A Microeconomic Explanation of the EPK Paradox

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Financial Market

Riskless bond with interest rate r, stock price process $(S_t)_{t\in[0,T]}$
S_t

- Market models
  - Black-Scholes model (Nobel prize 1997)
  - GARCH model (Nobel prize 2003)
  - non-parametric diffusion model (Aït-Sahalia & Lo, 2000)
- **risk neutral valuation principle for pay offs $\psi(S_T)$:**

$$\int_0^\infty e^{-Tr}\psi(s_T)\, \mathcal{K}(s_T) p(s_T)\, ds_T$$

p pdf of S_T, $\mathcal{K}$ pricing kernel (PK).

Pricing Kernels & Preferences

- representative investor with strictly increasing, concave, indirect von Neumann Morgenstern utility u
- **relationship between preferences and pricing kernel:**

$$\frac{du}{dx} \propto \mathcal{K}$$

Empirical Pricing Kernel (EPK)

- EPK: any estimation of pricing kernel $\mathcal{K}$
- different estimation methods and models for stock prices, Ait-Sahalia & Lo (2000), Engle & Rosenberg (2002), Brown & Jackwerth (2004), Detlefsen, Härdle & Moro (2010)

$$\downarrow$$

the paradox

EPK paradoxon: across maturities and time

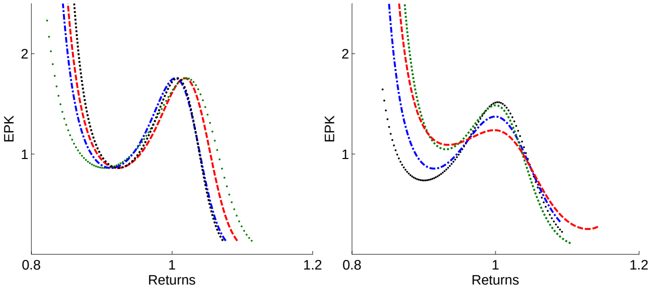


Figure 1: Examples of intertemporal pricing kernels for various maturities (left) and monthly pricing kernels for the first 6 months in 2006 for 1M maturity (right). Grith, Härdle and Park (2010)

EPK paradoxon: across maturities

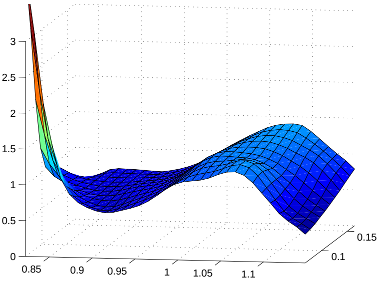


Figure 2: Estimated PK across moneyness κ and maturity τ, DAX on 20010710, Giacomini & Härdle (2008)

EPK paradoxon: across time

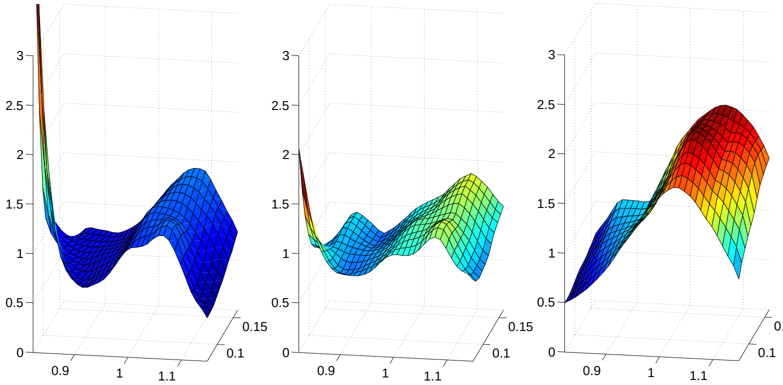


Figure 3: Empirical PK across κ and τ, estimated form DAX on 20010710, 20010904 and 20011130, Giacomini & Härdle (2008)

