Problem 1 **(22 Points)**

Consider the linear regression model

$$y_t = \beta_1 + \beta_2 x_t + e_t, \qquad e_t \sim i.i.d.\ N(0, \sigma^2), \qquad t = 1, \ldots, 80. \tag{1}$$

In period $t = 21$ there is structural break, which affects both, the intercept and the slope parameter β_1 and β_2. In addition, another structural break affects β_1 and β_2 in period $t = 61$.

a) Define dummy variables that can be used to capture both structural breaks in the parameters. Using these variables formulate the model in dummy variable representation. In addition, write down the model in matrix notation by defining appropriate vectors and matrices.

b) Give the sets of equation representation of the model defined before. As before, also give the matrix version of the model.

c) Use the relation between parameters in (a) and (b) to express the parameters of the sets of equation representation in terms of the dummy variable model parameters in (a).

d) Show that OLS estimation of the sets of equation model defined in (b) leads to the same parameter estimates as three separate OLS regressions of model (1) for the three subperiods $t = 1, \ldots 20$; $t = 21, \ldots, 60$; $t = 61, \ldots 80$.

e) Suppose only the first 60 observations are analyzed, i.e. consider a model with only one possible structural break. Using only observations for $t = 1, \ldots, 60$, an OLS estimation of (1) yields $\hat{e}'\hat{e} = 39$. In addition, using the same observations estimation of a dummy variable model which allows a change in β_1 and β_2 gives $\hat{\sigma}^2 = 0.6$. Test the hypothesis of no parameter change in model (1).

Problem 2 **(26 Points)**

Consider the following model

$$y_t = x_t \beta + e_t \tag{2}$$

$$x_t = x_t^* + \epsilon_t \tag{3}$$

$$\epsilon_t = u_t + m u_{t-1} \tag{4}$$

with $e_t \sim i.i.d.(0, \sigma_e^2)$ and $u_t \sim i.i.d.(0, \sigma_u^2)$. In the model x_t is measured with an error ϵ_t and x_t^* is assumed to be the unobservable true value of x_t without measurement errors. It is only known that $E(x_t^{*2}) = \sigma_x^2$ exists and that $E(x_t^* x_s^*) = 0$ for $t \neq s$. The measurement error ϵ_t follows a *moving average* process. ϵ_t, u_t and e_t are all *independent* of x_t^*.

a) Assume that $E(u_t e_t) = \sigma_{ue} > 0$, $E(u_t e_s) = 0\ \forall\ t \neq s$ and b being the OLS estimator of equation (2). Compute plim(b) and investigate whether b is a consistent estimator of (2).

b) Suggest an instrument variable for the estimation of (2) and show that the assumptions for a suitable instrument are fulfilled.

c) Estimate β with the IV estimator given that $x = (2, 3, -4, 1, -2, -1)'$ and $y = (3, 5, -1, 2, 0, 1)'$, where $x = (x_1, \cdots, x_T)'$ and $y = (y_1, \cdots, y_T)'$.

d) What is the best possible instrument if $E(u_t e_t) = 0$? Explain shortly.

Problem 3 (25 Points)

In a simple linear regression model

$$y_t = \beta_1 x_t + e_t, \quad t = 1, \ldots, T, \tag{5}$$

the following structure for the error terms is assumed:

$$E[e_t] = 0, \quad E[e_t^2] = \sigma_t^2, \quad E[e_t e_s] = 0 \;\; \forall \;\; t \neq s.$$

a) Write down the covariance matrix of the error vector $e = (e_1, \ldots, e_T)'$.

b) Transform equation (5), such that the transformed model satisfies the standard assumptions of the linear regression model.

c) Estimate the parameter β_1 efficiently. Express the estimator using sums rather than matrix notation. Briefly explain the properties of your estimator.

d) For model (5), compute the variance of the GLS estimator $\hat{\beta}_1$ ($\Sigma_{\hat{\beta}}$) and the true variance of the OLS estimator b (Σ_b). Now let $\sigma_t^2 = \sigma^2 x_t^2$ and $\sum x_t^2 = T$. Show that $\Sigma_{\hat{\beta}} \Sigma_b^{-1} = T / \sum x_t^4$.

e) The model

$$y = X\beta + e, \qquad e \sim (0, \sigma^2 \Psi),$$

can be transformed, such that it satisfies the standard assumptions. Let $y^* = X^*\beta + e^*$ denote the transformed model in matrix notation. Show that

$$\hat{\sigma}_g^2 = \frac{(y^* - X^*\hat{\beta})'(y^* - X^*\hat{\beta})}{T - K}$$

is an unbiased estimator for σ^2. Note: $\hat{\beta}$ denotes the OLS estimator of the transformed model.

Problem 4 (27 Points)

The following model is given

$$\gamma_{11} y_{t1} = \gamma_{12} y_{t2} + \beta_{11} x_{t1} + u_{t1} \tag{6}$$

$$\gamma_{22} y_{t2} = -\gamma_{23} y_{t3} + \beta_{21} x_{t1} + \beta_{22} x_{t2} + u_{t2} \tag{7}$$

$$y_{t3} = y_{t2} + y_{t1} \tag{8}$$

with y_i being endogenous and x_i being exogenous variables of the system. For the errors it is assumed that $u \sim (0, \Sigma \otimes I_T)$.

a) Write the model in the form $Y\Gamma + XB + E = 0$. Why should you eliminate one equation before you do that? Apply an appropriate normalization to the equations as well. Explain shortly, why you do the normalization.

b) Estimate the reduced form parameters Π using the following moments $x_1'x_1 = 1$, $x_1'x_2 = 2$, $x_2'x_2 = 2$, $x_1'y_1 = 1$, $x_2'y_1 = 4$, $x_1'y_2 = 3$, $x_2'y_2 = -1$.

c) Check the identification for both equations by applying the rank and order criteria.

d) Estimate the structural form parameters Γ, B using ILS where possible.

e) Write the 2nd structural equation in the form $\Pi\Gamma_2 = -B_2$. Assume now that $\beta_{21} = \beta_{11}$ and $\gamma_{11} = \gamma_{22} + \gamma_{23}$. Do you see a possibility to estimate the structural parameters in this equation from the reduced form parameters now? Explain briefly how you would proceed.