Yield Curve Modeling and Forecasting using Semiparametric Factor Dynamics

Wolfgang Karl Härdle
Piotr Majer

Ladislaus von Bortkiewicz Chair of Statistics
C.A.S.E. – Center for Applied Statistics and Economics
Humboldt–Universität zu Berlin
http://lvb.wiwi.hu-berlin.de
http://www.case.hu-berlin.de

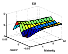
Yield Curve

Figure 1: Yield curves of Greece, Portugal, Spain, Italy; 199901-201011

Objectives

- Modeling the interest rate term structure using Dynamic Factor Model
  - Estimating and predicting factors and factor loadings
  - Understanding the dynamics of factor loadings
  - Impact of macroeconomy variables
  - Panel data analysis

- Forecasting the term structure
  - Forecasting evaluation against (dynamic) Nelson-Siegel Model



Statistical Challenges

- Time-varying high-dimensional object
- Dimension reduction: extraction of common factors
- Factors estimated non-parametrically
- Capturing dynamics by parametric time series models for factor loadings

Smooth in space and parametric in time

Economic Implications

- Understanding macroeconomic activity
- Conducting monetary policy
- Pricing and hedging interest rate products
- Portfolio allocation

Outline

1. Motivation ✓
2. Data
3. Dynamic Semiparametric Factor Model (DSFM)
4. Nelson-Siegel Model
5. Yield Curve Modeling and Forecasting
6. Conclusion

Data

- Zero-coupon bonds returns (observed monthly) - $Y_t \in \mathbb{R}^{11}$:
  - Italy (IT), Greece (GR), Portugal (PT), Spain (ES)
- Period covered: 199901-201203
- Dependence on Maturity - $X_{t,1} \in \mathbb{R}$
- Further explanatory variables - $X_{t,2} \in \mathbb{R}$
  - inflation rate (INF), manufacturing capacity utilization (CU)
  - unemployment rate (EMP), industrial production (IP), ΔGDP

Summary statistics

	Mean	Median	SD	Skewness	Kurtosis
1-year	2.8736	2.7843	1.1719	0.0523	2.1771
3-year	3.5394	3.4493	0.9836	0.4591	3.0203
5-year	3.9361	3.8228	0.8649	0.6655	3.5370
10-year	4.6039	4.4694	0.6801	0.5987	3.4534

Table 1: Statistical summary of the Italian 1, 3, 5, 10-year zero-coupon bond yields. Sample period 199901-201203; SD denotes Standard Deviation.

Notation

$$\underbrace{(X_{1,1}, Y_{1,1}), \ldots, (X_{J_1,1}, Y_{J_1,1})}_{t=1}, \ldots, \underbrace{(X_{1,T}, Y_{1,T}), \ldots, (X_{J_T,T}, Y_{J_T,T})}_{t=T},$$

where:

$X_{j,t} \in \mathbb{R}^d,\ Y_{j,t} \in \mathbb{R}$

T - the number of observed time periods (days)

J_t - the number of the observations in (day) t

$\mathrm{E}(Y_t|X_t) = F_t(X_t)$.

Quantify $F_t(X_t)$

Dynamic Semiparametric Factor Model

- Park et al. (2009), Fengler et al. (2007)

$$\mathsf{E}(Y_t|X_t) = \sum_{l=0}^{L} Z_{t,l}\, m_l(X_t) = Z_t^\top m(X_t) = Z_t^\top A^* \Psi(X_t)$$

$$\begin{aligned} Z_t &= (\mathbf{1}, Z_{t,1}, \ldots, Z_{t,L})^\top \text{low dim (stationary) time series} \\ m &= (m_0, m_1, \ldots, m_L)^\top, \text{tuple of functions} \\ \Psi(X_t) &= \{\psi_1(X_t), \ldots, \psi_K(X_t)\}^\top, \psi_k(x) \text{ space basis} \\ A^* &: (L+1) \times K \text{ coefficient matrix} \end{aligned}$$