Aims

- Microeconomic explanation of the EPK paradox
- Switching behaviour in terms of inverse problems

Outline

1. Motivation ✓
2. Pricing Kernels
3. Microeconomic Explanation for the EPK Paradox
4. Inverse Problem
5. References

The Financial Market

1. time interval $[0, T]$ of investment with finite horizon T
2. one riskless bond with deterministic Riemannian integrable process $(r_t)_{0 \leq t \leq T}$ of interest rates
3. one risky asset with nonnegative price process $(S_t)_{0 \leq t \leq T}$, semimartingale, S_0 constant

Risk Neutral Valuation Principle

Assumption
Arbitrage free market, there exists at least one equivalent martingale measure with density π

Risk neutral price of a non-negative pay off $\psi(S_T)$ (w.r.t. π):

$$\mathsf{E}\left[e^{-\int_0^T r_t dt}\ \psi(S_T)\ \pi\ \right] \quad = \quad \mathsf{E}\left[e^{-\int_0^T r_t dt}\ \psi(S_T)\ \ \mathsf{E}[\pi|S_T]\ \right]$$

The Pricing Kernel(s)

1. **pricing kernel** (w.r.t. π), positive random variable $\mathcal{K}_\pi$. s.t.

$$\mathrm{E}[\pi | S_T] \; = \; \mathcal{K}_\pi(S_T)$$

2. **rescaled pricing kernel** (w.r.t. π)

$$\tilde{\mathcal{K}}_\pi(R_T) \stackrel{\text{def}}{=} \mathcal{K}_\pi\left(R_T S_0\right)$$

with normalized return

$$R_T = \frac{S_T}{S_0}$$

Static Consumption Model

Consumer $i = 1, \ldots, m$ with a random endowment $e_i(R_T)$

1. chooses among nonnegative random consumption $c_i(R_T)$ under the **budget constraint**

$$\mathsf{E}\left[c_i(R_T)\tilde{\mathcal{K}}_\pi(R_T)\right] \quad \leq \quad \mathsf{E}\left[e_i(R_T)\tilde{\mathcal{K}}_\pi(R_T)\right]$$

2. has extended expected utility preferences

$$U^i\{c_i(R_T)\} \quad = \quad \mathsf{E}\left[\mathbf{1}_{[0,x_i]}(R_T)u_1^i\{c_i(R_T)\} + \mathbf{1}_{]x_i,1]}(R_T)u_2^i\{c_i(R_T)\}\right]$$

where $x_i \in]0,1[$, $u_j^i : [0,\infty[\to \mathbb{R} \cup \{-\infty\}$ satisfies

$$u_j^i(c) \in \mathbb{R} \text{ for } c > 0$$

$$u_j^i \text{ nondecreasing and concave}$$

Individual utility function

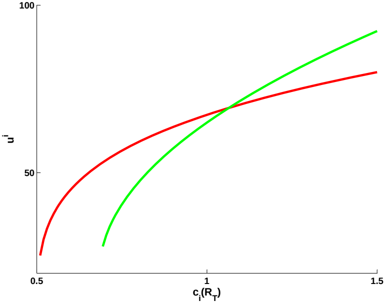


Figure 4: Regime dependent individual utility functions: bearish market (red) and bullish market (green)

Equilibrium

Contingent Arrow Debreu equilibrium $[(\bar{c}_1(R_T), ..., \bar{c}_m(R_T)); \tilde{\mathcal{K}}_\pi]$, in particular:

1. **individual optimization:** $\bar{c}_i(R_T)$ solves the optimization problem

$$\max \ U^i\{c_i(R_T)\}$$

   s.t. $c_i(R_T)$ satisfies individual budget constraint

2. **market clearing:** $\sum_{i=1}^{m} \bar{c}_i(R_T) = \sum_{i=1}^{m} e_i(R_T)$

