Shape Invariant Modelling and Risk Patterns

Maria Grith
Wolfgang Härdle
Juhyun Park

Ladislaus von Bortkiewicz Chair of Statistics
CASE - Center for Applied Statistics and Economics
Humboldt-Universität zu Berlin
http://ise.wiwi.hu-berlin.de
http://www.case.hu-berlin.de

Financial Market

Riskless bond with constant interest rate r, stock price $(S_t)_{t \in [0,T]}$ follows a diffusion process

- **risk neutral valuation principle**

$$e^{-Tr} \int_0^\infty \psi(s_T) \, \frac{q(s_T)}{p(s_T)} p(s_T) \, ds_T$$

where q is a risk neutral density and p is the probability density function of S_T.

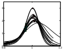
Pricing Kernels & Preferences

- the pricing kernel at time 0

$$\mathcal{K}_0(S_T) = \frac{q(S_T)}{p(S_T)}$$

- relationship between representative investor's preferences and pricing kernel (e.g. Leland 1980):

$$ARA(S_T) = \frac{p'(S_T)}{p(S_T)} - \frac{q'(S_T)}{q(S_T)} = \frac{-\mathcal{K}'(S_T)}{\mathcal{K}(S_T)}$$

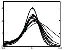


Empirical Pricing Kernel (EPK)

- EPK: any estimation of pricing kernel $\frac{q}{p}$
- under Black-Scholes model the EPK is decreasing in wealth
- other estimation methods and models for stock prices, Ait-Sahalia & Lo 2000, Engle & Rosenberg 2002, Brown & Jackwerth 2004

$\downarrow$

some paradoxa



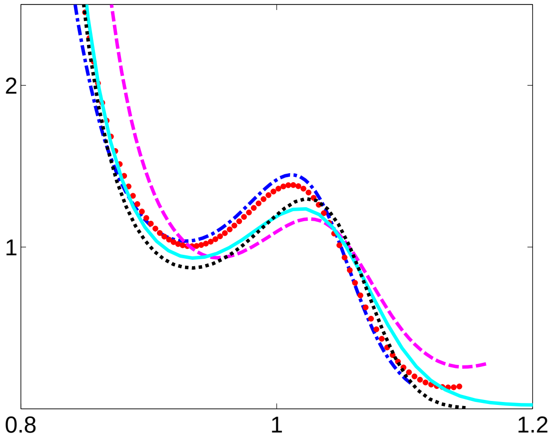


Figure 1: EPK/Moneyness K/S_t for maturities: $\tau =$0.097 (blue), 0.083 (red), 0.069 (magenta), 0.061 (cyan), 0.047 (black). Expiration date: 02-Jun-2006

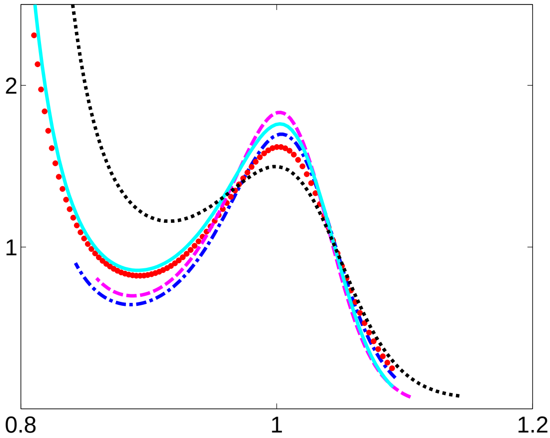
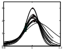


Figure 2: EPK/Moneyness for maturity τ =0.083, observed in 2006: 18-Jan (blue), 15-Feb (red), 22-Mar (magenta), 19-Apr (cyan), 17-May (black)



Literature

Multiple curves

- Gasser et al. (1984) Zurich Longitudinal Studies on Growth
- Härdle and Marron (1990) Automobile side impact data

Self Modelling

- Sylvestre et al. (1972)

EPK paradoxon: aims

Empirical pricing kernels are not monotone decreasing across returns, vary across maturities and observation time:

- **How to model the changes in the EPK functionals based on the common feature?**
- **Can EPK deformation explain the patterns in risk perception?**

Outline

1. Motivation ✓
2. Empirical Pricing Kernel
3. Shape Invariant Modelling
4. Pricing Kernel and Risk Aversion
5. Conclusions
6. Selected Bibliography

