

**Klausur / Exam**  
**Methoden der Ökonometrie / Econometric Methods**

**am / at 16.02.2009**

**Dauer/Examination Duration 150 Min.**

**Lesezeit/Reading Time 10 Min.**

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**Nachname/Last Name:**

**Vorname/First Name:**

**Immatrikulations–Nr/Enrolment No.:**

**Erlaubte Hilfsmittel: Formelsammlung (ausgeteilt), Taschenrechner (nicht programmierbar)**  
**Allowed Resources: Formulary (provided), Pocket Calculator (nonintelligent)**

**Bitte bestätigen Sie durch Ihre Unterschrift, dass Ihnen bekannt ist, dass jeder Täuschungsversuch zum Abbruch der Arbeit mit der Bewertung "ungenügend" (5.0) führt.**

**Please confirm by signing that you are aware that any attempt to deceive results in a grade "failed" (5.0).**

.....  
**Unterschrift/Signature**

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**(nur für die Prüfer/only for examiner)**

Punkte/Points	Problem 1	Problem 2	Problem 3	Problem 4	Gesamt/Total
mögliche / possible	36	25	30	32	123
erreichte / attained					

**Note / Grade:**

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**1. Prüfer/1st examiner**

.....  
**2. Prüfer/2nd examiner**

### Problem 1 (36 Points)

1. Consider the following generalized least-squares (GLS) framework:

$$y = X\beta + \varepsilon, \quad \varepsilon \sim N(0, \sigma^2\Omega),$$

where  $\sigma^2\Omega = \Psi$ . The deterministic  $(n \times k)$ -matrix  $X$  has  $\text{rank}(X) = k$ .

- (a) **(5 Points)** Show that the ordinary least-squares estimator (OLSE),  $\hat{\beta}_{OLS}$ , and the generalized least-squares estimator (GLSE),  $\hat{\beta}_{GLS}$ , are unbiased.
- (b) **(4 Points)** Derive the covariance matrices of  $\hat{\beta}_{GLS}$ , and of  $\hat{\beta}_{OLS}$ .
- (c) **(3 Points)** Briefly discuss  $\mathbb{V}[\hat{\beta}_{OLS}] - \mathbb{V}[\hat{\beta}_{GLS}]$ . Explain the implications in terms of efficiency.
- (d) **(12 Points)** Derive the GLS estimator from the GLS criterion

$$GLS(\beta^*) = (y - X\beta^*)' \Psi^{-1} (y - X\beta^*).$$

Matrix differentiation rules are given in the formulary. You may assume that the second order conditions are satisfied.

2. Now suppose we want to test some restrictions on the estimates for a classical regression model. Consider the case

$$y = X\beta + \varepsilon, \quad \varepsilon|X \sim N(0, \sigma^2 I_n).$$

with a constant and two explanatory variables, i.e.  $k = 3$  and  $\beta = (\beta_1, \beta_2, \beta_3)'$ . Suppose an economic theory implies that  $\beta_1 - \beta_3 = 0$ . Moreover, for a sample of size  $n = 20$  let the following data be given:

$$(X'X)^{-1} = \begin{bmatrix} 0.58 & 0.31 & -0.33 \\ 0.31 & 0.28 & -0.27 \\ -0.33 & -0.27 & 0.27 \end{bmatrix}, \quad X'y = \begin{bmatrix} 2.4 \\ -9.17 \\ -3.74 \end{bmatrix}, \quad e'e = 26.95.$$

Furthermore, define:

$\hat{\beta}_U$  as the OLS estimator of the unrestricted model,

$\hat{\beta}_R$  is the OLS estimator of the restricted model,

$e$  as the residuals of the unrestricted OLS model:  $e = y - X\hat{\beta}_U$ .

- (a) **(12 Points)** Use the F-test for linear restrictions to test  $H_0 : \beta_1 - \beta_3 = 0$  against  $H_1 : \beta_1 - \beta_3 \neq 0$  using a significance level of  $\alpha = 0.05$ . The critical value is given by  $F_{(1, 17, 95\%)} = 4.45$ . Based on the test decision which estimator would you use,  $\hat{\beta}_U$  or  $\hat{\beta}_R$ ?
- (b) **(2 Points)** Comparing Restricted and Unrestricted models with one regressor. Fill in the following two statements with either (Unrestricted or Restricted):
  - i. "The sum of squared residuals of the ..... model is greater."
  - ii. "The variance of  $\hat{\beta}_{OLS}$  of the ..... model is greater."

