Adaptive Interest Rate Modelling

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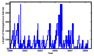
Interest Rate

- Essential for pricing derivatives and hedging corresponding risk.
- A signal of macroeconomic activity.
- Influenced by macroeconomic variables.
- Follow unstable dynamic process.



Classical One-factor Short Rate Models

Vasicek Model

$$dr(t) = a\{b - r(t)\}dt + \sigma dW_t$$

CIR Model

$$dr(t) = a\{b - r(t)\}dt + \sigma \sqrt{r(t)} dW_t$$

- $r_t \geq 0$
- Mean reversion

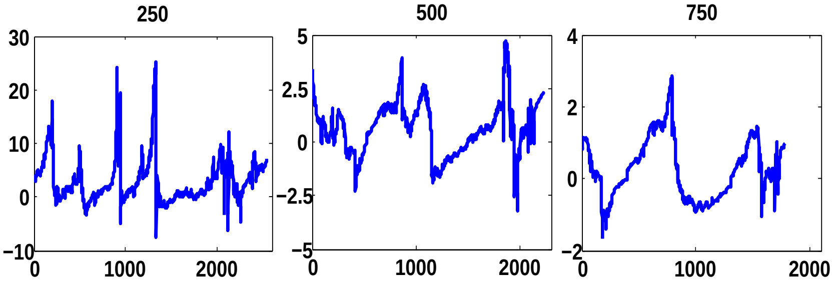


Figure 1: Moving window estimator $\hat{a}$ with window sizes 250, 500 and 750

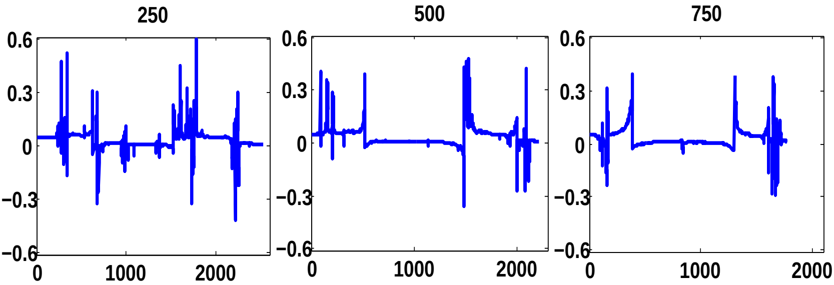


Figure 2: Moving window estimator $\hat{b}$ with window sizes 250, 500 and 750

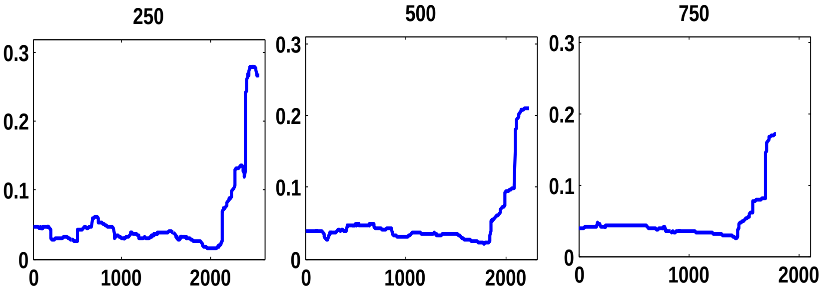
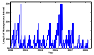


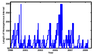
Figure 3: Moving window estimator $\hat{\sigma}$ with window sizes 250, 500 and 750



Extended Short Rate Models

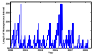
Three strands of extended interest rate models:

- Jump-Diffusion Models
- Regime-Switching Models
- Time-varying Coefficients Models.



Outline

1. Motivation ✓
2. Local Adaptive Approach for CIR model
3. Simulations
4. Empirical Study
5. Conclusion



Model

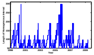
The time-varying CIR model with $\theta_t = (a_t, b_t, \sigma_t)^\top$:

$$dr(t) = a_t\{b_t - r(t)\}dt + \sigma_t \sqrt{r(t)}dW_t \tag{1}$$

Discretization:

$$r_{t_{i+1}} - r_{t_i} = a_t\{b_t - r_{t_i}\}\Delta t + \sigma_t \sqrt{r_{t_i}} Z_i \tag{2}$$

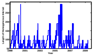
$\{Z_i\} \sim N(0, \Delta t)$



Local Parametric Approach (LPA)

Given time point t, go back and split time series into K intervals, $I_0 \subset I_1 \subset \cdots \subset I_K$, with $I_k = [t - m_k + 1, t]$.