The Financial Market I

In an arbitrage-free market, the European call price is given by

$$C_t(K,\tau,r,S_t) = e^{-r\tau}\int_0^{\infty}[S_T - K]^+ q(S_T|\tau,r,S_t)\,dS_T$$

- S_t the underlying asset price at time t,
- K the strike price,
- τ the time to maturity,
- $T = t + \tau$ the expiration date,
- r constant risk free interest,

The Financial Market II

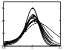
The price can be written as:

$$
\begin{aligned}
C_t(S_T) &= e^{-r}\,\mathsf{E}^{Q}\left\{(S_T - K)^{+}|S_t\right\} \\
&= e^{-r}\,\mathsf{E}^{P}\left\{(S_T - K)^{+}\mathcal{K}_t(S_t, S_T)|S_t\right\}
\end{aligned}
$$

with $\mathcal{K}_t(S_t, S_T)$ the pricing kernel at time t, s.t conditional risk neutral distributions $Q_{S_T|S_t}$:

$$
Q_{S_T|S_t=s_t}(S_T \le x) \stackrel{\text{def}}{=} \int_{-\infty}^{x} \mathcal{K}_t(s_t, \cdot)\; dP_{S_T|S_t=s_t}
$$

where $P_{S_T|S_t=s_t}$ is the conditional distribution of S_T under S_t.



Data

- **Source**: Reseach Data Center (RDC)
  http://sfb649.wiwi.hu-berlin.de

- Datastream DAX 30 Price Index;
  2 years worth of daily returns in a sliding window

- EUREX European Option Data; daily tick observations;
  selected 38 days of cross-sectional data: 200304:200605

Estimation of PK

- estimate PK as the ratio between 2 estimated densities:

$$\hat{\mathcal{K}}_t(s_t, S_T) = \frac{\hat{q}_t(S_T)}{\hat{p}_t(S_T)}$$

- $\hat{q}_t(S_T)$ by Rookley (1997) method based on the results of Breeden and Litzenberger (1978)

$$q_t(S_T) = e^{r\tau} \frac{\partial^2 C_t(\cdot)}{\partial K^2}\bigg|_{K=S_T}.$$

- $\hat{p}_t(S_T)$ historical density by kernel method

Estimation of RND

Rookley used a scaled version of the Black-Scholes call price formula that depends only on moneyness and maturity $\sigma_{IV}(K/S_t, \tau)$.

- local polynomial smoothing of degree 3
- quartic kernel

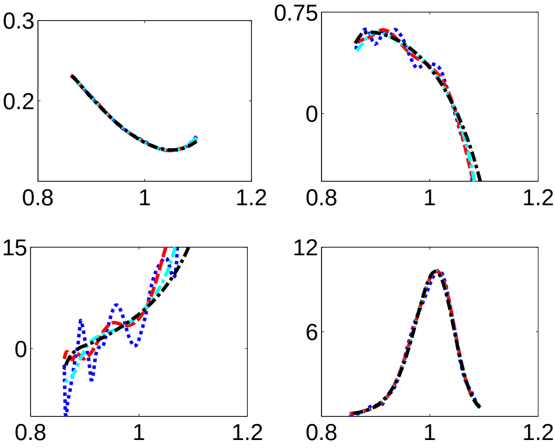


Figure 3: Implied volatility/Moneyness, its first and second derivative, q estimates with varying bandwidths (0.05, 0.10, 0.15, 0.20)

The Model

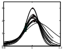
- Y_{tj} is be a noisy sample of T curves at design points u_j, with $j \in \{1, \ldots, n\}$:

$$Y_{tj} = \mathcal{K}_t(u_j) + \varepsilon_{tj}\,, \quad \text{with} \quad \varepsilon_{tj} \sim \mathrm{N}(0, \sigma_t^2).$$

- The smooth curves are of the form:

$$\mathcal{K}_t(u) = \theta_{t1}\mathcal{K}_0\left(\frac{u - \theta_{t3}}{\theta_{t2}}\right) + \theta_{t4}\,.$$

- $\mathcal{K}_0$ is a reference curve and $\theta = (\theta_{t1}, \theta_{t2}, \theta_{t3}, \theta_{t4})$ are horizontal and vertical deviation parameters



