1 Varying Coefficient Models

The general form of a varying coefficient model allowed for in *MulTi* is

$$y_t = x_t\gamma_t + z_t\beta + u_t, \tag{9.1}$$

where y_t is a scalar dependent variable, $x_t = (x_{1t}, \ldots, x_{Kt})$ is a $(1 \times K)$ vector of exogenous variables, $\gamma_t = (\gamma_{1t}, \ldots, \gamma_{Kt})'$ is a $(K \times 1)$ vector of possibly time varying coefficients, $z_t = (z_{1t}, \ldots, z_{Mt})$ is a $(1 \times M)$ regressor vector with constant coefficients $\beta = (\beta_1, \ldots, \beta_M)'$ and u_t is a scalar error term with mean zero and variance σ_u^2. Thus, the analysis of varying coefficient models is confined to single equation models. *MulTi* offers two types of methods for estimating the time paths $\gamma_1, \ldots, \gamma_T$ of the coefficients, namely the flexible least squares (FLS) method and maximum likelihood (ML) methods based on different types of state space models. When you choose the

Model type:	Varying coeff.

and the entry

C	Model Specification

in the main menu (see Fig. 2.3) you obtain the menu shown in Fig. 9.1. It allows you to choose between FLS and ML analysis. The two methods and their implementation will be considered in turn in the following. Since forecasting and structural analysis are not available for varying coefficient models, choosing the options

D	Forecasting

or

E	Structural Analysis

in the main menu in Fig. 2.3, an error message appears.

1.1 Flexible Least Squares Analysis

Flexible least squares (FLS) analysis was proposed by R. Kalaba & L. Tesfatsion (1989), "Time-Varying Linear Regression via Flexible Least Squares," *Computers and Mathematics with Applications*, 17, 1215-1245 (see also R. Kalaba, & L. Tesfatsion (1990), "Flexible Least Squares for Approximately Linear Systems," *IEEE Transactions on Systems, Man, and Cybernetics*, SMC-20, 978-989). In the FLS analysis the constant coefficient regressors z_t are not present in the model (9.1), that is, the model

$$y_t = x_t\gamma_t + u_t$$

is analyzed. FLS is a method for recursively computing the time paths $\gamma_1, \ldots, \gamma_T$ which minimize the objective function

$$\sum_{t=1}^{T}(y_t - x_t\gamma_t)^2 + \lambda \sum_{t=1}^{T-1}(\gamma_{t+1} - \gamma_t)'D(\gamma_{t+1} - \gamma_t) \tag{9.2}$$

Figure 9.1: Menu for choosing the method for analyzing a varying coefficient model.

where λ is a prespecified positive constant and D is a fixed positive definite $(K \times K)$ matrix which is also specified by the user of the method. The first sum in the objective function penalizes equation errors u_t while the second sum in (9.2) penalizes coefficient variation. The relative weight assigned to these two terms is controlled by λ. A small value of λ reduces the importance of coefficient variation in the objective function and hence leads to more volatile coefficient time paths. On the other hand, a large value of λ penalizes coefficient variation heavily and, hence, results in smooth or almost constant time paths. *MulTi* automatically computes time paths of the coefficients for a range of λ values. The weighting matrix D in (9.2) may be chosen so as to compensate for different scalings of the regressors. *MulTi* offers the choice of three types of D matrices.

The main menu for a flexible least squares analysis is shown in Fig. 9.2. The cursor appears on the right-hand side when you enter the FLS menu. On that side you may choose the sample stretch which you would like to analyze by selecting

A	Beginning of sample

and

B	End of sample

and specifying the desired numbers. In contrast to some other menus you don't have to allow for presample values even if you include lags of some of your variables as regressors

Figure 9.2: Menu for FLS analysis.

in the model. The program takes care of the presample values automatically by reducing the sample period appropriately. You cannot use 1 as `Beginning of sample`, however, if you have differenced one or more variables in the preliminary analysis.

You may include all variables from your data set and lags of these variables as independent regressor variables in the model. If you want to include an intercept term in addition, set

`C` `Constant term`

to `yes`.