B-Splines

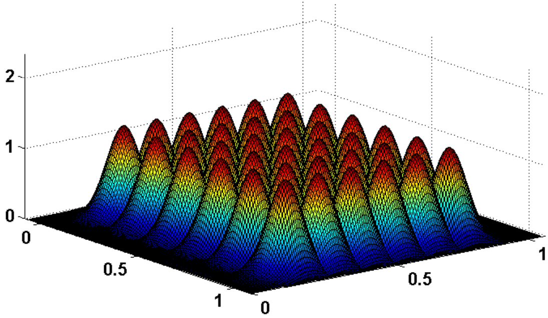


Figure 2: B-splines basis functions; order of B-splines: quadratic; number of knots: $6 \times 6 = 36$ ▸ B-Splines

Estimation

Recall

$$Y_{t,j} = \sum_{l=0}^{L} Z_{t,l} m_l(X_{t,j}) + \varepsilon_{t,j} = Z_t^\top A\psi(X_{t,j}) + \varepsilon_{t,j}$$

$\psi(x) = \{\psi_1(x), \ldots, \psi_K(x)\}^\top$ tensor B-spline basis

$$(\widehat{Z}_t, \widehat{A}) = \arg\min_{Z_t, A} \sum_{t=1}^{T} \sum_{j=1}^{J} \left\{ Y_{t,j} - Z_t^\top A\psi(X_{t,j}) \right\}^2$$

- Selection of L by explained variance

Implementation

- Explained variance

$$EV(L) = 1 - \frac{\sum_{t=1}^{T} \sum_{j=1}^{J} \left\{ Y_{t,j} - \sum_{l=0}^{L} Z_{t,l} m_l(X_{t,j}) \right\}^2}{\sum_{t=1}^{T} \sum_{j=1}^{J} \left\{ Y_{t,j} - \bar{Y} \right\}^2}$$

Statistical Inference

- Asymptotically negligible difference between $\widehat{Z}_t$ and Z_t
- Time series modeling of $\widehat{Z}_t$

Panel DSFM

$$Y^i_{t,j} = \sum_{l=0}^{L} Z^i_{t,l} \widetilde{m}_l(X_{t,j}) + \varepsilon^i_{t,j}, \quad 1 \leq j \leq J,\ 1 \leq t \leq T,$$

$1 \leq i \leq I$ - country index

- Domestics specific effects and time evolution captured by Z^i_t

$\widetilde{m}_l$ - common factors

$$(\widehat{Z}^1_t, \dots, \widehat{Z}^I_t, \widehat{A}) = \arg \min_{Z^1_t, \dots, Z^I_t, A} \sum_{i=1}^{I} \sum_{t=1}^{T} \sum_{j=1}^{J} \left\{ Y^i_{t,j} - Z^{i\top}_t A \psi(X_{t,j}) \right\}^2 \tag{1}$$

Dynamic Nelson-Siegel Model

$$
\begin{aligned}
Y_{t,j} =& L_t + S_t \left\{ \frac{1-\exp(-\lambda X_{t,j})}{\lambda X_{t,j}} \right\} + C_t \left\{ \frac{1-\exp(-\lambda X_{t,j})}{\lambda X_{t,j}} - \exp(-\lambda X_{t,j}) \right\} + \varepsilon_{t,j} \\
=& Z_t^\top m(X_{t,j}) + \varepsilon_{t,j},
\end{aligned}
$$

- $Z_t = (L_t, S_t, C_t)^\top$ loadings: L_t - level, S_t - slope, C_t - curvature
- $m(\cdot) = (\mathbf{1}, \frac{1-\exp(-\lambda(\cdot))}{\lambda(\cdot)}, \frac{1-\exp(-\lambda(\cdot))}{\lambda(\cdot)} - \exp(-\lambda(\cdot)))$ common factors

Dynamic Nelson-Siegel Model

- Z_t estimated day-by-day by OLS (for fixed λ)
- VAR(1) model for $Z_t \in \mathbb{R}^3$

$$Z_t = \mu + \mathcal{A} Z_{t-1} + \eta_t, \qquad (2)$$

- Ability to reproduce (historical) yield curve properties
  - mean reversion
  - basket of different shapes
- Parsimonious structure

Yield Curve Modeling

- DSFM approaches
  - yield curves modeled domestically (DSFM)
  - panel term structure model (PDSFM)
- Explanatory variables $X_{t,j}$
  - time to maturity
  - inflation Rate

Explained Variance

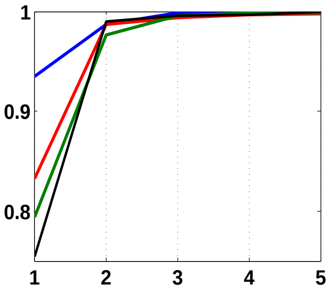


Figure 3: Explained variance for the first 5 factors for domestic DSFM: PT, GR, ES and IT (left panel) and PDSFM (right panel).

