Advanced Econometrics

Prof. Bernd Fitzenberger, Ph.D.

Humboldt-University Berlin – Summer Semester 2019

Course Outline

0. Introductory Material
1. Review Linear Regression Model for Cross-Sectional Data
2. System Estimation, Linear Panel Data Models
3. Nonlinear Least Squares and Maximum Likelihood
4. Binary Response Models and Limited Dependent Variables
5. Linear Quantile Regression

0. Introductory Material

Section Outline

0.1. Matrix Algebra

Reference: Greene (2008) App. A

Matrix: Rectangular array of numbers

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nk} \end{pmatrix} \qquad n \times k \text{ matrix}$$

Transpose:

$$A' = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ \vdots & & \ddots & \vdots \\ a_{1k} & a_{2k} & \cdots & a_{nk} \end{pmatrix} \qquad k \times n \text{ matrix}$$

$$(A + B)' = A' + B'$$

Inner Product:

for $a' = (a_1, \ldots, a_n)$ and $b' = (b_1, \ldots, b_n)$

$$a'b = a_1 b_1 + \ldots + a_n b_n = b'a$$

Matrix Multiplication:

$$\underbrace{C}_{n\times m} = \underbrace{A}_{n\times k}\underbrace{B}_{k\times m} \qquad \Rightarrow \qquad c_{ik} = \underbrace{a_{i.}}_{\nearrow}{}' \ \underbrace{b_{.k}}_{\nwarrow}$$

ith row of A kth column of B

Identity matrix for $n \in \mathbb{N}$:

$$I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} \qquad I_n A = A$$

Rules for Matrix Multiplication:

$$\begin{aligned} (AB)C &= A(BC) \\ A(B+C) &= AB+AC \\ (AB)' &= B'A' \end{aligned}$$

Example: n data points for $1 \times k$ vector $x_i = (x_{1i}, \ldots, x_{ki})$ (WO convention)

$$X = \begin{pmatrix} x_{11} & \cdots & x_{k1} \\ \ldots & & \ldots \\ x_{1n} & \cdots & x_{kn} \end{pmatrix} \qquad \text{n rows} \mathrel{\hat{=}} \text{observations}$$

Matrix product:

$$X'X = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \cdots & & \cdots \\ x_{k1} & \cdots & x_{kn} \end{pmatrix} \cdot \begin{pmatrix} x_{11} & \cdots & x_{k1} \\ \cdots & & \cdots \\ x_{1n} & \cdots & x_{kn} \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{i=1}^n {x_{1i}}^2 & \cdots & \sum_{i=1}^n x_{1i}x_{ki} \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^n x_{ki}x_{1i} & \cdots & \sum_{i=1}^n {x_{ki}}^2 \end{pmatrix}$$

$$= \sum_{i=1}^n \begin{pmatrix} x_{1i} \\ \vdots \\ x_{ki} \end{pmatrix} (x_{1i}, \ldots, x_{ki}) = \sum_{i=1}^n x_i' x_i \quad \leftarrow \text{summation notation}$$

Let $j_n = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ be a $n \times 1$ vector of ones, then $j_n j_n' = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{pmatrix}$,

and $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ $n \times 1$ vector, then

$$\frac{1}{n} j_n j_n' x = \frac{1}{n} \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \frac{1}{n} \begin{pmatrix} \sum x_i \\ \vdots \\ \sum x_i \end{pmatrix} = \begin{pmatrix} \bar{x} \\ \vdots \\ \bar{x} \end{pmatrix} = j_n \bar{x}$$

where $\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$ sample average.

Deviations from sample average

$$x - j_n\bar{x} = \begin{pmatrix} x_1 - \bar{x} \\ \vdots \\ x_n - \bar{x} \end{pmatrix} = x - \frac{1}{n} j_n j_n' x = \Bigg(\underbrace{I_n}_{\text{identity matrix}} - \frac{1}{n} j_n j_n' \Bigg) x = M^0 x$$

where $M^0 = I - \frac{1}{n} j_n j_n'$ is the matrix generating deviations from the mean (example of a projection matrix)

with

$$M^0 j_n = \left(I_n - \frac{1}{n} j_n j_n' \right) j_n = j_n - \frac{1}{n} j_n j_n' j_n = j_n - j_n = 0$$

since $\frac{1}{n} j_n' j_n = \frac{1}{n} n = 1$.

M^0 is an example of a so called idempotent matrix, i.e. a square matrix M with $M^2 = M\,M = M$.

When M is symmetric, it follows that $M'M = M$.

Verify:

$$\begin{aligned} M^0 M^0 &= \left(I - \frac{1}{n} j_n j_n'\right)\left(I - \frac{1}{n} j_n j_n'\right) \\ &= I - \frac{1}{n} j_n j_n' - \frac{1}{n} j_n j_n' + \frac{1}{n^2} j_n \underbrace{j_n' j_n}_{n} j_n' \\ &= I - \frac{1}{n} j_n j_n' = M^0 \end{aligned}$$

Sum of squared deviations:

$$\sum_{i=1}^{n} (x_i - \bar{x})^2 = (M^0x)'(M^0x) = x'M^{0\prime}M^0x = x'M^0x = \sum_{i=1}^{n} x_i\,(x_i - \bar{x})$$

Product of deviations of x_i and y_i:

$$\begin{aligned}\sum_{i=1}^{n} (x_i - \bar{x})\,(y_i - \bar{y}) &= (M^0x)'(M^0y) = x'M^{0\prime}M^0y \\ &= x'M^0y \\ &= \sum x_i(y_i - \bar{y}) \\ &= \sum (x_i - \bar{x})\,y_i\end{aligned}$$

Empirical Variance-Covariance-Matrix of x, y

$$\begin{aligned}
\text{Cov}\left[(x,y)\right] &= \begin{pmatrix} \frac{1}{n}\sum(x_i-\bar{x})^2 & \frac{1}{n}\sum(x_i-\bar{x})(y_i-\bar{y}) \\ \frac{1}{n}\sum(x_i-\bar{x})(y_i-\bar{y}) & \frac{1}{n}\sum(y_i-\bar{y})^2 \end{pmatrix} \\
&= \frac{1}{n}\begin{pmatrix} x'M^0x & x'M^0y \\ y'M^0x & y'M^0y \end{pmatrix} \\
&= \frac{1}{n}\begin{pmatrix} x'M^0 \\ y'M^0 \end{pmatrix}\begin{pmatrix} M^0x & M^0y \end{pmatrix} \\
&= \frac{1}{n}\begin{pmatrix} x' \\ y' \end{pmatrix} M^0 \begin{pmatrix} x & y \end{pmatrix}
\end{aligned}$$

Rank of a matrix A

$=$ maximum number of linearly independent columns

$=$ dimension of vector space spanned by column vectors

$=$ maximum number of linearly independent rows

$=$ dimension of vector space spanned by row vectors

A: $n \times k$ matrix $\rightarrow$ $\text{rank}(A) \leq \min(n, k)$

Properties:

i) rank $(AB) \leq$ min (rank (A), rank(B))

ii) rank (A) $=$ rank $(A'A) =$ rank (AA')

- Square $k \times k$ matrix A has full rank if $rank(A) = k$.
- $n \times k$ matrix A with $n \geq k$ has full column rank if $rank(A) = k$.
- $n \times k$ matrix A with $n \leq k$ has full row rank if $rank(A) = n$.

Inverse of a square matrix:

Let A be a $k \times k$ matrix

Inverse A^{-1} defined by $AA^{-1} = I$ or equivalently $A^{-1}A = I$

A^{-1} exists, i.e. A is invertible (or nonsingular) $\Leftrightarrow$ A has full rank.

Example: Diagonal matrix

$$A := \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a_k \end{pmatrix} = \text{diag}(a_1, \ldots, a_k) \Rightarrow \quad A^{-1} = \begin{pmatrix} \frac{1}{a_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{a_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \frac{1}{a_k} \end{pmatrix}$$

Inverse A^{-1} exists if all $a_j \neq 0$ for $j = 1, \ldots, k$.