Estimation of SIM I

- Synchronisation

$$\mathcal{K}_t(\theta_{t2}u + \theta_{t3}) = \theta_{t1}\mathcal{K}_0(u) + \theta_{t4}\,, \qquad \theta_{t1} > 0\,, \quad \theta_{t2} > 0\,.$$

- Normalizing conditions:

$$T^{-1}\sum_{t=1}^{T}\theta_{t1} = T^{-1}\sum_{t=1}^{T}\theta_{t2} = 1, \quad T^{-1}\sum_{t=1}^{T}\theta_{t3} = T^{-1}\sum_{t=1}^{T}\theta_{t4} = 0\,.$$

- Common curve

$$T^{-1}\sum_{t=1}^{T}\mathcal{K}_t(\theta_{t2}u + \theta_{t3}) = \mathcal{K}_0(u)\,.$$

Common Shape

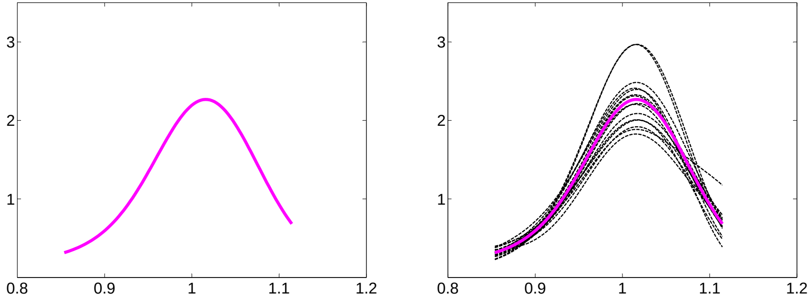
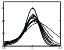


Figure 4: Estimated common shape function $\mathcal{K}_0$ (left) and transformed curves $\mathcal{K}_t(\theta_{t2}u + \theta_{t3})$ of those in Figure 5 on the common domain (right)



Estimation of SIM II

- Peak identification

$$0 = \mathcal{K}_t'(u) = \frac{\theta_{t1}}{\theta_{t2}} \mathcal{K}_0' \left(\frac{u - \theta_{t3}}{\theta_{t2}} \right).$$

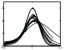
- Inflection point

$$0 = \mathcal{K}_t''(u) = \frac{\theta_{t1}}{\theta_{t2}^2} \mathcal{K}_0'' \left(\frac{u - \theta_{t3}}{\theta_{t2}} \right),$$

- The solutions also satisfy

$$u_t = \theta_{t2} u_0 + \theta_{t3}.$$

- This gives the starting values of θ_{t3} and θ_{t2}.



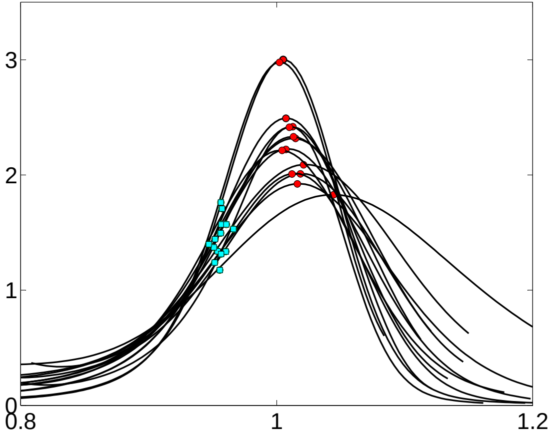
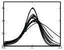


Figure 5: Landmark identification. Pricing kernels with fixed maturity 1 month between 200304:200605



Estimation of SIM III

- Estimation procedure :

$$\min_{\theta} \int \{\hat{\mathcal{K}}_t(\theta_2 u + \theta_3) - \theta_1 \hat{\mathcal{K}}_0(u) - \theta_4\}^2 w(u)\, du\,, \qquad (1)$$

where $\hat{\mathcal{K}}_i$ are nonparametric estimates of the curves and the common region is defined for some $a \geq inf(u_{1,t})$ and $b \leq sup(u_{n,t})$

$$w(u) = \prod_t 1_{[a,b]}\{(u - \theta_{t3})/\theta_{t2}\}\,.$$

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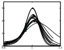


