A Microeconomic Explanation of the EPK Paradox

Maria Grith
Wolfgang Karl Härdle
Volker Krätschmer

Ladislaus von Bortkiewicz Chair of Statistics
C.A.S.E. – Center for Applied Statistics and Economics
Humboldt–Universität zu Berlin
http://lvb.wiwi.hu-berlin.de

Financial Market

Riskless bond with interest rate r, stock price process $(S_t)_{t\in[0,T]}$
S_t

- Market models
  - Black-Scholes model (Nobel prize 1997)
  - GARCH model (Nobel prize 2003)
  - non-parametric diffusion model (Aït-Sahalia & Lo, 2000)
- **risk neutral valuation principle for pay offs $\psi(S_T)$:**

$$\int_0^\infty e^{-Tr}\psi(s_T)\,\mathcal{K}(s_T)p(s_T)\,ds_T$$

p pdf of S_T, $\mathcal{K}$ pricing kernel (PK).

Pricing Kernels & Preferences

- representative investor with strictly increasing, concave, indirect von Neumann Morgenstern utility u
- **relationship between preferences and pricing kernel:**

$$\frac{du}{dx} \propto \mathcal{K}$$

Empirical Pricing Kernel (EPK)

- EPK: any estimation of pricing kernel $\mathcal{K}$
- different estimation methods and models for stock prices, Ait-Sahalia & Lo (2000), Engle & Rosenberg (2002), Brown & Jackwerth (2004), Detlefsen, Härdle & Moro (2010)

$$\downarrow$$

the paradox

EPK paradoxon: across maturities and time

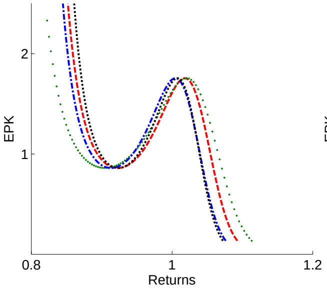


Figure 1: Examples of intertemporal pricing kernels for various maturities (left) and monthly pricing kernels for the first 6 months in 2006 for 1M maturity (right). Grith, Härdle and Park (2010)

EPK paradoxon: across maturities

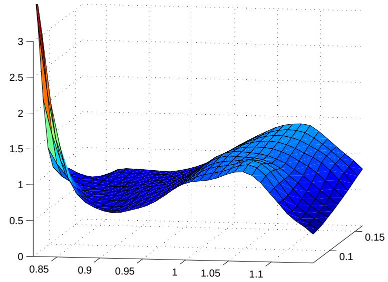


Figure 2: Estimated PK across moneyness κ and maturity τ, DAX on 20010710, Giacomini & Härdle (2008)

EPK paradoxon: across time

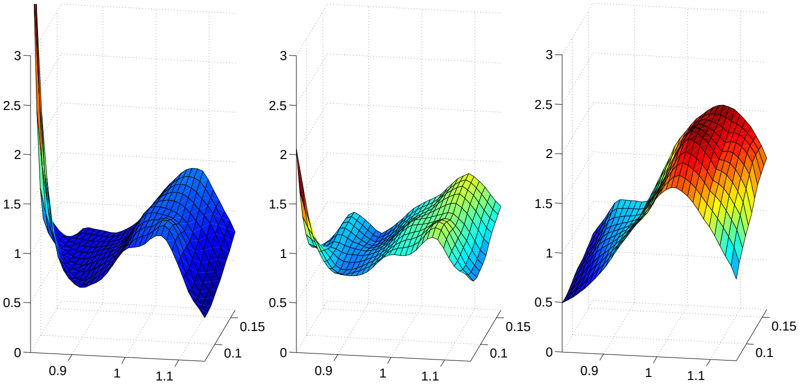


Figure 3: Empirical PK across κ and τ, estimated form DAX on 20010710, 20010904 and 20011130, Giacomini & Härdle (2008)

Aims

- Microeconomic explanation of the EPK paradox
- Switching behaviour in terms of inverse problems