Properties:

i) $(A^{-1})^{-1} = A$

ii) $(A^{-1})' = (A')^{-1}$

iii) If A is symmetric, then A^{-1} is symmetric

iv) $(AB)^{-1} = B^{-1}A^{-1}$

v) $A = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} \Leftrightarrow A^{-1} = \begin{pmatrix} {A_{11}}^{-1} & 0 \\ 0 & {A_{22}}^{-1} \end{pmatrix}$ block diagonal

vi) Nonsingular matrix $B \;\rightarrow\; rank(AB) = rank(A)$

Eigenvalues (Characteristic Roots) and Eigenvectors:

Eigenvalues λ (scalars) and nonzero eigenvectors c are the solution of $Ac = \lambda c$ for square $k \times k$ matrix A.

$$Ac = \lambda c \Leftrightarrow (A - \lambda I_n)c = 0$$

We are looking for the nontrivial solutions $c \neq 0$ which can be found by solving the characteristic equation involving the determinant

$$\det(A - \lambda I_n) = |A - \lambda I_n| = 0$$

for λ and then finding some $c \neq 0$ for which $Ac = \lambda c$ (note c is not unique!)

Properties:

i) A has full rank (A^{-1} exists) is equivalent to all eigenvalues are nonzero ($\lambda \neq 0$)

ii) If A^{-1} exists, then its eigenvalues are the inverses of the eigenvalues of A

iii) Diagonal matrix

$$A = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a_k \end{pmatrix}$$

Eigenvalues $\lambda_1 = a_1, \ldots, \lambda_k = a_k$

Eigenvectors $\begin{pmatrix} 1 \\ 0 \\ 0 \\ \ldots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \ldots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \\ \ldots \\ 0 \\ 1 \end{pmatrix}$

iv) $det(A) = |A| = \prod_{j=1}^{k} \lambda_j$

Definition:

- A is called positive definite, if all eigenvalues are strictly positive ($\lambda_j > 0$)
- A is called positive semidefinite, if all eigenvalues are nonnegative ($\lambda_j \geq 0$)
- A is called negative definite, if all eigenvalues are strictly negative ($\lambda_j < 0$)
- A is called negative semidefinite, if all eigenvalues are nonpositive ($\lambda_j \leq 0$)

Spectral decomposition of a symmetric matrix:

A $k \times k$ symmetric matrix A has k distinct orthogonal eigenvectors $c_1, c_2, \dots, c_k$ and k not necessarily distinct, real eigenvalues $\lambda_1, \dots, \lambda_k$.

We have $Ac_j = \lambda_j c_j$ which is summarized in $AC = C\Lambda$ where $C = [c_1 \cdots c_k]$ eigenvectors as columns

and $\Lambda = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_k \end{pmatrix}$ diagonal matrix with eigenvalues.

Orthogonality of eigenvectors: $c_i' c_j = 0$ for $i \neq j$ and normalization $c_i' c_i = 1$

$$CC' = C'C = I_n \quad \text{and} \quad C' = C^{-1}$$

This implies:

Diagonalization $\qquad C'AC = C'C\Lambda = \Lambda$

Spectral Decomposition $\qquad A = CC'ACC' = C\Lambda C' = \sum_{j=1}^{k} \lambda_j \, c_j c_j'$

The Generalized Inverse of a Matrix

- Case when A is not invertible because A is not a square matrix or A is not singular!

Definition: A generalized inverse of A is another matrix A^+ that satisfies

1. $AA^+A = A$
2. $A^+AA^+ = A^+$
3. A^+A is symmetric
4. AA^+ is symmetric

Note:

- A unique matrix that satisfies 1.–4. is called the Moore-Penrose inverse
- If A^{-1} exists, then $A^+ = A^{-1}$

Two cases: **Case A** (no square matrix $k < n$) and **Case B** (symmetric square matrix)

Case A: Let A be an $n \times k$ matrix with $k < n$ and $rank(A) = r \leq k$

1.) $r = k \Leftrightarrow A$ does have full column rank $\Leftrightarrow (A'A)^{-1}$ exists

Moore-Penrose inverse is

$$A^+ = (A'A)^{-1}A'$$

Verify 1.–4.:

1. $AA^+A = A(A'A)^{-1}A'A = A$
2. $A^+AA^+ = (A'A)^{-1}A'AA^+ = A^+$
3. $A^+A = (A'A)^{-1}A'A = I$ symmetric
4. $(A(A'A)^{-1}A')' = A''(A'A)^{-1}A' = A(A'A)^{-1}A'$ symmetric

2.) $rank(A) = r < k$

Use r nonzero characteristic roots of $A'A$ and associated eigenvectors in matrix C_1, then

$$A'A = C_1\Lambda_1^{-1}C_1' \quad \text{spectral decompose}$$

The Moore-Penrose inverse is

$$A^+ = C_1\Lambda_1^{-1}C_1'A'$$

where $r \times r$ diagonal matrix $\Lambda_1 = diag(\lambda_1, \ldots, \lambda)$ of nonzero eigenvalues.

Case B: If A is symmetric ($n = k$), then

$$A^+ = C_1 \Lambda_1^{-1} C_1'$$

where Λ_1 is a diagonal matrix containing the nonzero eigenvalues of A and C_1 the associated orthonormalized eigenvectors.

Quadratic Form: $x'Ax$

- A positive definite $\iff x'Ax > 0$ for all $x \neq 0$
- A positive semidefinite $\iff x'Ax \geq 0$ for all $x \neq 0$
- A negative definite $\iff x'Ax < 0$ for all $x \neq 0$
- A negative semidefinite $\iff x'Ax \leq 0$ for all $x \neq 0$

Example:

x, y random variables with variance-covariance matrix

$$V = \begin{pmatrix} \text{Var}(x) & \text{Cov}(x,y) \\ \text{Cov}(x,y) & \text{Var}(y) \end{pmatrix}$$

- V is always positive semidefinite.
- If x and y are not perfectly correlated, then V is positive definite.
- If x, y are jointly normally distributed $\begin{pmatrix} x \\ y \end{pmatrix} \sim \text{N}\left[\begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix}, V \right]$ then quadratic form $\begin{pmatrix} x & y \end{pmatrix} V^{-1} \begin{pmatrix} x \\ y \end{pmatrix} \sim \chi^2_2$-distributed, if V has full rank.
- V^{-1}: multivariate standardization.
- Since V is positive definite also V^{-1} is positive definite and therefore $\begin{pmatrix} x & y \end{pmatrix} V^{-1} \begin{pmatrix} x \\ y \end{pmatrix} > 0$ unless $\begin{pmatrix} x \\ y \end{pmatrix} = 0$.

