\int_0^\infty e^{-\alpha t} X^2(t)\,dt + c\sum_{j=1}^\infty e^{-\alpha \tau_j^{(0)}} + c\sum_{j=1}^\infty e^{-\alpha \tau_j^{(1)}}\right] \quad (1)$$

is minimized. The process X satisfies the equation

$$dX(t) = \big(\mu Y(t) - \mu(1 - Y(t))\big)dt + \sigma dW(t), \quad (2)$$

$X(0) = x_0$, $Y(0) = y_0$, and Y is the process – determined by the switching times $\tau^{(0)}$ and $\tau^{(1)}$ – which records if the controlled process X is running with positive drift μ, i. e. $Y(t) = 1$, or is running at rate $-\mu$ and $Y(t) = 0$.

For ease of presentation, we restrict attention to the "natural" initial values (x_0, y_0) in which both $x_0 \le 0$ and $y_0 = 1$ or both $x_0 > 0$ and $y_0 = 0$.

Let

$$\rho = \rho(\alpha) := \frac{1}{\sigma^2}\left(\sqrt{\mu^2 + 2\alpha\sigma^2} - \mu\right), \quad (3)$$

and define

$$\psi(x) := e^{\rho x} \quad \text{and} \quad \phi(x) := e^{-\rho x}.$$

Let $R = -\frac{4\mu}{\alpha^2}$. It follows from [11] that the nonlinear optimization problem

$$\max_{u<0<v}\left\{\frac{Z(u,v)}{N(u,v)}\right\},$$

where

$$Z(u,v) = (Ru - c)\phi(v) - (Rv + c)\phi(u)$$

and

$$N(u,v) = \psi(v)\phi(u) - \phi(v)\psi(u) = 2\sinh\left(\rho\,(v-u)\right)$$

characterizes the optimal switching times for (1), (2).

It is instructive to consider the special (symmetric) case when $u = -v$ as the initial analysis of the two-dimensional optimization problem.

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Theorem 2.1: Set $R := -\frac{4\mu}{\alpha^2} < 0$ and observe ρ is positive.

(i) Define $\mathscr{H}(v) = \frac{Z(-v,v)}{N(-v,v)}$ for $v > 0$. The unique solution of $\max_{v>0}\{\mathscr{H}(v)\}$ is determined by the transcendental equation

$$\rho v + \rho \frac{c}{R} = \tanh(\rho v). \tag{4}$$

(ii) The optimization problem $\max_{u<0<v}\left\{\frac{Z(u,v)}{N(u,v)}\right\}$ has a unique solution (u^*, v^*) with $u^* = -v^* < 0$.

(iii) Let v^* satisfy (4). For $x_0 \in \mathbb{R}$, recall $y_0 = I_{(-\infty,0)}(x_0)$. Then the optimal value $\hat{M}^{(sw)}(x_0, y_0)$ of the control problem (1), (2) is given by the expression,

$$\begin{aligned}\hat{M}^{(sw)}(x_0,y_0) &= \left(\frac{x_0^2}{\alpha} + \frac{\sigma^2 - 2\mu|x_0|}{\alpha^2} + \frac{2\mu^2}{\alpha^3}\right) \\ &\quad + \frac{R}{2\rho}\cdot\frac{1}{\cosh(\rho v^*)} e^{-\rho|x_0|} \\ &= \left[\left(\frac{x_0^2}{\alpha} - \frac{2\mu|x_0|}{\alpha^2} + \frac{2\mu^2}{\alpha^3}\right) + \frac{\sigma^2}{\alpha^2}\right] \\ &\quad - \frac{2\mu}{\rho\alpha^2}\frac{e^{-\rho|x_0|}}{\cosh(\rho v^*)}. \end{aligned} \tag{5}$$

Moreover, the optimal switching times are the hitting times of the process (X,Y) at $(-v^*, 0)$ and $(v^*, 1)$.