Upon choosing

`D` `Choice of dependent variable`

a pop-up menu appears listing all the variables of your system. You may choose the dependent variable from this list. Then the regressor variables must be specified by selecting

`E` `Choice of independent variables`

Once you have chosen this entry *MulTi* wants you to specify the lags of each variable that you would like to be used as regressors. Specify -1 if you want to exclude a particular variable and its lags from the set of regressors. At this stage you must not choose lag 0

of the dependent variable because that variable cannot be an independent variable at the same time. You may, however, include other lags of your dependent variable, such as 1 or 2, in the list of regressors.

The following three options are available for the weighting matrix D:

$$(X'X)/T, \quad \text{diag}(X'X)/T \quad \text{and} \quad \alpha I_K. \tag{9.3}$$

Here T denotes the sample size,

$$X = \begin{bmatrix} x_1 \\ \vdots \\ x_T \end{bmatrix}$$

in the $(T \times K)$ regression matrix, $\text{diag}(X'X)/T$ denotes the diagonal matrix with the same diagonal elements as $(X'X)/T$ and zero off-diagonal elements. The first two weighting matrices given in (9.3) compensate for different scalings of the regressors. Thus, they are useful if different regressors have quite different magnitudes. The third weighting matrix, αI_K, assigns equal weight to each coefficient. If you choose this weighting matrix you are asked to specify α. This allows you to override to some extent the range of λ values which is automatically used by *MulTi*. Once you have selected

F	Choice of weighting matrix

you may choose one of the three weighting matrices given in (9.3) by highlighting it with the cursor keys and pressing the [↩] key.

The time paths for values

$$\lambda = .001, .01, .1, 1, 10, 100, 1000, 10000$$

are computed when you select

Q	Return to results page

Plots of the time paths for each of the coefficients for all the different λ values are displayed upon choosing

A	Time paths of coefficients

on the left-hand page of the FLS menu. Press the [↩] key to display the graphs for the next coefficient. An example is given in Fig. 9.3.

Under

B	Summary statistics

on the left-hand side of the FLS menu the means, standard deviations and coefficients of variation of the FLS time paths are available for each of the coefficients and for each value of λ. An example is shown in Fig. 9.4.

A pop-up menu for a residual analysis appears when

C	Residual analysis

Figure 9.3: FLS time paths of a coefficient.

is selected. The full range of diagnostics is available including autocorrelations and plots of residuals and squared residuals. Rough $1/\sqrt{T}$ standard errors are provided for the autocorrelations. The degrees of freedom for the approximate χ^2-distribution of the portmanteau test are determined as the maximum lag of the involved autocorrelations minus the number of regressors K. It is unclear whether this is a reasonable approximation to the actual distribution of the portmanteau statistic. Therefore the given values of the distribution function should be interpreted cautiously.

If at some stage you are not sure which model you are currently analyzing select

E Model

to display the names and lags of the dependent and independent variables. To modify your model you may choose

P Change model ⟶

which takes you back to the right-hand side of the menu.

Figure 9.4: Summary statistics for FLS time paths.

1.2 Maximum Likelihood Analysis

In the maximum likelihood (ML) analysis a state space model of the following general form is estimated:

$$y_t = x_t\gamma_t + z_t\beta + u_t \qquad \text{(measurement equation)} \tag{9.4}$$

$$\gamma_t = B\gamma_{t-1} + w_{t-1} \qquad \text{(transition equation)} \tag{9.5}$$

where y_t, x_t, z_t, γ_t and β are defined as in (9.1), u_t is normally distributed white noise with variance σ_u^2, that is $u_t \sim N(0, \sigma_u^2)$, $w_t = (w_{1t}, \ldots, w_{Kt})'$, $t = 0, 1, \ldots$, is a Gaussian, K-dimensional, zero mean white noise process with diagonal covariance matrix

$$\Sigma_w = \begin{bmatrix} \sigma_{w_1}^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_{w_K}^2 \end{bmatrix},$$

that is, $w_t \sim N(0, \Sigma_w)$. The process w_t is assumed to be independent of the process u_t. Furthermore, B is the $(K \times K)$ *transition matrix*. *MulTi* allows two possible B matrices. In the *random walk model*, $B = I_K$ and in the *return-to-normality model*,

$$B = \begin{bmatrix} b_1 & & 0 \\ & \ddots & \\ 0 & & b_K \end{bmatrix}$$

is a diagonal matrix with unknown diagonal elements. Finally, we assume that the initial state γ_0 is also normally distributed,