Trace of a matrix:

Square $k \times k$ matrix A

$$\text{tr}(A) = \sum_{j=1}^{k} a_{jj} \qquad \text{sum of diagonal elements}$$

Properties:

i) $\text{tr}(cA) = c \cdot \text{tr}(A)$ for scalar c

ii) $\text{tr}(A') = \text{tr}(A)$

iii) $\text{tr}(A + B) = \text{tr}(A) + \text{tr}(B)$

iv) $\text{tr}(AB) = \text{tr}(BA)$

v) $\text{tr}(A) = \sum_{j=1}^{k} \lambda_j$ trace of matrix equals the sum of its eigenvalues

Kronecker Product:

For $n \times k$ matrix A, $l \times m$ matrix B

$$\underbrace{A \otimes B}_{(nl)\times(km)} \text{ matrix} = \begin{bmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nk} \end{bmatrix} \otimes B$$

$$= \underbrace{\left.\begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1k}B \\ a_{21}B & a_{22}B & \cdots & a_{2k}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}B & a_{n2}B & \cdots & a_{nk}B \end{bmatrix}\right\} n \cdot l \quad \text{rows}}_{k \cdot m \quad \text{columns}}$$

Properties:

i) $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$

ii) $(A \otimes B)' = A' \otimes B'$

iii) $\text{tr}(A \otimes B) = \text{tr}(A) \cdot \text{tr}(B)$

iv) $(A \otimes B)(C \otimes D) = AC \otimes BD$ if AC, BD is possible

Calculus and Matrix Algebra:

First and second order Taylor series approximation

- y scalar
- $x = (x_1, \ldots, x_n)'$ $\quad n \times 1$ vector
- $y = f(x)$ twice differentiable

Gradient:

$$\nabla_x y := \underbrace{\frac{\partial y}{\partial x}}_{n\times 1 \text{ vector}} = \frac{\partial f(x)}{\partial x} = \begin{pmatrix} \frac{\partial y}{\partial x_1} \\ \vdots \\ \frac{\partial y}{\partial x_n} \end{pmatrix} = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$$

column vector as convention

Hessian:

$$H = \frac{\partial^2 y}{\partial x \partial x'} = \begin{bmatrix} \frac{\partial^2 y}{\partial x_1^2} & \frac{\partial^2 y}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 y}{\partial x_1 \partial x_n} \\ \frac{\partial^2 y}{\partial x_2 \partial x_1} & \frac{\partial^2 y}{\partial x_2^2} & \cdots & \frac{\partial^2 y}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 y}{\partial x_n \partial x_1} & \frac{\partial^2 y}{\partial x_2 \partial x_n} & \cdots & \frac{\partial^2 y}{\partial x_n^2} \end{bmatrix} = [f_{ij}]$$

First order Taylor series approximation in $x = (x_{10}, \ldots, x_{n0})$

$$y = f(x) \approx f(x_0) + \sum_{i=1}^{n} f_i(x_0)(x_i - x_{i0}) = f(x_0) + \left(\left. \frac{\partial y}{\partial x} \right|_{x_0} \right)' (x - x_0)$$

Second order approximation

$$\begin{aligned} y = f(x) &\approx f(x_0) + \sum_{i=1}^{n} f_i(x_0)(x_i - x_{i0}) + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} f_{ij}(x_0) \cdot (x_i - x_{i0}) \cdot (x_j - x_{j0}) \\ &= f(x_0) + \underbrace{\left(\left. \frac{\partial y}{\partial x} \right|_{x_0} \right)' (x - x_0)}_{\text{inner product}} + \frac{1}{2} \underbrace{(x - x_0)' H(x_0)(x - x_0)}_{\text{quadratic form}} \end{aligned}$$

Differentiation of inner products and quadratic forms:

i) $y = a'x = \sum_{i=1}^{n} a_i x_i = x'a$

$$\frac{\partial y}{\partial x} = \frac{\partial a'x}{\partial x} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = a$$

ii) $z = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = Ax = \begin{pmatrix} \sum_{i=1}^{k} a_{1i}\, x_i \\ \vdots \\ \sum_{i=1}^{k} a_{ni}\, x_i \end{pmatrix}$

A $n \times k$ matrix, x $k \times 1$ vector, z $n \times 1$ vector

$$\frac{\partial z}{\partial x} = \left(\frac{\partial z_1}{\partial x}, \ldots, \frac{\partial z_n}{\partial x} \right) = A' \quad \leftarrow \text{columnwise gradients of } z_1, \ldots, z_n$$

iii) $y = x'Ax = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j a_{ij}$ quadratic form

a) $\frac{\partial y}{\partial x} = (A + A')x$
If A is symmetric ($A = A'$), then $\frac{\partial y}{\partial x} = 2Ax$

b) $\frac{\partial y}{\partial A} = xx' = \begin{pmatrix} x_1^2 & \cdots & x_1 x_n \\ \vdots & \ddots & \vdots \\ x_1 x_n & \cdots & x_n^2 \end{pmatrix}$ outer product, $n \times n$ matrix

Expected values and variances:

Let

- a be a $k \times 1$ vector of constants
- A a $n \times k$ matrix of constants, and
- x a $k \times 1$ vector of random variables

then

$$E\, a'x = a'(E\, x) = \sum_{i=1}^{k} a_i\, Ex_i$$

$$E\, Ax = A(E\, x) = \begin{bmatrix} \sum_{i=1}^{k} a_{1i}\, Ex_i \\ \dots \\ \sum_{i=1}^{k} a_{1i}\, Ex_i \end{bmatrix}$$

$$Var(a'x) = a'\, Var(x) a = \sum_{i=1}^{k} \sum_{j=1}^{k} a_i a_j Cov(x_i, x_j) \geq 0 \quad \leftarrow \text{ quadratic form}$$

$Var(x)$ must be positive semidefinite

$$Var(Ax) = A\, Var(x)\, A'$$

0.2 Statistics and Probability Theory

Reference: WO 2+3, Greene App. B-D

Random Variable (RV) x taking values x_i

Probability distribution: $f(x_i) = Prob(x = x_i)$ for discrete RV

i) $0 \leq Prob(x = x_i) \leq 1$

ii) $\sum_{x_i} f(x_i) = 1$

Continuous RV: Density $f(x_i) \geq 0$

i) $Prob(a \leq x \leq b) = \int\limits_a^b f(t)dt$

ii) $\int\limits_{-\infty}^{\infty} f(t)dt = 1$

Cumulative Distribution Function CDF

$$Prob(x \leq x_i) = F(x_i) = \begin{cases} \Sigma_{t \leq x_i} f(t) & : \text{discrete} \\ \int_{-\infty}^{x_i} f(t)dt & : \text{continuous} \end{cases}$$

For continuous case: $f(x_i) = \frac{dF(x_i)}{dx_i}$

Expected value (Mean):

$$\mu \equiv Ex = \begin{cases} \sum_{x_i} x_i f(x_i) & : \text{discrete} \\ \int_{-\infty}^{\infty} tf(t)dt & : \text{continuous} \end{cases}$$

Variance:

$$\sigma^2 \equiv Var(x) = E[(x - \mu)]^2$$

$$\sigma^2 = \begin{cases} \sum_{x_i} (x_i - \mu)^2 f(x_i) & : \text{discrete} \\ \int_{-\infty}^{\infty} (t - \mu)^2 f(t)dt & : \text{continuous} \end{cases}$$

Standard deviation:

$$\sigma = \sqrt{\sigma^2} = \sqrt{Var(x)}$$

Chebychev's Inequality:

$$Prob(|x - \mu| \geq \delta) \leq \frac{\sigma^2}{\delta^2}$$

$$Eg(x) = \begin{cases} \sum_{x_i} g(x_i)f(x_i) & : \quad \text{discrete} \\ \int_{-\infty}^{\infty} g(t)f(t)dt & : \quad \text{continuous} \end{cases}$$

In general: $Eg(x) \neq g(E(x))$

Jensen's inequality:

$$Eg(x) \leq g(E(x)) \quad \text{for} \quad \underbrace{g''(x) < 0}_{concave}$$

$$Eg(x) \geq g(E(x)) \quad \text{for} \quad \underbrace{g''(x) > 0}_{convex}$$

E.g. $\quad E\log(x) \leq \log(E(x))$

Normal distribution

$$x \sim N(\mu, \sigma^2) \quad \text{with density} \quad f(x_i) = \frac{1}{\sqrt{2\pi}\sigma} \quad e^{-\frac{(x_i-\mu)^2}{2\sigma^2}}$$

$$Ex = \mu \quad \text{and} \quad Var(x) = \sigma^2$$

Standard Normal $z \sim N(0, 1)$

$$\text{Define density :} \qquad \phi(z_i) = \frac{1}{\sqrt{2\pi}} \quad e^{-\frac{z_i^2}{2}}$$

$$F(z_i) = \Phi(z_i) = \int_{-\infty}^{z_i} \phi(t)dt \quad = \int_{-\infty}^{z_i} \frac{1}{\sqrt{2\pi}} \quad e^{-\frac{t^2}{2}} dt$$

$$F_x(x_i) = Prob(x \leq x_i) = Prob\left(\frac{x-\mu}{\sigma} \leq \frac{x_i-\mu}{\sigma}\right)$$

$$= Prob\left(z \leq \frac{x_i-\mu}{\sigma}\right) = \Phi\left(\frac{x_i-\mu}{\sigma}\right)$$