Indirect Utilities of Representative Investor

The equilibrium guarantees nonnegative weights $\alpha_1, ..., \alpha_m$ (summing up to 1)

$$
\begin{aligned}
\sum_{i=1}^{m} \alpha_i U^i \{\bar{c}_i(R_T)\} &= U_\alpha \left\{ \sum_{i=1}^{m} e_i(R_T) \right\} \\
&\stackrel{\text{def}}{=} \max_{c_i} \left\{ \sum_{i=1}^{m} \alpha_i U^i \{c_i(R_T)\} \middle| \sum_{i=1}^{m} c_i(R_T) \leq \sum_{i=1}^{m} e_i(R_T) \right\}
\end{aligned}
$$

Extended expected utility representation

$$U_\alpha \left\{ \sum_{i=1}^{m} e_i(R_T) \right\} = \mathsf{E} \left[u_\alpha \left\{ R_T, \sum_{i=1}^{m} e_i(R_T) \right\} \right]$$

where for $r_T, e \geq 0$

$$u_\alpha(r_T, e) = \mathbf{1}_{[0,x_1]}(r_T) u_\alpha^1(e) + \sum_{i=1}^{m} \mathbf{1}_{]x_i, x_{i+1}]}(r_T) u_\alpha^{i+1}(e) \text{ for } r_T, e \geq 0$$

with $\underline{z} \stackrel{\text{def}}{=} x_0 \leq x_1 \leq \ldots \leq x_m < x_{m+1} \stackrel{\text{def}}{=} \bar{z}$, and

- $u_\alpha^k : [0, \infty[\to \mathbb{R} \cup \{-\infty\}$ is nondecreasing and concave

A Simple Solution

Let $\sum_{i=1}^{m} e_i(R_T) = R_T$.

Theorem:

Let $u_j^i|]0,\infty[$ be twice continuously differentiable satisfying **Inada conditions** for every $i \in \{1,...,m\}$ and $j \in \{1,2\}$.

Then $u_\alpha^k|]0,\infty[$ is continuously differentiable for $k \in \{1,\ldots,m+1\}$ and there is some $y > 0$ such that for any $r_T > 0$:

$$\mathbf{1}_{[0,x_1]}(r_T)\frac{du_\alpha^1}{de}\Big|_{e=r_T} + \sum_{k=1}^{m} \mathbf{1}_{]x_k,x_{k+1}]}(r_T)\frac{du_\alpha^{k+1}}{de}\Big|_{e=r_T} = y\,\tilde{\mathcal{K}}_\pi(r_T)$$

PK and Risk Aversion

$\tilde{\mathcal{K}}_\pi \; : \,]0,\infty] \to [0,\infty[$ left-continuous and piecewise nonincreasing C^1-mapping with $\lim\limits_{x\to 0} \tilde{\mathcal{K}}_\pi(x) = \infty$ and $\lim\limits_{x\to\infty} \tilde{\mathcal{K}}_\pi(x) = 0$.

Arrow-Pratt coefficients of absolute risk aversion at unknown jump points $x_1 < \ldots < x_m$ from the left and from the right:

$$\lim_{\delta\to 0_+} \frac{\frac{\widetilde{\mathcal{K}}_\pi(x_i-\delta)-\widetilde{\mathcal{K}}_\pi(x_i)}{\delta}}{\widetilde{\mathcal{K}}_\pi(x_i)}, \quad \lim_{\delta\to 0_+} \frac{\frac{\widetilde{\mathcal{K}}_\pi(x_i+\delta)-\widetilde{\mathcal{K}}_\pi(x_i)}{\delta}}{\widetilde{\mathcal{K}}_\pi(x_i)}$$

for i$\in \{1, ..., m\}$.

