Problem 1 (25 points)

Consider the model:

$$y_t = x'_{(t)}\beta + e_t, \qquad t = 1, \ldots, T, \tag{1}$$

where the regressors are non–stochastic and $\beta = (\beta_1, \ldots, \beta_K)'$. Assume that the error term $e = (e_1, \ldots, e_T)'$ has the following structure:

$$e \sim (0, \Phi), \qquad \text{where} \quad \Phi \quad \text{is positive definite.}$$

1. **(15 points)**

   a) Show that both, the OLS estimator $b = (X'X)^{-1}X'y$ and the GLS estimator $\hat{\beta}_G = (X'\Phi^{-1}X)^{-1}X'\Phi^{-1}y$ are unbiased?

   b) The covariance matrices of the OLS estimator b and the GLS estimator $\hat{\beta}_G$ (under the assumption of a general error covariance matrix Φ) are given by

   $$Cov(b) = (X'X)^{-1}(X'\Phi X)(X'X)^{-1} \tag{2}$$

   and

   $$Cov(\hat{\beta}_G) = (X'\Phi^{-1}X)^{-1}, \tag{3}$$

   respectively. Using $A = (X'X)^{-1}X' - (X'\Phi^{-1}X)^{-1}X'\Phi^{-1}$ verify that the difference between both covariance matrices $Cov(b) - Cov(\hat{\beta}_G)$ can be expressed as $A\Phi A'$, and show that $A\Phi A'$ is positive semidefinite.

   c) How do you interpret the results in 1a) and 1b)?

2. **(6 points)** Assume now that $x_{(t)} = (1 \quad x_{t2})'$ and that for the error term e_t holds:

   $$e_t = x_{t2}u_t, \qquad u = (u_1, \ldots, u_T)' \sim (0, \sigma^2 I_T).$$

   a) Write down the expectation vector and the covariance matrix for the error vector e.

   b) Find a suitable matrix P such that $e^* = Pe \sim (0, \sigma^2 I_T)$ holds for some $\sigma^2 > 0$.

3. **(4 points)** Assume now that $e_t \sim i.i.d.\ N(0, \sigma^2)$. Give a $(1 - \alpha)$ confidence interval for β_1.

<u>Problem 2</u> (25 points)

Consider the following model:

$$y = Z\beta + e, \tag{4}$$

where Z is the $T \times K$ matrix of stochastic regressors.

a) **(4 points)** Show that the OLS estimator $b = (Z'Z)^{-1}Z'y$ is unbiased if Z is strictly exogenous, i.e. if $E[e|Z] = 0$.

b) **(6 points)** Suppose that there exists a positive definite matrix Σ_{ZZ} such that $\dfrac{Z'Z}{T} \xrightarrow{P} \Sigma_{ZZ}$ for $T \to \infty$. Verify that b is inconsistent if Z is not weakly exogenous, i.e. if the following does **not** hold: $\dfrac{Z'e}{T} \xrightarrow{P} 0$ for $T \to \infty$.

c) **(5 points)** Show that β can be consistently estimated by $b_{iv} = (X'Z)^{-1}X'y$, if there exists a $(T \times K)$–regressor matrix X containing K instrumental variables such that for $T \to \infty$

$$\frac{X'Z}{T} \xrightarrow{P} \Sigma_{XZ}, \quad \text{with} \quad \Sigma_{XZ} \quad \text{positive definite,} \tag{5}$$

$$\frac{X'e}{T} \xrightarrow{P} 0 \tag{6}$$

hold.

d) **(4 points)** What is the intuition behind the two instrument conditions (5) and (6)?

e) **(6 points)** Let $\hat{Z} = X(X'X)^{-1}X'Z$ be the predicted value of Z. Establish that the OLS regression of y on $\hat{Z}$ gives the instrumental variable estimator, i.e. show that $b_{iv} = (\hat{Z}'\hat{Z})^{-1}\hat{Z}'y$.

Problem 3 **(25 Points)**

Consider the linear regression model

$$y_t = \beta_1 + \beta_2 x_t + e_t, \qquad e_t \sim i.i.d.\ N(0, \sigma^2), \qquad t = 1, \ldots, 75. \tag{7}$$

A structural break in period $t = 26$ leads to a shift in the intercept, while the slope parameter is not affected. There is another break in period $t = 51$ that now affects both, the intercept and the slope parameter.

a) **(6 points)** Define dummy variables that can be used to capture both structural breaks in the parameters. Using these variables formulate the model in dummy variable representation. In addition, write down the model in matrix notation by defining appropriate vectors and matrices.

b) **(6 points)** Give the sets of equation representation of the model defined before. As before, also give the matrix version of the model.

c) **(5 points)** Use the relation between parameters in a) and b) to express the parameters of the sets of equation representation in terms of the dummy variable model parameters in a).

d) **(8 points)** Suppose only the last 50 observations are analyzed, i.e. consider a model with only one potential structural break. Using only observations for $t = 26, \ldots, 75$, an OLS estimation of (7) yields $\hat{e}'\hat{e} = 39$. In addition, using the same observations estimation of a dummy variable model which allows a change in β_1 and β_2 gives $s^2 = 0.7$. Test the hypothesis of no parameter change in model (7) at a significance level of $\alpha = 0.05$.

Problem 4 **(25 Points)**

Consider the following simultaneous equation model

$$y_{t1} = [\gamma_{21}\delta - (1-\delta)]y_{t2} + (1-\delta)y_{t3} + \delta\beta_{11}x_{t1} + \delta e_{t1} \tag{8}$$

$$y_{t2} = \gamma_{12}y_{t1} + \beta_{22}x_{t2} + e_{t2} \tag{9}$$

$$y_{t3} = y_{t2} + y_{t1} \tag{10}$$

where y_i are the endogenous and x_i are exogenous variables and $\delta \neq 0$. For the errors it is assumed that $e \sim (0, \Sigma \otimes I_T)$.

a) **(6 points)** Plug-in the identity (10) into equation (8) and simplify the model. Apply a normalization to the equations where necessary. Write the model in the form $Y\Gamma + XB + E = 0$.

b) **(6 points)** Check the identification of the structural equations using the order and the rank criteria.

c) **(6 points)** The reduced form estimates are given by

$$\hat{\Pi} = \begin{pmatrix} \hat{\pi}_{11} & \hat{\pi}_{12} \\ \hat{\pi}_{21} & \hat{\pi}_{22} \end{pmatrix}.$$

Estimate the structural parameters Γ and B using indirect least squares (ILS) where possible.

d) **(7 points)** Now suppose a specific theory suggests $\beta_{22} = 0$ for the model considered above. Reexamine the identification using the order and the rank criteria. Give a brief intuitive explanation why the results change. Explain how you could estimate the structural parameters γ_{21}, γ_{12} and β_{11}.