Skewness: $S \equiv E[(x-\mu)^3] = 0$ for normal distribution

Kurtosis: $E[(x-\mu)^4] = 3\sigma^4$ for normal distribution

Excess Kurtosis (relative to normal):

$$\frac{E[(x-\mu)^4]}{\sigma^4} - 3 \quad = \quad 0 \quad \text{for normal distribution}$$

Chi-squared– (χ^2) , t– and F–distributions

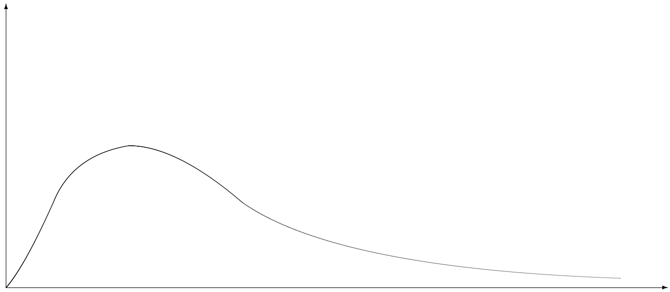
χ^2–distribution: $z_1, ..., z_n$ independent $N(0, 1)$

$$y = \sum_{j=1}^{n} z_j^2 \quad \sim \quad \chi_n^2\text{–distributed with } n \text{ degrees of freedom}$$

F- Distribution:

- $y_1 \sim \chi^2_{n_1}$, $y_2 \sim \chi^2_{n_2}$
- y_1 and y_2 independent

$F(n_1, n_2) = \frac{y_1/n_1}{y_2/n_2}$ $\sim$ F–distributed with n_1 degrees of freedom in numerator and n_2 degrees of freedom in denominator

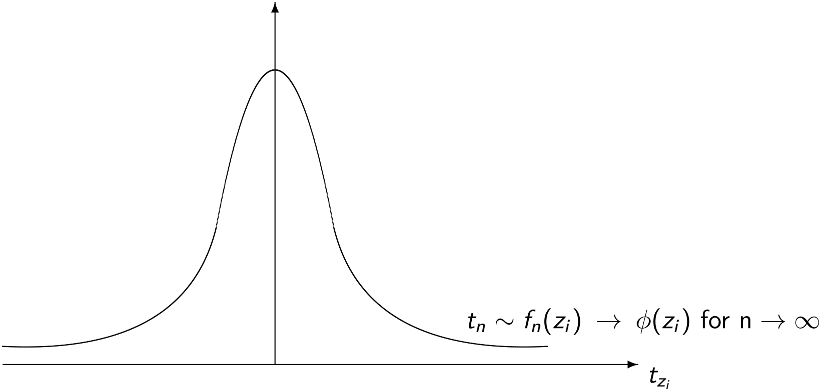


stylized shape of probability density function of χ^2_n or $F(n_1, n_2)$

t–distribution:

$$t = \frac{z}{\sqrt{\frac{y}{n}}} \quad \sim \quad t_n \quad \text{distributed (t-distribution with n degrees of freedom)}$$

$$z \sim N(0,1)\,,\; y \sim \chi^2_n\,,\; \text{and } y, z \text{ independent}$$



Note: $t^2 \sim F(1, n)$

Joint distribution: x, y RV

$$Prob(a \leq x \leq b, c \leq y \leq d) = \begin{cases} \sum_{a \leq x_i \leq b} \sum_{c \leq y_j \leq d} \quad f(x_i, y_j) & : \quad \text{discrete} \\ \int_a^b \int_c^d f(t,s) \quad ds \quad dt & : \quad \text{continuous} \end{cases}$$

Probability density function: $f(t,s) \geq 0$

$$\sum_{x_i} \sum_{y_j} f(x_i, y_j) = 1 \quad \text{discrete}$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t,s)\, ds\, dt = 1 \qquad \text{continuous}$$

Distribution function:

$$F(x_i, y_j) = Prob(x \leq x_i, y \leq y_j) = \begin{cases} \sum_{x \leq x_i} \sum_{y \leq y_j} f(x_i, y_i) & : \quad \text{discrete} \\ \int_{-\infty}^{x_i} \int_{-\infty}^{y_j} f(t,s) ds \quad dt & : \quad \text{continuous} \end{cases}$$

Expected value of function of (x, y):

$$E\ g(x,y) = \begin{cases} \sum\sum g(x_i, y_j) f(x_i, y_j) & : \quad \text{discrete} \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(t,s) f(t,s)\, ds\, dt & : \quad \text{continuous} \end{cases}$$

Covariance between x and y:

$$\sigma_{xy} \equiv Cov(x,y) = E[(x - Ex)(y - Ey)] = E\,xy - (Ex)(Ey)$$

x, y independent :

$$f(x_i, y_i) \quad = \quad f(x_i) f(y_i) \quad \substack{\Rightarrow \\ \not\Leftarrow} \quad Cov(x,y) = 0$$

Correlation:

$$r_{xy} \;=\; \frac{Cov(x,y)}{\sqrt{Var(x) \cdot Var(y)}} = \frac{\sigma_{xy}}{\sigma_x \sigma_y}$$

Rules:

a, b, c, d = constants

$$
\begin{aligned}
E(ax + by + c) &= a\,Ex + b\,Ey + c \\
Var(ax + by + c) &= a^2 Var(x) + b^2 Var(y) + 2abCov(x, y) \\
Cov(ax + by, cx + dy) &= acVar(x) + bdVar(y) + (ad + bc)Cov(x, y)
\end{aligned}
$$

Conditional distribution:

$$f(y = y_j|x = x_i) \equiv f(y_j|x_i) = \frac{f(x_i, y_j)}{f(x_i)}$$

Conditional expectation:

$$E(y|x = x_i) = \int_{-\infty}^{\infty} s \underbrace{f(y = s|x_i)}_{\equiv f(s|x_i)} ds$$

Conditional variance:

$$\begin{aligned} Var(y|x = x_i) &= E[(y - E(y|x = x_i))^2|x = x_i] \\ &= \int_{-\infty}^{\infty} (s - E(y|x = x_i))^2 \, f(s|x_i) ds \end{aligned}$$

0.3 Asymptotics

Motivation:

For many econometric problems, the analytical properties of the estimator can only be determined asymptotically.

Probability Limit and Consistency of an Estimator

Definition 1:

The **probability limit** θ of a sequence of random variables $\hat{\theta}_N$ results as the limit for N going to infinity such that the probability that the absolute difference between $\hat{\theta}_N$ and θ is less than some small positive ε goes to one. Mathematically this is expressed by

$$\underset{N\to\infty}{lim} \quad P\{|\hat{\theta}_N - \theta| < \varepsilon\} \;=\; 1 \qquad \text{for every} \quad \varepsilon > 0$$

and abbreviated by $\underset{N\to\infty}{plim} \;\; \hat{\theta}_N \;=\; \theta$ (or $\hat{\theta}_N \overset{P}{\to} \theta$).

Definition 2:

An estimator $\hat{\theta}_N$ for the true parameter value θ is (weakly) **consistent**, if

$$\underset{N\to\infty}{plim} \;\; \hat{\theta}_N \;=\; \theta \; .$$

Remarks:

1. The sample mean $\bar{Y}_N$ of a sequence of random variables Y_i with expected value $E(Y_i) = \mu_Y$ is under very general conditions a consistent estimator of μ_Y, d.h. $plim\, \bar{Y}_N = \mu_Y$.