Outline

1. Motivation ✓
2. Pricing Kernels
3. Microeconomic Explanation for the EPK Paradox
4. Inverse Problem
5. References

The Financial Market

1. time interval $[0, T]$ of investment with finite horizon T
2. one riskless bond with deterministic Riemannian integrable process $(r_t)_{0 \leq t \leq T}$ of interest rates
3. one risky asset with nonnegative price process $(S_t)_{0 \leq t \leq T}$, semimartingale, S_0 constant

Risk Neutral Valuation Principle

Assumption

Arbitrage free market, there exists at least one equivalent martingale measure with density π

Risk neutral price of a non-negative pay off $\psi(S_T)$ (w.r.t. π):

$$\mathsf{E}\left[e^{-\int_0^T r_t dt}\ \psi(S_T)\ \pi\ \right] \;=\; \mathsf{E}\left[e^{-\int_0^T r_t dt}\ \psi(S_T)\ \mathsf{E}[\pi|S_T]\ \right]$$

The Pricing Kernel(s)

1. **pricing kernel** (w.r.t. π), positive random variable $\mathcal{K}_\pi$. s.t.

$$\mathsf{E}[\pi | S_T] \;=\; \mathcal{K}_\pi(S_T)$$

2. **rescaled pricing kernel** (w.r.t. π)

$$\tilde{\mathcal{K}}_\pi(R_T) \stackrel{\text{def}}{=} \mathcal{K}_\pi\left(R_T S_0\right)$$

with normalized return

$$R_T = \frac{S_T}{S_0}$$

Static Consumption Model

Consumer $i = 1, \ldots, m$ with a random endowment $e_i(R_T)$

1. chooses among nonnegative random consumption $c_i(R_T)$ under the **budget constraint**

$$\mathsf{E}\left[c_i(R_T)\tilde{\mathcal{K}}_\pi(R_T)\right] \quad \leq \quad \mathsf{E}\left[e_i(R_T)\tilde{\mathcal{K}}_\pi(R_T)\right]$$

2. has extended expected utility preferences

$$U^i\{c_i(R_T)\} \quad = \quad \mathsf{E}\left[\mathbf{1}_{[0,x_i]}(R_T)u_1^i\{c_i(R_T)\} + \mathbf{1}_{]x_i,1]}(R_T)u_2^i\{c_i(R_T)\}\right]$$

where $x_i \in]0,1[$, $u_j^i : [0,\infty[\to \mathbb{R} \cup \{-\infty\}$ satisfies

$$u_j^i(c) \in \mathbb{R} \text{ for } c > 0$$

$$u_j^i \text{ nondecreasing and concave}$$

Individual utility function

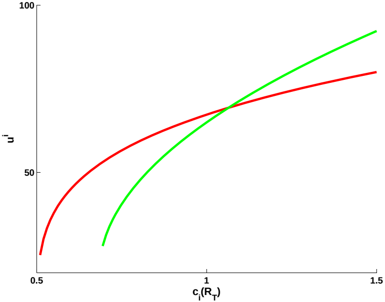
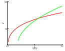


Figure 4: Regime dependent individual utility functions: bearish market (red) and bullish market (green)



Equilibrium

Contingent Arrow Debreu equilibrium $[(\bar{c}_1(R_T), ..., \bar{c}_m(R_T)); \tilde{\mathcal{K}}_\pi]$, in particular:

1. **individual optimization:** $\bar{c}_i(R_T)$ solves the optimization problem

$$\max \; U^i\{c_i(R_T)\}$$

   s.t. $c_i(R_T)$ satisfies individual budget constraint

2. **market clearing:** $\sum_{i=1}^{m} \bar{c}_i(R_T) = \sum_{i=1}^{m} e_i(R_T)$

Indirect Utilities of Representative Investor

The equilibrium guarantees nonnegative weights $\alpha_1, ..., \alpha_m$ (summing up to 1)

$$\sum_{i=1}^{m} \alpha_i U^i\{\bar{c}_i(R_T)\} = U_\alpha \left\{ \sum_{i=1}^{m} e_i(R_T) \right\}$$

$$\stackrel{\text{def}}{=} \max_{c_i} \left\{ \sum_{i=1}^{m} \alpha_i U^i\{c_i(R_T)\} \middle| \sum_{i=1}^{m} c_i(R_T) \le \sum_{i=1}^{m} e_i(R_T) \right\}$$

