Advanced Econometrics

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Prof. Dr. Nikolaus Hautsch
Chair of Econometrics
Humboldt-Universität zu Berlin

You have to answer 2 out of 3 problems within 90 minutes (plus 10 minutes "reading time"). If you answer all questions, only the first 2 problems will be taken into account.

You may answer in English or in German. But please stick to one language.

You find necessary underlying results and formulas in the appendix.

Do your best to write legibly. Exams or parts of exams which cannot be read with reasonable effort will not be graded.

Good luck!

Problem 1: GMM Estimation

Let $z_i \equiv (y_i, w_i)$ be a vector of endogenous variables and instruments for observations $i = 1, \dots, n$. Define θ_0 as the true $(k \times 1)$ parameter vector.

a) Give the definition of a conditional moment function. Show how it can be used to construct appropriate unconditional moment functions. How many moment functions and instruments do we need for an (over-)identified model?

b) Give the objective function of GMM estimation and explain the underlying principle. In which sense is it a generalization of the method of moment objective function?

c) Assume a linear model with conditionally heteroscedastic errors

$$y_i = x_i'\theta_0 + \varepsilon_i, \quad \varepsilon_i \sim IID(0, \Omega(w_i)), \quad i = 1, \dots, n,$$

where x_i is a $(k \times 1)$ vector of regressors with $\mathrm{E}[\varepsilon_i x_i] \neq 0$, and w_i is a $(q \times 1)$ vector of instruments with $\mathrm{E}[\varepsilon_i w_i] = 0$ and $q > k$.

Derive the GMM estimator and its asymptotic covariance matrix based on the moment function $\psi(z_i, \theta) = w_i \varepsilon_i$ and an identity weighting matrix.

d) Which (asymptotic) properties of sample moments of the data do you need for the estimator derived in (c) to be consistent? Prove the consistency.

e) Assume that $\Omega(w_i) = \sigma^2 I_k$, where I_k denotes an $(k \times k)$ identity matrix. Show that the GMM estimator for θ_0 using an *optimal* instrument matrix is given by

$$\hat{\theta}_n = (X'W(W'W)^{-1}W'X)^{-1}X'W(W'W)^{-1}W'Y,$$

where X and W denote the corresponding $(n \times k)$ and $(n \times q)$ matrices of the regressors and instruments, respectively. How do we call this estimator? Why does it essentially correspond to a method of moments estimator?

Hint: Substitute $E[x_i|w_i]$ by the corresponding linear projection

$$E[x_i|w_i] = X'W(W'W)^{-1}w_i.$$

f) Consider the (G)MM estimator for θ_0 derived in (e) for $X = W$. By assuming an appropriate distribution for ε_i, suggest an ML estimator which is identical with that (G)MM estimator. What does this result mean for the robustness of your ML estimator? How do we call such an ML estimator?

Problem 2: Nonlinear Regression

Assume a latent regression model of the form

$$Y_i^* = x_i'\beta_0 + \varepsilon_i, \qquad i = 1, \ldots, n,$$

with the following observation rule for the observable dependent variable:

$$Y_i = \begin{cases} 1, & \text{if } Y_i^* > 0, \\ 0, & \text{otherwise.} \end{cases}$$

It is assumed that the error term ε_i follows a logistic distribution with $\mathrm{E}[\varepsilon_i | x_i] = 0$ and $\mathrm{V}[\varepsilon_i | x_i] = 1$.

a) Derive the log likelihood function of the model.

b) Derive the score function.

c) Show that the Hessian has the form

$$H \equiv \frac{\partial^2 \ln L(\beta)}{\partial \beta \partial \beta'} = -\sum_{i=1}^{n} F\left(x_i'\beta\right)\left(1 - F\left(x_i'\beta\right)\right) x_i x_i',$$

where $F(\cdot)$ denotes the distribution function of the logistic distribution.

d) Formulate the model as a nonlinear regression model

$$y_i = \Psi(x_i, \beta_0) + \xi_i,$$

where $\Psi(\cdot)$ is some function of β_0 and x_i, and ξ_i is an error term with $\mathrm{E}[\xi_i | x_i] = 0$. Give the form of $\Psi(\cdot)$.

e) Show that the error terms ξ_i are conditionally heteroscedastic given the regressors.

f) You want to estimate β_0 by GMM. Formulate possible moment functions using exclusively the results in (e).

g) Show that the ML estimator considered above can be represented as

  (i) a GMM estimator based on appropriate moment functions,

  (ii) a pseudo ML estimator based on appropriate distributional assumptions for ξ_i.

