Problem 1 (22 Points)

Consider the following linear regression model

$$y_t = \beta_1 + \beta_2 x_{t2} + e_t, \quad e_t \sim N(0, \sigma_t^2) \text{ independent for } t = 1, ..., T, \tag{1}$$

where x_{t2} is a non-stochastic regressor. Let $\Phi = diag[\sigma_1^2, ..., \sigma_T^2]$ be a positive definite matrix.

1. **(7 Points)** Assume that Φ is known.
   (a) Derive the covariance matrix of the OLS estimator b for $\beta = (\beta_1, \beta_2)'$ in model (1).
   (b) Propose an efficient estimator for β and derive its expectation.
   (c) What does efficiency of an estimator mean?
2. **(9 Points)** The following data are available:

$$X'X = \begin{bmatrix} 50.000 & 5.410 \\ 5.410 & 43.315 \end{bmatrix}, \quad (X'X)^{-1} = \begin{bmatrix} 0.020 & -0.003 \\ -0.003 & 0.023 \end{bmatrix},$$

$$X'y = \begin{bmatrix} 19.690 \\ 46.213 \end{bmatrix}, \quad y'y = 1995.$$

   Let Φ be unknown and proceed with the wrong assumption that $\Phi = \sigma^2 I_T$.
   (a) Calculate the OLS estimate b of the vector of coefficients $\beta = (\beta_1, \beta_2)'$.
   (b) Estimate the variance of the error term. (Hint: $\hat{e}'\hat{e} = y'y - b'X'y$.)
   (c) Estimate the covariance matrix of the OLS estimator b.
   (d) Compute the test statistic for testing the null hypothesis

$$H : \beta_2 = 0. \tag{2}$$

   Which critical value would you need at a 5% significance level?
3. **(6 Points)** In addition to the data above, let

$$X'\hat{\Phi}X = \sum_{t=1}^{T} \hat{e}_t^2 x_{(t)} x_{(t)}' = \begin{bmatrix} 1941.775 & 175.990 \\ 175.990 & 737.029 \end{bmatrix}$$

   be given, where the $\hat{e}_t$'s are the OLS residuals from part 2 and $\hat{\Phi} = diag[\hat{e}_1^2, ..., \hat{e}_T^2]$.
   (a) Compute the White heteroscedasticity consistent covariance matrix estimator.
   (b) For testing the null hypothesis (2) compute the test statistic based on the OLS estimate of β_2 and its White standard error. What can you conclude when comparing your results with that of part 2(c)?

Solution to problem 1 (22 Points)

1. (a) Note first that $b - \beta = (X'X)^{-1}X'e$. Then

$$\begin{aligned}
\mathrm{Cov}(b) &= \mathrm{E}[(b-\beta)(b-\beta)'] = \mathrm{E}[((X'X)^{-1}X'e)((X'X)^{-1}X'e)'] \\
&= \mathrm{E}[(X'X)^{-1}X'ee'X(X'X)^{-1}] \\
&= (X'X)^{-1}X'\mathrm{E}[ee']X(X'X)^{-1} \\
&= (X'X)^{-1}X'\Phi X(X'X)^{-1}
\end{aligned}$$

(b) The GLS estimator is efficient, because the standard assumptions for the error terms are violated and Φ is known. The OLS estimator won't be efficient anymore except for $\sigma_t^2 \equiv \sigma^2 \ \forall t$.

$$\begin{aligned}
\hat{\beta}_{GLS} &= (X'\Phi^{-1}X)^{-1}X'\Phi^{-1}y \\
\mathrm{E}[y] &= \mathrm{E}[X\beta + e] = X\beta + \underbrace{E[e]}_{=0} = X\beta \\
\Longrightarrow \mathrm{E}[\hat{\beta}_{GLS}] &= \mathrm{E}[(X'\Phi^{-1}X)^{-1}X'\Phi^{-1}y] \\
&= (X'\Phi^{-1}X)^{-1}X'\Phi^{-1}\underbrace{\mathrm{E}[y]}_{=X\beta} \\
&= \underbrace{(X'\Phi^{-1}X)^{-1}X'\Phi^{-1}X}_{=I_2}\beta \\
&= \beta
\end{aligned}$$

$\Longrightarrow$ The GLS estimator is unbiased.

(c) An estimator $\hat{\theta}$ of θ is called efficient if it is BLUE (or, as here, BUE under the normality assumption). I.e., the estimator is unbiased and $\mathrm{Cov}(\tilde{\theta}) - \mathrm{Cov}(\hat{\theta})$ is non-negative definite for any unbiased estimator $\tilde{\theta}$ of θ.