Example 1

Assume that

- investors have an identical switching point x_0.
- each investor i switches between CRRA utilities $u_i^j(y) = y^{\gamma_i^j}/\gamma_i^j$ $(j = 0, 1)$ with $0 < \gamma_i^1 < \gamma_i^0 < 1$,

Then

$$\mathbf{1}_{[0,x_0]}(r_T)\frac{du_\alpha^1(r_T,\cdot)}{de}\Big|_{e=r_T} + \mathbf{1}_{]x_0,\infty[}(r_T)\frac{du_\alpha^{m+1}(r_T,\cdot)}{de}\Big|_{e=r_T} = y\mathcal{K}_\pi(r_T)$$

for every realization r_T of R_T.

$$r_T = F^0\left(\frac{du_\alpha^1(r_T,\cdot)}{dy}\ \Big|_{y=r_T}\right) = F^1\left(\frac{du_\alpha^m(r_T,\cdot)}{dy}\ \Big|_{y=r_T}\right)$$

for any positive realization r_T, where

$$F^j :]0,\infty[\to]0,\infty[,\ z \mapsto \sum_{\substack{i=1 \\ \alpha_i>0}}^{m} \left(\frac{z}{\alpha_i}\right)^{\frac{1}{\gamma_i^j - 1}} \qquad (j = 0, 1)$$

are decreasing bijective mappings. If $x_0 > \max\{F^0(1), F^1(1)\}$, then

$$\frac{du_\alpha^{m+1}(r_T,\cdot)}{dy}\ \Big|_{y=r_T} > \frac{du_\alpha^1(r_T,\cdot)}{dy}\ \Big|_{y=r_T} \text{ for } r_T \geq x_0.$$

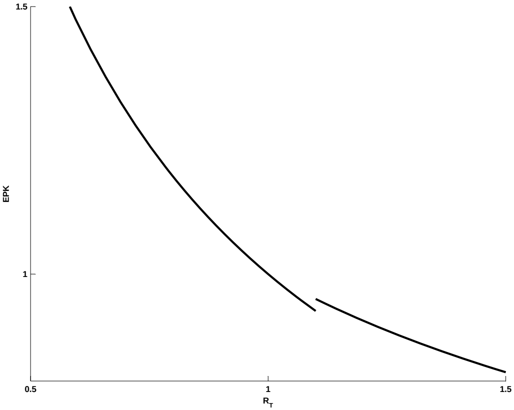


Figure 5: One switching point in the PK with $\gamma_\alpha^1 = 0.50 < \gamma_\alpha^0 = 0.75$

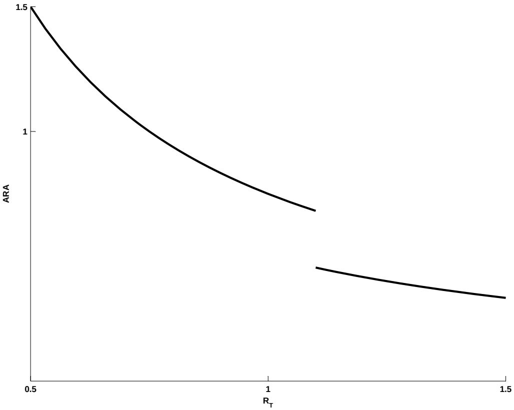


Figure 6: Implied ARA for one switching point in the PK with $\gamma_\alpha^1 = 0.50 < \gamma_\alpha^0 = 0.75$

Example 2

Assume that

- investors may differ in their switching point x_i.
- each investor i switches between the same u^0 and u^1
- $\omega(r_T)$ share of agents with preferences u^0 for r_T, ($\omega \in [0,1]$)

Then if $\alpha_1 = \alpha_2 = \cdots = \alpha_m = \alpha$ it holds in equilibrium

$$\omega(r_T)\bar{c}^0 + \{1 - \omega(r_T)\}\bar{c}^1 = \bar{e}(R_T) \stackrel{\text{def}}{=} R_T, \text{ for every } r_T,$$

with

$$\frac{du^0(r_T,\cdot)}{dy}\Big|_{y=\bar{c}^0(r_T)} = \frac{du^1(r_T,\cdot)}{dy}\Big|_{y=\bar{c}^1(r_T)}.$$