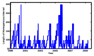
- Accept the smallest interval I_0 without change point (i.e. homogeneous interval).
- Sequentially check the historical intervals to search a change point.
- Two methods of LPA: Local model selection(LMS) and Local change point (LPC).



Algorithm of LPC

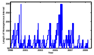
Goal: Find an unknown change point τ in the interval I_k.

1. Determine $\hat{\theta} = \tilde{\theta}_0$.
2. Increase interval to I_k, $k \geq 1$. Get $\tilde{\theta}_{I_k}$.
3. Compare test statistics with critical value. If test statistic is accepted go to step 4, otherwise go to step 5.
4. Let $\hat{\theta} = \tilde{\theta}_{I_k}$, and set $k = k + 1$, repeat step 2.
5. Detect the change point τ in I_k, $I_{\hat{k}} = I_{k-1}$ without change point.



Why we use LPA?

- Find the longest stable interval for each t.
- Allow for structural breaks and jumps in parameter values.
- Distinguish blooming and declining regimes.



Test Statistic

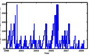
Test statistics $T_{I_{k+1},\tau}$:

$$T_{I_{k+1},\tau} = L_J(\tilde{\theta}_J) + L_{J^c}(\tilde{\theta}_{J^c}) - L_{I_{k+1}}(\tilde{\theta}_{I_{k+1}}) \tag{3}$$

where $J = [\tau + 1, t]$, and $J^c = [t - m_{k+1}, \tau]$, and $\tau \in J_k = I_k \backslash I_{k-1}$.

Consider the supremum of the test statistics over interval J_k:

$$T_k = sup_{\tau \in J_k} T_{I_{k+1},\tau} \tag{4}$$



Test Algorithm

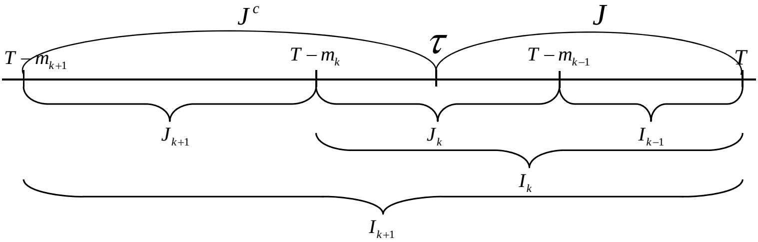
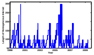


Figure 4: Construction of the Test Statistics in the Local Change Point Test

- The criteria for testing homogeneous intervals:

$$T_k \leq \mathfrak{z}_k, \qquad \text{for} \quad k \leq \hat{k} \tag{5}$$

  and $T_{k+1} > \mathfrak{z}_{\hat{k}+1}$.
- $I_{\hat{k}}$ is the longest time homogeneous interval for time point t,
- $\hat{\theta}_t = \tilde{\theta}_{I_{\hat{k}}}$.
- $\mathfrak{z}_k$ is the critical value, obtained by Monte-Carlo simulations.



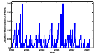
Risk Bound

Parametric risk bound $\mathfrak{R}_r(\theta_0)$, given the true value θ_0, for any interval I_k ,

$$\mathsf{E}_{\theta_0} \left| L_{I_k}(\tilde{\theta}_{I_k}, \theta_0) \right|^r \leq \mathfrak{R}_r(\theta_0) \tag{6}$$

where

$L_{I_k}(\tilde{\theta}_{I_k}, \theta_0) = L_{I_k}(\tilde{\theta}_{I_k}) - L_{I_k}(\theta_0)$ is the likelihood ratio between the two parameters.