2. For two sequences of random variables $\hat{\theta}_{1,N}$ and $\hat{\theta}_{2,N}$ it follows:

$$plim\,(\hat{\theta}_{1,N} + \hat{\theta}_{2,N}) \;=\; plim\,\hat{\theta}_{1,N} \;+\; plim\,\hat{\theta}_{2,N}$$

$$plim\,(\hat{\theta}_{1,N} \cdot \hat{\theta}_{2,N}) \;=\; plim\,\hat{\theta}_{1,N} \;\cdot\; plim\,\hat{\theta}_{2,N}$$

$$plim\,\left(\frac{\hat{\theta}_{1,N}}{\hat{\theta}_{2,N}}\right) \;=\; \frac{plim\,\hat{\theta}_{1,N}}{plim\,\hat{\theta}_{2,N}}$$

Slutzky's Theorem:

$plim\, g\left(\hat{\theta}_N\right) \;=\; g\left(plim\,\hat{\theta}_N\right)$ at continuity points of $g(.)$

Convergence and Asymptotic Orders of Magnitude

Motivation:

For many semiparametric problems it is important to determine the speed of convergence, i.e. the asymptotic order of magnitude.

Definition 1 (Fixed Sequences):

The sequence $\{X_N\}$ of real numbers is said to be at most of order N^k and is denoted by

$$X_N = O(N^k) \quad \text{if} \quad \lim_{N\to\infty} \frac{X_N}{N^k} = c$$

for some constant c .

Definition 2 (Fixed Sequences):

The sequence $\{X_N\}$ of real numbers is said to be of smaller order than N^k and is denoted by

$$X_N = o(N^k) \quad \text{if} \quad \lim_{N\to\infty} \frac{X_N}{N^k} = 0 \quad .$$

Definition 3 (Stochastic Sequences):

The sequence of random variables $\{X_N\}$ is said to be at most of order N^k and is denoted by

$$X_N \; = \; O_p(N^k)$$

if for every $\varepsilon > 0$ there exist numbers C and $\tilde{N}$ such that

$$P\left\{ \frac{|X_N|}{N^k} > C \right\} < \varepsilon \qquad \text{for all} \quad N > \tilde{N}.$$

Definition 4 (Stochastic Sequences):

The sequence of random variables $\{X_N\}$ is said to be of smaller order than N^k and is denoted by

$$X_N \; = \; o_p(N^k) \quad \text{if} \quad \underset{N \to \infty}{plim} \; \frac{X_N}{N^k} \; = \; 0 \quad .$$

Chebychev's Law of Large Numbers:

Let the random variables $\{X_i\}$ be uncorrelated with $EX_i = \mu_i$ and $Var(X_i) = \sigma_i^2 < \infty$ in a sample of size N $(i = 1, \ldots, N)$. Then

$$\bar{X}_N - \bar{\mu}_N \ \xrightarrow{P}\ 0$$

if $\bar{\sigma}^2 \to 0$, as N goes to infinity where $\bar{X}_N = \frac{1}{N}\sum_{i=1}^N X_i$ denotes the sample mean, $\bar{\mu}_N = \frac{1}{N}\sum_{i=1}^N \mu_i$ and $\bar{\sigma}^2 = \frac{1}{N^2}\sum_{i=1}^N \sigma_i^2 = \frac{1}{N}\left(\frac{1}{N}\sum_{i=1}^N \sigma_i^2\right)$.

Alternative Representation:

Under the above assumptions it follows that $\left(\bar{X}_N - \bar{\mu}_N\right) = o_p(1)$

Special Case: If $\quad \mu_i = \mu \quad$ then $\quad plim \bar{X}_N = \mu$.

Lindberg–Levy's Central Limit Theorem:

Let $\{X_i\}$ be a sequence of i.i.d. random variables such that $EX_i = \mu$ and $Var(X_i) = \sigma^2 < \infty$ in a sample of size N $(i = 1, \ldots, N)$. Then

$$\sqrt{N}\,\frac{(\bar{X}_N - \mu)}{\sigma} \ \xrightarrow{d}\ \mathcal{N}(0,1) \qquad \text{(i.e. } \bar{X}_N \text{ is } \sqrt{N} - \text{consistent).}$$

Implication:

Under the above assumptions it follows that $\left(\bar{X}_N - \mu\right) = O_p(N^{-1/2})$.

Liapounov's Central Limit Theorem:

Let $\{X_{N,i}\}$ be a sequence of independently distributed random variables with $EX_{N,i} = \mu_{N,i}$ and $Var(X_{N,i}) = \sigma^2_{N,i} < \infty$ in a sample of size N $(i = 1, \ldots, N)$.

Let $E|X_{N,i}|^{2+\delta} < \infty$ for some $\delta > 0$. If $\lim_{N\to\infty} \sum_{i=1}^{N} \frac{E|X_{N,i}-\mu_{N,i}|^{2+\delta}}{\tilde{\sigma}_N^{2+\delta}} = 0$, then $\frac{\sum_{i=1}^{N}(X_{N,i}-\mu_{N,i})}{\tilde{\sigma}_N} \xrightarrow{d} \mathcal{N}(0,1)$ for $\tilde{\sigma}^2_N = \sum_{i=1}^{N} \sigma^2_{N,i}$.

Implication:

Under the above assumptions it follows that $\frac{\sum_{i=1}^{N}(X_{N,i}-\mu_{N,i})}{\tilde{\sigma}_N} = O_p(1)$

1. Review: Linear Regression Model for Cross-Sectional Data

Section Outline

1.1 Preliminaries: Conditional Expectations, Causal Analysis, Linear Projections

- y explained/dependent/response variable
- $x = (x_1, ..., x_k)$ explanatory / independent variables, regressors, control variables, covariates (x is observed)

Structural conditional expectation (CE): $E(y|w, c)$

Based on random sample of (y, w, c) we can estimate the effect of w on y holding c constant.

Complications arise when there is no random sample of (y, w, c)

- $\rightarrow$ measurement error
- $\rightarrow$ simultaneous determination of y, w, c
- $\rightarrow$ some variables we would like to control for (elements of c) cannot be observed

$\Rightarrow$ CE of interest involves data for which the econometrician cannot collect data or requires an experiment that cannot be carried out.

Identification assumptions:

$\rightarrow$ Can recover structural CE of interest

Definition CE:

y (random variable) explained variable, $x \equiv (x_1, x_2, ..., x_k)$ $(1 \times k)$-vector of explanatory variables, $E(|y|) < \infty$

then function $\mu : \mathbb{R}^k \rightarrow \mathbb{R}$

(CE) $E(y|x_1, x_2, ..., x_k) = \mu(x_1, x_2, ..., x_k)$ or $E(y|x) = \mu(x)$

Distinguish

$E(y|x)$: random variable because x is a random variable

from

$E(y|x = x_0)$: conditional expectation when x takes specific value x_0

$\rightarrow$ Distinction most of the time not important
$\rightarrow$ Use $E(y|x)$ as short hand notation

Parametric model for $E(y|x)$ where $\mu(x)$ depends on a finite set of unknown parameters

Examples:

(i) $E(y|x_1, x_2) = \beta_0 + \beta_1 x_1 + \beta_2 x_2$

(ii) $E(y|x_1, x_2) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_2^2 + \beta_4 x_1 x_2$

(iii) $E(y|x_1, x_2) = \exp[\beta_0 + \beta_1 \log(x_1) + \beta_2 x_2] \quad \text{with } \; y \geq 0, \; x_1 > 0$

(i) is linear in parameters and explanatory variables

(ii) is linear in parameters and nonlinear in explanatory variables

(iii) is nonlinear in both

Partial Effect:

- Continuous x_i, and differentiable μ

$$\Delta E(y|x) = \frac{\partial \mu}{\partial x_j} \Delta x_j \quad \text{holding } \; x_1, ..., x_{j-1}, x_{j+1}, ..., x_k \text{ fixed}$$