What do these results mean for the optimality of the GMM estimator in (f)?

h) Derive the asymptotic covariance matrix of the GMM/PML estimator considered in (g). Which condition must be fulfilled such that the GMM/PML covariance matrix coincides with that of the ML estimator considered above? In which situations does this condition not hold?

Problem 3: Essays

Write short essays on <u>two</u> of the following topics:

(i) Asymptotic efficiency of GMM estimators based on over-identifying restrictions.

(ii) Fundamental principles of Bayesian vs. frequentist inference.

(iii) Single-equation GMM vs. multiple-equation GMM.

(iv) Bayesian inference for the parameter of a Bernoulli distribution.

(v) Reduced form and structural form estimation of simultaneous equations models.

Appendix

(i) The asymptotic distribution of the GMM estimator $\hat{\theta}_n(W_n)$ of parameter θ_0 with unconditional moment function $\psi(z_i, \theta)$ and weighting matrix W_n is given by

$$\sqrt{n}(\hat{\theta}_n(W_n) - \theta_0) \xrightarrow{d} N(0, \Delta_0 V_0 \Delta_0'),$$

where

$$\begin{aligned}
\Delta_0 &\equiv (A_0' W_0 A_0)^{-1} A_0' W_0, \\
A_0 &\equiv \mathrm{E}\left[\frac{\partial \psi(z_i, \theta_0)}{\partial \theta'}\right], \\
V_0 &\equiv \mathrm{V}[\psi(z_i, \theta_0)] = \mathrm{E}[\psi(z_i, \theta_0)\psi(z_i, \theta_0)'], \\
\theta_0 &\equiv \operatorname{plim} \hat{\theta}_n(W_n), \\
W_0 &\equiv \operatorname{plim} W_n.
\end{aligned}$$

(ii) The asymptotic distribution of the ML estimator of parameter θ_0 is given by

$$\sqrt{n}(\hat{\theta}_n - \theta_0) \xrightarrow{d} N(0, \mathcal{I}_1(\theta_0)^{-1}),$$

where

$$\mathcal{I}_1(\theta) \equiv -\mathrm{E}\left[\frac{\partial^2 \ln f(y_1; \theta)}{\partial \theta \partial \theta'}\right],$$

and $f(\cdot)$ denotes the p.d.f. of the (ergodic stationary) variables y_i, $i = 1, \ldots, n$.

(iii) Optimal conditional moment estimation: The GMM estimator of θ_0 based on the conditional moment function $\rho(z_i, \theta)$ and the instrument matrix

$$A^*(w_i) = \Omega(w_i)^{-1} D(w_i),$$

where

$$\begin{aligned}
\Omega(w_i) &\equiv \mathrm{E}[\rho(z_i, \theta_0)\rho(z_i, \theta_0)' | w_i], \\
D(w_i) &\equiv \mathrm{E}\left[\left.\frac{\partial \rho(z_i, \theta_0)}{\partial \theta'}\right| w_i\right]
\end{aligned}$$

is efficient relative to all GMM estimators using the same conditional moment function.

(iv) The distribution function $F(z)$ and the density function $f(z)$ of the logistic distribution are given by

$$F(y) = \frac{1}{1 + \exp(-y)}, \qquad f(y) = F(y)(1 - F(y)).$$

(v) The density function of a Beta distribution is given by

$$f(y;\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} y^{\alpha-1}(1-y)^{\beta-1}, \quad \alpha,\beta > 0,$$

with

$$\begin{aligned} \mathrm{E}[y] &= \frac{\alpha}{\alpha+\beta}, \\ \mathrm{V}[y] &= \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}. \end{aligned}$$