## Problem 2 (25 Points)

1. Consider the linear regression model:

$$y_i = x_i' \beta + \varepsilon_i, \quad \varepsilon_i \sim (0, \sigma^2), \quad i = 1, \dots, n, \quad (1)$$

where  $x_i$  and  $\beta$  are  $(k \times 1)$ -vectors. Moreover, it is assumed that:

A1)  $\{(y_i, x_i')'\}$  is a sequence of i.i.d.  $(k + 1 \times 1)$ -random vectors,

A2)  $\mathbb{E}[x_i x_i'] = \Omega_{XX}$  is finite and strictly positive definite,

A3) 4th moments of regressors exist,

A4)  $\mathbb{E}[\varepsilon_i x_i] = 0$ ,

A5)  $\mathbb{E}[\varepsilon_i^2 x_i x_i'] = \sigma^2 \Omega_{XX}$

Denote the  $(n \times k)$ -matrix  $X = [x_1, \dots, x_n]'$  and the  $(n \times 1)$ -vector  $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)'$ .  $\hat{\beta}_n$  is the OLSE of  $\beta$ .

(a) **(1 Point)** Fill in the gap:

“Under assumptions A1), A2) and A3) it can be shown that  $(X'X)/n \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \dots\dots$ ”

(b) **(3 Points)** Which central limit theorem holds for  $X'\varepsilon/\sqrt{n}$ ? Justify briefly and give the limiting distribution.

(c) **(4 Points)** Using 1(b), show that  $\sqrt{n}(\hat{\beta}_n - \beta) \xrightarrow{\mathbb{D}} N(0, \sigma^2 \Omega_{XX})$ .

(d) **(2 Points)** Approximate the distribution of  $\hat{\beta}_n$  using the asymptotic result from part 1(c).

(e) **(5 Points)** Justify that the WLLN holds for  $X'\varepsilon/n$ . Give the probability limit.

(f) **(6 Points)** Using your results from 1(a) and 1(e) and the hints given below, show that the OLS estimator for  $\sigma^2$ ,

$$\hat{\sigma}^2 = \frac{1}{n-k} \varepsilon' M \varepsilon = \frac{1}{n-k} (\varepsilon' \varepsilon - \varepsilon' X (X'X)^{-1} X' \varepsilon),$$

where  $M$  is the residual-maker, is consistent.

*Hints:* Multiply the variance estimator with  $\frac{n}{n-k}$  and reformulate it. Furthermore, you can use that  $\text{plim}(X') = (\text{plim}(X))'$  for a random matrix  $X$ . The following probability limit is given:

$$\frac{1}{n} \sum_{i=1}^n \varepsilon_i^2 \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \sigma^2.$$

2. **(4 points)** Consider equation (1) above with only one regressor,  $x_i = i$ ,  $i = 1, \dots, n$ , i.e.

$$y_i = i \cdot \beta + \varepsilon_i, \quad i = 1, \dots, n. \quad (2)$$

One assumption often required for the consistency of the OLSE with deterministic regressors is  $\lim_{n \rightarrow \infty} (X'X)/n = \Omega_{XX}$ , where  $\Omega_{XX}$  is finite. Compute  $\lim_{n \rightarrow \infty} X'X/n$  for equation (2). Does the assumption hold? All necessary sum expressions can be found in the formulary.

### **Problem 3 (30 Points)**

Consider

$$y_i = \mu + \varepsilon_i, \tag{3}$$

$\varepsilon_i, i = 1, \dots, n$ , i.i.d. and follows a Normal distribution with  $\mathbb{E}[\varepsilon_i] = 0$  and  $\mathbb{V}[\varepsilon_i] = 1$ .

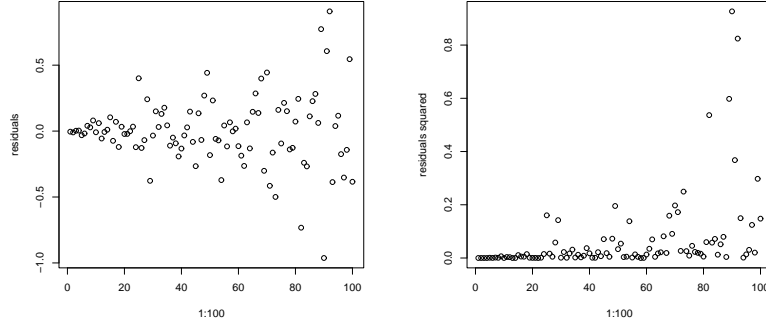
1. **(3 Points)** Calculate the log likelihood function.
2. **(2 Points)** Derive the score function of the log likelihood function. Interpret this function.
3. **(3 Points)** Derive the maximum likelihood estimator (MLE) and its finite sample distribution.
4. **(2 Points)** Show that the MLE  $\hat{\lambda}_{ML}$  is unbiased.
5. **(2 Points)** What is the invariance principle in the ML context?
6. **(8 Points)** Show that the Wald and LM test statistics are equal in the given setting (3) for the null hypothesis  $H_0 : \mu = \mu^*$ , where  $\mu^*$  is a known scalar.
7. **(3 points)** Suppose you want to analyse a sample of size  $n = 12$  with  $\sum_{i=1}^{12} y_i = 1312$ . Use the LM-test to test your null hypothesis for  $\mu^* = 120$  at a 5 % level. The critical value is  $\chi^2_{(1,95\%)} = 3.842$ .
8. **(4 Points)** Point out some advantages and disadvantages of the Wald and LR-tests.
9. **(3 Points)** What is wrong with the following statement? Explain briefly.  
*The Maximum Likelihood Estimation (MLE) method provides consistent, unbiased and efficient estimators. The variances of MLEs are equal to the Cramer-Rao-lower-bound.*

