Exercise 1 (20 Points)

Consider the two-equation model

$$y_{t1} = x_t\beta_1 + e_{t1}, \tag{1}$$
$$y_{t2} = x_t\beta_2 + e_{t2}, \tag{2}$$

where y and x denote endogenous and exogenous variables,respectively:
y_{t1} : industrial investment,
y_{t2} : investment in the remaining sectors,
x_t : costs of capital utilization.

It is assumed that the errors (1) and (2) are jointly normally distributed and serially uncorrelated with expectation zero and covariance matrix Σ. The following data are available:

$$\sum_{t=1}^{30} {x_t}^2 = 100, \quad \sum_{t=1}^{30} x_t y_{t1} = -30, \quad \sum_{t=1}^{30} x_t y_{t2} = -10,$$
$$\widehat{\sum} = \begin{bmatrix} 0.6 & 0.4 \\ 0.4 & 0.8 \end{bmatrix}.$$

i) Estimate the coefficients β_1 and β_2, where you should take into account that both equations have the same regressor. What are the properties of the estimator which you chose?

ii) Show that for the 2-equation model

$$y_{t1} = z_{t1}\beta_1 + e_{t1}, \tag{3}$$
$$y_{t2} = z_{t2}\beta_2 + e_{t2}, \tag{4}$$

the SUR estimate is identical to the OLS estimate of each equation if the covariance matrix is assumed to be diagonal.

iii) Compute the covariance matrix of the estimated coefficients and test the hypothesis $H_0 : \beta_1 = \beta_2$ at a significance level of 0.05.

Exercise 2 (20 Points)

For the regression model:

$$y_t = \beta_1 x_{t-1} + e_t, \tag{5}$$

it is assumed that the regressor x_t is stochastic and $E(x_t x_{t-1}) = c$ for all t. Furthermore, it holds that

i) $e_t \sim (0, \sigma^2)$, $E(e_t e_s) = 0$ for $t \neq s$, $E(x_t e_t) = a^2$ and $E(x_t e_s) = 0$ for $t \neq s$,

   or, alternatively,

ii) $e_t = \rho e_{t-1} + v_t$, $v_t \sim (0, \sigma^2)$, $E(v_t v_s) = 0$ for $t \neq s$, $E(x_t v_t) = a^2$ and $E(x_t v_s) = 0$ for $t \neq s$.

Suggest for each case an appropriate estimator and show its consistency.

Exercise 3 (20 Points)

For the linear regression model:

$$y_t = \beta_1 + x_t \beta_2 + e_t, \quad e_t \sim i.i.d.(0, \sigma^2), \quad t = 1, \ldots, 40, \tag{6}$$

it is assumed that β_2 is subject to a structural break at period $t = 16$.

i) Write the model in the "sets-of-equations format". Use also the matrix notation.

ii) Estimate the parameters of the model with OLS by using the following data:

$$\sum_{t=1}^{15} x_t = \sum_{t=16}^{4} 0 x_t = 0, \quad \sum_{t=1}^{15} {x_t}^2 = 20, \quad \sum_{t=16}^{40} {x_t}^2 = 30,$$
$$\sum_{t=1}^{40} y_t = 60, \quad \sum_{t=1}^{15} x_t y_t = 16, \quad \sum_{t=16}^{40} x_t y_t = 18.$$

iii) Write down the model in the "dummy-variables format". How can you derive its parameters from the estimates of the "sets-of-equations format"?

Exercise 4 (20 Points)

For the relationship between y_t and x_t, the following nonlinear model is specified:

$$y_t = (x_t + \beta_1)^{\beta_2} + e_t$$

i) Write down the "linear pseudomodel". Describe (without formulae) how a Gauss-Newton algorithm can be constructed by using the linear pseudomodel.

ii) Compute the Hessian matrix that is needed for the Newton-Raphson algorithm.

iii) Assume that the coefficients are estimated as $\widehat{\beta}_1 = 0.5$ and $\beta_2 = 1.0$ with the estimated covariance matrix:

$$\widehat{\Sigma}_{\widehat{\beta}} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.2 \end{bmatrix}.$$

Compute the Wald statistic for a test of the hypothesis $\beta_1^{\beta_2} = 1$.

Exercise 5 (20 Points)

Consider the following system of simultaneous equations:

$$\begin{aligned} y_{t1} &= \gamma_{31} y_{t3} + \beta_{11} x_{t1} + \beta_{21} x_{t2} + e_{t1}, && (7)\\ y_{t2} &= \gamma_{32} y_{t3} + \beta_{32} x_{t3} + e_{t2}, && (8)\\ y_{t3} &= \gamma_{23} x_{t2} + e_{t3}, && (9)\\ & && (10) \end{aligned}$$

where y and x indicate dependent and predetermined variables, respectively.

i) Write down the model in the form $Y\Gamma + XB + \mathbf{E} = 0$ and check the identification of the parameters in each equation by using the order criterion.

ii) Estimate equation (8) with the 2SLS method using the data:

$$\sum \hat{y}_{t3} y_{t3} = 100, \quad \sum \hat{y}_{t3} y_{t2} = 5, \quad \sum \hat{y}_{t2} y_{t3} = 6,$$
$$\sum x_{t3}{}^2 = 2, \quad \sum x_{t3} \hat{y}_{t2} = 11, \quad \sum x_{t2} \hat{y}_{t2} = 12,$$
$$\sum \hat{y}_{t3} x_{t3} = -10, \quad \sum x_{t3} y_{t2} = 15, \quad \sum x_{t2} y_{t2} = 10.$$

Here, $\hat{y}$ denotes the estimate of y resulting from the estimated reduced form of the model.

iii) Derive the reduced form for equation (8).

iv) Give conditions under which the OLS estimator of the structural equation (8) is consistent.