Klausur / Exam
Methoden der Ökonometrie / Econometric Methods

am / at 06.04.2009
Dauer/Examination Duration 150 Min.

Lesezeit/Reading Time 10 Min.

Nachname/Last Name: **Vorname/First Name:**

Immatrikulations–Nr/Enrolment No.:

Erlaubte Hilfsmittel: Formelsammlung (ausgeteilt), Taschenrechner (nicht programmierbar)
Allowed Resources: Formulary (provided), Pocket Calculator (nonintelligent)

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Please confirm by signing that you are aware that any attempt to deceive results in a grade "failed" (5.0).

....
Unterschrift/Signature

(nur für die Prüfer/only for examiner)

Punkte/Points	Problem 1	Problem 2	Problem 3	Problem 4	Gesamt/Total
mögliche / possible	32	28	31	29	120
erreichte / attained					

Note / Grade:

....
1. Prüfer/1st examiner   **2. Prüfer/2nd examiner**

Problem 1 (32 Points)

1. Consider the linear regression model:

$$y = X\beta + \varepsilon \qquad \varepsilon \sim N(0, \sigma^2 I)$$

where X is a $(n \times k)$ deterministic matrix with $rank(X) = k$ and $\lim_{n \to \infty} \frac{1}{n} X'X$ is a finite, nonsingular matrix.

   (a) **(2 Points)** Show that the OLS estimator $\widehat{\beta}_{OLS} = (X'X)^{-1} X'y$ is unbiased.

   (b) **(2 Points)** Derive the covariance matrix of $\widehat{\beta}_{OLS}$.

2. Let there be two subsamples, (1) and (2), in which the parameters are not necessarily the same. An unrestricted regression that allows the coefficients to be different in the two periods is

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix}$$

   (a) **(3 Points)** Determine the unrestricted least squares estimators for β_1 and β_2.

   (b) **(2 Points)** Write the formulae of the residuals e_1 and e_2 for this unrestricted regression. Use the $\hat{\beta}_1$ and $\hat{\beta}_2$ derived in part (a).

   (c) **(2 Points)** We can consider the restriction $\beta_1 = \beta_2$. A resticted estimator can be obtained by stacking the data and estimating a single regression:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \beta + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix}$$

The resulting restricted least squares estimator is:

$$\widehat{\beta}_R = (X_1'X_1 + X_2'X_2)^{-1} (X_1'y_1 + X_2'y_2)$$

Write the formulae for the residuals e_1 and e_2 for this restricted regression using the given $\widehat{\beta}_R$.

   (d) **(8 Points)** There are 70 observations in subsample (1) and 30 observations in subsample (2). You are given the following residuals:

$$\begin{array}{lll} \text{Unrestricted Model:} & e_1'e_1 = 314 & e_2'e_2 = 107 \\ \text{Restricted Model:} & e_1'e_1 = 492 & e_2'e_2 = 530 \end{array}$$

Use the F-test for linear restrictions to test $H_0 : \beta_1 = \beta_2$ against $H_0 : \beta_1 \neq \beta_2$ using a significance level of $\alpha = 0.05$. The critical value is given by $F(1, 98, 95\%) = 3.94$. Based on the test decision which estimator would you use, $\widehat{\beta}_U$ or $\widehat{\beta}_R$?

3. The Gauss-Markov Theorem states that under the full ideal conditions the $\widehat{\beta}_{OLS}$ is the best linear unbiased estimator (BLUE).

   (a) **(5 Points)** State the five conditions.

   (b) **(8 Points)** Consider an alternative Linear Unbiased estimator: $\widetilde{\beta} = \widehat{\beta}_{OLS} + Cy$. Show that $\widehat{\beta}_{OLS}$ is more efficient than $\widetilde{\beta}$. *Hint: You may assume that $E[CX] = 0$.*

Problem 2 (28 Points)

Consider the linear regression model:

$$y_i = x_i'\beta + \varepsilon_i, \quad \varepsilon_i \sim (0, \sigma^2), \qquad i = 1, ..., n ,$$ (1)

where x_i and β are $(k \times 1)$–vectors. Moreover, it is assumed that:

A1) $\{(y_i, x_i')'\}$ is a sequence of i.i.d. $(k + 1 \times 1)$–random vectors,

A2) $\mathbb{E}[x_i x_i'] = \Omega_{XX}$ is finite and strictly positive definite,

A3) 4th moments of regressors exist,

A4) $\mathbb{E}[\varepsilon_i x_i] = 0$,

A5) $\mathbb{E}[\varepsilon_i^2 x_i x_i'] = \sigma^2 \Omega_{XX}$.