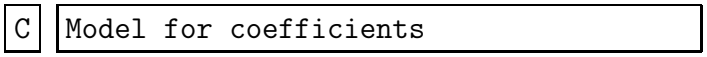
$$\gamma_0 \sim N(\bar{\gamma}_0, \Sigma_0),$$

and is independent of the u_t and w_t processes. For $t = 1, \ldots, T$ the likelihood function of this model is optimized with respect to the parameters $\sigma_u^2, \sigma_{w_1}^2, \ldots, \sigma_{w_K}^2, \beta$ and, if applicable, $b_1, \ldots, b_K$. State space models are discussed in Chapter 13 of *IMTSA*.

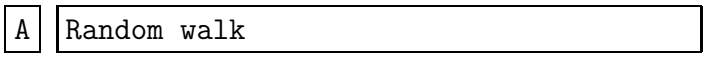
Figure 9.5: Menu for maximum likelihood analysis of a state space model.

On the right-hand side of the ML analysis menu shown in Fig. 9.5 the beginning and the end of the sample to be used in the estimation must be specified. Since *MulTi* internally adjusts the sample beginning if presample values are required in the following analysis you don't have to worry about allowance for such values at this stage.

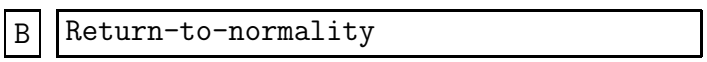
If you choose



a pop-up menu appears which gives you the choice between



and



If you decide to go with the random walk model the transition matrix $B = I_K$ and you may specify the dependent variable y_t of the measurement equation by choosing

D	Choice of dependent variable

Furthermore under

E	Choice of constant coeff.

and

F	Choice of varying coeff.

the regressors z_t and x_t with constant and varying coefficients, respectively, must be chosen. *MulTi* asks you to specify the desired lags of all variables in your data set. No regressor may be chosen for z_t and x_t at the same time. Also, you may not choose lag zero of the dependent variable as a regressor. Choose -1 if you want to exclude a variable and its lags.

In the return-to-normality model the measurement equation has the form

$$y_t = x_t\gamma_t + x_t\beta + u_t \tag{9.6a}$$

and the transition equation is

$$\gamma_t = B\gamma_{t-1} + w_{t-1} \tag{9.6b}$$

Defining $\eta_t = \gamma_t + \beta$ this state space model is equivalent to:

$$\begin{aligned} y_t &= x_t\eta_t + u_t \\ (\eta_t - \beta) &= B(\eta_{t-1} - \beta) + w_{t-1} \end{aligned}$$

Thus, the regressors with varying coefficients in (9.6) are the same as those with constant coefficients and, hence, $z_t = x_t$. Therefore, you just have the choice of regressors with varying coefficients under

F	Choice of varying coeff.

They will automatically be incorporated as z_t regressors as well.

Once you have fully specified the model you may proceed to the left-hand side of the menu by choosing

Q	Return to results page

Two different iterative algorithms are available in *MulTi* for optimizing the log-likelihood function, the full EM algorithm and the scoring algorithm. Both algorithms are described by W. Schneider (1988), "Analytical uses of Kalman filtering in econometrics – a survey", *Statistical Papers*, 29, 3-33. The scoring algorithm is also discussed in Sec. 13.4 of *IMTSA*. Both algorithms make extensive use of the Kalman filter. The EM algorithm performs the individual iterations much faster than the scoring algorithm. It usually needs many more iterations to reach the optimum, though. A good strategy in practice is to

perform some EM iterations first and then switch to the scoring algorithm. t-ratios and standard errors are available only if at least one scoring iteration has been performed.

If you choose

A	Full EM-algorithm

a pop-up menu appears which allows you to modify the default termination criteria for the algorithm. Possible termination criteria are the maximum number of iterations, an insufficient increase of the log-likelihood function and if the relative change of the parameters falls short of a prespecified bound. The relative change of the parameter values in the i-th iteration is defined as

$$\sqrt{\frac{1}{N}\sum_{n=1}^{N}\left(\frac{\delta_n^{(i)}-\delta_n^{(i-1)}}{\delta_n^{(i-1)}}\right)^2}$$

where N is the number of free parameters in the model and $\delta_n^{(i)}$ is the n-th element of the parameter vector obtained in the i-th iteration. The quantity

$$\left(\frac{\delta_n^{(i)}-\delta_n^{(i-1)}}{\delta_n^{(i-1)}}\right)^2$$

is replaced by zero if $\delta_n^{(i-1)}=0$.

