<u>Problem 1</u> (25 Points)

Consider the model:

$$y = X\beta + e, \tag{1}$$

where the $(T \times K)$–regressor matrix is non–stochastic. Assume that the error term $e = (e_1, \dots, e_T)'$ has the following structure:

$$e \sim (0, \Phi), \qquad \text{with} \quad \Phi \quad \text{positive definite.}$$

1. **(7 points)**
   a) Show that the OLS estimator $b = (X'X)^{-1}X'y$ is still unbiased.
   b) Show that the covariance matrix of the OLS estimator $b = (X'X)^{-1}X'y$ (under the assumption of a general error covariance matrix Φ) is given by

   $$Cov(b) = (X'X)^{-1}(X'\Phi X)(X'X)^{-1}.$$

   c) Discuss briefly why in this framework the GLS estimator $\hat{\beta}_G = (X'\Phi^{-1}X)^{-1}X'\Phi^{-1}y$ should be preferred to the OLS estimator b.
2. **(10 points)** Assume now that the error term e_t has the following structure:

   $$E[e_t] = 0, \qquad E[e_t^2] = \sigma_t^2, \qquad E[e_t e_s] = 0 \quad \forall t \neq s.$$

   a) Write down the expectation vector and the covariance matrix for the error vector e.
   b) Find a suitable matrix P, such that $e^* = Pe \sim (0, \sigma^2 I_T)$ holds for some $\sigma^2 > 0$.
   c) Show that the OLS estimator of the model transformed by P corresponds to the GLS estimator of model (1).
3. **(8 points)** Assume now that $e_t \sim N(0, \sigma^2) i.i.d.$ Give a $(1 - \alpha)$ confidence interval for $x'_{(1)}\beta$.

Problem 2 (25 Points)

Consider the following model:

$$y_1 = X_1\beta_1 + e_1,$$
$$y_2 = X_2\beta_2 + e_2,$$

where X_1 and X_2 are $(T \times K)$–matrices of non–stochastic regressors. The error term is assumed to have the following structure:

$$e = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \sim (0, \Sigma \otimes I_T) \quad \text{and } \Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}.$$

a) **(2 points)** Show that separate OLS estimations of β_1, β_2 are unbiased, i.e. $E[b_1] = \beta_1$ and $E[b_2] = \beta_2$.

b) **(3 points)** Show that the GLS estimator $\hat{\beta}_G$ of $\beta = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$ in the system is also unbiased, i.e.

$$E[\hat{\beta}_G] = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}.$$

c) **(9 points)** Show that the GLS estimator equals the OLS estimator when $X_1 = X_2 = \bar{X}$. Under what other condition do GLS and OLS coincide?

d) **(4 points)** Explain briefly one difference and one commonality between seemingly unrelated regression models and simultaneous equation models?

e) **(7 points)** Show that

$$\frac{(y_1 - X_1 b_1)'(y_1 - X_1 b_1)}{T - K}$$

is an unbiased estimator for σ_{11}. Hint: You may use e.g. that $E[z'Az] = \mu' A\mu + \text{tr}(A\Omega)$ if $z \sim (\mu, \Omega)$ and A is some symmetric matrix.

Problem 3 (25 Points)

Consider the following nonlinear model

$$y_t = f(x_t, \beta) + e_t, \tag{2}$$

where y_t, x_t and β are scalars.

a) **(7 points)** Derive the *linear pseudomodel* for equation (2) by using a first order Taylor approximation for $f(x_t, \beta)$ around β_n. Explain how to derive the Gauss-Newton recursion formula from the *linear pseudomodel*.

b) **(7 points)** Suppose a specific form of (2) represents a nonlinear consumption function

$$y_t = x_t^{\beta} + e_t, \tag{3}$$

where y_t denotes a measure of consumption and x_t is a measure of income. Derive the Newton-Raphson recursion formula for estimating equation (3) by nonlinear least squares (NLS).

c) **(7 points)** A more general form of a consumption function may be written as

$$y_t = \beta_1 x_t^{\beta_2} + e_t. \tag{4}$$

With the economically plausible restriction $\beta_2 = 1$, this model becomes linear. How would you obtain reasonable starting values for NLS estimation of β_1 and β_2 in this case? How would you obtain starting values without any information about β_2? Discuss the role of using different starting values in NLS estimation.

d) **(4 points)** Suppose you want to test $H : \beta_2 = 1$ in model (4). Would you prefer a Wald, a Lagrange Multiplier or a Likelihood Ratio test for this hypothesis? Explain your decision. (You don't have to compute the chosen test!)

Problem 4 (25 Points)

Consider the following simultaneous equation model

$$\begin{aligned}
y_{t1} &= \gamma_{21} y_{t2} + \gamma_{31} y_{t3} + \beta_{11} x_{t1} + e_{t1} \\
y_{t2} &= \gamma_{32} y_{t3} + \beta_{12} x_{t1} + e_{t2} \\
y_{t3} &= \beta_{23} x_{t2} + e_{t3}
\end{aligned}$$

where y_i are the endogenous and x_i are exogenous variables. For the errors it is assumed that $e \sim (0, \Sigma \otimes I_T)$.

a) **(4 points)** Write the model in the form $Y\Gamma + XB + E = 0$.

b) **(9 points)** Check the identification of the 3 structural equations using the order and the rank criteria. Suggest a suitable estimation procedure for each structural equation.

c) **(6 points)** The reduced form estimates are given by

$$\hat{\Pi} = \begin{pmatrix} \hat{\pi}_{11} & \hat{\pi}_{12} & \hat{\pi}_{13} \\ \hat{\pi}_{21} & \hat{\pi}_{22} & \hat{\pi}_{23} \end{pmatrix}.$$

Give efficient estimates for the structural parameters Γ and B using indirect least squares (ILS) for those equations where it is possible.

d) **(6 points)** Check whether the first structural equation is identified when $\beta_{11} = 0$. What happens when $\beta_{12} = 0$ in addition? Explain your result intuitively.