The Algorithm

- Iterative scheme based on (1)
- Given prior estimates for $(\theta_{t2}, \theta_{t3})$ and $\hat{\mathcal{K}}_0$

$$\min_{\theta} \sum_{j} \left\{ \hat{\mathcal{K}}_t(\theta_{t2} u_j + \theta_{t3}) - \theta_{t1} \hat{\mathcal{K}}_0(u_j) - \theta_{t4} \right\}^2 w(u_j) \,. \qquad (2)$$

- Update $(\theta_{t2}, \theta_{t3})$ and $\hat{\mathcal{K}}_0$
- Repeat procedure until convergence is reached

Asymptotics: EPK

$$
\begin{aligned}
\hat{\mathcal{K}}(u) - \mathcal{K}(u) &= \frac{\hat{q}(u)}{\hat{p}(u)} - \frac{q(u)}{p(u)} \\
&\simeq \frac{\hat{q}(u) - q(u)}{p(u)} - \frac{q(u)}{p(u)} \frac{\hat{p}(u) - p(u)}{p(u)} .
\end{aligned}
$$

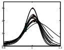
Bias

$$
\mathsf{E}\left\{ \hat{\mathcal{K}}(u) - \mathcal{K}(u) \right\} \simeq \mathcal{O}(h_q^4) + \mathcal{O}(h_p^2) ,
$$

Variance

$$
\mathrm{Var}\left\{ \hat{\mathcal{K}}(u) - \mathcal{K}(u) \right\} \simeq \mathcal{O}(M h_q)^{-1} + \mathcal{O}(m h_p)^{-1} .
$$

with M sample size for q, m sample size for p,
h_q and h_p the corresponding bandwidths.



Asymptotics: SIM Parameters

- based on standard non-linear least square methods

$$\hat{\theta}_t \approx \mathrm{N}(\theta_t, \Sigma_t)\,.$$

$$\hat{\Sigma}_t = \hat{\sigma}^2 \left[n^{-1} \sum_{j=1}^{n} \left\{ \bigtriangledown_{\boldsymbol{\theta}} \tilde{\mathcal{K}}_t(u_j; \tilde{\boldsymbol{\theta}}) \right\} \left\{ \bigtriangledown_{\boldsymbol{\theta}} \tilde{\mathcal{K}}_t(u_j; \tilde{\boldsymbol{\theta}}) \right\}^\top \right]^{-1},$$

where $\bigtriangledown_{\boldsymbol{\theta}} \mathcal{K}(u; \boldsymbol{\theta})$ is the first derivative of the function $\mathcal{K}(u; \boldsymbol{\theta})$ and

$$\hat{\sigma}_t^2 = n^{-1} \sum_{j=1}^{n} \hat{e}_{tj}^2\,.$$

with $\hat{e}_{tj} = \hat{\mathcal{K}}_t(u_j) - \tilde{\mathcal{K}}_t(u_j)$ where $\hat{\mathcal{K}}$ is the initial estimates and $\tilde{\mathcal{K}}$ is the SIM estimates.

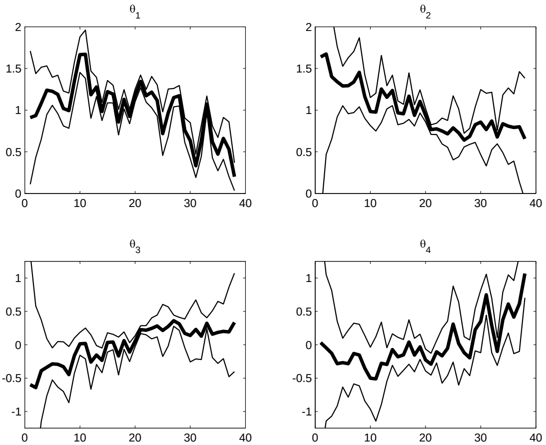


Figure 6: Parameter estimates of the SIM and their confidence intervals at 95% confidence level for the EPK 200304:200605

ARA and SIM

Under SIM specifications the ARA measure is given by:

$$ARA_t(u) = \frac{-\frac{\theta_{t1}}{\theta_{t2}}\mathcal{K}_0'\left(\frac{u-\theta_{t3}}{\theta_{t2}}\right)}{\theta_{t1}\mathcal{K}_0\left(\frac{u-\theta_{t3}}{\theta_{t2}}\right)+\theta_{t4}}.$$