$\widehat{=}$ ceteris paribus effect for propertly specified population model

- Discrete $x_j : x_{j,0} \to x_{j,1}$

$$\Delta E(y|x) = E(y|x_1, ..., x_{j-1}, x_{j,1}, x_{j+1}, ..., x_k) - E(y|x_1, ..., x_{j-1}, x_{j,0}, x_{j+1}, ..., x_k)$$

Examples:

ad i) $\frac{\partial E(y|x)}{\partial x_1} = \beta_1 = \text{constant}$

ad ii) $\frac{\partial E(y|x)}{\partial x_1} = \beta_1 + \beta_4 x_2$, i.e. partial effect of x_1 varies with x_2

ad iii) $\frac{\partial E(y|x)}{\partial x_1} = exp[\beta_0 + \beta_1 \log(x_1) + \beta_2 x_2] \frac{\beta_1}{x_1}$ $\rightarrow$ highly nonlinear

(Partial) Elasticity (only continuous case)

$$\frac{\partial E(y|x)}{\partial x_j} \cdot \frac{x_j}{E(y|x)} = \frac{\partial \log E(y|x)}{\partial \log x_j}$$

(Partial) Semielasticity:

$$\frac{\partial E(y|x)}{\partial x_j} \cdot \frac{1}{E(y|x)} = \frac{\partial \log E(y|x)}{\partial x_j}$$

Average Partial Effect (APE, 'integrate out distribution of x'):

$$E_x \{\Delta E(y|x)\} = E_x \left\{ \frac{\partial \mu}{\partial x_i} \Delta x_j \right\}$$

Examples:

ad i) APE$= \beta_1$

ad ii) APE$= \beta_1 + \beta_4 E x_2$

ad iii) APE$= E \left\{ exp[\beta_0 + \beta_1 \log(x_1) + \beta_2 x_2] \frac{\beta_1}{x_1} \right\}$

APE's in cases ii and iii can be estimated by sample averages of the expressions evaluated at the sample estimates of the coefficients $\hat{\beta}$

Error form of models of conditional expectations

We can always write

(1) $y = E(y|x) + u \quad$ where $u = y - E(y|x)$

and it follows by definition:

(2) $E(u|x) = 0$

Implications:

(i) $E(u) = 0$

(ii) u is uncorrelated with any function of $x_1, ..., x_k$

Implication (i) and (ii) follows from the law of iterated expectations

$$\underline{\text{LIE}} : E(y|x) = E[E(y|w)|x] \quad \text{if } x = f(w)$$

i.e. $\{\text{Information set incorporated in } x\} \subseteq \{\text{Information set incorporated in } w\}$

i) $E(y|x) = E[E(y|w)|x]$
$\rightarrow$integrating out w wrt x: $\int yf(y|x)dy = \int[\int yf(y|w,x)dy]f(w|x)dw$

ii) $E(y|x) = E[E(y|x)|w]$
Knowing w implies knowing x
$\rightarrow$ Routinely used in the course

'The smaller information set always dominates'

Therefore

$$E(u) = E_x[E(u|x)] = E_x 0 = 0$$

which gives implication (i) and

$$E(u|f(x)) = E[E(u|x)|f(x)] = E[0|f(x)] = 0$$

which gives implication (ii).

Example:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + u$$

with

$$E(u|x_1, x_2) = 0$$

implies:

$E(u) = 0, Cov(x_1, u) = 0, Cov(x_2, u) = 0$ and u is also uncorrelated with $x_1^2, x_2^2, x_1 x_2, \exp(x_1)$ *etc.*

i.e. the functional form of $E(y|x)$ is properly specified.

We have $\beta_2 = \frac{\partial E(y|x_1,x_2)}{\partial x_2}$ because $E(u|x_1, x_2) = 0$, i.e. u is uncorrelated with any function of x_2. Thus β_2 describes the mean impact of x_2 on y.

$E(u|x_1, x_2) = 0$ sometimes called mean independence

We have:

$$\text{Independence} \underset{\not\Leftarrow}{\Rightarrow} \text{Mean Independence} \underset{\not\Leftarrow}{\Rightarrow} \text{Uncorrelatedness}$$

Mean independence defines a Conditional Expectation

Uncorrelatedness defines a Linear Projection

Different nested sets of conditioning variables

Important special case: $w = (x, z)$

$$\underbrace{E(y|x)}_{\mu_1(x)} = E[\underbrace{E(y|x,z)}_{\mu_2(x,z)}|x]$$

$$\underbrace{\mu_1}_{\text{observed}} = E[\mu_2(x, \underbrace{z}_{\text{unobserved}})|x]$$

Identification problem: Can we link the estimable $\mu_1(x)$ to the structural $\mu_2(x, z)$ which is the causal relationship of interest?

$$\underbrace{x, z}_{\text{more information}} \quad \text{versus} \quad \underbrace{x}_{\text{less information}}$$

$$\begin{aligned} \mu_1(x, z) &= E(y|x, z) \\ \mu_2(x) &= E(y|x) \end{aligned}$$

By LIE, we have ('integrating z out')

$$\mu_2(x) = E(y|x) = E[E(y|x, z)|x] = E[\mu_1(x, z)|x]$$

$\rightarrow$ allows to study effects of omitted regressors/unobserved components z on the relationship between y and x.

Example: Wage Equation

$$
\begin{aligned}
&E(wage|educ, exper) \\
&= \beta_0 + \beta_1 educ + \beta_2 exper + \beta_3 exper^2 + \beta_4 educ \cdot exper \\
&= E(wage|educ, exper, exper^2, educ \cdot exper)
\end{aligned}
$$

by LIE, i.e. it is redundant to condition on $exper^2$ and $educ \cdot exper$.

Conditional Variance

The conditional variance of y given x is defined as

$$\begin{aligned} Var(y|x) = E(u^2|x) &\equiv \sigma^2(x) \equiv E[(y - E(y|x))^2|x] \\ &= E(y^2|x) - [E(y|x)]^2 \end{aligned}$$

Note: $\sigma^2(x)$ is a random variable when x is viewed as a random vector.

Properties:

$$Var(a(x)y + b(x)|x) = [a(x)]^2 Var(y|x)$$

Decomposition of variance (corresponds to LIE)

$$\begin{aligned} Var(y) &= E[Var(y|x)] + Var(E(y|x)) \\ &= \underbrace{E[\sigma^2(x)]}_{\text{average conditional variance}} + \underbrace{Var(\mu(x))}_{\text{variance of condtional expectation}} \end{aligned}$$

$$\text{where } \mu(x) = E(y|x).$$

Extension (further conditioning variable z)

$$Var(y|x) = E[Var(y|x,z)|x] + Var[E(y|x,z)|x]$$

Consequently:

$$E[Var(y|x)] \geq E[Var(y|x,z)]$$

$\rightarrow$ further conditioning variables z reduce the average conditional variances.