2. (a)

$$\begin{aligned}
b &= (X'X)^{-1}X'y \\
&= \begin{bmatrix} 0.020 & -0.003 \\ -0.003 & 0.023 \end{bmatrix}\begin{bmatrix} 19.690 \\ 46.213 \end{bmatrix} = \begin{bmatrix} 0.020 * 19.690 - 0.003 * 46.213 \\ -0.003 * 19.690 + 0.023 * 46.213 \end{bmatrix} \\
&\approx \begin{bmatrix} 0.255 \\ 1.004 \end{bmatrix}
\end{aligned}$$

(b)

$$\begin{aligned}
s^2 &= \frac{\hat{e}'\hat{e}}{T-K} = \frac{y'y - b'X'y}{T-K} \\
&= \frac{1995 - [0.255 \;\; 1.004]\begin{bmatrix} 19.690 \\ 46.213 \end{bmatrix}}{50-2} \approx \frac{1943.581}{48} \\
&\approx 40.491
\end{aligned}$$

(c)

$$\widehat{\text{Cov}(b)} = s^2(X'X)^{-1}$$
$$= 40.491\begin{bmatrix} 0.020 & -0.003 \\ -0.003 & 0.023 \end{bmatrix}$$
$$\approx \begin{bmatrix} 0.810 & -0.121 \\ -0.121 & 0.931 \end{bmatrix}$$

(d)

$$t = \frac{b_2}{\hat{\sigma}_{b_2}} = \frac{1.004}{\sqrt{0.931}} \approx 1.041$$

Under H it holds $t \sim t_{T-K}$, so that an α-test rejects H if $|t| > t_{T-K}^{1-\frac{\alpha}{2}}$. Therefore we would need the 0.975-quantile of the t-distribution with 48 degrees of freedom.

3. (a)

$$\widetilde{\text{Cov}(b)} = (X'X)^{-1}X'\hat{\Phi}X(X'X)^{-1}$$
$$= \begin{bmatrix} 0.020 & -0.003 \\ -0.003 & 0.023 \end{bmatrix}\begin{bmatrix} 1941.775 & 175.990 \\ 175.990 & 737.029 \end{bmatrix}\begin{bmatrix} 0.020 & -0.003 \\ -0.003 & 0.023 \end{bmatrix}$$
$$\approx \begin{bmatrix} 0.762 & -0.085 \\ -0.085 & 0.383 \end{bmatrix}$$

(b)

$$t = \frac{b_2}{\tilde{\sigma}_{b_2}} = \frac{1.004}{\sqrt{0.383}} \approx 1.622$$

When we use the White standard error for b_2 the t-statistic increases, and thus $H : \beta_2 = 0$ is more likely to be rejected.

Problem 2 (23 Points)

Consider the following model with two equations

$$\begin{aligned} y_{t1} &= \beta_1 x_{t1} + e_{t1}, \\ y_{t2} &= \beta_2 x_{t2} + e_{t2}, \end{aligned}$$

where $x_{t1} = 1$ for all $t = 1, ..., T$ and x_{t2} is non-stochastic. For $T = 50$ observations the following sample moments are given:

$$x_2' x_2 = 100, \quad x_2' y_2 = 50, \quad x_2' y_1 = 60, \quad y_2' y_2 = 90, \quad y_1' y_1 = 500, \quad y_1' y_2 = 40,$$

$$\sum_{t=1}^{50} x_{t2} = 100, \quad \sum_{t=1}^{50} y_{t1} = 150, \quad \sum_{t=1}^{50} y_{t2} = 50.$$

1. **(2 Points)** Compute the OLS estimates of β_1 and β_2.
2. **(6 Points)** Assume that the error terms have the following structure:

$$e = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \sim (0, \Sigma \otimes I_T) \quad \text{with} \quad \Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}.$$

   Estimate Σ using the OLS residuals $(\hat{e}_1, \hat{e}_2)$ from part 1 and

$$\hat{\sigma}_{ij} = \frac{\hat{e}_i' \hat{e}_j}{T} \quad \text{for} \quad i, j = 1, 2.$$

3. **(8 Points)** Compute the feasible GLS estimator for β_1 and β_2 under the assumption of part 2:
   - Let $\hat{\Sigma}^{-1} = \begin{pmatrix} \hat{\sigma}^{11} & \hat{\sigma}^{12} \\ \hat{\sigma}^{21} & \hat{\sigma}^{22} \end{pmatrix}$. Rewrite the formula for $\hat{\hat{\beta}}_G$ in terms of $\hat{\sigma}^{ij}$ and the sample moments.
   - Calculate $\hat{\Sigma}^{-1}$.
   - Give the estimates for β_1 and β_2.
4. **(2 Points)** What is the motivation behind using the feasible GLS estimator?
5. **(5 Points)** Show that the GLS estimator of $\beta = (\beta_1, \beta_2)'$ is equivalent to the OLS estimator if Σ is a diagonal matrix.

