

# Shape Invariant Modelling and Risk Patterns

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## Financial Market

Riskless bond with constant interest rate  $r$ , stock price  $(S_t)_{t \in [0, T]}$  follows a diffusion process

□ **risk neutral valuation principle**

$$e^{-Tr} \int_0^\infty \psi(s_T) \frac{q(s_T)}{p(s_T)} p(s_T) ds_T$$

where  $q$  is a risk neutral density and  $p$  is the probability density function of  $S_T$ .



## Pricing Kernels & Preferences

- the pricing kernel at time 0

$$\mathcal{K}_0(S_T) = \frac{q(S_T)}{p(S_T)}$$

- relationship between representative investor's preferences and pricing kernel (e.g. Leland 1980):

$$ARA(S_T) = \frac{p'(S_T)}{p(S_T)} - \frac{q'(S_T)}{q(S_T)} = \frac{-\mathcal{K}'(S_T)}{\mathcal{K}(S_T)}$$



## Empirical Pricing Kernel (EPK)

- EPK: any estimation of pricing kernel  $\frac{q}{p}$
- under Black-Scholes model the EPK is decreasing in wealth
- other estimation methods and models for stock prices, Ait-Sahalia & Lo 2000, Engle & Rosenberg 2002, Brown & Jackwerth 2004



some paradoxa



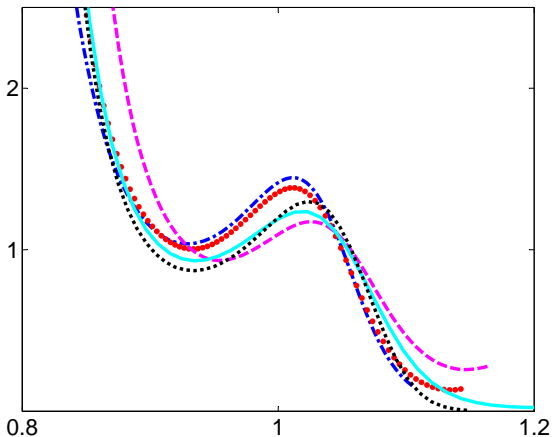


Figure 1: EPK/Moneyiness  $K/S_t$  for maturities:  $\tau = 0.097$  (blue),  $0.083$  (red),  $0.069$  (magenta),  $0.061$  (cyan),  $0.047$  (black). Expiration date: 02-Jun-2006



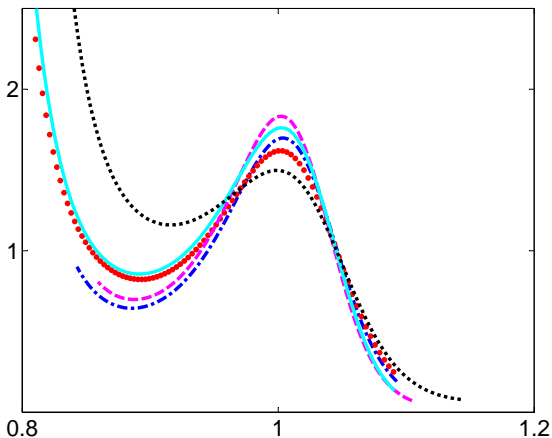


Figure 2: EPK/Moneyness for maturity  $\tau = 0.083$ , observed in 2006: 18-Jan (blue), 15-Feb (red), 22-Mar (magenta), 19-Apr (cyan), 17-May (black)



## Literature

### Multiple curves

- Gasser et al. (1984) Zurich Longitudinal Studies on Growth
- Härdle and Marron (1990) Automobile side impact data

### Self Modelling

- Sylvestre et al. (1972)



## EPK paradoxon: aims

Empirical pricing kernels are not monotone decreasing across returns, vary across maturities and observation time:

- How to model the changes in the EPK functionals based on the common feature?
- Can EPK deformation explain the patterns in risk perception?