Extended expected utility representation

$$U_\alpha\left\{\sum_{i=1}^{m} e_i(R_T)\right\} = \mathsf{E}\left[u_\alpha\left\{R_T, \sum_{i=1}^{m} e_i(R_T)\right\}\right]$$

where for $r_T, e \geq 0$

$$u_\alpha(r_T, e) = \mathbf{1}_{[0,x_1]}(r_T)u_\alpha^1(e) + \sum_{i=1}^{m} \mathbf{1}_{]x_i,x_{i+1}]}(r_T)u_\alpha^{i+1}(e) \text{ for } r_T, e \geq 0$$

with $\underline{z} \overset{\text{def}}{=} x_0 \leq x_1 \leq \ldots \leq x_m < x_{m+1} \overset{\text{def}}{=} \bar{z}$, and

- $u_\alpha^k : [0,\infty[\to \mathbb{R} \cup \{-\infty\}$ is nondecreasing and concave

A Simple Solution

Let $\sum_{i=1}^{m} e_i(R_T) = R_T$.

Theorem:

Let $u_j^i |]0, \infty[$ be twice continuously differentiable satisfying **Inada conditions** for every $i \in \{1, ..., m\}$ and $j \in \{1, 2\}$.

Then $u_\alpha^k |]0, \infty[$ is continuously differentiable for $k \in \{1, \ldots, m+1\}$ and there is some $y > 0$ such that for any $r_T > 0$:

$$\mathbf{1}_{[0,x_1]}(r_T) \frac{du_\alpha^1}{de} \Big|_{e=r_T} + \sum_{k=1}^{m} \mathbf{1}_{]x_k,x_{k+1}]}(r_T) \frac{du_\alpha^{k+1}}{de} \Big|_{e=r_T} = y \, \tilde{\mathcal{K}}_\pi(r_T)$$

PK and Risk Aversion

$\tilde{\mathcal{K}}_\pi :]0,\infty] \to [0,\infty[$ left-continuous and piecewise nonincreasing C^1-mapping with $\lim\limits_{x\to 0} \tilde{\mathcal{K}}_\pi(x) = \infty$ and $\lim\limits_{x\to\infty} \tilde{\mathcal{K}}_\pi(x) = 0$.

Arrow-Pratt coefficients of absolute risk aversion at unknown jump points $x_1 < \ldots < x_m$ from the left and from the right:

$$\lim_{\delta\to 0_+} \frac{\frac{\widetilde{\mathcal{K}}_\pi(x_i-\delta)-\widetilde{\mathcal{K}}_\pi(x_i)}{\delta}}{\widetilde{\mathcal{K}}_\pi(x_i)}, \quad \lim_{\delta\to 0_+} \frac{\frac{\widetilde{\mathcal{K}}_\pi(x_i+\delta)-\widetilde{\mathcal{K}}_\pi(x_i)}{\delta}}{\widetilde{\mathcal{K}}_\pi(x_i)}$$

for i$\in \{1, ..., m\}$.

Example 1

Assume that

- investors have an identical switching point x_0.
- each investor i switches between CRRA utilities $u_i^j(y) = y^{\gamma_i^j}/\gamma_i^j$ $(j = 0, 1)$ with $0 < \gamma_i^1 < \gamma_i^0 < 1$,

Then

$$\mathbf{1}_{[0,x_0]}(r_T)\frac{du_\alpha^1(r_T,\cdot)}{de}\Big|_{e=r_T} + \mathbf{1}_{]x_0,\infty[}(r_T)\frac{du_\alpha^{m+1}(r_T,\cdot)}{de}\Big|_{e=r_T} = y\mathcal{K}_\pi(r_T)$$

for every realization r_T of R_T.