Solution to problem 2 (23 Points)

1.

$$b_1 = \frac{\sum_{t=1}^{50} y_{t1}}{T} = \frac{150}{50} = 3$$
$$b_2 = (x_2'x_2)^{-1}x_2'y_2 = (100)^{-1}50 = 0.5$$

2.

$$\hat{\sigma}_{11} = \frac{\hat{e}_1'\hat{e}_1}{T} = \frac{(y_1 - x_1b_1)'(y_1 - x_1b_1)}{T} = \frac{y_1'y_1 - b_1'x_1'y_1}{T}$$
$$= \frac{500 - 3 * 150}{50} = 1$$
$$\hat{\sigma}_{22} = \frac{\hat{e}_2'\hat{e}_2}{T} = \frac{(y_2 - x_2b_2)'(y_2 - x_2b_2)}{T} = \frac{y_2'y_2 - b_2'x_2'y_2}{T}$$
$$= \frac{90 - 0.5 * 50}{50} = 1.3$$
$$\hat{\sigma}_{12} = \frac{\hat{e}_1'\hat{e}_2}{T} = \frac{(y_1 - x_1b_1)'(y_2 - x_2b_2)}{T} = \frac{y_1'y_2 - y_1'x_2b_2 - b_1'x_1'y_2 + b_1'x_1'x_2b_2}{T}$$
$$= \frac{40 - 60 * 0.5 - 3 * 50 + 3 * 100 * 0.5}{50} = \frac{10}{50} = 0.2$$
$$\hat{\sigma}_{21} = \frac{\hat{e}_2'\hat{e}_1}{T} = \frac{\hat{e}_1'\hat{e}_2}{T} = \hat{\sigma}_{12} = 0.2$$
$$\hat{\Sigma} = \begin{bmatrix} \hat{\sigma}_{11} & \hat{\sigma}_{12} \\ \hat{\sigma}_{21} & \hat{\sigma}_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0.2 \\ 0.2 & 1.3 \end{bmatrix}$$

3.

$$\begin{aligned}
\hat{\hat{\beta}}_G =& [X'(\hat{\Sigma}^{-1} \otimes I_T)X]^{-1}X'(\hat{\Sigma}^{-1} \otimes I_T)y \\
=& \left[\begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}' \left(\begin{pmatrix} \hat{\sigma}_{11} & \hat{\sigma}_{12} \\ \hat{\sigma}_{21} & \hat{\sigma}_{22} \end{pmatrix}^{-1} \otimes I_T\right) \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}\right]^{-1} \\
& \begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix}' \left[\begin{pmatrix} \hat{\sigma}_{11} & \hat{\sigma}_{12} \\ \hat{\sigma}_{21} & \hat{\sigma}_{22} \end{pmatrix}^{-1} \otimes I_T\right] \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\
=& \left[\begin{bmatrix} x_1' & 0 \\ 0 & x_2' \end{bmatrix} \begin{bmatrix} \hat{\sigma}^{11}I_T & \hat{\sigma}^{12}I_T \\ \hat{\sigma}^{21}I_T & \hat{\sigma}^{22}I_T \end{bmatrix} \begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix}\right]^{-1} \\
& \begin{bmatrix} x_1' & 0 \\ 0 & x_2' \end{bmatrix} \begin{bmatrix} \hat{\sigma}^{11}I_T & \hat{\sigma}^{12}I_T \\ \hat{\sigma}^{21}I_T & \hat{\sigma}^{22}I_T \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\
=& \begin{bmatrix} \hat{\sigma}^{11}x_1'x_1 & \hat{\sigma}^{12}x_1'x_2 \\ \hat{\sigma}^{21}x_2'x_1 & \hat{\sigma}^{22}x_2'x_2 \end{bmatrix}^{-1} \begin{bmatrix} \hat{\sigma}^{11}x_1'y_1 + \hat{\sigma}^{12}x_1'y_2 \\ \hat{\sigma}^{21}x_2'y_1 + \hat{\sigma}^{22}x_2'y_2 \end{bmatrix}
\end{aligned}$$

$$\hat{\Sigma}^{-1} = \begin{bmatrix} 1 & 0.2 \\ 0.2 & 1.3 \end{bmatrix}^{-1} = \frac{1}{1 * 1.3 - (0.2)^2} \begin{bmatrix} 1.3 & -0.2 \\ -0.2 & 1 \end{bmatrix} \approx \begin{bmatrix} 1.032 & -0.159 \\ -0.159 & 0.794 \end{bmatrix}$$

$$\begin{aligned}
\hat{\hat{\beta}}_G =& \begin{bmatrix} 1.032 * 50 & -0.159 * 100 \\ -0.159 * 100 & 0.794 * 100 \end{bmatrix}^{-1} \begin{bmatrix} 1.032 * 150 - 0.159 * 50 \\ -0.159 * 60 + 0.794 * 50 \end{bmatrix} \\
\approx& \begin{bmatrix} 3.158 \\ 1.012 \end{bmatrix}
\end{aligned}$$