## Outline

1. Motivation ✓
2. Empirical Pricing Kernel
3. Shape Invariant Modelling
4. Pricing Kernel and Risk Aversion
5. Conclusions
6. Selected Bibliography



## The Financial Market I

In an arbitrage-free market, the European call price is given by

$$C_t(K, \tau, r, S_t) = e^{-r\tau} \int_0^{\infty} [S_T - K]^+ q(S_T | \tau, r, S_t) dS_T$$

- $S_t$  the underlying asset price at time  $t$ ,
- $K$  the strike price,
- $\tau$  the time to maturity,
- $T = t + \tau$  the expiration date,
- $r$  constant risk free interest,



## The Financial Market II

The price can be written as:

$$\begin{aligned}C_t(S_T) &= e^{-r} E^Q \{(S_T - K)^+ | S_t\} \\ &= e^{-r} E^P \{(S_T - K)^+ \mathcal{K}_t(S_t, S_T) | S_t\}\end{aligned}$$

with  $\mathcal{K}_t(S_t, S_T)$  the pricing kernel at time  $t$ , s.t conditional risk neutral distributions  $Q_{S_T|S_t}$  :

$$Q_{S_T|S_t=s_t}(S_T \leq x) \stackrel{\text{def}}{=} \int_{-\infty}^x \mathcal{K}_t(s_t, \cdot) dP_{S_T|S_t=s_t}$$

where  $P_{S_T|S_t=s_t}$  is the conditional distribution of  $S_T$  under  $S_t$ .



## Data

- **Source:** Reseach Data Center (RDC)  
<http://sfb649.wiwi.hu-berlin.de>
- Datastream DAX 30 Price Index;  
2 years worth of daily returns in a sliding window
- EUREX European Option Data; daily tick observations;  
selected 38 days of cross-sectional data: 200304:200605



## Estimation of PK

- estimate PK as the ratio between 2 estimated densities:

$$\hat{\mathcal{K}}_t(s_t, S_T) = \frac{\hat{q}_t(S_T)}{\hat{p}_t(S_T)}$$

- $\hat{q}_t(S_T)$  by Rookley (1997) method based on the results of Breeden and Litzenberger (1978)

$$q_t(S_T) = e^{rT} \frac{\partial^2 C_t(\cdot)}{\partial K^2} \Big|_{K=S_T} .$$

- $\hat{p}_t(S_T)$  historical density by kernel method



## Estimation of RND

Rookley used a scaled version of the Black-Scholes call price formula that depends only on moneyness and maturity

$\sigma_{IV}(K/S_t, \tau)$ .

- ▣ local polynomial smoothing of degree 3
- ▣ quartic kernel



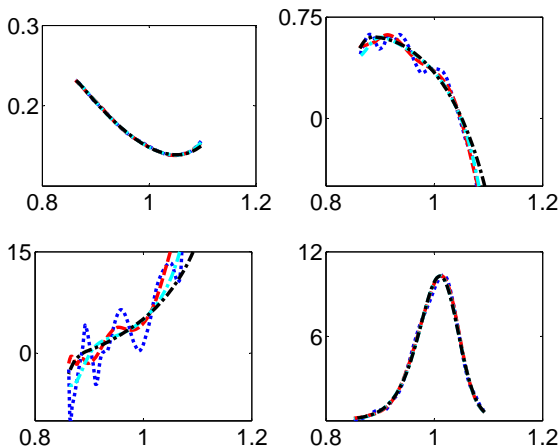


Figure 3: Implied volatility/Moneyness, its first and second derivative,  $q$  estimates with varying bandwidths (0.05, 0.10, 0.15, 0.20)



## The Model

- $Y_{tj}$  is be a noisy sample of  $T$  curves at design points  $u_j$ , with  $j \in \{1, \dots, n\}$ :

$$Y_{tj} = \mathcal{K}_t(u_j) + \varepsilon_{tj}, \quad \text{with} \quad \varepsilon_{tj} \sim \text{N}(0, \sigma_t^2).$$

- The smooth curves are of the form:

$$\mathcal{K}_t(u) = \theta_{t1} \mathcal{K}_0\left(\frac{u - \theta_{t3}}{\theta_{t2}}\right) + \theta_{t4}.$$

- $\mathcal{K}_0$  is a reference curve and  $\theta = (\theta_{t1}, \theta_{t2}, \theta_{t3}, \theta_{t4})$  are horizontal and vertical deviation parameters





## Estimation of SIM I

- Synchronisation

$$\mathcal{K}_t(\theta_{t2}u + \theta_{t3}) = \theta_{t1}\mathcal{K}_0(u) + \theta_{t4}, \quad \theta_{t1} > 0, \quad \theta_{t2} > 0.$$