$$r_T = F^0\left(\frac{du_\alpha^1(r_T,\cdot)}{dy}\,\Big|_{y=r_T}\right) = F^1\left(\frac{du_\alpha^m(r_T,\cdot)}{dy}\,\Big|_{y=r_T}\right)$$

for any positive realization r_T, where

$$F^j :]0,\infty[\to]0,\infty[, z \mapsto \sum_{\substack{i=1\\ \alpha_i>0}}^{m} \left(\frac{z}{\alpha_i}\right)^{\frac{1}{\gamma_i^j-1}} \qquad (j=0,1)$$

are decreasing bijective mappings. If $x_0 > \max\{F^0(1), F^1(1)\}$, then

$$\frac{du_\alpha^{m+1}(r_T,\cdot)}{dy}\,\Big|_{y=r_T} > \frac{du_\alpha^1(r_T,\cdot)}{dy}\,\Big|_{y=r_T} \text{ for } r_T \geq x_0.$$

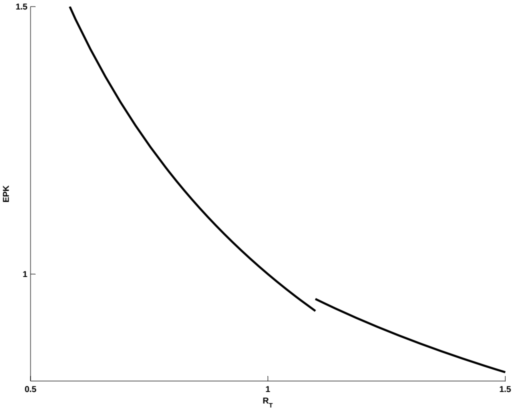


Figure 5: One switching point in the PK with $\gamma_\alpha^1 = 0.50 < \gamma_\alpha^0 = 0.75$

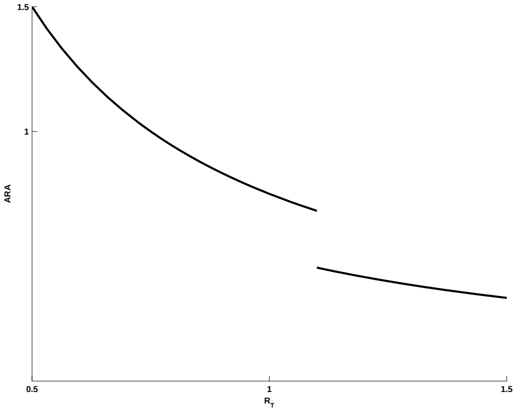


Figure 6: Implied ARA for one switching point in the PK with $\gamma_\alpha^1 = 0.50 < \gamma_\alpha^0 = 0.75$

Example 2

Assume that

- investors may differ in their switching point x_i.
- each investor i switches between the same u^0 and u^1
- $\omega(r_T)$ share of agents with preferences u^0 for r_T, ($\omega \in [0,1]$)

Then if $\alpha_1 = \alpha_2 = \cdots = \alpha_m = \alpha$ it holds in equilibrium

$$\omega(r_T)\bar{c}^0 + \{1 - \omega(r_T)\}\bar{c}^1 = \bar{e}(R_T) \stackrel{\text{def}}{=} R_T, \text{ for every } r_T,$$

with

$$\frac{du^0(r_T, \cdot)}{dy}\Big|_{y=\bar{c}^0(r_T)} = \frac{du^1(r_T, \cdot)}{dy}\Big|_{y=\bar{c}^1(r_T)}.$$

Let

$$u^0 = b_0 \frac{x^{1-\gamma^0}}{1-\gamma^0} + a_0 \quad \text{and} \quad u^1 = b_1 \frac{x^{1-\gamma^1}}{1-\gamma^1} + a_1,$$

for some constants b_0, $b_1 > 0$ and a_0, a_1 and u^1 is more concave than u^0 at each x. Then

$$\mathcal{K}_\pi(r_T) = 1_{[0,x_m]}(r_T) b_0 x^{\gamma_0}|_{x=\bar{c}^0(r_T)} + 1_{]x_m,\infty[}(r_T) b_1 x^{\gamma_1}|_{x=\bar{c}^1(r_T)}.$$