The Effect of θ_1 on $\mathcal{K}_0$ and ARA_0

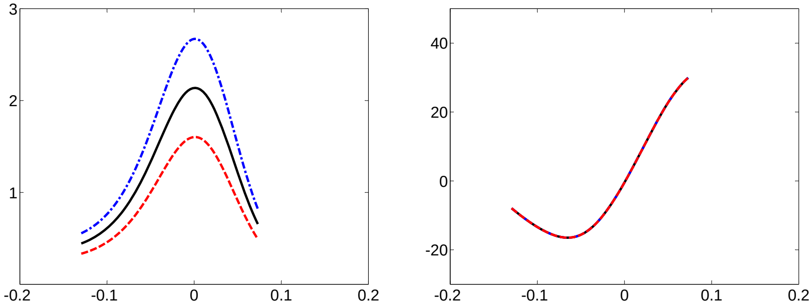


Figure 7: EPK (left) and ARA (right) $\theta_1 = 0.75$ (red), $\theta_1 = 1.25$ (blue) compared to the baseline model $\theta_0 = (1, 1, 0, 0)$ (black)

The Effect of θ_2 on $\mathcal{K}_0$ and ARA_0

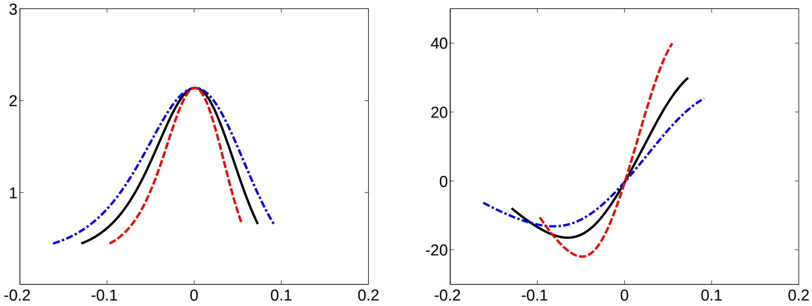


Figure 8: EPK (left) and ARA (right) $\theta_2 = 0.75$ (red), $\theta_2 = 1.25$ (blue) compared to the baseline model $\theta_0 = (1, 1, 0, 0)$ (black)

The Effect of θ_3 on $\mathcal{K}_0$ and ARA_0

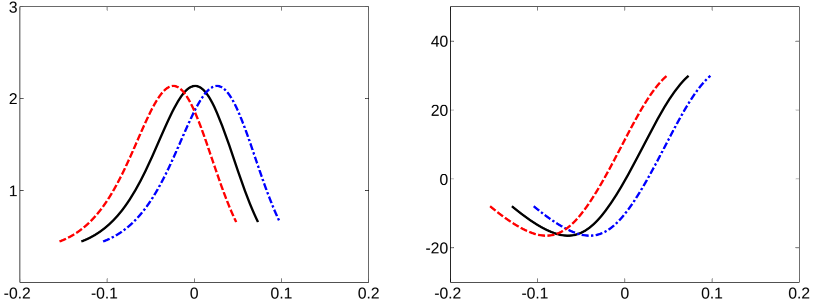


Figure 9: EPK (left) and ARA (right) $\theta_3 = -0.025$ (red), $\theta_3 = 0.025$ (blue) compared to the baseline model $\theta_0 = (1, 1, 0, 0)$ (black)

The Effect of θ_4 on $\mathcal{K}_0$ and ARA_0

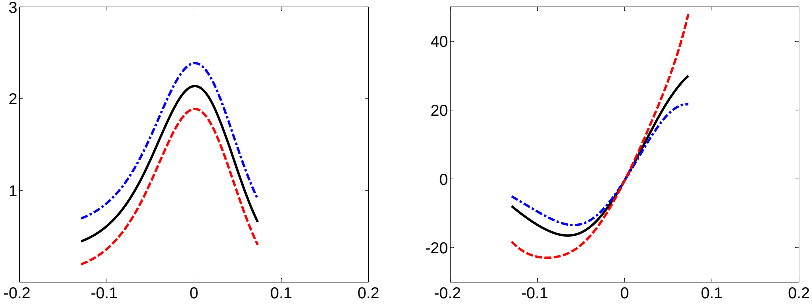
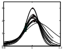


Figure 10: EPK (left) and ARA (right) $\theta_4 = -0.25$ (red), $\theta_4 = 0.25$ (blue) compared to the baseline model $\theta_0 = (1, 1, 0, 0)$ (black)



Risk Aversion and Business Cycle Indicators

- **Data**: Daily observations. German market.