Small Modeling Bias Condition

The SMB condition for the interval I_k, and given some $\theta \in \Theta$:

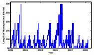
$$\mathsf{E}\,\Delta_{I_k}(\theta) \le \Delta \tag{7}$$

and

$$\Delta_{I_k} = \sum_{t \in I_k} \mathcal{K}\{r(t), r(t;\theta)\}$$

$\mathcal{K}\{r_t, r_t(\theta)\}$: Kullback-Leibler divergence between $P_{r(t)}$ and $P_{r(t;\theta)}$.

Oracle Choice k^*: the largest I_{k^*} s.t. (7) holds.



Propagation Condition and Stability

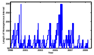
Propagation:

$$\mathsf{E}_{\theta_0} |L_{I_k}(\tilde{\theta}_{I_k}, \hat{\theta}_{I_k})|^r \leq \rho \Re_r(\theta_0) \tag{8}$$

Stability: I_k is accepted interval, then $\hat{\theta}_{I_k} = \tilde{\theta}_{I_k}$

$$L(\hat{\theta}_{I_k}, \hat{\theta}_{I_{k+1}}) \leq \mathfrak{z}_k \tag{9}$$

For fixed Δ, the loss $|L_{I_k}(\tilde{\theta}_{I_k}, \theta_0)|^r$ stochastically bounded by a constant proportional to e^{Δ}

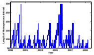


Critical Value

- Sequential choice of critical values $\mathfrak{z}_k$.
- Change point detected at step $\ell \leq k$
- $\mathfrak{B}_\ell$: rejection at step ℓ.

$$\mathfrak{B}_\ell = \{T_1 \leq \mathfrak{z}_1, \cdots, T_{\ell-1} \leq \mathfrak{z}_{\ell-1}, T_\ell > \mathfrak{z}_\ell\}$$

and $\hat{\theta}_{I_k} = \tilde{\theta}_{I_{\ell-1}}$ on $\mathfrak{B}_\ell, \quad \ell = 1, 2, \cdots, k$.



Critical Value

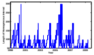
To determine $\mathfrak{z}_1$,

$$\max_{k=1,\cdots,K} \mathsf{E}_{\theta_0} \left|L(\tilde{\theta}_{I_k}, \tilde{\theta}_{I_0})\right|^r \mathbf{1}(\mathfrak{B}_1) \leq \rho \mathfrak{R}_r(\theta_0)/K \tag{10}$$

$\mathfrak{B}_\ell$ only depends on $\mathfrak{z}_1, \cdots, \mathfrak{z}_\ell$, controlled by $\mathfrak{z}_\ell$. The minimal value ensures

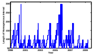
$$\max_{k \geq l} \mathsf{E}_{\theta_0} \left|L(\tilde{\theta}_{I_k}, \tilde{\theta}_{I_{l-1}})\right|^r \mathbf{1}(\mathfrak{B}_l) = \rho \mathfrak{R}_r(\theta_0)/K \tag{11}$$

$\mathfrak{R}_r(\theta_0)$ is parametric risk bound.



Choice of the Length of Interval

- I_0 with length m_0.
- Interval I_k: $m_k = [m_0 a^k]$ with $a > 1$.
- Results not sensitive to a.
- r=0.5, power of the loss function.
- ρ= 0.2, level of the test.
- $m_0 = 40$, $a = 1.25$, and K= 15, $m_K = 1136$.



Oracle Property

I_{k^*} is the oracle interval, $\mathsf{E}\,\Delta_{I_{k^*}(\theta)} \leq \Delta$, $k \leq k^*$, and $\hat{\theta}_{I_{\hat{k}}}$ close to the oracle estimate $\tilde{\theta}_{I_{k^*}}$

$$\mathsf{E} \log\{1 + \frac{|L_{I_{k^*}}(\tilde{\theta}_{I_{k^*}}, \hat{\theta}_{I_{\hat{k}}})|^r}{\mathfrak{R}_r(\theta)}\} \leq \rho + \Delta \tag{12}$$

For $k > k^*$, the adaptive estimator $\hat{\theta}_{I_{\hat{k}}}$ satisfies

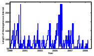
$$\mathsf{E} \log\{1 + \frac{|L_{I_{k^*}}(\tilde{\theta}_{I_{k^*}}, \hat{\theta}_{I_{\hat{k}}})|^r}{\mathfrak{R}_r(\theta)}\} \leq \rho + \Delta + \log\{1 + \frac{\mathfrak{z}_{k^*}^r}{\mathfrak{R}_r(\theta)}\} \tag{13}$$