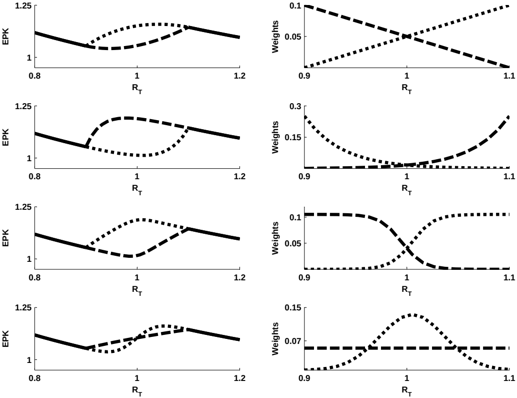


Figure 7: The relationship between the shape of the pricing kernel and the weight function $w(r_T) = d\omega(r_T)/dr_T$ for linear, exponential, logistic, constant and bell shaped specifications for $\gamma^0 = \gamma^1 = 0.5$, $b_0 = 1$ and $b_1 = 1.2$

Data Fit

Use a test function approach

- family $\mathcal{V}$ of strictly nonincreasing C^1-mappings $v :]0, \infty[\to \mathbb{R}$ with $\lim_{x \to 0} v(x) = \infty$ and $\lim_{x \to \infty} v(x) = 0$.
- test functions

$$\sum_{i=1}^{N+1} \mathbf{1}_{]x_{i-1}, x_i]}(x) v_i(x), \quad v_1, \ldots, v_{N+1} \in \mathcal{V}$$

with N switching points.

Grid-based Approach

- Use Korovkin approximation results for mappings on continuous intervals

Proposition

The mapping $\sum_{i=1}^{N} \widetilde{\mathcal{K}}_\pi\{a + \frac{i}{N}(b-a)\}\mathbf{1}_{]a+\frac{(i-1)}{N}(b-a),a+\frac{i}{N}(b-a)]}(x)$ *converges to* $\widetilde{\mathcal{K}}_\pi(x)$ *on* $[a,b]$ *uniformly on compacta of continuity points of* $\widetilde{\mathcal{K}}_\pi|[a,b]$ *for any nondegenerated interval* $[a,b] \subseteq]0,\infty[$.

The Inverse Problem

$\mathcal{Z}_N$ set of partitions $x_0 \leq x_1 \leq \ldots \leq x_{N+1}$
Find

$$\min_{(x_1,\ldots,x_N)\in\mathcal{Z}_N} \min_{v_1,\ldots,v_{N+1}\in\mathcal{V}} \int \left\{ \widetilde{\mathcal{K}}_\pi(x) - \sum_{i=1}^{N} v_i(x)\mathbf{1}_{]x_{i-1},x_i]}(x) \right\}^2 \hat{p}(x)\, dx$$

where $\hat{p}$ is an approximation of the density function p.

Grid-based Approach. Operationalization

Solve $\gamma_0^*, ...\gamma_{N+1}^*, \beta_0^*, ..., \beta_{N+1}^*, x_1^*, ..., x_N^* = \arg\min F_N$,

$$F_N = \sum_{j=1}^{n} \left\{ \hat{\mathcal{K}}_\pi(s_j) - \sum_{i=1}^{N} v_i(s_j) \mathbf{1}] x_{i-1}, x_i] (s_j) \right\}^2 \hat{p}(s_j) \Delta_j$$

$\hat{\mathcal{K}}$ estimates $\tilde{\mathcal{K}}$, n gridpts and $\Delta_j = s_j - s_{j-1}$.

$$v_i(x) = \beta_i x^{-\gamma_i} \quad \text{if} \quad x_{i-i} < x \leq x_i$$

Distcrete Switching Points

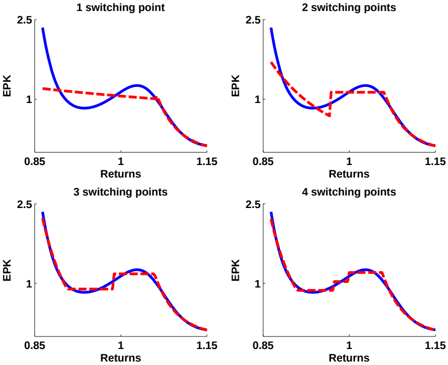


Figure 8: Nonparametric EPK (blue) and fitted PK specified by ?? (red) on 20060621