4. GLS is preferable to OLS for general $\mathrm{Cov}(e) = \Phi$. But the GLS estimator depends on Φ. Therefore, in case of an unknown Φ one should replace Φ in the GLS formula by some estimator, which gives the feasible GLS estimator.

5. If Σ is a diagonal matrix, then $\sigma_{12} = \sigma_{21} = 0$, i.e.

$$\Sigma = \begin{bmatrix} \sigma_{11} & 0 \\ 0 & \sigma_{22} \end{bmatrix}$$

$$\begin{aligned}
\hat{\beta}_G &= (X'(\Sigma^{-1} \otimes I_T)X)^{-1} X'(\Sigma^{-1} \otimes I_T)y \\
&= \left[\begin{pmatrix} x_1' & 0 \\ 0 & x_2' \end{pmatrix} \begin{pmatrix} \frac{1}{\sigma_{11}} I_T & 0 \\ 0 & \frac{1}{\sigma_{22}} I_T \end{pmatrix} \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} \right]^{-1} \begin{bmatrix} x_1' & 0 \\ 0 & x_2' \end{bmatrix} \begin{bmatrix} \frac{1}{\sigma_{11}} I_T & 0 \\ 0 & \frac{1}{\sigma_{22}} I_T \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{\sigma_{11}} x_1' x_1 & 0 \\ 0 & \frac{1}{\sigma_{22}} x_2' x_2 \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{\sigma_{11}} x_1' y_1 \\ \frac{1}{\sigma_{22}} x_2' y_2 \end{bmatrix} \\
&= \begin{bmatrix} (\frac{1}{\sigma_{11}} x_1' x_1)^{-1} & 0 \\ 0 & (\frac{1}{\sigma_{22}} x_2' x_2)^{-1} \end{bmatrix} \begin{bmatrix} \frac{1}{\sigma_{11}} x_1' y_1 \\ \frac{1}{\sigma_{22}} x_2' y_2 \end{bmatrix} \\
&= \begin{bmatrix} \sigma_{11} (x_1' x_1)^{-1} \frac{1}{\sigma_{11}} x_1' y_1 \\ \sigma_{22} (x_2' x_2)^{-1} \frac{1}{\sigma_{22}} x_2' y_2 \end{bmatrix} = \begin{bmatrix} (x_1' x_1)^{-1} x_1' y_1 \\ (x_2' x_2)^{-1} x_2' y_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{\sum_{t=1}^{50} y_{t1}}{T} \\ (x_2' x_2)^{-1} x_2' y_2 \end{bmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}
\end{aligned}$$

Problem 3 (29 Points)

The following model is given:

$$y \;=\; Z\beta + e, \qquad e \sim N(0, \sigma^2 I_T), \tag{3}$$

where Z is a stochastic $(T \times K)$ matrix. Furthermore it holds, for $T \to \infty$, that $\frac{\sum_{t=1}^{T} \mathbb{E}\left[z_{(t)} z'_{(t)}\right]}{T} \to A$, where A is a constant $(K \times K)$ matrix.

1. **(12 Points)** Assume that e and Z are stochastically independent. Show that then $\operatorname{plim}_{T\to\infty} \frac{Z'e}{T} = 0$:
   - Write $\frac{Z'e}{T}$ as a sum.
   - Show that $\mathbb{E}\left[z_{(t)} e_t\right] = 0$.
   - How does $\operatorname{Cov}\left[z_{(t)} e_t\right]$ look like?
   - Compute $\operatorname{Cov}\left[z_{(t)} e_t, z_{(s)} e_s\right]$, $s \neq t$.
   - Show that, for $T \to \infty$, $\mathbb{E}\left[\frac{Z'e}{T}\right] \to 0$.
   - Show that, for $T \to \infty$, $\operatorname{Cov}\left[\frac{Z'e}{T}\right] \to 0$.
   - Why can you conclude now that $\operatorname{plim}_{T\to\infty} \frac{Z'e}{T} = 0$?