(iv) For given $c \geq 0$ let $v := v(c)$ satisfy (4). For $x_0 \in \mathbb{R}$, recall $y_0 = I_{(-\infty,0)}(x_0)$, and let X satisfy (2) with Y switching at the hitting times of (X,Y) at $(-v, 0)$ and $(v, 1)$. Then

$$\begin{aligned} &E_{x_0}\left[\int_0^\infty e^{-\alpha t} X_t^2\, dt\right] \\ &= \left[\left(\frac{x_0^2}{\alpha} - \frac{2\mu|x_0|}{\alpha^2} + \frac{2\mu^2}{\alpha^3}\right) + \frac{\sigma^2}{\alpha^2}\right] \\ &\quad - \frac{2\mu}{\rho\alpha^2}\,\frac{\rho v}{\sinh(\rho v)}\, e^{-\rho|x_0|}. \end{aligned} \tag{6}$$

Proof: (i) If $u = -v$ the expression $\frac{Z(-v,v)}{N(-v,v)}$ simplifies to

$$\mathscr{H}(v) = -\frac{(Rv + c)\cosh(\rho v)}{2\sinh(\rho v)\cosh(\rho v)} = \frac{(-c + (-R)v)}{2\sinh(\rho v)}.$$

The necessary optimality condition becomes

$$0 = \mathscr{H}'(v) = -\tanh(\rho v) + \left(\rho\frac{c}{R} + \rho v\right). \tag{7}$$

There are several ways to see that (7), and hence (4), has a unique positive solution. Perhaps the easiest way is a "proof by picture;" simply graph both functions of the identity (4). Rigorous ways are to analyze the function (4) and, besides applying the Intermediate Value Theorem, show that the function is strictly monotone increasing on $(0,\infty)$; an alternative is to verify that the assumptions of Banach's Fixed Point Theorem are satisfied for the function $x \mapsto \tanh(x) - \rho\frac{c}{R}$.

(ii) In the general case when $u < 0 < v$ the necessary optimality conditions $\nabla \frac{Z}{N}(u,v) = 0$ can be rewritten as

$$N(u,v)\cdot\frac{\partial Z}{\partial u}(u,v) + Z(u,v)\cdot 2\rho\cosh(\rho(v-u)) = 0$$

and

$$N(u,v)\cdot\frac{\partial Z}{\partial v}(u,v) - Z(u,v)\cdot 2\rho\cosh(\rho(v-u)) = 0.$$

Both equations involve the parameter R. Solving for R, the first equation yields the expression

$$R = \frac{c\rho\left(e^{\rho v} + e^{\rho u}\right)^2}{e^{2\rho v} - e^{2\rho u} + \rho u\left(e^{2\rho v} + e^{2\rho u}\right) - 2\rho v e^{\rho(v+u)}}.$$

From the second equation we derive the formula

$$R = \frac{c\rho\left(e^{\rho v} + e^{\rho u}\right)^2}{e^{2\rho u} - e^{2\rho v} + \rho v\left(e^{2\rho v} + e^{2\rho u}\right) - 2\rho u e^{\rho(v+u)}}.$$

Equating the right hand sides of these two expressions we obtain after a series of simple but somewhat tedious algebraic manipulations the identity

$$(v+u)\left(e^{\rho v} - e^{\rho u}\right) = 0.$$

Since the second factor is positive the unique solution of the optimality conditions satisfies $u = -v$.

(iii) Formula (5) is an immediate consequence of the general characterization theorems proved in [11].

(iv) Assume $x_0 <$ and $y_0 = 1$. By (1), the weighted squared L^2-norm of X equals $M^{(sw)}(x_0, y_0)$ *minus* the total expected discounted switching costs. The characterization theorems of [11] together with identity (4) and formula (5) provide a way to express these costs in terms of $v(c)$, μ, α, σ^2, x_0 and ρ. Observe that the expressions for the cost in (4) hold with $a_1^* = -v$ and $b_1^* = v$. Using once more the characterization results along with $\phi(x) = e^{-\rho x}$ and $\psi(x) = e^{\rho x}$, these expected discounted switching costs equal