- Normalizing conditions:

$$T^{-1} \sum_{t=1}^T \theta_{t1} = T^{-1} \sum_{t=1}^T \theta_{t2} = 1, \quad T^{-1} \sum_{t=1}^T \theta_{t3} = T^{-1} \sum_{t=1}^T \theta_{t4} = 0.$$

- Common curve

$$T^{-1} \sum_{t=1}^T \mathcal{K}_t(\theta_{t2}u + \theta_{t3}) = \mathcal{K}_0(u).$$



## Common Shape

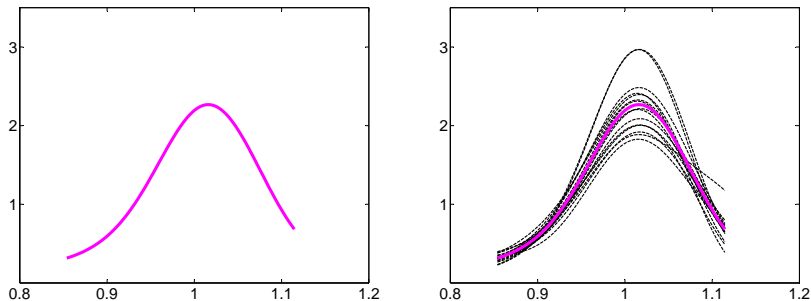


Figure 4: Estimated common shape function  $\mathcal{K}_0$  (left) and transformed curves  $\mathcal{K}_t(\theta_{t2}u + \theta_{t3})$  of those in Figure 5 on the common domain (right)



## Estimation of SIM II

- Peak identification

$$0 = \mathcal{K}'_t(u) = \frac{\theta_{t1}}{\theta_{t2}} \mathcal{K}'_0\left(\frac{u - \theta_{t3}}{\theta_{t2}}\right).$$

- Inflection point

$$0 = \mathcal{K}''_t(u) = \frac{\theta_{t1}}{\theta_{t2}^2} \mathcal{K}''_0\left(\frac{u - \theta_{t3}}{\theta_{t2}}\right),$$

- The solutions also satisfy

$$u_t = \theta_{t2}u_0 + \theta_{t3}.$$

- This gives the starting values of  $\theta_{t3}$  and  $\theta_{t2}$ .



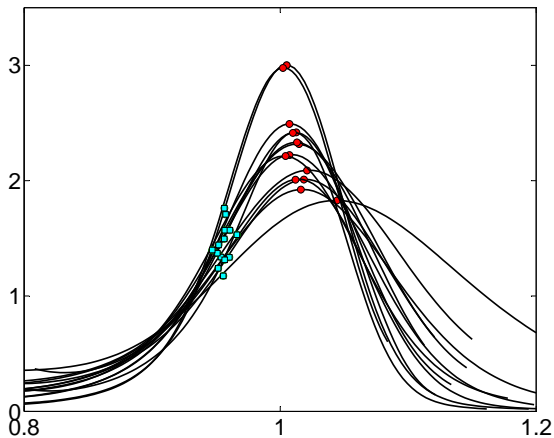


Figure 5: Landmark identification. Pricing kernels with fixed maturity 1 month between 200304:200605



## Estimation of SIM III

- Estimation procedure :

$$\min_{\theta} \int \{ \hat{\mathcal{K}}_t(\theta_2 u + \theta_3) - \theta_1 \hat{\mathcal{K}}_0(u) - \theta_4 \}^2 w(u) du, \quad (1)$$

where  $\hat{\mathcal{K}}_i$  are nonparametric estimates of the curves and the common region is defined for some  $a \geq \inf(u_{1,t})$  and  $b \leq \sup(u_{n,t})$

$$w(u) = \prod_t 1_{[a,b]} \{ (u - \theta_{t3}) / \theta_{t2} \}.$$



## The Algorithm

- Iterative scheme based on (1)
- Given prior estimates for  $(\theta_{t2}, \theta_{t3})$  and  $\hat{\mathcal{K}}_0$

$$\min_{\theta} \sum_j \{ \hat{\mathcal{K}}_t(\theta_{t2}u_j + \theta_{t3}) - \theta_{t1}\hat{\mathcal{K}}_0(u_j) - \theta_{t4} \}^2 w(u_j). \quad (2)$$