2. **(15 Points)** Consider now the following model:

$$y_t = \beta y_{t-1} + e_t, \qquad e_t = \nu_t + \rho \nu_{t-1}, \quad t = 1, \ldots, T, \tag{4}$$

$$|\beta| < 1,\ |\rho| < 1,\ \nu_t \sim N(0, \sigma_\nu^2)\ i.i.d.$$

   In addition to this, y_{-1} and y_0 are observable. From the assumptions above it follows that for all t, $\mathbb{E}[y_t] = 0$, $\operatorname{Var}[y_t] = \sigma_y^2$ and $\mathbb{E}[\nu_t y_{t-s}] = 0$ for $s > 0$.

   (a) Write the matrix Z for model (4) in terms of the observations of which it is composed. Write $\frac{Z'e}{T}$ as a sum which depends on the regressor variable y_{t-1}.
   (b) Show that $\mathbb{E}[y_{t-1} e_t] \neq 0$, if $\rho \neq 0$. What do you conclude?
   (c) Show that $\mathbb{E}[y_{t-2} e_t] = 0$ and in general $\mathbb{E}[y_{t-2} y_{t-1}] \neq 0$.
   (d) Give the elements of the instrument matrix X for model (4), if y_{t-2} will be used as an instrument for y_{t-1}. Represent $\frac{X'Z}{T}$ and $\frac{X'e}{T}$ as sums depending on the instrumental variable y_{t-2}.
   (e) How would you proceed to show that $\operatorname{plim}_{T\to\infty} \frac{X'e}{T} = 0$ and $\operatorname{plim}_{T\to\infty} \frac{X'Z}{T} \neq 0$?

3. **(2 Points)** What is wrong with the following text? Explain shortly.

   The instrumental variable estimator is a special case of the OLS estimator, and it is consistent if the instruments are uncorrelated with the stochastic regressors of the model.

Solution to problem 3 **(29 Points)**

1. •

$$\frac{Z'e}{T} \text{ with } Z = \begin{bmatrix} z'_{(1)} \\ \vdots \\ z'_{(T)} \end{bmatrix} \quad \Rightarrow \quad Z'e = (z_{(1)} \dots z_{(T)}) \begin{pmatrix} e_1 \\ \vdots \\ e_T \end{pmatrix} = \sum_{t=1}^{T} z_{(t)} e_t$$

$$\frac{Z'e}{T} = \frac{1}{T} \sum_{t=1}^{T} z_{(t)} e_t$$

• Note: $\overset{(*)}{=}$ indicates that we use the fact that $z_{(t)}$ and e_s are independent $\forall t \neq s$.

$$\mathbb{E}[z_{(t)} e_t] \overset{(*)}{=} \mathbb{E}[z_{(t)}] \underbrace{\mathbb{E}[e_t]}_{=0} = 0$$

•

$$\begin{aligned} \mathrm{Cov}[z_{(t)} e_t] &= \mathbb{E}[(z_{(t)} e_t)(z_{(t)} e_t)'] = \mathbb{E}[e_t^2 z_{(t)} z'_{(t)}] \overset{(*)}{=} \mathbb{E}[e_t^2] \mathbb{E}[z_{(t)} z'_{(t)}] \\ &= \sigma^2 \mathbb{E}[z_{(t)} z'_{(t)}] \end{aligned}$$

•

$$\begin{aligned} \mathrm{Cov}[z_{(t)} e_t, z_{(s)} e_s] &= \mathbb{E}[(z_{(t)} e_t)(z_{(s)} e_s)'] = \mathbb{E}[e_t e_s z_{(t)} z'_{(s)}] \\ &\overset{(*)}{=} \underbrace{\mathbb{E}[e_t e_s]}_{=0} \mathbb{E}[z_{(t)} z'_{(s)}] = 0, \quad \text{for } t \neq s \text{ since } \mathbb{E}[e_t e_s] = 0 \end{aligned}$$

•

$$\mathbb{E}\left[\frac{Z'e}{T}\right] = \frac{\sum \mathbb{E}[z_{(t)} e_t]}{T} = 0 \quad \forall T \qquad \Rightarrow \mathbb{E}\left[\frac{Z'e}{T}\right] = 0 \text{ for } T \to \infty$$

•

$$\begin{aligned} \mathrm{Cov}\left[\frac{Z'e}{T}\right] &= \mathrm{Cov}\left[\frac{\sum z_{(t)} e_t}{T}\right] = \frac{\sum \mathrm{Cov}[z_{(t)} e_t]}{T^2} \quad \Big(\text{Recall: } \mathrm{Cov}[z_{(t)} e_t, z_{(s)} e_s] = 0\Big) \\ &= \frac{\sigma^2}{T} \frac{\sum \mathbb{E}[z_{(t)} z'_{(t)}]}{T} \longrightarrow 0 \cdot A = 0 \text{ for } T \to \infty \end{aligned}$$

• The above establishes mean square convergence which in turn implies convergence in probability.