Let

$$u^0 = b_0 \frac{x^{1-\gamma^0}}{1-\gamma^0} + a_0 \quad \text{and} \quad u^1 = b_1 \frac{x^{1-\gamma^1}}{1-\gamma^1} + a_1,$$

for some constants b_0, $b_1 > 0$ and a_0, a_1 and u^1 is more concave than u^0 at each x. Then

$$\mathcal{K}_\pi(r_T) = 1_{[0,x_m]}(r_T) b_0 x^{\gamma_0}|_{x=\bar{c}^0(r_T)} + 1_{]x_m,\infty[}(r_T) b_1 x^{\gamma_1}|_{x=\bar{c}^1(r_T)}.$$

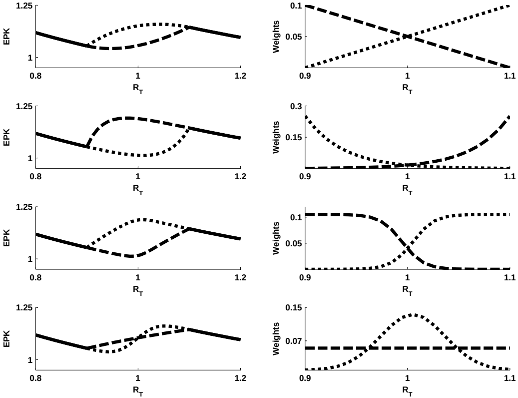


Figure 7: The relationship between the shape of the pricing kernel and the weight function $w(r_T) = d\omega(r_T)/dr_T$ for linear, exponential, logistic, constant and bell shaped specifications for $\gamma^0 = \gamma^1 = 0.5$, $b_0 = 1$ and $b_1 = 1.2$

Data Fit

Use a test function approach

- family $\mathcal{V}$ of strictly nonincreasing C^1-mappings $v :]0,\infty[\to \mathbb{R}$ with $\lim_{x\to 0} v(x) = \infty$ and $\lim_{x\to\infty} v(x) = 0$.
- test functions

$$\sum_{i=1}^{N+1} \mathbf{1}_{]x_{i-1},x_i]}(x) v_i(x), \quad v_1, \dots, v_{N+1} \in \mathcal{V}$$

with N switching points.

Grid-based Approach

- Use Korovkin approximation results for mappings on continuous intervals

Proposition

The mapping $\sum_{i=1}^{N} \widetilde{\mathcal{K}}_\pi\{a + \frac{i}{N}(b-a)\}\mathbf{1}_{]a+\frac{(i-1)}{N}(b-a),a+\frac{i}{N}(b-a)]}(x)$ *converges to* $\widetilde{\mathcal{K}}_\pi(x)$ *on* $[a,b]$ *uniformly on compacta of continuity points of* $\widetilde{\mathcal{K}}_\pi|[a,b]$ *for any nondegenerated interval* $[a,b] \subseteq]0,\infty[$.

The Inverse Problem

$\mathcal{Z}_N$ set of partitions $x_0 \leq x_1 \leq \ldots \leq x_{N+1}$
Find

$$\min_{(x_1,\ldots,x_N)\in\mathcal{Z}_N} \; \min_{v_1,\ldots,v_{N+1}\in\mathcal{V}} \int \left\{ \widetilde{\mathcal{K}}_\pi(x) - \sum_{i=1}^{N} v_i(x)\mathbf{1}_{]x_{i-1},x_i]}(x) \right\}^2 \hat{p}(x)\, dx$$

where $\hat{p}$ is an approximation of the density function p.