- Credit spread (CD): 5Y Corporate - Gov. bond yield;
- Yield curve (YC): 3M - 10Y Gov. bond yield;
- Short term interest rate (IR): 3M Gov. bond yield;
- Datastream DAX 30 stock index (I_{Dax}).

- P_x, P_y EPK peak coordinates

Business Cycle Indicators

Indicator	Expansion	Recession
CS	$\Downarrow$	$\Uparrow$
YC	$\Downarrow$	$\Uparrow$
IR	$\Downarrow$	$\Uparrow$
I_{Dax}	$\Uparrow$	$\Downarrow$

Table 1: Behaviour of the economic indicators under the BC phases

	$\Delta\theta_1$	$\Delta\theta_2$	$\Delta\theta_3$	$\Delta\theta_4$	ΔP_x	ΔP_y
$\Delta\theta_1$	1.00					
$\Delta\theta_2$	-0.71*	1.00				
$\Delta\theta_3$	0.71*	-0.99*	1.00			
$\Delta\theta_4$	-0.93*	0.45*	-0.45*	1.00		
ΔP_x	-0.27***	0.41**	-0.38**	0.1	1.00	
ΔP_y	0.96*	-0.83*	0.83*	-0.82*	-0.31***	1.00
ΔCS	-0.30***	0.19	-0.19	0.31***	0.13	-0.28***
ΔYC	-0.02	-0.15	0.15	0.11	-0.21	0.03
ΔIR	0.01	0.10	-0.08	-0.02	0.53*	-0.00
R_{Dax}	0.68*	-0.57*	0.56*	-0.59*	-0.52*	0.69

Table 2: Correlation coefficients between SIM parameters, EPK peak coordinates and BC indicators (sig at 1% = *, 5% = **, 10% = ***)

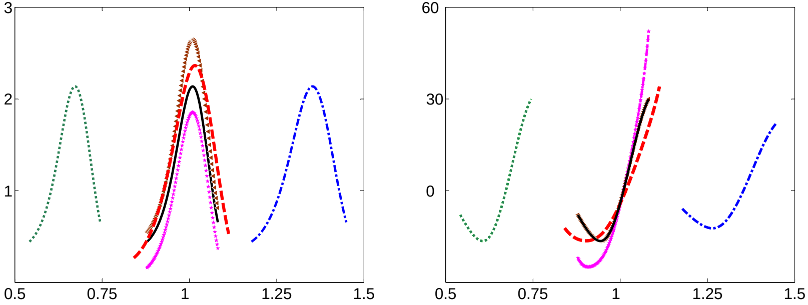


Figure 11: EPK_t and ARA_t(red/dashed) in individual effects: θ_1 (brown/triangle), θ_2 (blue/dash-dotted), θ_3 (green/dotted), θ_4 (magenta/star) compared to the baseline model (black), on 20030716

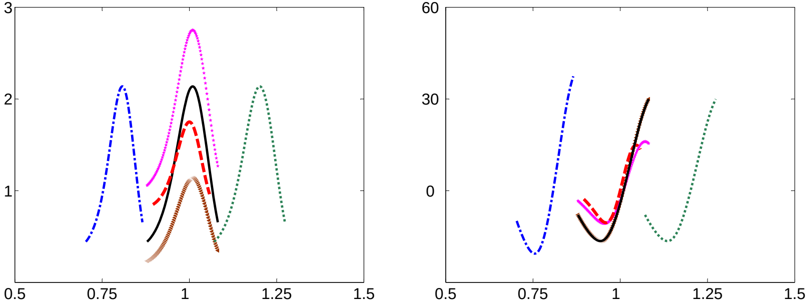


Figure 12: EPK_t and ARA_t (red/dashed) in individual effects: θ_1 (brown/triangle), θ_2 (blue/dash-dotted), θ_3 (green/dotted), θ_4 (magenta/star) compared to the baseline model (black), on 20060419

Conclusions

- An increase in the EPK peak comes with a decrease in duration
- Changes in risk proclivity are negatively correlated with the credit spread growth and positively correlated with R_{Dax}
- Agents update the expected value of the risky bets in the same sense with the ΔIR and in the oposite sense with R_{Dax}
- **Local risk proclivity is pro-cyclical**

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