Simulation Setup

We simulate CIR process with 1500 observations and 100 times.

t	a	b	σ
$t \in [1, 500]$	0.2	0.04	0.03
$t \in [501, 1000]$	0.5	0.06	0.1
$t \in [1001, 1500]$	0.8	0.01	0.07

Table 1: The parameter settings for simulations of the CIR process



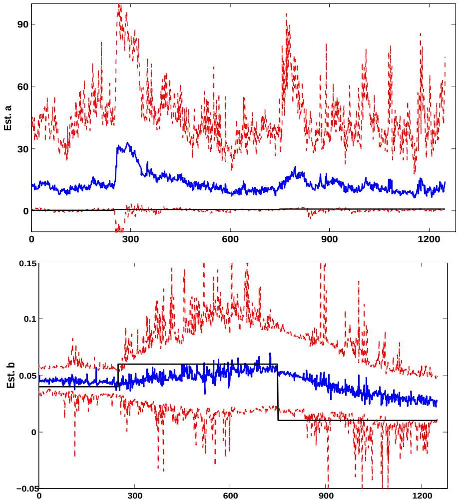
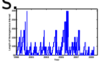


Figure 5: LPA estimator $\hat{a}$ and $\hat{b}$ with simulated CIR paths.



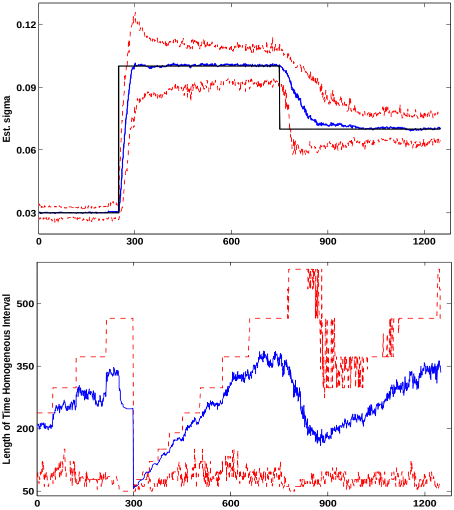


Figure 6: LPA estimator $\hat{\sigma}$ and selected time homogeneous intervals.

Data

Yield of 3M US T-Bill from the Federal Reserve Bank of St. Louis from 19980102 to 20090513.

	Mean	*SD*	*Skewness*	*Kurtosis*
r_t	0.0319	0.0176	-0.1159	-1.4104
dr_t	-1.764×10^{-5}	0.0006	-0.7467	34.4856

Table 2: Statistical Summary of 3-month T-Bill

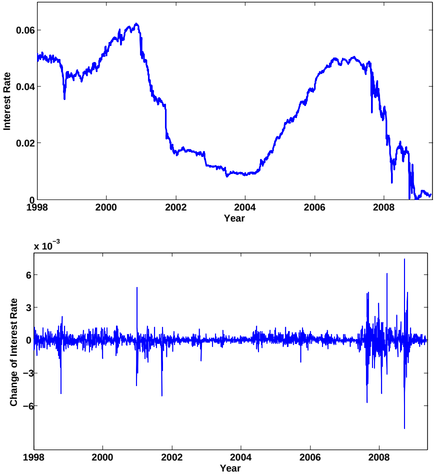
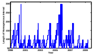


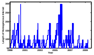
Figure 7: 3-month Treasure Bill Rate: 19980102—20090513. Top panel: Daily yields. Bottom panel: Changes of daily yields.