Linear Projections

Even though a structural CE (conditional expectation) $E(y|x)$ is typically not a linear function of x, it is possible to use the linear projection of y on the random variables $(x_1, ..., x_k) =: x$

$$\underbrace{L(y|1, x_1, ..., x_k)}_{\text{(including an intercept)}} = L(y|1, x) = \beta_0 + \beta_1 x_1 + ... + \beta_k x_k$$

$$= \beta_0 + x\beta$$

where

$$\beta := [Var(x)]^{-1} Cov(x, y)$$

$$\beta_0 = E(y) - E(x)\beta = E(y) - \beta_1 E(x_1) - ... - \beta_k E(x_k)$$

Variance–Covariance matrix is the $(k \times k)$-matrix

$$Var(x) = \begin{pmatrix} Var(x_1) & \dots & Cov(x_k, x_1) \\ Cov(x_2, x_1) & \ddots & \\ \vdots & & \\ Cov(x_k, x_1) & \dots & Var(x_k) \end{pmatrix} = E[(x - E(x))(x - E(x))']$$

Note:

$$x - E(x) = \begin{pmatrix} x_1 - E(x_1) \\ \vdots \\ x_k - E(x_k) \end{pmatrix}$$

and

$$(x - E(x))' = (x_1 - E(x_1), ..., x_k - E(x_k))$$

$$Cov(x, y) = \begin{pmatrix} Cov(x_1, y) \\ \vdots \\ Cov(x_k, y) \end{pmatrix} \qquad (k \times 1)\text{-vector}$$

Linear projection with a zero intercept

$$\begin{aligned} L(y|x) &= L(y|x_1, ..., x_k) = \gamma_1 x_1 + ... + \gamma_k x_k = x\gamma \\ \text{where} \quad & \gamma := [E(x'x)]^{-1} E(x'y) \end{aligned}$$

The linear projection can be derived as the linear predictor minimizing the mean square prediction error ($\equiv$ Best linear predictor or least squares linear predictor), i.e.

$$\min_{b_0, b \in \mathbb{R}^k} E[(y - b_0 - xb)^2]$$

yields β and β_0 as defined.

Using the linear projection

$$L(y|x) = \beta_0 + \beta_1 x_1 + ... + \beta_k x_k$$

define the error term u by

$$u := y - L(y|x)$$

or

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + ... + \beta_k x_k + u$$

By definition of linear projections:

$$Eu = 0 \text{ and } Cov(x_j, u) = 0 \quad (j = 1, \ldots, k)$$

Note: This does not imply independence between x and u or mean independence $E(u|x) = 0$

Primary use of linear projections: Obtaining estimable equations involving the parameters of an underlying conditional expectation of interest. Appendix of WO Chapter 2 contains more results on conditional expectations etc. which will be useful later.

1.2 Derivation of the OLS Estimator and its Asymptotic Properties

Population equation of interest:

$$y = x\beta + u$$

where: x is a $1 \times K$ vector
$\beta = (\beta_1, \ldots, \beta_K)'$ is a $K \times 1$ vector
$x_1 \equiv 1$: with intercept

Sample of size N: $\{(x_i, y_i) : i = 1, \ldots, N\}$
i.i.d. random variables where x_i is $1 \times K$ and y_i is a scalar.

For each observation

$$y_i = x_i\beta + u_i$$

Consistency

Assumption OLS.1: $E(x'u) = 0$

Assumption OLS.2: $rank(Ex'x) = K$
$\rightarrow$ expected outer product matrix has full rank, i.e.

$$Ex'x = \begin{pmatrix} 1 & Ex_2 & \dots & Ex_K \\ Ex_2 & Ex_2^2 & \dots & Ex_2x_K \\ \vdots & \vdots & \ddots & \vdots \\ Ex_K & Ex_Kx_2 & \dots & Ex_K^2 \end{pmatrix} \quad \text{is invertible}$$

Under OLS.1 and OLS.2, the parameter vector β is identified, which is equivalent to saying that β can be written in terms of population moments (and of course be solved for!)

To see this:

$$
\begin{aligned}
y &= x\beta + u \\
x'y &= x'x\beta + x'u \\
Ex'y &= Ex'x\beta + \underbrace{Ex'u}_{=0} \quad \text{by OLS.1} \\
\beta &= (Ex'x)^{-1}Ex'y \quad \text{by OLS.2}
\end{aligned}
$$

Because (x, y) is observed $\rightarrow \beta$ is identified.

Analogy principle:

Choose an estimator by turning the population relationship (based on the probability distribution for the data generating process) into its sample counterpart (based on the empirical distribution for the sample).

Here, the analogy principle implies the method-of-moments: Replace the population moments $E(x'y)$ and $E(x'x)$ (expected values) by their corresponding sample moments (averages).

$$E(x'y) \rightarrow \frac{1}{N} \sum_{i=1}^{N} x_i' y_i$$

$$E(x'x) \rightarrow \frac{1}{N} \sum_{i=1}^{N} x_i' x_i$$

$$\widehat{\beta} = \left(\frac{1}{N}\sum_{i=1}^{N} x_i' x_i\right)^{-1} \left(\frac{1}{N}\sum_{i=1}^{N} x_i' y_i\right) \qquad \text{with } y_i = x_i\beta + u_i$$

$$= \left(\frac{1}{N}\sum_{i=1}^{N} x_i' x_i\right)^{-1} \left(\frac{1}{N}\sum_{i=1}^{N} x_i' (x_i\beta + u_i)\right)$$

$$= \left(\frac{1}{N}\sum_{i=1}^{N} x_i' x_i\right)^{-1} \left(\frac{1}{N}\sum_{i=1}^{N} x_i' x_i\right)\beta + \left(\frac{1}{N}\sum_{i=1}^{N} x_i' x_i\right)^{-1} \left(\frac{1}{N}\sum_{i=1}^{N} x_i' u_i\right)$$

$$= \beta + \left(\frac{1}{N}\sum_{i=1}^{N} x_i' x_i\right)^{-1} \left(\frac{1}{N}\sum_{i=1}^{N} x_i' u_i\right)$$

OLS Estimator in Matrix Notation

$$\widehat{\beta} = (X'X)^{-1}X'Y$$

where $X = \begin{pmatrix} X_1 \\ \vdots \\ X_N \end{pmatrix} = \begin{pmatrix} 1 & x_{21} & \dots & x_{K1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{2N} & \dots & x_{KN} \end{pmatrix}$ and $Y = \begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix}$

Under OLS.2: $X'X$ is nonsingular with probability approaching one

<u>and</u> $plim \left[\left(\frac{1}{N} \sum_{i=1}^{N} x_i' x_i \right)^{-1} \right] = A^{-1}$ where $A = E(x'x)$

(Corollary 3.1 in WO Chapter 3)

Under OLS.1: $plim \left(\frac{1}{N} \sum_{i=1}^{N} x_i' u_i \right) = E(x'u) = 0$

By **Slutzky's theorem** (WO Lemma 3.4): plim $\widehat{\beta} = \beta + A^{-1} \cdot 0 = \beta$

<u>WO Theorem 4.1:</u>

Under assumptions OLS.1 and OLS.2, the OLS estimator $\widehat{\beta}$ obtained from a random sample following the population model (5) is consistent for β.

$\rightarrow$ Simplicity should not undermine usefulness.
$\rightarrow$ Whenever estimable equation is of the form then consistency follows.

Under the assumption of theorem 4.1, $x\beta$ is the linear projection of y on x.

$\rightarrow$ OLS estimates linear projection consistently (also in cases such as y being a binary variable) ...and conditional expectations that are linear in parameters.

If either OLS.1 or OLS.2 fail, β is not identified
$\rightarrow$ typically because x and u are correlated.

OLS estimator not necessarily unbiased under OLS.1 and OLS.2 (Jensen's Inequality)

$$E\left[\left(\frac{1}{N}\sum_{i=1}^{N} x_i' x_i\right)^{-1}\left(\frac{1}{N}\sum_{i=1}^{N} x_i' u_i\right)\right]$$

$$\neq E\left[\frac{1}{N}\sum_{i=1}^{N} x_i' x_i\right]^{-1} \underbrace{E\left[\frac{1}{N}\sum_{i=1}^{N} x_i' u_i\right]}_{=0}$$

$\rightarrow E(u|x) = 0 \quad$ implies $\quad E\hat{\beta} = \beta$ (unbiasedness) because of LIE.

We do not need to assume independence
$\rightarrow Var(u|x)$ unrestricted.