$$\begin{aligned} &c e^{\rho x_0} e^{-\rho v}\left[1 + e^{-2\rho v}\right]\frac{1}{1 - e^{-4\rho v}} \\ &\quad = c e^{-\rho|x_0|}\frac{\cosh(\rho v)}{\sinh(2\rho v)} \\ &\quad = e^{-\rho|x_0|}\left[-\frac{2\mu}{\rho\alpha^2}\cdot\frac{1}{\cosh(\rho v)} + \frac{2\mu}{\rho\alpha^2}\cdot\frac{v\rho}{\sinh(\rho v)}\right]; \end{aligned}$$

the last equality replaces c by its expression obtained from (4). Hence, (6) follows. ∎

Remark 2.2: Observe that the solution of (5) when $c = 0$ is $v^* = 0$. This value for v^* indicates that, in the absence of costs for switching, an optimal control policy is to use a positive rate when the process X is negative and a negative rate when X is positive. We note that due to the properties of the diffusion process X in this situation, there will be infinitely many switches shortly after X hits level 0. The imposition of a positive cost for switching avoids this control policy.

III. Analysis of process X with drift $-\mu\,\mathrm{sgn}(x)$

Formula (5) for the value function of the switching problem allows one to deduce a formula for the Laplace transform of the second moment of the process X defined by

$$dX(t) = -\mu\,\mathrm{sgn}(X(t))\,dt + \sigma\,dW(t), \quad X(0) = x. \tag{8}$$

We denote the initial value by x rather than x_0 for the remainder of the paper and will sometimes take this to be a parameter and sometimes to be a variable. This process and some modifications of this process have been studied by many authors (see e.g., [3] and [13, Section 6.5]). The Laplace transform treats the discount parameter $\alpha > 0$ as its variable so we explicitly state the dependence on α in the sequel.

Proposition 3.1: Let $\rho(\alpha)$ be defined by (3). Then the Laplace transform of the second moments of the process X of (8) is

$$\begin{aligned} V(\alpha;x) &= E\left[\int_0^\infty e^{-\alpha t}X^2(t)dt\right] \\ &= \left[\left(\frac{x^2}{\alpha} - \frac{2\mu|x|}{\alpha^2} + \frac{2\mu^2}{\alpha^3}\right) + \frac{\sigma^2}{\alpha^2}\right] \\ &\quad - \frac{2\mu}{\alpha^2\rho(\alpha)}\,e^{-\rho(\alpha)|x|} \\ &= \left[\left(\frac{x^2}{\alpha} - \frac{2\mu|x|}{\alpha^2} + \frac{2\mu^2}{\alpha^3}\right) + \frac{\sigma^2}{\alpha^2}\right] \\ &\quad - 2\mu e^{-\frac{\mu|x|}{\sigma^2}}\,\frac{1}{\alpha^2\rho(\alpha)}\,e^{-|x|\sqrt{\frac{2}{\sigma^2}}\sqrt{\alpha+\frac{\mu^2}{2\sigma^2}}}. \end{aligned} \tag{9}$$

Proof: Formula (9) follows from Theorem 2.1(iii) by putting $v = 0$ in (5) due to the observation in Remark 2.2. The second formula for $V(\alpha;x)$ displays $e^{-\rho|x|}$ in a form that facilitates the determination of its inverse Laplace transform in (10). ■

Remark 3.2: Consider the process $\tilde{X}$ with $\tilde{X}(t) = x + \mu t + \sigma W(t)$ for $t \geq 0$ when $x < 0$ which has constant drift rate μ or $\tilde{X}(t) = x - \mu t + \sigma W(t)$ when $x \geq 0$ which has constant drift $-\mu$. A simple calculation shows

$$E_x\left[\int_0^\infty e^{-\alpha t}\tilde{X}^2(t)\,dt\right] = \left(\frac{x^2}{\alpha} - \frac{2\mu|x|}{\alpha^2} + \frac{2\mu^2}{\alpha^3}\right) + \frac{\sigma^2}{\alpha^2}.$$

Thus the remaining term in (9) compensates for the overestimation by the first term from using constant drift rate $\pm\mu$ and therefore equals the value of the option to switch any time in $[0,\infty)$ between the drift rates $+\mu$ and $-\mu$.