Grid-based Approach. Operationalization

Solve $\gamma_0^*, ...\gamma_{N+1}^*, \beta_0^*, ..., \beta_{N+1}^*, x_1^*, ..., x_N^* = \arg\min F_N$,

$$F_N = \sum_{j=1}^{n} \left\{ \hat{\mathcal{K}}_\pi(s_j) - \sum_{i=1}^{N} v_i(s_j) \mathbf{1}]x_{i-1}, x_i](s_j) \right\}^2 \hat{p}(s_j)\Delta_j$$

$\hat{\mathcal{K}}$ estimates $\tilde{\mathcal{K}}$, n gridpts and $\Delta_j = s_j - s_{j-1}$.

$$v_i(x) = \beta_i x^{-\gamma_i} \quad \text{if} \quad x_{i-i} < x \le x_i$$

Distcrete Switching Points

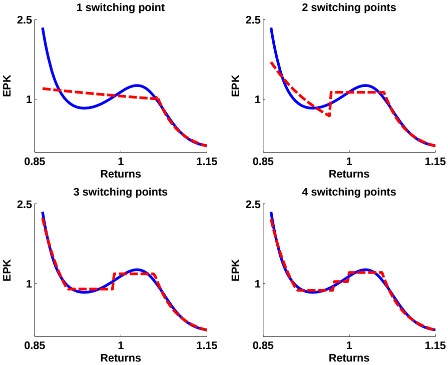


Figure 8: Nonparametric EPK (blue) and fitted PK specified by ?? (red) on 20060621

1 switching point

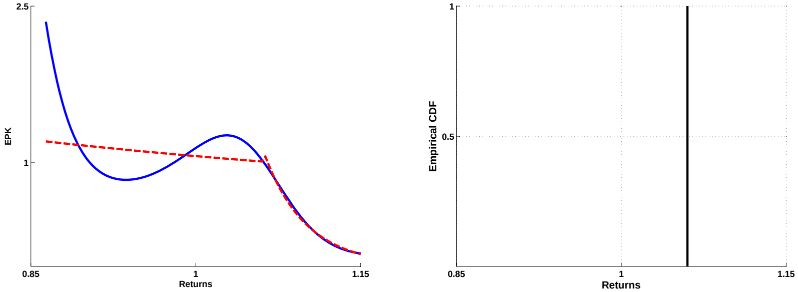


Figure 9: β_1=1.06, γ_1=0.85, β_2=5.88, γ_2=27.99

2 switching points

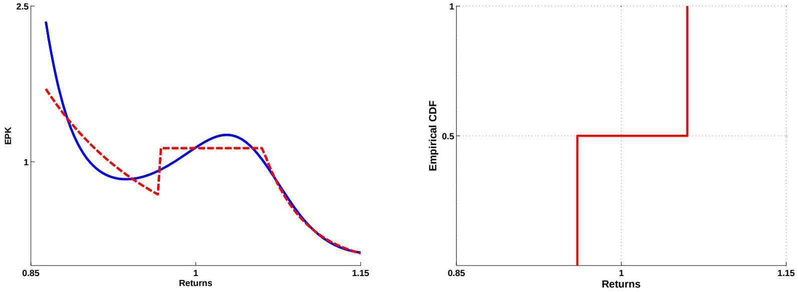


Figure 10: β_1=0.51, γ_1=0, β_2=1.13, γ_2=0, β_3=5.88, γ_3=27.99

3 switching points

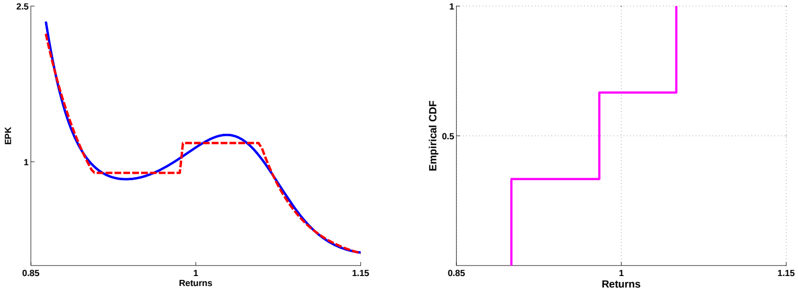


Figure 11: β_1=0.14, γ_1=0, β_2=0.89, γ_2=0, β_3=1.18, γ_3=27.74, β_4=5.71, γ_4=27.74