- Update  $(\theta_{t2}, \theta_{t3})$  and  $\hat{\mathcal{K}}_0$
- Repeat procedure until convergence is reached



## Asymptotics: EPK

$$\begin{aligned}\hat{\mathcal{K}}(u) - \mathcal{K}(u) &= \frac{\hat{q}(u)}{\hat{p}(u)} - \frac{q(u)}{p(u)} \\ &\simeq \frac{\hat{q}(u) - q(u)}{p(u)} - \frac{q(u)}{p(u)} \frac{\hat{p}(u) - p(u)}{p(u)}.\end{aligned}$$

### Bias

$$E \left\{ \hat{\mathcal{K}}(u) - \mathcal{K}(u) \right\} \simeq \mathcal{O}(h_q^4) + \mathcal{O}(h_p^2),$$

### Variance

$$\text{Var} \left\{ \hat{\mathcal{K}}(u) - \mathcal{K}(u) \right\} \simeq \mathcal{O}(Mh_q)^{-1} + \mathcal{O}(mh_p)^{-1}.$$

with  $M$  sample size for  $q$ ,  $m$  sample size for  $p$ ,  
 $h_q$  and  $h_p$  the corresponding bandwidths.



## Asymptotics: SIM Parameters

- based on standard non-linear least square methods

$$\hat{\theta}_t \approx N(\theta_t, \Sigma_t).$$

$$\hat{\Sigma}_t = \hat{\sigma}_t^2 \left[ n^{-1} \sum_{j=1}^n \left\{ \nabla_{\theta} \tilde{\mathcal{K}}_t(u_j; \tilde{\theta}) \right\} \left\{ \nabla_{\theta} \tilde{\mathcal{K}}_t(u_j; \tilde{\theta}) \right\}^{\top} \right]^{-1},$$

where  $\nabla_{\theta} \mathcal{K}(u; \theta)$  is the first derivative of the function  $\mathcal{K}(u; \theta)$  and

$$\hat{\sigma}_t^2 = n^{-1} \sum_{j=1}^n \hat{e}_{tj}^2.$$

with  $\hat{e}_{tj} = \hat{\mathcal{K}}_t(u_j) - \tilde{\mathcal{K}}_t(u_j)$  where  $\hat{\mathcal{K}}$  is the initial estimates and  $\tilde{\mathcal{K}}$  is the SIM estimates.





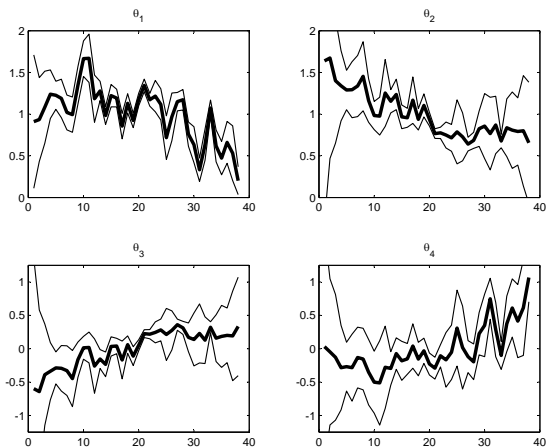


Figure 6: Parameter estimates of the SIM and their confidence intervals at 95% confidence level for the EPK 200304:200605



## ARA and SIM

Under SIM specifications the ARA measure is given by:

$$ARA_t(u) = \frac{-\frac{\theta_{t1}}{\theta_{t2}} \mathcal{K}'_0\left(\frac{u-\theta_{t3}}{\theta_{t2}}\right)}{\theta_{t1} \mathcal{K}_0\left(\frac{u-\theta_{t3}}{\theta_{t2}}\right) + \theta_{t4}}.$$