Formula (9) can be used in many ways. One possibility is to use (9) as a benchmark when evaluating other control policies or when deciding on the magnitute of drift rates μ or switching locations v. The following collection of formulae are easily derived from (6) and (9).

Corollary 3.3: Let $dX(t) = u^*(X(t))dt + \sigma dW(t)$, $X(0) = 0$, where u^* is the switching control where the drift rates $\mp\mu$, $\mu > 0$, change at the locations $\pm v$, $v > 0$.

(i) Let $\mu = 0$, i. e. $dX(t) = \sigma dW(t)$, $X(0) = 0$. Then

$$E_0\left[\int_0^\infty e^{-\alpha t}X^2(t)dt\right] = \frac{\sigma^2}{\alpha^2}.$$

(ii) Let $\mu > 0$ and $v = 0$, i. e. $dx(t) = -\mu\,\mathrm{sgn}(X(t))dt + \sigma dW(t)$, $X(0) = 0$. Then $\exists\xi$, $0 < \xi < h := \frac{2\alpha\sigma^2}{\mu^2}$ such that

$$\begin{aligned} BV &:= E_0\left[\int_0^\infty e^{-\alpha t}X_t^2dt\right] \\ &= \frac{\sigma^2}{\alpha^2}\left[1 - \frac{\mu^2}{\alpha\sigma^2}\left(\sqrt{1+\frac{2\alpha\sigma^2}{\mu^2}} - 1\right)\right] \\ &= \frac{\sigma^2}{\alpha^2}\left(\frac{h}{2}\cdot\frac{1}{(1+\xi)^{3/2}}\right). \end{aligned}$$

(iii) Let $\mu > 0$ and $v > 0$, then

$$\begin{aligned} &E_0\left[\int_0^\infty e^{-\alpha t}X_t^2dt\right] \\ &= \frac{\sigma^2}{\alpha^2} - \frac{2\mu}{\alpha^2}\left(\frac{1}{\rho}\cdot\frac{\rho v}{sinh(\rho v)} - \frac{\mu}{\alpha}\right) \\ &= \frac{\sigma^2}{\alpha^2} - \frac{\sigma^2}{\alpha^2}\cdot\frac{\mu^2}{\alpha\sigma^2}\left[\left(\sqrt{1+\frac{2\alpha\sigma^2}{\mu^2}} + 1\right)\frac{\rho v}{sinh(\rho v)} - 2\right] \\ &= \frac{\sigma^2}{\alpha^2}\left[1 - \frac{\mu^2}{\alpha\sigma^2}\left(\sqrt{1+\frac{2\alpha\sigma^2}{\mu^2}} - 1\right)\right. \\ &\quad \left. + \frac{\mu^2}{\alpha\sigma^2}\left(\sqrt{1+\frac{2\alpha\sigma^2}{\mu^2}} + 1\right)\cdot\left(1 - \frac{\rho v}{sinh(\rho v)}\right)\right] \\ &= BV + \frac{\mu^2}{\alpha^3}\left(\sqrt{1+\frac{2\alpha\sigma^2}{\mu^2}} + 1\right)\left(1 - \frac{\rho v}{sinh(\rho v)}\right). \end{aligned}$$

To illustrate how this corollary might be used, let us assume that time is measured in years, $\sigma = 0.05$, $\alpha = 0.03$ and, for ease of computation, assume $x_0 = 0$. This choice of x_0 indicates that there is initially no deviation between the state of the system and the desired equilibrium level.

First consider the process in which no drift can be applied. While the system on average is in equilibrium, the weighted L_2-norm of X is $\sigma^2/\alpha^2 = 25/9 \approx 2.78$, see Cor. 3.3 (i). We observe that the 99 %-confidence interval for the total difference between production and demand at the end of one year is $(-0.15, 0.15)$.