4 switching points

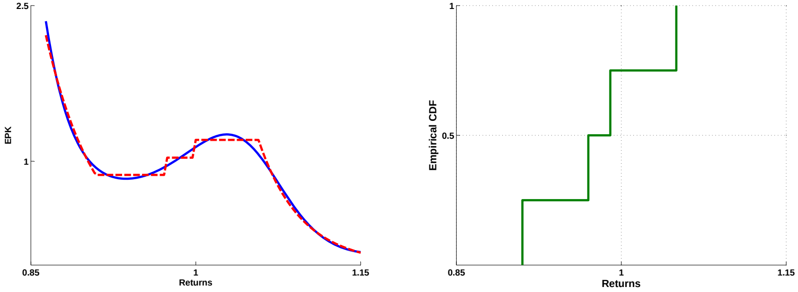


Figure 12: β_1=0.15, γ_1=18.28, β_2=0.87, γ_2=0, β_3=1.03, γ_3=0, β_4=1.21, γ_4=0, β_5=5.53, γ_5=27.45

5 switching points

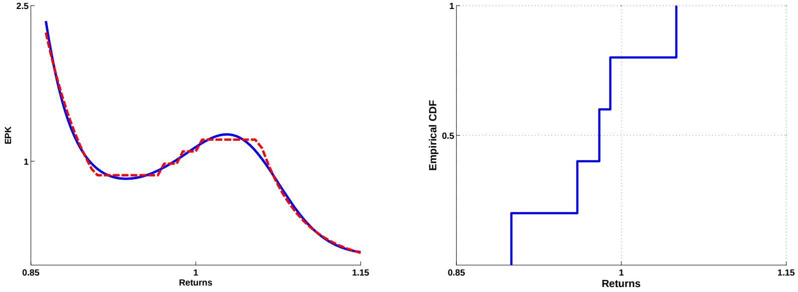


Figure 13: β_1=0.14, γ_1=18.76, β_2=0.87, γ_2=0, β_3=0.98, γ_3=0, β_4=1.10, γ_4=0, β_5=1.21, γ_5=0, β_6=5.71, γ_6=27.79

5 switching points

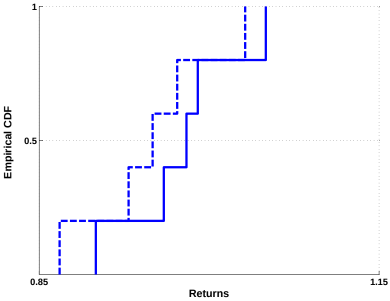
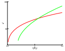


Figure 14: Empirical CDF on 20000920 (dashed) and 20060621 (solid)



Continuous Switching Points

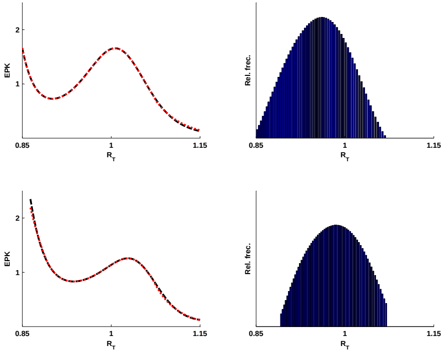


Figure 15: Left: $\hat{\mathcal{K}}_\pi$ (dashed, black) and v (red, dotted) on 20000920 (upper panel) with b_0=0.01, γ_0=30.10, b_1=3.58, γ_1=23.15 and 20060621 (lower panel) with b_0=0.14, γ_0=18.76, b_1=5.71, γ_1=27.79. Right: Estimated weighting functions w

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