MLE Estimator of CIR model

Sample Size	$\hat{a}$	$\hat{b}$	$\hat{\sigma}$
19980102–20090513	0.2657	0.0153	0.0944
19980102–20070731	0.1424	0.0252	0.0428
20070801–20090513	3.6792	0.0081	0.2280

Table 3: Estimated parameters of CIR model using MLE



Critical Value

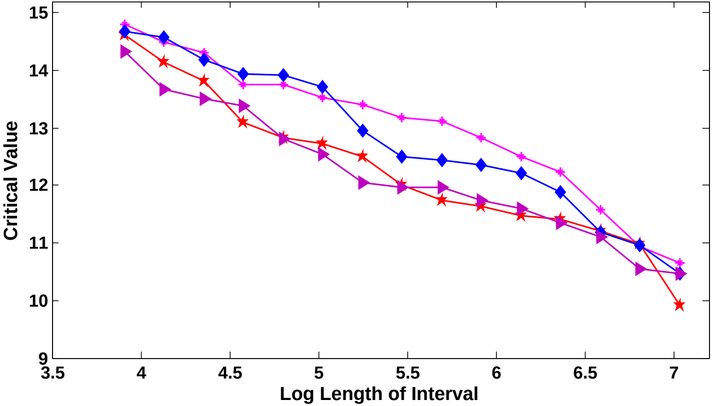
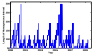


Figure 8: Critical values with $m_0 = 40$, K=15



Estimator $\hat{a}$

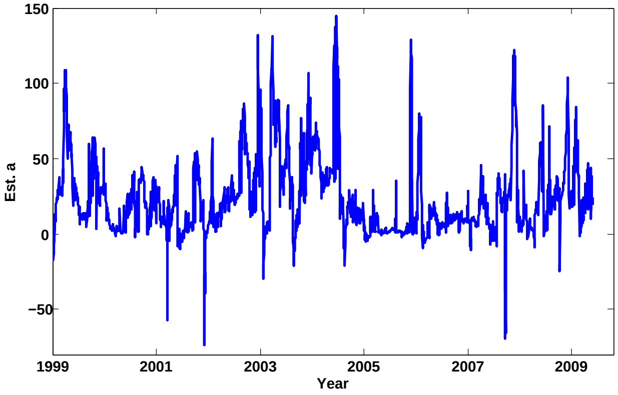
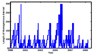


Figure 9: Estimated $\hat{a}$ by LPA



Estimator $\hat{b}$

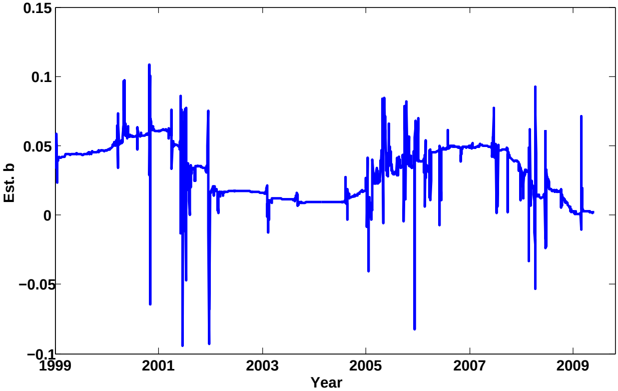
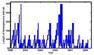


Figure 10: Estimated $\hat{b}$ by LPA



Estimator $\hat{\sigma}$

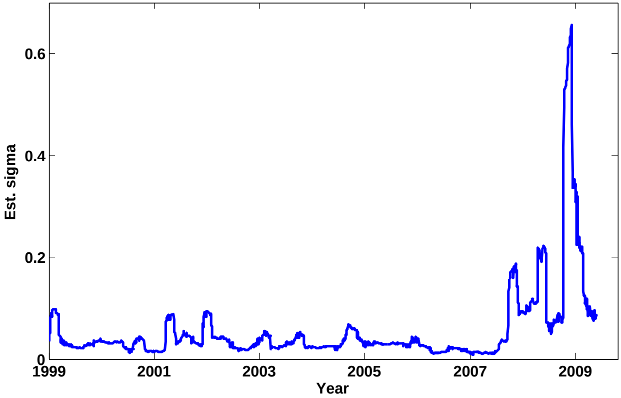
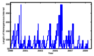