Now consider the policy which switches between levels of positive und negative drift value $\mu = 0.3$. Use x_0, σ, α as above and set $y_0 = 1$. The choice of y_0 means the process is initially experiencing a positive drift. Assume (soft) state constraints of $\pm v = \pm 0.15$ are desirable so the policy which switches between $\mp\mu$ whenever X hits $\pm v$ is adopted. Then the weighted L_2-norm, see Cor. 3.3 (iii) is approximately 0.25, that is, approximately 9 % of the uncontrolled level. One can also check that the average time between changing drift rates is one year. The value 0.25 should be compared with the benchmark value BV for this example which is easily computed to be approximately 0.001.

The calculations for more general situations, such as asymmetric production rates, are more involved but the general results apply and modifications of the formulas of Corollary 3.3 can be derived.

Another possibility how to use (9) is to compute the inverse Laplace transform of $V(\alpha;x)$ and in this way derive an explicit formula for the second moment of each $X(t)$, $t \geq 0$. Let $x \in \mathbb{R}$, $t \geq 0$, and let Φ denote the standard Normal cumulative distribution function. The inverse Laplace transform is

$$\begin{aligned}\tilde{V}(t,x) &:= E_x[X_t^2] \\ &= \frac{\sigma^4}{2\mu^2} + t\sigma^2\left(|x| - \mu t - \frac{\sigma^2}{\mu}\right)\frac{1}{\sqrt{2\pi t\sigma^2}}e^{-\frac{(|x|-\mu t)^2}{2t\sigma^2}} \\ &\quad + \left((|x|-\mu t)^2 + t\sigma^2 - \frac{\sigma^4}{2\mu^2}\right)\Phi\left(\frac{|x|-\mu t}{\sigma\sqrt{t}}\right) \\ &\quad + \left(\frac{|x|\sigma^2}{\mu} + t\sigma^2 - \frac{\sigma^4}{2\mu^2}\right)e^{\frac{2\mu|x|}{\sigma^2}} \\ &\quad \cdot \left(1 - \Phi\left(\frac{|x|+\mu t}{\sigma\sqrt{t}}\right)\right). \qquad (10)\end{aligned}$$

When $\mu = \sigma^2 = 1$ formula (10) reduces to the expression given by Karatzas and Shreve [13, p. 441].

IV. New solution to the Beneš-Problem

Now, viewing $\tilde{V}$ as a function of time t and initial position x, it can be easily verified that $\tilde{V}(t,x)$ satisfies the partial differential equation

$$-\tilde{V}_t' + \frac{\sigma^2}{2}\tilde{V}_{xx}'' - \mu \cdot \operatorname{sgn}(x)\tilde{V}_x' = 0, \qquad (11)$$

as well as the conditions

$$\lim_{t\searrow 0}\tilde{V}(t,x) = x^2, \qquad \lim_{t\to\infty}\tilde{V}(t,x) = \frac{\sigma^4}{2\mu^2}.$$

However, rather than work with the formula (10) for $\tilde{V}$ to show it satisfies (11) along with the initial and terminal conditions, it is easier to work in the frequency domain and use the formula for $V(\alpha;x)$ directly. One can verify that

$$\frac{\sigma^2}{2}V_{xx}''(\alpha;x) - \mu \cdot \operatorname{sgn}(x)V_x'(\alpha;x) - \alpha V(\alpha;x) + x^2 \equiv 0. \qquad (12)$$

Observe that

$$\begin{aligned}\int_0^\infty e^{-\alpha s}\tilde{V}_t(s,x)\,ds &= e^{-\alpha s}\tilde{V}(s,x)\Big|_0^\infty + \alpha\int_0^\infty e^{-\alpha s}\tilde{V}(s,x)\,ds \\ &= -\tilde{V}(0,x) + \alpha V\end{aligned}$$

and also

$$V_x'(\alpha;x) = \int_0^\infty e^{-\alpha t}\tilde{V}_x'(t,x)\,dt$$

and

$$\tilde{V}_{xx}''(\alpha;x) = \int_0^\infty e^{-\alpha t}\tilde{V}_{xx}''(t,x)\,dt.$$

Hence, it immediately follows that $\tilde{V}$ satisfies (11). The boundary conditions follow from the Initial Value Theorem and the Final Value Theorem of Laplace transforms.