Denote the $(n \times k)$–matrix $X = [x_1, ..., x_n]'$ and the $(n \times 1)$–vector $\varepsilon = (\varepsilon_1, ..., \varepsilon_n)'$. $\widehat{\beta}_n$ is the OLSE of β.

1. **(1 Point)** Fill in the gap:
   "Under assumptions A1), A2) and A3) it can be shown that $(X'X)/n \xrightarrow{\mathbb{P}} \text{.......}$"

2. **(3 points)** Give the definitions for an unbiased, an asymptotically unbiased and a consistent estimator of β.

3. **(2 points)** What is the difference between an unbiased and an asymptotically unbiased estimator?

4. **(2 points)** Give one reason why an asymptotically unbiased estimator is not necessarily consistent.

5. **(3 Points)** Does the central limit theorem hold for $X'\varepsilon/\sqrt{n}$? Justify briefly and give the limiting distribution.

6. **(4 Points)** Using 5, show that $\sqrt{n}(\widehat{\beta}_n - \beta) \xrightarrow{\mathbb{D}} N(0, \sigma^2 \Omega_{XX}^{-1})$.

7. **(3 Points)** Approximate the distribution of $\widehat{\beta}_n$ using the asymptotic result from part 6.

8. **(4 Points)** Justify that the WLLN holds for $X'\varepsilon/n$. Give the probability limit.

9. **(6 Points)** Using your results from 1 and 8 and the hints given below, show that the OLS estimator for σ^2,
   $$\widehat{\sigma}^2 = \frac{1}{n-k}\varepsilon'M\varepsilon = \frac{1}{n-k}(\varepsilon'\varepsilon - \varepsilon'X(X'X)^{-1}X'\varepsilon),$$
   where M is the residual–maker, is consistent.

   Hints: Multiply the variance estimator with $\frac{n}{n}$ and reformulate it. Furthermore, you can use that $plim(X') = (plim(X))'$ for a random matrix X. The following probability limit is given:
   $$\frac{1}{n}\sum_{i=1}^{n}\varepsilon_i^2 \xrightarrow{\mathbb{P}} \sigma^2.$$

3

Problem 3 (31 Points)

Consider y_i, $i = 1, \ldots, n$, i.i.d. and following a Bernoulli distribution

$$f(y|p) = p^y(1-p)^{1-y}. \tag{2}$$

with $\mathbb{E}[y_i] = p$ and $\mathbb{V}[y_i] = p(1-p)$.

1. **(2 Points)** Show that the log likelihood function is given by

$$\ln L(p|y) = \ln(p) \sum_{i=1}^{n} y_i + \ln(1-p) \sum_{i=1}^{n} (1 - y_i) \tag{3}$$

2. **(2 Points)** Derive the score function of the log likelihood function. Interpret this function.

3. **(3 Points)** Derive the maximum likelihood estimator (MLE).

4. **(1 Point)** Show that the MLE $\hat{p}_{ML}$ is unbiased.

5. **(2 Points)** Derive the variance of $\hat{p}_{ML}$.

6. **(5 Points)** Determine the second derivative of $\ln L(p|y)$ and the (scalar) Fisher Information I(p). Show that $\hat{p}_{ML}$ is efficient. Which theorem do you use to show that $\hat{p}_{ML}$ is efficient?

7. **(13 Points)** Suppose you observed a random sample of size $n = 25$ with $\sum_{i=1}^{25} y_i = 8$. Use the Wald-test to test your null hypothesis for $p^* = 0.5$ against the alternative that $p^* \neq 0.5$ at a 5 % level. The critical value is $\chi^2_{(1,95\%)} = 3.842$. Proceed as follows:

   (a) Calculate $\hat{p}_{ML}$.

   (b) Formulate your null hypothesis in terms of $r(p)$ and derive $R(p)$, where r is the restriction function and R the derivative of the restriction function.

   (c) Estimate the Fisher Information by $I(\hat{p}_{ML})$.

   (d) Calculate the value of the Wald test statistic and make your decision.

   (e) Let the subscript U denote the unrestriced model and R denote the restricted model. Give the Likelihood Ratio test statistic.

   (f) Calculate the value of the LR test statistic and make your decision.