2. (a)

$$Z = \begin{bmatrix} y_o \\ y_1 \\ \vdots \\ y_{T-1} \end{bmatrix} \quad \Rightarrow \quad z_{(t)} = y_{t-1} \quad \text{and} \quad \frac{Z'e}{T} = \frac{[y_0 \dots y_{T-1}] \begin{bmatrix} e_1 \\ \vdots \\ e_T \end{bmatrix}}{T} = \frac{\sum_{t=1}^{T} y_{t-1} e_t}{T}$$

(b)

$$\begin{aligned}\mathbb{E}[y_{t-1}e_t] &= \mathbb{E}[y_{t-1}(v_t+\rho v_{t-1})] = \underbrace{\mathbb{E}[y_{t-1}v_t]}_{=0} + \rho\mathbb{E}[y_{t-1}v_{t-1}] \\ &= \rho\mathbb{E}[(\beta y_{t-2}+e_{t-1})v_{t-1}] = \rho\beta\underbrace{\mathbb{E}[y_{t-2}v_{t-1}]}_{=0} + \rho\mathbb{E}[e_{t-1}v_{t-1}] \\ &= \rho\mathbb{E}[(v_{t-1}+\rho v_{t-2})v_{t-1}] = \rho\underbrace{\mathbb{E}[v_{t-1}^2]}_{=\sigma_v^2} + \rho^2\underbrace{\mathbb{E}[v_{t-2}v_{t-1}]}_{=0} \\ &= \rho\sigma_v^2 \neq 0 \ \text{ if } \ \rho \neq 0\end{aligned}$$

If $\frac{Z'e}{T}$ convergences in probability, then the probability limit is likely to be different from zero (at least if $\mathrm{Var}(Z'e/T) \to 0$ for $T \to \infty$), so that Z is not weakly exogenous.

(c)

$$\mathbb{E}[y_{t-2}e_t] = \mathbb{E}[y_{t-2}(v_t+\rho v_{t-1})] = \underbrace{\mathbb{E}[y_{t-2}v_t]}_{=0} + \rho\underbrace{\mathbb{E}[y_{t-2}v_{t-1}]}_{=0} = 0$$

$$\begin{aligned}\mathbb{E}[y_{t-2}y_{t-1}] &= \mathbb{E}[y_{t-2}(\beta y_{t-2}+e_{t-1})] = \beta\mathbb{E}[y_{t-2}^2] + \mathbb{E}[y_{t-2}e_{t-1}] \\ &= \beta\sigma_y^2 + \rho\sigma_v^2 \neq 0 \ \text{ in general}\end{aligned}$$

(d) Consider the model:

$$y_t = \beta y_{t-2} + e_t \quad t = 1, \ldots, T.$$

Thus:

$$X = \begin{bmatrix} y_{-1} \\ y_0 \\ \vdots \\ y_{T-2} \end{bmatrix}.$$

$$\frac{X'Z}{T} = \frac{[y_{-1}\ y_0\ \ldots\ y_{T-2}]\begin{bmatrix} y_0 \\ \vdots \\ y_{T-1}\end{bmatrix}}{T} = \frac{\sum_{t=1}^T y_{t-2}y_{t-1}}{T}$$

$$\frac{X'e}{T} = \frac{[y_{-1}\ y_0\ \ldots\ y_{T-2}]\begin{bmatrix} e_1 \\ \vdots \\ e_T\end{bmatrix}}{T} = \frac{\sum_{t=1}^T y_{t-2}e_t}{T}$$

(e) Find an appropriate law of large numbers (for serially correlated random variables) such that its applications to the sequences $y_{t-2}y_{t-1}$ and $y_{t-2}e_t$, respectively, provide $1/T\sum_{t=1}^T y_{t-2}y_{t-1} \xrightarrow{\mathbb{P}} \beta\sigma_y^2 + \rho\sigma_v^2 \quad (= \mathbb{E}[y_{t-2}y_{t-1}]) \neq 0$ and $1/T\sum_{t=1}^T y_{t-2}e_t \xrightarrow{\mathbb{P}} 0 \quad (= \mathbb{E}[y_{t-2}e_t])$.

3. LS is special case of the IVE (or the IVE in a particular model can be seen as special case of the LS). For consistency, there is at least some correlation between regressors and instruments necessary.