In-Sample Fit

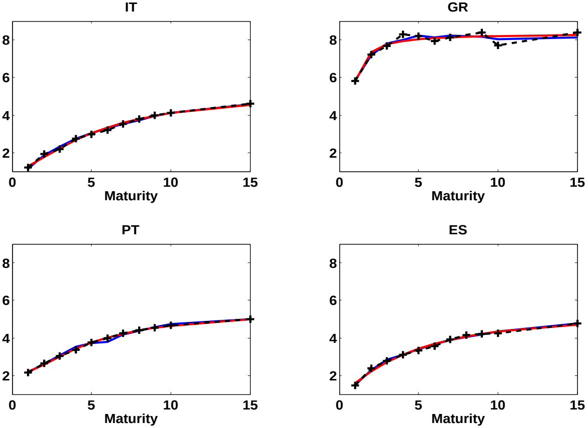


Figure 4: The term structure of interest rates (dotted black) observed on 20100331, DSFM (blue) and the Nelson-Siegel fitted data.

DSFM Factor $\widehat{m}_1$

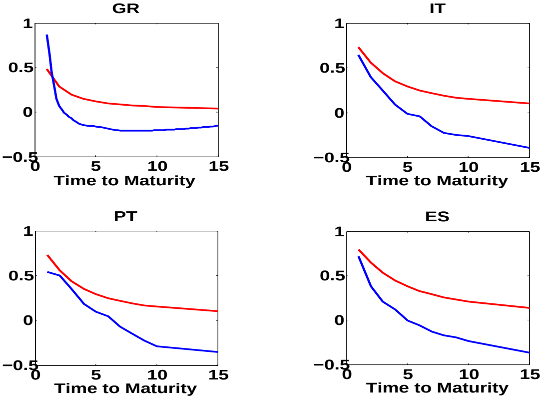


Figure 5: Estimated $\widehat{m}_1$ depending on time to maturity [Years] using domestic DSFM and Nelson-Sigel slope factor with $\lambda_{GR} = 0.049$, $\lambda_{IT} = 0.127$, $\lambda_{PT} = 0.109$ and $\lambda_{ES} = 0.174$, respectively.

DSFM Factor $\widehat{m}_2$

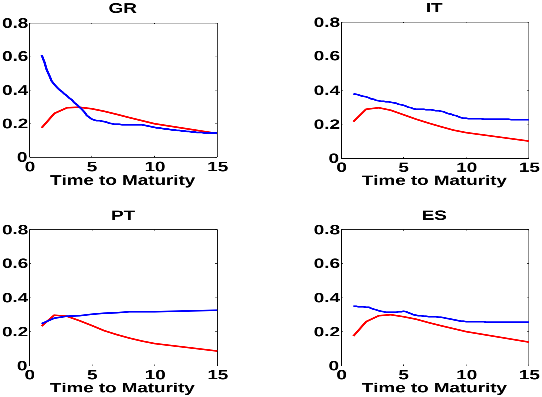


Figure 6: Estimated $\widehat{m}_2$ depending on time to maturity [Years] using domestic DSFM and Nelson-Sigel slope factor with $\lambda_{GR} = 0.049$, $\lambda_{IT} = 0.127$, $\lambda_{PT} = 0.109$ and $\lambda_{ES} = 0.174$, respectively.