Figure 11: Estimated $\hat{\sigma}$ by LPA



Homogeneous Intervals

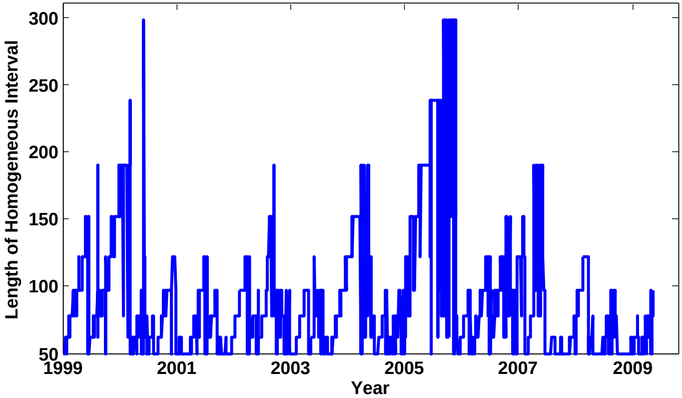
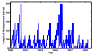


Figure 12: Selected time homogeneous intervals with $\rho = 0.2$, and $r = 0.5$



In Sample Fitting

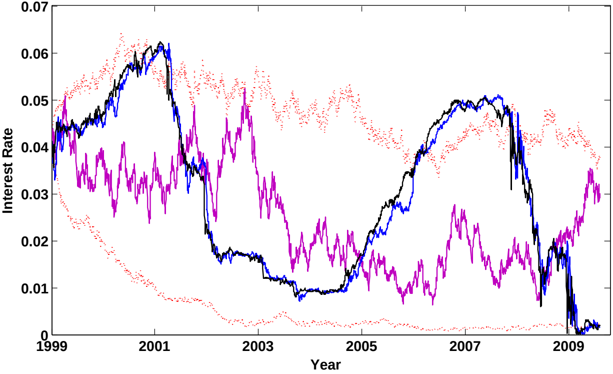
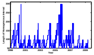


Figure 13: Confidence Interval (Red); Real Data (Black); LPA CIR (Blue); CIR (Purple)



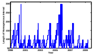
Forecasting

$\mathcal{H}$: the prediction period horizon, then the absolute prediction error(APE):

$$\text{APE}(t) = \sum_{h \in \mathcal{H}} |r_{t+h} - \hat{r}_{t+h|t}| / |\mathcal{H}| \tag{14}$$

$$\text{APE Ratio} = \frac{APE_{LPA}(t)}{APE_{MW}(t)}$$

LPA: Local Parametric Approach.
MW: Moving Window Estimation.



Forecasting

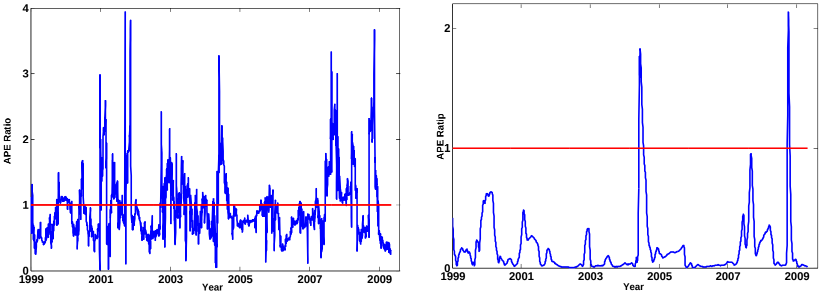
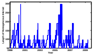


Figure 14: The APE ratio between LPA and MW with window size 250. Left: 1-day ahead forecasting; Right: 10-day ahead forecasting.

Performance of Forecasting

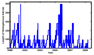
Horizon		MAE		
		$l = 250$	$l = 500$	$l = 750$
One Day	LPA	4.74×10^{-4}	4.85×10^{-4}	4.96×10^{-4}
	MW	4.78×10^{-4}	4.41×10^{-4}	4.16×10^{-4}
Ten Days	LPA	0.0201	0.0215	0.0232
	MW	0.1868	1.0032	1.8054

Table 4: The MAE of 1 day and 10 days ahead forecasting of the short rate based on the LPA and MW.



Conclusion

- Interest rate in recession is more volatile.
- The selected time homogeneous intervals can not last long due to the complexities of macroeconomy.
- The LPA can detect jumps and structural break points in the interest rate dynamics.
- The LPA outperforms the moving window estimation especially in long horizon forecasting.



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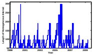

Adaptive Interest Rate Modelling

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