1 switching point

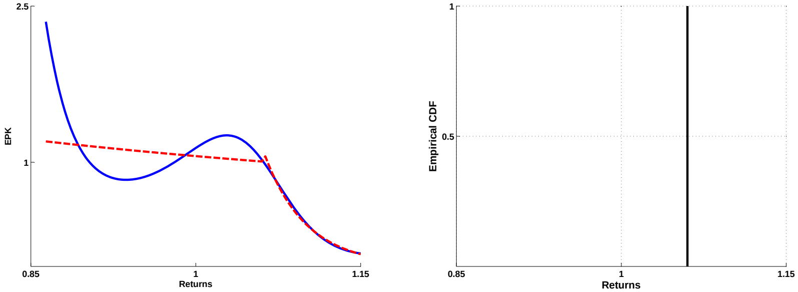


Figure 9: β_1=1.06, γ_1=0.85, β_2=5.88, γ_2=27.99

2 switching points

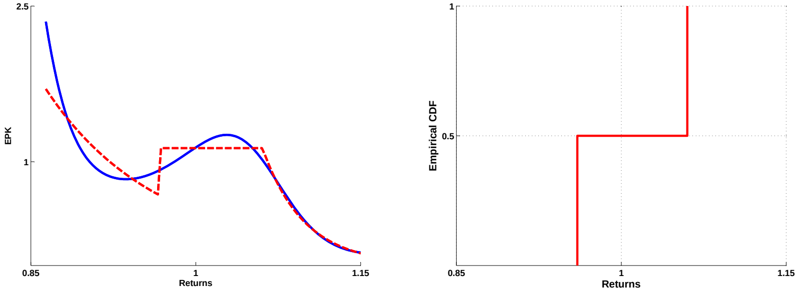


Figure 10: β_1=0.51, γ_1=0, β_2=1.13, γ_2=0, β_3=5.88, γ_3=27.99

3 switching points

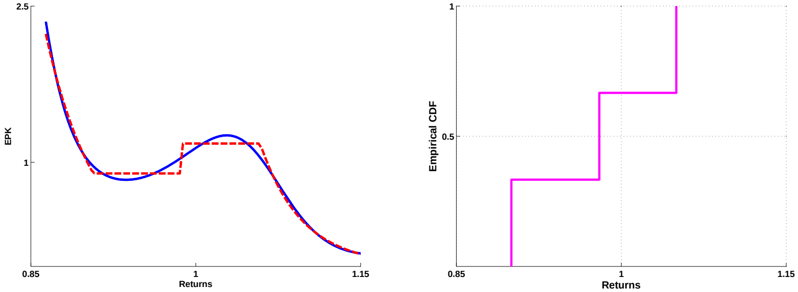


Figure 11: β_1=0.14, γ_1=0, β_2=0.89, γ_2=0, β_3=1.18, γ_3=27.74, β_4=5.71, γ_4=27.74

4 switching points

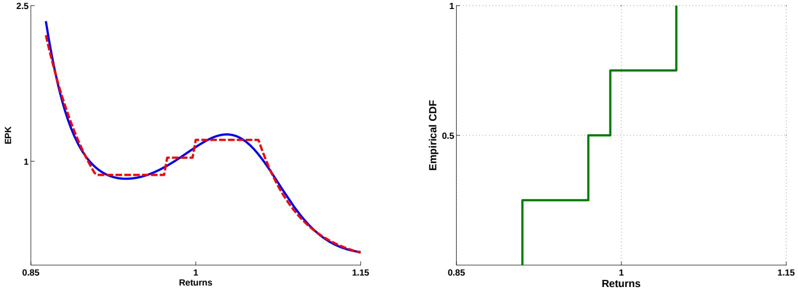


Figure 12: β_1=0.15, γ_1=18.28, β_2=0.87, γ_2=0, β_3=1.03, γ_3=0, β_4=1.21, γ_4=0, β_5=5.53, γ_5=27.45