<u>Problem 4</u> (26 Points)

The following model is given:

$$
\begin{aligned}
y_1 &= \gamma_1 y_2 + \beta_{11} x_1 + e_1 && (5)\\
y_2 &= \gamma_2 y_1 + \beta_{22} x_2 + \beta_{32} x_3 + e_2 && (6)
\end{aligned}
$$

where y_1, y_2 are endogenous and x_1, x_2, x_3 are exogenous $(T \times 1)$ vectors.

1. **(3 Points)** Write the model in the form $Y\Gamma + XB + E = 0$.

2. **(8 Points)** Check the identification of both equations using the order and the rank conditions.

3. **(6 Points)** Assume that an estimation $\hat{\Pi}$ of the reduced form parameters is known. Use the relation $\hat{\Pi}\Gamma_2 = -B_2$ (see the formulary) to estimate the structural parameters $\gamma_2, \beta_{22}, \beta_{32}$.

4. **(7 Points)** Show for equation (6) that $\hat{\delta}_{2(2SLS)} = \hat{\delta}_{2(ILS)}$:

   - Rewrite $\hat{Z}_2$ in dependence on Y_2 and X_2.
   - How many rows and columns does $X'Z_2$ have? Hint: Think about the variables, which appear in X, X_2, Y_2 and Z_2.
   - Assume that $X'Z_2$ has full column rank. Starting with $\hat{\delta}_{2(2SLS)}$ show that $\hat{\delta}_{2(2SLS)} = \hat{\delta}_{2(ILS)}$, i.e. $(\hat{Z}_2'\hat{Z}_2)^{-1}\hat{Z}_2' y_2 = (X'Z_2)^{-1}X'y_2$.

5. **(2 Points)** How does the identifiability of the model change, when you assume for economic reasons that $\beta_{22} = \beta_{32}$?

Solution to problem 4 **(26 Points)**

1.

$$
\begin{aligned}
0 &= -y_1 + \gamma_1 y_2 + \beta_{11} x_1 + e_1 \\
0 &= \gamma_2 y_1 - y_2 + \beta_{22} x_2 + \beta_{32} x_3 + e_2
\end{aligned}
$$

$\Rightarrow$

$$
0 = \underbrace{[y_1 \;\; y_2]}_{=Y} \underbrace{\begin{bmatrix} -1 & \gamma_2 \\ \gamma_1 & -1 \end{bmatrix}}_{=\Gamma} + \underbrace{[x_1 \;\; x_2 \;\; x_3]}_{=X} \underbrace{\begin{bmatrix} \beta_{11} & 0 \\ 0 & \beta_{22} \\ 0 & \beta_{32} \end{bmatrix}}_{=B} + \underbrace{[e_1 \;\; e_2]}_{=E}
$$

2. Let

$$
\Delta = \begin{bmatrix} \Gamma \\ B \end{bmatrix} = \begin{bmatrix} -1 & \gamma_2 \\ \gamma_1 & -1 \\ \beta_{11} & 0 \\ 0 & \beta_{22} \\ 0 & \beta_{32} \end{bmatrix} = [\Delta_1 \, \Delta_2].
$$

Equation (5):

$$
\begin{aligned}
R_1 \Delta_1 = 0 \quad \text{with} \quad R_1 &= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \Rightarrow \quad rank(R_1) = 2 > M - 1 \\
R_1 \Delta &= \begin{bmatrix} 0 & \beta_{22} \\ 0 & \beta_{32} \end{bmatrix} \quad \Rightarrow \quad rank(R_1 \Delta) = 1 = M - 1 \\
&\Rightarrow \quad \text{overidentified!}
\end{aligned}
$$

Equation (6):

$$
\begin{aligned}
R_2 \Delta_2 = 0 \quad \text{with} \quad R_2 &= \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad rank(R_2) = 1 = M - 1 \\
R_2 \Delta &= \begin{bmatrix} \beta_{11} & 0 \end{bmatrix} \quad \Rightarrow \quad rank(R_2 \Delta) = 1 = M - 1 \\
&\Rightarrow \quad \text{exactly identified!}
\end{aligned}
$$

3.