## The Effect of $\theta_1$ on $\mathcal{K}_0$ and $ARA_0$

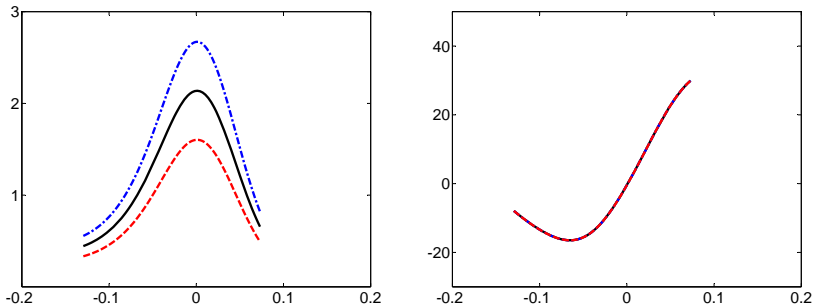


Figure 7: EPK (left) and ARA (right)  $\theta_1 = 0.75$  (red),  $\theta_1 = 1.25$  (blue) compared to the baseline model  $\theta_0 = (1, 1, 0, 0)$  (black)



## The Effect of $\theta_2$ on $\mathcal{K}_0$ and $ARA_0$

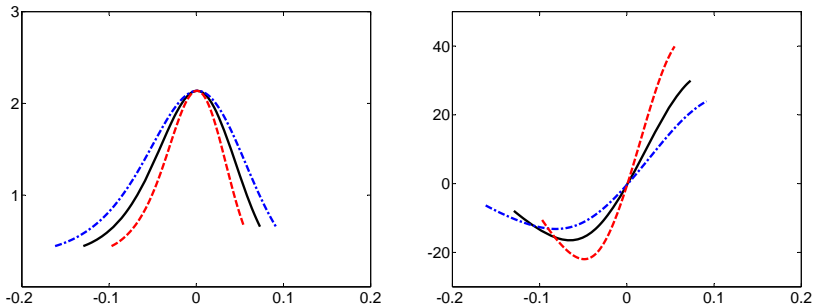


Figure 8: EPK (left) and ARA (right)  $\theta_2 = 0.75$  (red),  $\theta_2 = 1.25$  (blue) compared to the baseline model  $\theta_0 = (1, 1, 0, 0)$  (black)



## The Effect of $\theta_3$ on $\mathcal{K}_0$ and $ARA_0$

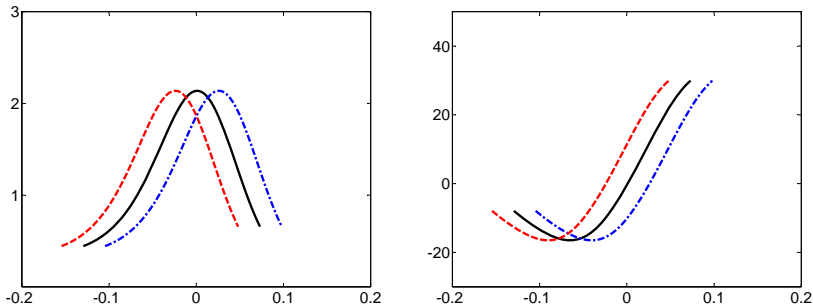


Figure 9: EPK (left) and ARA (right)  $\theta_3 = -0.025$  (red),  $\theta_3 = 0.025$  (blue) compared to the baseline model  $\theta_0 = (1, 1, 0, 0)$  (black)



## The Effect of $\theta_4$ on $\mathcal{K}_0$ and $ARA_0$

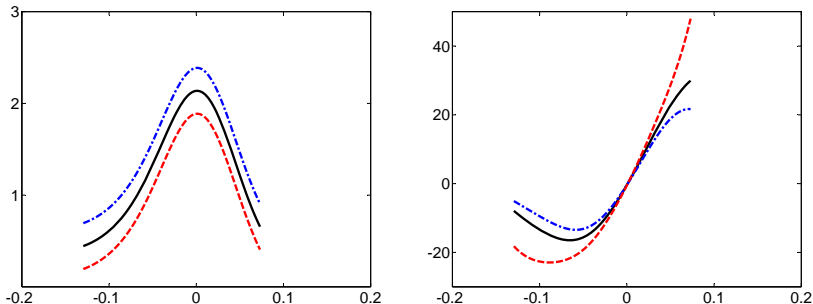


Figure 10: EPK (left) and ARA (right)  $\theta_4 = -0.25$  (red),  $\theta_4 = 0.25$  (blue) compared to the baseline model  $\theta_0 = (1, 1, 0, 0)$  (black)



## Risk Aversion and Business Cycle Indicators

- **Data:** Daily observations. German market.
- Credit spread ( $CD$ ): 5Y Corporate - Gov. bond yield;
- Yield curve ( $YC$ ): 3M - 10Y Gov. bond yield;
- Short term interest rate ( $IR$ ): 3M Gov. bond yield;
- Datastream DAX 30 stock index ( $I_{Dax}$ ).
  