Estimated Factor Loadings, $\widehat{Z}_t$

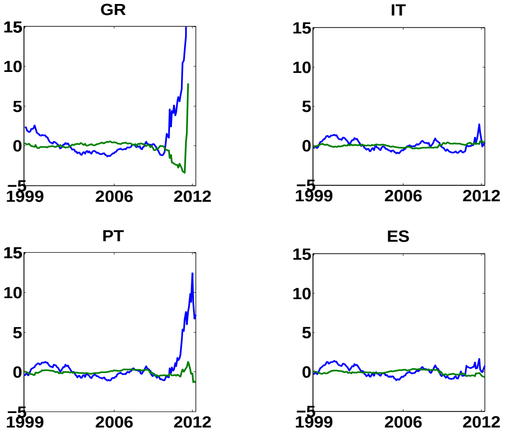


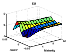
Figure 7: Estimated factor loadings $\widehat{Z}_t$ of the yield curve over whole sample using domestic DSFM; blue line corresponds to $\widehat{Z}_{t,1}$, green - $\widehat{Z}_{t,2}$.

$\widehat{Z}_t$ Properties

- Unit root hypothesis is not rejected at significance 5%
- Highly persistent processes
- Stationarity of $\Delta\widehat{Z}_t \equiv \widehat{Z}_t - \widehat{Z}_{t-1}$ not rejected
- $VAR(p)$ models for $\Delta\widehat{Z}_t \in \mathbb{R}^2$

$$\Delta\widehat{Z}_t = c + \alpha_1 \Delta\widehat{Z}_{t-1} + \ldots + \alpha_p \widehat{Z}_{t-p} + \varepsilon_t$$

- Model selection based on HQ and SC information criteria



Including Further Explanatory Variables

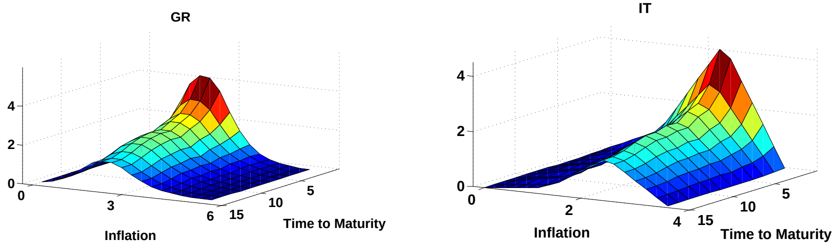


Figure 8: Estimated first factor depending on time to maturity and inflation rate for Greece and Italy

Including Further Explanatory Variables

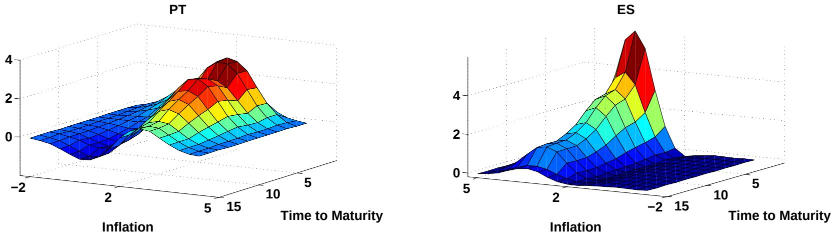


Figure 9: Estimated first factor depending on time to maturity and inflation rate for Portugal and Spain

Including Further Explanatory Variables

- Impact of the Inflation Rate
  - stronger on short rates
  - similar across countries
  - peak at the 2% rate - central bank target

Estimated PDSFM Factors

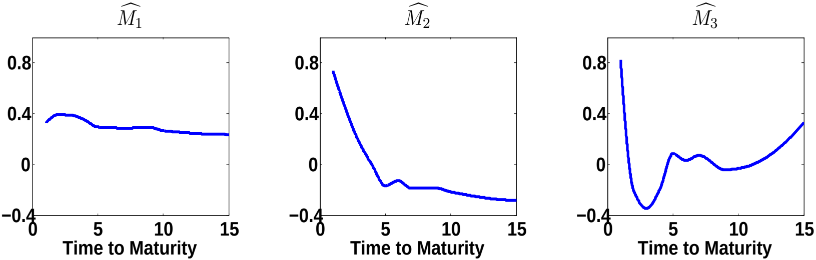


Figure 10: Estimated factors $\widehat{M}$ depending on time to maturity (years), using PDSFM with 3 factors

RMSE

	GR	IT	PT	ES
NS	0.5600	0.0685	0.2009	0.0636
DSFM	0.2886	0.0872	0.4195	0.0695
DSFM(INF)	0.6813	0.1550	0.5520	0.2110
PDSFM	0.6810	0.1474	0.4516	0.1434

Table 2: RMSE derived by Nelson-Siegel model (NS), domestic DSFM, DSFM with inflation rate and PDSFM (3 factors) in dependence on time to maturity.