$$
\hat{\Pi} \Gamma_2 = -B_2
$$

$$
\begin{bmatrix} \hat{\pi}_{11} & \hat{\pi}_{12} \\ \hat{\pi}_{21} & \hat{\pi}_{22} \\ \hat{\pi}_{31} & \hat{\pi}_{32} \end{bmatrix} \begin{bmatrix} \hat{\gamma}_2 \\ -1 \end{bmatrix} = - \begin{bmatrix} 0 \\ \hat{\beta}_{22} \\ \hat{\beta}_{32} \end{bmatrix} \quad \Leftrightarrow \quad \begin{bmatrix} \hat{\pi}_{11} \hat{\gamma}_2 - \hat{\pi}_{12} \\ \hat{\pi}_{21} \hat{\gamma}_2 - \hat{\pi}_{22} \\ \hat{\pi}_{31} \hat{\gamma}_2 - \hat{\pi}_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ -\hat{\beta}_{22} \\ -\hat{\beta}_{32} \end{bmatrix}
$$

$$
\begin{array}{rcc}
 & \hat{\gamma}_2 = \frac{\hat{\pi}_{12}}{\hat{\pi}_{11}} & \frac{\hat{\pi}_{12}}{\hat{\pi}_{11}} = \hat{\gamma}_2 \\
\Leftrightarrow & \hat{\pi}_{21} \hat{\gamma}_2 = -\hat{\beta}_{22} + \hat{\pi}_{22} \quad \Leftrightarrow & -\hat{\pi}_{21} \frac{\hat{\pi}_{12}}{\hat{\pi}_{11}} + \hat{\pi}_{22} = \hat{\beta}_{22} \\
 & -\hat{\pi}_{31} \hat{\gamma}_2 + \hat{\pi}_{32} = \hat{\beta}_{32} & \hat{\pi}_{32} - \hat{\pi}_{31} \frac{\hat{\pi}_{12}}{\hat{\pi}_{11}} = \hat{\beta}_{32}
\end{array}
$$

4. •

$$
\hat{Z}_2 = [\hat{Y}_2 \quad X_2] = [X(X'X)^{-1} X' Y_2 \quad X(X'X)^{-1} X' X_2] = X(X'X)^{-1} X' \underbrace{[Y_2 \quad X_2]}_{=Z_2}
$$

Remark: $X(X'X)^{-1}X'$ is a projection matrix and $X(X'X)^{-1}X'X_2 = X_2$.

- $$\begin{aligned} &X'Z_2 \text{ is a } (3\times 3)\text{–matrix}, && Y_2 = [y_1], \\ &X = [x_1 \;\; x_2 \;\; x_3], && X_2 = [x_2 \;\; x_3], \\ &Z_2 = [Y_2 \;\; X_2] = [y_1 \;\; x_2 \;\; x_3] \end{aligned}$$

- $$\begin{aligned} \hat{\delta}_{2(2SLS)} &= (\hat{Z}_2'\hat{Z}_2)^{-1}\hat{Z}_2' y \\ &= [Z_2'X(X'X)^{-1}\underbrace{X'X(X'X)^{-1}}_{=I_3}X'Z_2]^{-1}Z_2'X(X'X)^{-1}X'y_2 \\ &= [X'Z_2]^{-1}[(X'X)^{-1}]^{-1}[Z_2'X]^{-1}[Z_2'X][X'X]^{-1}X'y_2 \\ &= [X'Z_2]^{-1}X'y_2 = \hat{\delta}_{2(ILS)} \end{aligned}$$

  Remark: As $X'Z_2$ is a (3×3)–matrix with $rank(X'Z_2) = 3$, the inverse $[X'Z_2]^{-1}$ exists.

5.

$$\beta_{22} = \beta_{32} \Rightarrow \Delta = \begin{bmatrix} -1 & \gamma_2 \\ \gamma_1 & -1 \\ \beta_{11} & 0 \\ 0 & \beta_{22} \\ 0 & \beta_{22} \end{bmatrix}$$

No changes to equation (5).

And equation (6):

$$R_2\Delta_2 = 0 \quad \text{with} \quad R_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \text{ and } rank(R_2) = 2$$

$$rank(R_2\Delta) = rank\begin{bmatrix} \beta_{11} & 0 \\ 0 & 0 \end{bmatrix} = 1$$

$\Rightarrow$ Now overidentified!

Problem 5 (10 Points) Multiple Choice

For the following statements indicate whether they are True (T) or False (F). For each true answer you will get 2 points, for each false answer you will lose 1 point and for unanswered questions you will get 0 points. For the whole problem you will not get less than zero points.

Statements

T F	
[ ] [X]	1. In a linear regression model the sum of the OLS residuals ($\sum_{t=1}^{T} \hat{e}_t$) is always zero.
[ ] [X]	2. The OLS residuals $\hat{e}_t$ are uncorrelated, if this is the case for the errors e_t.
[X] [ ]	3. A break in the intercept of a linear regression model can be modelled by an appropriately defined dummy variable.
[X] [ ]	4. If the regressor matrix X in the linear model $y = X\beta + e$ is not a full rank, then an unbiased estimator for β does not exist.
[ ] [X]	5. The Gauss-Newton method for estimating the parameters in a nonlinear regression model is based on a second order Taylor series approximation of the least squares criterion.