If you choose

Q	Quit

in the pop-up menu containing the termination criteria, an initial estimator is computed automatically and some intermediate estimation results appear on the screen. They enable you to monitor the performance of the algorithm. An example screen is depicted in Fig. 9.6. The algorithm stops if either the prespecified maximum number of iterations has been performed or if the two other termination criteria are jointly satisfied.

Once the iterations have terminated you may return to the ML analysis menu shown in Fig. 9.5 and select

B	Scoring algorithm

Again a pop-up menu appears that allows you to modify the termination criteria. It is useful to choose a small maximum number of iterations at this stage because scoring iterations may be very time consuming and you may want to check how long they take. The derivatives required in this algorithm are computed numerically with the Kalman filter. When you leave the pop-up menu the iterations start from the last available parameter vector, for instance, the one reached by the EM algorithm. If no previous parameter values are available *MulTi* computes its own initial estimates. Like for the EM algorithm summary results of the scoring iterations are displayed on the screen while the algorithm is in progress. Once the iterations have terminated and you have returned to the ML menu you may again choose

Figure 9.6: Iterations of the EM algorithm.

B Scoring algorithm

The following iterations will then start from the parameter values obtained in the final iteration of the previous call of the scoring algorithm. Thus, you don't lose much by choosing a small number of iterations in each call of the algorithm. Note, however, that the algorithm starts from fresh initial values if you modify the model.

If at least one iteration of the scoring or EM algorithm has been made, estimation results are available under

C Parameter estimates

Estimates of the mean of the initial state γ_0 and the covariance matrix Σ_0 are given in addition to estimates of the other parameters. In other words, in the random walk model estimates of β, σ_u^2 and $\sigma_{w_1}^2, \ldots, \sigma_{w_K}^2$ are provided and for the return-to-normality model estimates of $\sigma_u^2, b_1, \ldots, b_K$ and $\sigma_{w_1}^2, \ldots, \sigma_{w_K}^2$ are given. If at least one iteration of the scoring algorithm has been performed estimated standard errors and t-ratios are also provided. Since the actual asymptotic distributions of the t-ratios are unknown in general they should be interpreted cautiously. Plots of the implied time paths of the coefficients with estimated two-standard error bounds are available under

D Time paths of coefficients

Press ↩ to display the next time path. An example is shown in Fig. 9.7. The time

paths are the estimated conditional expectations $\gamma_{t|T}$ based on the full sample $y_1, \ldots, y_T$. Thus, they are computed with the Kalman smoother on the basis of the estimated state space model. The two-standard error bounds shown in Fig. 9.7 are obtained from the associated covariance matrices $\Sigma_\gamma(t|T)$ of the conditional distributions.

Figure 9.7: Plot of a time path of a coefficient from a return-to-normality model.

A pop-up menu with some diagnostics for the innovations from the state space model appears upon selecting

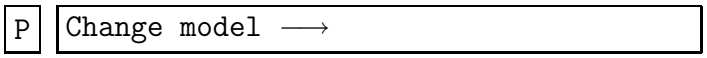
E	Innovation analysis

The innovations are defined as $u_{t|t-1} = y_t - y_{t|t-1}$, where $y_{t|t-1}$ denotes the conditional expectation of y_t given $y_1, \ldots, y_{t-1}$. The standard errors of the autocorrelations of the innovations and squared innovations are estimated as $1/\sqrt{T}$. The distribution of the portmanteau statistic is approximated by a χ^2-distribution with degrees of freedom equal to the maximum lag of the involved autocorrelations minus the number of regressors in the measurement equation (9.4).

If at some stage of your analysis you are not sure about the model you are working on (for instance, if you are not sure which variables and lags you have specified as regressor variables) you may select

F	Model

to get the information. If you want to change the model choose



to return to the right-hand page of the ML analysis menu.

In *MulTi* forecasting and structural analysis are not possible with varying coefficient models. Therefore, if you quit from the ML menu all results and models will be erased. æ