5 switching points

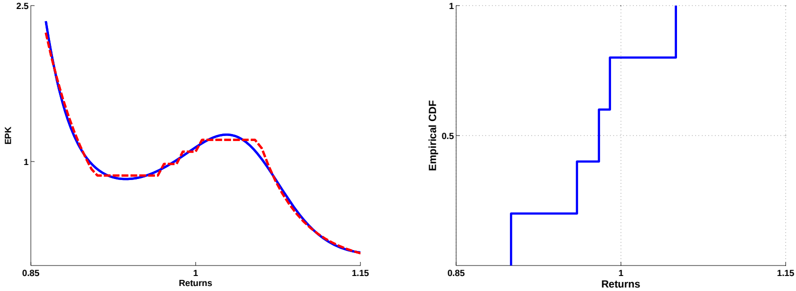


Figure 13: β_1=0.14, γ_1=18.76, β_2=0.87, γ_2=0, β_3=0.98, γ_3=0, β_4=1.10, γ_4=0, β_5=1.21, γ_5=0, β_6=5.71, γ_6=27.79

5 switching points

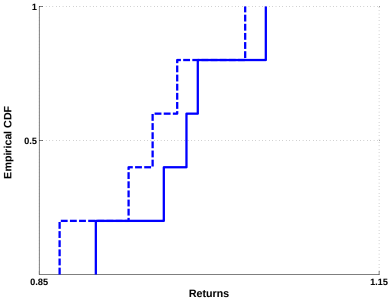
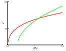


Figure 14: Empirical CDF on 20000920 (dashed) and 20060621 (solid)



Continuous Switching Points

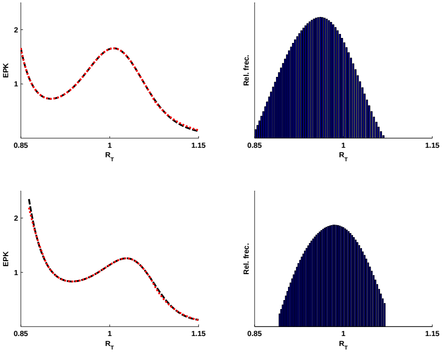


Figure 15: Left: $\hat{\mathcal{K}}_\pi$ (dashed, black) and v (red, dotted) on 20000920 (upper panel) with b_0=0.01, γ_0=30.10, b_1=3.58, γ_1=23.15 and 20060621 (lower panel) with b_0=0.14, γ_0=18.76, b_1=5.71, γ_1=27.79. Right: Estimated weighting functions w

References

Aït-Sahalia, Y. and A. Lo
Nonparametric Risk Management and Implied Risk Aversion
Journal of Econometrics, 2000, 94, 9-51.

Berger, J. O. and G. Salinetti
Approximations of Bayesian decision problems: the epigraphical approach
Ann. Op. Res., 1995, 56, 1-13.

Brown, D. and J. Jackwerth
The Pricing Kernel Puzzle: Reconciling Index Option Data and Economic Theory
University of Konstanz, working paper, 2004

References

Christoffersen, P., Jacobs, K. and Heston, S. L.
A GARCH Option Model with Variance-Dependent PK
AFA 2011 Denver Meetings Paper, 2011

Engle, R. and J. Rosenberg
Empirical Pricing Kernels
Journal of Financial Economics, 2002, 64(3), 341-372.

Detlefsen, K., W. Härdle and R. Moro
Empirical Pricing Kernels and Investor Preferences
Mathematical Methods in Economics and Finance , 2010.

References

Dal Maso, G.
An introduction to $\Gamma-$convergence
Birkhäuser, Boston, 1993.

Giacomini, E. and Härdle, W.
Dynamic Semiparametric Factor Models in Pricing Kernel Estimation
Functional and Operational Statistics. Eds. Dabo-Niang, S. and Ferraty, F., Springer Verlag, , 2008.

Grith, M., Härdle, W. K. and Park, J.
Shape Invariant Modelling Pricing Kernels and Risk Aversion

Resubmitted to Journal of Financial Econometrics, 17 December 2010.

References

Hens, T. and Reichlin, Ch.
Three Solutions to the Pricing Kernel Puzzle
Working Paper, Universität Zürich, 2010.