## Problem 4 (32 Points)

1. Consider:

$$Y = X\beta + \varepsilon, \quad \varepsilon \sim N(0, \Psi).$$

Figure 1: Residual Plots



(a) Residuals of OLS against X

(b) Squared Residuals of OLS against X

(a) **(2 Points)** The residuals of the OLS are plotted against X in Figure 2(a). What does the residual plot suggest?

(b) **(8 Points)** Assume that  $n = 100$  and  $\Psi = \text{diag}(\sigma_1^2, \dots, \sigma_{100}^2)$  with

$$\sigma_i^2 = \exp(\gamma_0 + \gamma_1 D_i) \quad \text{and} \quad D_i = \begin{cases} 0 & \text{for } i = 1, \dots, 50 \\ 1 & \text{for } i = 51, \dots, 100 \end{cases}$$

Consider the auxiliary regression:

$$e_i^2 = \delta_1 + \delta_2 X_i + \xi_i,$$

where  $e_i$  is the OLS residual. Let the following data be given:

$$\sum_{i=1}^{100} (e_i^2 - \bar{e}^2)^2 = 2.3 \quad \text{and} \quad \sum_{i=1}^{100} \hat{\xi}_i^2 = 2.083,$$

Conduct the Breusch-Pagan test for the constancy of the error variance at  $\alpha = 0.05$ , which is given as  $\chi_{1,\alpha=.1}^2 = 3.84$ . State the hypothesis, decision rule, and conclusion.

(c) **(4 Points)** Figure 2(b) shows the squared residuals against X. What does the plot suggest about the relationship between the variance of error terms and X? Suggest a way of using Figure 2(b) to implement weighted least squares.

(d) **(4 Points)** Table 1 provides the estimated OLS estimates  $\hat{\beta}_0$  and  $\hat{\beta}_1$ , and the GLS estimates  $b_{g0}$  and  $b_{g1}$ . Compare the OLS and GLS estimates. What do you find? Explain the reason behind the results.

Table 1: OLS and GLS Estimates					
OLS	estimate	standard error	GLS	estimate	standard error
$\hat{\beta}_0$	0.0035	0.0536	$b_{g0}$	0.0035	0.0014
$\hat{\beta}_1$	0.5485	0.0921	$b_{g1}$	0.5458	0.0045

2. Production in a factory was modeled by the following regression equation using 100 observations, 50 from the day shift and 50 from the night shift:

$$W_t = 7 + 35E_t + 50M_t - 1.5D_t + \varepsilon_t \quad (4)$$

where  $W$  is the number of units produced,  $E$  is the number of employees working during the shift,  $M$  is the number of machines working during the shift, and  $D$  and  $N$  are dummy variables for the day and night shifts respectively:

$$D_t = \begin{cases} 1 & \text{for } t = 1, \dots, 50 \\ 0 & \text{for } t = 51, \dots, 100 \end{cases} \quad N_t = \begin{cases} 0 & \text{for } t = 1, \dots, 50 \\ 1 & \text{for } t = 51, \dots, 100 \end{cases}$$

The expected paths of  $E$  and  $M$  for the next year are as follows:

$$\frac{1}{50} \sum_{t=1}^{50} E_t = 110 \quad \frac{1}{50} \sum_{t=1}^{50} M_t = 100 \quad \frac{1}{50} \sum_{t=51}^{100} E_t = 80 \quad \frac{1}{50} \sum_{t=51}^{100} M_t = 130$$

- (a) **(3 Points)** Compute the expected number of units produced in each shift of the next year.  
(b) **(3 Points)** Explain why we should **not** use the following model (i.e. both dummies + intercept):

$$W_t = \beta_0 + \beta_1 E_t + \beta_2 M_t + \beta_3 D_t + \beta_4 N_t + \varepsilon_t$$

- (c) **(4 Points)** Suppose you use the same data to estimate an equation of the same form as Equation (4), except employ dummy variable  $N$ . Determine  $\beta_0$  and  $\beta_4$ .

$$W_t = \beta_0 + 35E_t + 50M_t + \beta_4 N_t + \varepsilon_t$$

- (d) **(4 Points)** Now suppress the intercept and use both shift dummies. Determine  $\beta_3$  and  $\beta_4$ .

$$W_t = 35E_t + 50M_t + \beta_3 D_t + \beta_4 N_t + \varepsilon_t$$