Since we are working in the "frequency" domain as indicated by formula (5) we can give a new proof of the Beneš-Problem which complements the other solutions refered to above.

Proposition 4.1: The function $u(x) = -\operatorname{sgn}(x)$ is the optimal control for the problem of minimizing, $T > 0$ given,

$$E\left[X^2(T)\right],$$

over processes X satisfying $dX(t) = u(X(t))\,dt + dW(t)$, with $X(0) = x$ and control functions u satifying the hard constraint $|u(x)| \leq 1$.

Proof: Put $\sigma = 1$ and $\mu = 1$ in all previous formulas. One can verify the function $\tilde{V}(t,x)$ not only satisfies (10) but also the partial differential equation

$$\tilde{V}_t' - \frac{1}{2}\tilde{V}_{xx}'' + \min_{-1\leq u\leq 1}\{u \cdot \tilde{V}_x'\} = 0.$$

This can be seen by either analyzing the sign of $\tilde{V}_x'$ using formula (10) or by exploiting the representation of $\tilde{V}_x'$ in terms of V_x'. This latter approach only argues in the frequency domain.

We indicate the "frequency" proof in the case $x \geq 0$. First, we need to show that the inverse Laplace transform of the last term in (9) exists. Let $g(t;x)$ be the Laplace transform of $\alpha \mapsto e^{-x\rho(\alpha)}$, where $\rho(\alpha)$ is defined by (3); here the dependence on α is explicitly expressed. Since $\alpha \mapsto x\rho(\alpha)$ is a positive function with a completely monotone derivative, $e^{-x\rho(\alpha)}$ is completely monotone as well [8, §13.4]. Hence, the inverse Laplace transform $g(t;x)$ exists; moreover, since $1 = e^{-x\rho(0)}$ the function g is a probability density. Now observe

$$\tilde{V}_x'(t,x) = 2x - 2t + 2\int_0^t (t-s)g(s;x)ds.$$

While g can be explicitly computed we stress the fact that our analysis takes place in the (α,x)-space. Since

$$2x = \lim_{\alpha\to\infty}\alpha V_x'(\alpha;x) = \lim_{t\searrow 0}\tilde{V}_x'(t,x)$$

and

$$0 = \lim_{\alpha\searrow 0}\alpha V_x'(\alpha;x) = \lim_{t\to\infty}\tilde{V}_x'(t,x),$$

the identity $\operatorname{sgn}(x) = \operatorname{sgn}(\tilde{V}_x')$ would follow from the property $\tilde{V}_{xt}'' < 0$. But $\tilde{V}_{xt}'' = -2 + 2\int_0^t g(s;x)ds < 0$ since g is a probability density. The case $x < 0$ is analyzed in a similar way. Finally, put

$$F(s,x) := \tilde{V}(T-s,x), \quad 0 \leq s \leq T,$$

and check that F satisfies the Hamilton-Jacobi-Bellman equation of the Beneš-Problem, i. e. $F(T,x) = x^2$ and

$$\min_{-1\leq u\leq 1}\{F_s' + \frac{1}{2}F_{xx}'' + uF_x'\} = 0.$$

■

V. Concluding Remarks

This note establishes the optimality of a simple two-point hitting rule for a particular entry-and-exit problem. It uses the novel approach of relating the control problem to a nonlinear optimization problem in two variables. This nonlinear optimization problem can be solved explicitely. The explicit solution provides a "frequency analysis" of the diffusion having drift rate $-\mu\,\mathrm{sgn}(x)$ and a new "frequency" proof of optimality for the solution of the Beneš control problem. This technique can also be used when analyzing similar kind of control problems.

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