Exercise 1 (25 Points)
Consider the following 2–equations model where y denotes endogenous and x denotes exogenous variables:

$$\begin{aligned} y_{t1} &= x_{t1}\beta_1 + e_{t1}, && (1) \\ y_{t2} &= x_{t2}\beta_2 + e_{t2}. && (2) \end{aligned}$$

It is assumed that the error terms in (1) and (2) are jointly normally distributed with mean zero and covariance matrix Σ. The following data are available:

$$\begin{bmatrix} x_1'x_1 & x_1'x_2 \\ x_2'x_1 & x_2'x_2 \end{bmatrix} = \begin{bmatrix} 10 & 6 \\ 6 & 6 \end{bmatrix}, \quad \begin{bmatrix} x_1'y_1 & x_1'y_2 \\ x_2'y_1 & x_2'y_2 \end{bmatrix} = \begin{bmatrix} 10 & 6 \\ 12.4 & 10 \end{bmatrix},$$

$$\hat{\Sigma} = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}, \quad T = 100.$$

1. Show that the OLS–estimator for β_1 in equation (1) is unbiased. Give reasons why joint estimation of equations (1) and (2) should nevertheless be preferred.
2. Compute the SUR–estimator for β_1 and β_2.
3. Perform a test of the null hypothesis $H_0: \ \beta_1 = \beta_2$ at a 5% significance level.

Exercise 2 (15 Points)
Consider the following regression model:

$$y_t = \beta y_{t-1} + e_t, \quad e_t = v_t + \rho v_{t-1}, \quad t = 1, \ldots, T, \tag{3}$$

with

$$|\beta| < 1, \quad |\rho| < 1, \quad v_t \sim i.i.d.(0, \sigma^2), \quad E(v_t y_{t-s}) = 0, \ s = 1, 2, \ldots$$

1. Show that the OLS–estimator $b = \sum_{t=1}^{T} y_t y_{t-1} / \sum_{t=1}^{T} y_{t-1}^2$ fulfills:
$$\text{plim}(b) = \beta + \rho \frac{\sigma^2}{a^2} \quad \text{with} \quad a^2 = \text{Var}(y_t) = \text{Var}(y_{t-1}).$$
2. Propose a suitable instrumental variable for y_{t-1}. Argue that your instrumental variable fulfills the conditions for a consistent IV–estimation under the present model assumptions.

Exercise 3 (15 Points)
Consider the following model:

$$y_t = \beta^{x_t} + e_t, \quad e_t \sim N(0, 1), \tag{4}$$

with x_t exogenous and non–stochastic. The parameter β shall be estimated by nonlinear least squares (NLS).

1. Determine the recursion formula of the Gauss–Newton algorithm.
2. What is the difference between the Newton–Raphson algorithm and the Gauss–Newton algorithm?
3. Explain briefly the relationship between NLS–estimation using the Gauss–Newton algorithm and ML–estimation using the scoring–algorithm.

Exercise 4 (27 Points)
Consider the following interdependent model where y denotes endogenous and x denotes exogenous variables:

$$\begin{aligned} y_{t1} &= \gamma_{21}\, y_{t2} + \beta_{11}\, x_{t1} + \beta_{21}\, x_{t2} + e_{t1}, & (5)\\ y_{t2} &= \gamma_{12}\, y_{t1} + \gamma_{32}\, y_{t3} + \beta_{22}\, x_{t2} + \beta_{32}\, x_{t3} + e_{t2}, & (6)\\ y_{t3} &= \gamma_{13}\, y_{t1} + \beta_{23}\, x_{t2} + e_{t3}. & (7)\end{aligned}$$

1. Write down equations (5) to (7) in the structural form $Y\Gamma + XB + \mathbf{E} = 0$.
2. What do you assume about the error terms? Explain your assumptions.
3. Check whether equations (5) to (7) are identified under the additional restriction $\gamma_{32} = \beta_{32}$.
4. Estimation of the reduced form model yields:

$$\hat{\Pi} = \begin{bmatrix} 3.8 & 2.4 & 2.8 \\ 3.4 & 2.2 & 2.7 \\ 2.0 & 2.0 & 1.8 \end{bmatrix}.$$

   Estimate the coefficients in (5).

Exercise 5 (18 Points)
In the following simple model y_t^d, y_t^s and p_t are assumed to be endogenous:

$$\begin{aligned} y_t^d &= \alpha_1 p_t + \alpha_2 q_t + e_{t1}, & (8)\\ y_t^s &= \alpha_3 p_t + \alpha_4 p_{t-1} + \alpha_5 t + e_{t2}, & (9)\\ y_t^d &= y_t^s = y_t, & (10)\end{aligned}$$

with

y_t^d demand for good X,
y_t^s supply of good X,
p_t price of good X in period t,
q_t disposable income in period t,
t time trend.

For the error terms it is assumed that at each time point $t = 1, \dots, T$

$$\begin{bmatrix} e_{t1} \\ e_{t2} \end{bmatrix} \sim i.i.d.(0, \Sigma), \quad \Sigma = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}.$$

1. Write down the model in the structural form $Y\Gamma + XB + \mathbf{E} = 0$.
2. Give arguments against estimating (8) and (9) separately by ordinary least squares (OLS).
3. Is there a difference between the 2SLS estimator and the ILS estimator in equation (9)? Explain your answer.
4. Would you prefer the 3SLS estimator to the 2SLS estimator in the present model? Explain your answer.