Factors and macroeconomic fundamentals

- Contemporaneous correlation between factor loadings and macroeconomic variables:

$$\Delta \widehat{Z}_t = C + \beta_1 INF_t + \beta_2 CU_t + \beta_3 EMP_t + \beta_4 IP_t + \beta_5 \Delta GDP_t + \varepsilon_t \quad (3)$$

- $\widehat{Z}_{t,1}$ driven by INF, ΔGDP and IP at significance level 5%
- $\widehat{Z}_{t,2}$ can not be explained by (3)

Forecasting Setup

- Period: 200701-201203
- Rolling windows shifted over 1 month grids
- Forecasting horizon: 12 months
- Forecasting approaches
  - domestic DSFM ($L = 2$)
  - dynamic NS

Forecasting

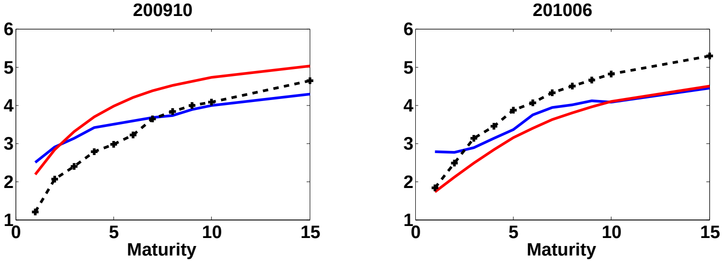


Figure 11: Term structure of interest rates (dotted black) observed on 20091030 (left) and 20100630 (right) for Italy with the DSFM (blue) and the dynamic Nelson-Siegel (red) forecasts.

RMSPE

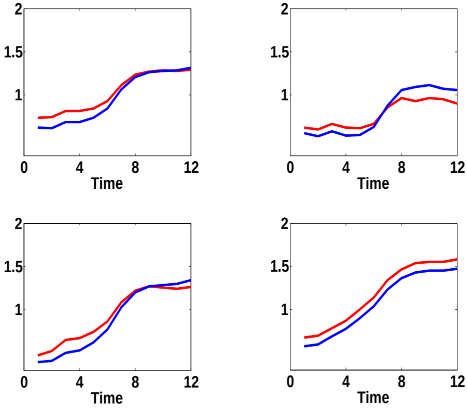


Figure 12: Root Mean Squared Prediction Error DSFM (2 factors) and NS for whole yield curve (top left), 2 year rate (top right), 7 year rate (bottom left) and 15 year rate (bottom right).

Conclusion

- Two factors are sufficient to model the term structure of interest rates domestically
- Panel term structure required 3 factors
- Estimated factor loadings are unit root, highly persistent processes
- Macroeconomic variables included
- Yield curves are modeled and forecasted successfully

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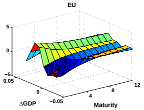

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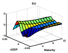
B-Splines

▸ B-Splines

Univariate **B-spline** basis $\Psi = \{\psi_1(X), \ldots, \psi_K(X)\}^\top$ is a series of $\psi_k(X)$ functions defined by $x_0 \leq x_2 \leq \ldots \leq x_{K-1}$, K knots and order p, i.e. for $p = 2$ (quadratic)

$$\psi_j(x) = \begin{cases} \frac{1}{2}(x - x_j)^2 & \text{if } x_j \leq x < x_{j+1} \\ \frac{1}{2} - (x - x_{j+1})^2 + (x - x_{j+1}) & \text{if } x_{j+1} \leq x < x_{j+2} \\ \frac{1}{2}\left\{1 - (x - x_{j+2})^2\right\} & \text{if } x_j \leq x < x_{j+1} \\ x & \text{otherwise} \end{cases}$$



B-Splines

▸ B-Splines

- Knots K and order p has to be specified in advance (EV criterion); K corresponds to bandwidth
- In higher dimensions, for $dim(X) = d > 1$

$$\Psi = \{\psi_1(X_1), \ldots, \psi_{K_1}(X_1)\} \times \ldots \times \{\psi_1(X_d), \ldots, \psi_{K_d}(X_d)\}$$

- Flexible and computationally efficient approach to capture various spatial structures