- $P_x, P_y$  EPK peak coordinates



## Business Cycle Indicators

Indicator	Expansion	Recession
$CS$	↓	↑
$YC$	↓	↑
$IR$	↓	↑
$I_{Dax}$	↑	↓

Table 1: Behaviour of the economic indicators under the BC phases





	$\Delta\theta_1$	$\Delta\theta_2$	$\Delta\theta_3$	$\Delta\theta_4$	$\Delta P_x$	$\Delta P_y$
$\Delta\theta_1$	1.00					
$\Delta\theta_2$	-0.71*	1.00				
$\Delta\theta_3$	0.71*	-0.99*	1.00			
$\Delta\theta_4$	-0.93*	0.45*	-0.45*	1.00		
$\Delta P_x$	-0.27***	0.41**	-0.38**	0.1	1.00	
$\Delta P_y$	0.96*	-0.83*	0.83*	-0.82*	-0.31***	1.00
$\Delta CS$	-0.30***	0.19	-0.19	0.31***	0.13	-0.28***
$\Delta YC$	-0.02	-0.15	0.15	0.11	-0.21	0.03
$\Delta IR$	0.01	0.10	-0.08	-0.02	0.53*	-0.00
$R_{Dax}$	0.68*	-0.57*	0.56*	-0.59*	-0.52*	0.69

Table 2: Correlation coefficients between SIM parameters, EPK peak coordinates and BC indicators (sig at 1% = \*, 5% = \*\*, 10% = \*\*\*)



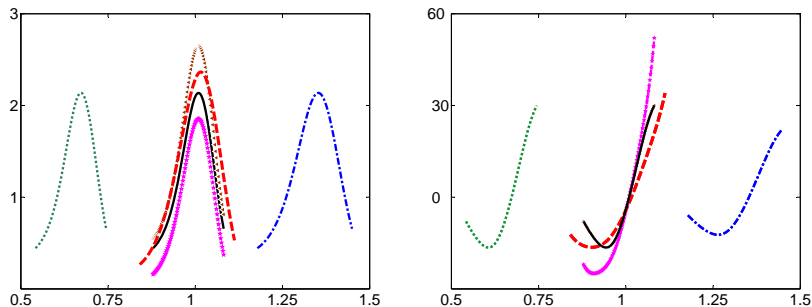


Figure 11:  $EPK_t$  and  $ARA_t$  (red/dashed) in individual effects:  $\theta_1$  (brown/triangle),  $\theta_2$  (blue/dash-dotted),  $\theta_3$  (green/dotted),  $\theta_4$  (magenta/star) compared to the baseline model (black), on 20030716



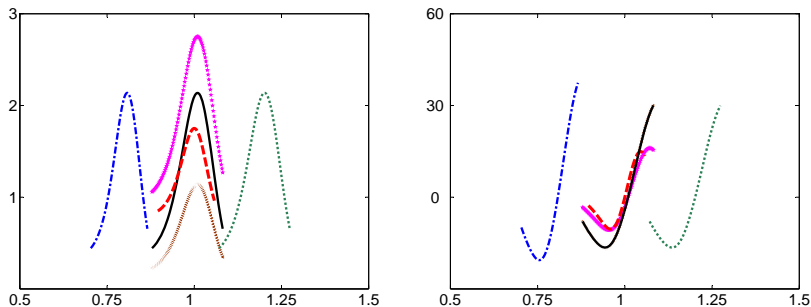


Figure 12:  $EPK_t$  and  $ARA_t$  (red/dashed) in individual effects:  $\theta_1$  (brown/triangle),  $\theta_2$  (blue/dash-dotted),  $\theta_3$  (green/dotted),  $\theta_4$  (magenta/star) compared to the baseline model (black), on 20060419



## Conclusions

- An increase in the EPK peak comes with a decrease in duration
- Changes in risk proclivity are negatively correlated with the credit spread growth and positively correlated with  $R_{Dax}$
- Agents update the expected value of the risky bets in the same sense with the  $\Delta IR$  and in the opposite sense with  $R_{Dax}$
- **Local risk proclivity is pro-cyclical**



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