Aside: Standard derivation of the OLS estimator $\hat{\beta}$ in matrix notation

Minimizing $\sum_{i=1}^{N} u_i^2 = U'U$ sum of squared residuals

$U' = (u_1, \ldots, u_N)$
$U = Y - X\beta$

$$\underset{\{\beta\}}{min} \; U'U = (Y - X\beta)'(Y - X\beta) = Y'Y - \beta'X'Y - Y'X\beta + \beta'X'X\beta$$

F.O.C.: $\frac{\partial U'U}{\partial \beta} = -X'Y - X'Y + 2X'X\widehat{\beta} = 0$

$$\Leftrightarrow \underbrace{X'X\widehat{\beta} = X'Y}_{\text{normal equations}} \Rightarrow \widehat{\beta} = (X'X)^{-1}X'Y$$

$$\Leftrightarrow X'(Y - X\widehat{\beta}) = X'\widehat{U} = 0$$

$$\Leftrightarrow \frac{1}{N}\sum_{i=1}^{N} \begin{pmatrix} x_{1i} \\ \vdots \\ x_{Ki} \end{pmatrix} \widehat{u}_i = 0$$

Covariance between x_i and u_i is set to zero to calculate the OLS estimator $\widehat{\beta}$. $\widehat{=}$ another way to interpret $\widehat{\beta}$ as a method-of-moment estimator ($\rightarrow$ analogy principle).

Asymptotic distribution of the OLS estimator

Rewrite

$$\widehat{\beta} = \beta + \left(\frac{1}{N} \sum_{i=1}^{N} x_i' x_i \right)^{-1} \left(\frac{1}{N} \sum_{i=1}^{N} x_i' u_i \right)$$

as

$$\sqrt{N}(\widehat{\beta} - \beta) = \left(\frac{1}{N} \sum_{i=1}^{N} x_i' x_i \right)^{-1} \left(\frac{1}{\sqrt{N}} \sum_{i=1}^{N} x_i' u_i \right)$$

We know $\left[\left(\frac{1}{N} \sum_{i=1}^{N} x_i' x_i \right)^{-1} - A^{-1} \right] = O_p(1)$

Also $\{(x_i' u_i) : i = 1, 2 \ldots\}$ is i.i.d. sequence with $E x_i' u_i = 0$ and we assume each element has a finite variance. Then the central limit theorem implies:

$$\frac{1}{\sqrt{N}} \sum_{i=1}^{N} x_i' u_i \xrightarrow{d} N(0, B)$$

where B is a $K \times K$ matrix: $B \equiv E(u^2 x' x)$

Recall: $x'x$ is the outer product of the $K \times 1$ row vector x

This implies

$$\sqrt{N}(\widehat{\beta} - \beta) = A^{-1} \left(\frac{1}{\sqrt{N}} \sum_{i=1}^{N} x_i' u_i \right) + o_p(1)$$

Under <u>Heteroscedasticity:</u> $\sqrt{N}(\widehat{\beta} - \beta) \overset{a}{\sim} N(0, A^{-1} B A^{-1})$

Under Heteroscedasticity: $\sqrt{N}(\widehat{\beta} - \beta) \overset{a}{\sim} N(0, A^{-1}BA^{-1})$

Under Homoskedasticity:

Assumption OLS.3: $E(u^2x'x) = \sigma^2 Ex'x$

where $\sigma^2 = Eu^2 = Var(u)$

WO Theorem 4.2 (Asymptotic Normality of OLS):

Under Assumptions OLS.1 - OLS.3: $\sqrt{N}(\widehat{\beta} - \beta) \overset{a}{\sim} N(0, \sigma^2 A^{-1})$

Proof: Use $B = \sigma^2 A$ q.e.d.

Practical usage:

Treat $\widehat{\beta}$ as approximately jointly normal with expected value β and Variance-Covariance-Matrix (VCOV) $V = \frac{\sigma^2}{N}[Ex'x]^{-1}$.

V is estimated by

$$\widehat{Avar(\widehat{\beta})} = \frac{\widehat{\sigma}^2}{N}\left[\frac{1}{N}\sum_{i=1}^{N} x_i' x_i\right]^{-1} = \widehat{\sigma}^2 \left(X'X\right)^{-1}$$

and

$$\widehat{\sigma}^2 \equiv s^2 = \frac{1}{N-K}\sum_{i=1}^{N} \widehat{u}_i^2$$

Heteroscedasticity

Failure of assumption OLS.3: $E(u^2x'x) = \sigma^2 E(x'x)$ has nothing to do with consistency of OLS estimator $\widehat{\beta}$ (WO theorem 4.1) and the proof of asymptotic normality is still valid but the final asymptotic variance is different.

Two options:

Option i): Weighted Least Squares to obtain a more efficient estimator

Specify a model for $Var(y|x)$ and 'estimate' this model (e.g. by regressing $\hat{u}_i^2$ on a flexible function of x_i or other covariates). This model povides an estimate (prediction) of $Var(u_i|x_i) = Var(y_i|x_i)$.

Then, use Weighted Least Squares (WLS) as follows:

Divide y_i and every element of x_i (including unity for the intercept) by $\sqrt{Var(y_i|x_i)}$ and apply OLS to the weighted data

$$\underbrace{\frac{y_i}{\sqrt{Var(y_i|x_i)}}}_{\tilde{y}_i} = \underbrace{\frac{1}{\sqrt{Var(y_i|x_i)}} x_i}_{\tilde{x}_i} \beta + \underbrace{\frac{u_i}{\sqrt{Var(y_i|x_i)}}}_{\tilde{u}_i}$$

$$Var(\tilde{u}_i|x_i) = \frac{Var(u_i|x_i)}{Var(y_i|x_i)} \equiv 1$$

Transformed Model:

$\tilde{y}_i = \tilde{x}_i\beta + \tilde{u}_i$ satisfies OLS.1-OLS.3 (homoskedastic)
$\Rightarrow$ Special case of Generalized Least Squares which we will cover later
$\Rightarrow$ leads to a different estimator of β which hinges on a correct specification of $Var(y_i|x_i)$
$\Rightarrow$ Efficiency gain possible with correct specification of $Var(y_i|x_i)$

Option ii): Heteroscedasticity robust inference

Often we want to stick to the consistent estimator $\widehat{\beta}$
$\rightarrow$ because no correct specification of $Var(y_i|x_i)$ available
$\rightarrow$ WLS generally inconsistent for linear projections (e.g. when OLS.1 holds but $E(u|x) \neq 0$)

Appropriate asymptotic variance

Without OLS.3 the asymptotic variance of $\widehat{\beta}$ is $Avar(\widehat{\beta}) = \frac{1}{N}A^{-1}BA^{-1}$

A^{-1} is consistently estimated by $\left(\frac{1}{N}\sum_{i=1}^{N} x_i'x_i\right)^{-1} = \widehat{A}^{-1}$

B is consistently estimated by $\left(\frac{1}{N}\sum_{i=1}^{N} u_i^2 x_i'x_i\right)$

We replace the unobserved error terms u_i by the estimated residuals $\widehat{u}_i = y_i - x_i\widehat{\beta}$

$$\widehat{B} = \frac{1}{N}\sum_{i=1}^{N} \widehat{u}_i^2 x_i'x_i \xrightarrow{p} B$$

Heteroscedasticity-robust variance estimator

$$\widehat{Avar}(\widehat{\beta}) = \frac{1}{N}\widehat{A}^{-1}\widehat{B}\widehat{A}^{-1} = (X'X)^{-1}\left(\sum_{i=1}^{N} \widehat{u}_i^2 x_i' x_i\right)(X'X)^{-1}$$

often called White standard errors, White-Eicker standard error, or Huber standard errors.

Typically with degrees-of-freedom adjustment to improve finite sample properties.

$$\widehat{Avar}(\widehat{\beta}) = \frac{1}{N-K}\widehat{A}^{-1}\widehat{B}\widehat{A}^{-1} = (X'X)^{-1}\left(\frac{N}{N-K}\sum_{i=1}^{N} \widehat{u}_i^2 x_i' x_i\right)(X'X)^{-1}$$

t-statistics, χ^2-statistics (but not F-statistics based on comparison of sums of squared residuals in restricted and unrestricted model!) can be used in the usual way.