8. **(3 Points)** Discuss the difference between Wald and LR tests. Specifically:

   (a) Which test statistic is larger or equal compared to the other in finite samples?

   (b) When will the two test statistics be identical?

   (c) What is the advantage of the Wald test over the LR test?

Problem 4 (29 Points)

1. Consider the following linear regression model:
$$y_t = \beta_1 + \beta_2 x_{2t}^* + \varepsilon_t, \qquad t = 1, .., T \qquad \varepsilon_t \sim N(0, \sigma_\varepsilon^2) \quad i.i.d., \tag{4}$$

where y_t denotes consumption expenditures in period t, x_{2t}^* denotes unobservable income in period t and $\beta = (\beta_1, \beta_2)'$ is a (2×1) parameter vector. x_{2t}^* and ε_t are independent. Assume that observable income x_{2t} can be expressed as unobservable income and an additional random component η_t:
$$x_{2t} = x_{2t}^* + \eta_t, \qquad \eta_t \sim (0, \sigma_\eta^2) \quad i.i.d., \tag{5}$$

where x_{2t}^* and η_t are independent. In addition, η_t and ε_t are independent.

   (a) **(2 Points)** Rewrite the model in equation (4) to obtain
   $$y_t = \beta_1 + \beta_2 x_{2t} + u_t. \tag{6}$$
   Give the representation of u_t in terms of β_2, η_t and ε_t.

   (b) **(5 Points)** Show that $\mathbb{E}[x_{2t} u_t] = -\beta_2 \sigma_\eta^2 \neq 0$. How do you interpret this result?

   (c) **(4 Points)** Let $X = (x_1, x_2)$ with x_1 denoting a $(T \times 1)$-vector of ones. What can you say about the consistency and unbiasedness of $\widehat{\beta}_{OLS} = (X'X)^{-1} X'y$ in model (6)? Show your results using plim $\left(\frac{X'X}{T}\right) = \Omega_{XX}$ (positive definite) and plim $\left(\frac{X'u}{T}\right) = \Omega_{Xu} \neq 0$.

2. Assume the following relationship between the unobservable income x_{2t}^* and investment w_{2t}:
$$x_{2t}^* = \gamma_1 + \gamma_2 w_{2t}, \tag{7}$$

where $(\gamma_1, \gamma_2)'$ is a parameter vector. Denoting $W = (w_1, w_2)$, with w_1 being a $(T \times 1)$-vector of ones, it can be shown that (i) plim $\left(\frac{W'W}{T}\right) = \Omega_{WW}$ (positive definite) and (ii) plim $\left(\frac{X'W}{T}\right) = \Omega_{XW}$ (nonsingular). Assume there exists a weak law, such that (iii) plim $\left(\frac{W'u}{T}\right) = 0$ holds.

   (a) **(2 Points)** How do you interpret (ii) and (iii)?

   (b) **(10 Points)** To conduct a simple IV estimation of model (4), take the following data as given:
   $$\sum_{t=1}^{T} x_{2t} = 360, \qquad \sum_{t=1}^{T} w_{2t} = 90 \qquad \sum_{t=1}^{T} w_{2t} x_{2t} = 600, \qquad \sum_{t=1}^{T} x_{2t} y_t = 660,$$
   $$\sum_{t=1}^{T} y_t = 310, \qquad \sum_{t=1}^{T} w_{2t} y_t = 525, \qquad T = 50.$$

   Compute the instrumental variable estimator $\widehat{\beta}_{IV}$. Provide the economic interpretion for the estimated coefficients.

3. **(1 Point)** Name the crucial condition for W that has to be fulfilled if $\widehat{\beta}_{IV} = \widehat{\beta}_{GIV}$, where $\widehat{\beta}_{GIV}$ denotes the generalized instrumental variable estimator.

4. **(5 Points)** What is wrong with the following text on the properties of the Ordinary Least Squares Estimator (OLSE) and the Instrumental Variable Estimator (IVE)? Explain briefly.

   The OLSE is inconsistent, but unbiased in case of a positive correlation between the explanatory variables and the error terms. The more correlated the instruments are with the explanatory variables, the larger will the covariance matrix of the IVE be (in terms of the elements of the matrix). Consistency of the IVE can be established by any variable which is correlated with the explanatory variables. The less correlated the instruments are with the explanatory variables, the less defensible is the claim that these same variables are uncorrelated with the disturbances. The OLSE is not a special case of the IVE if the explanatory variables and error terms are uncorrelated.