Extended Libor Models and Their Calibration

Denis Belomestny

Weierstraß Institute Berlin

Haindorf, 7 Februar 2008
Overview

1. Introduction
   - Forward Libor Models

2. Modelling
   - Modelling under Terminal Measure
   - Modelling under Forward Measures

3. Pricing and Calibration
   - Pricing of Caplets
   - Specification Analysis
   - Calibration Procedure

4. Calibration in work
Forward Libor Models

- **Tenor structure**: $0 = T_0 < T_1 < \ldots < T_M < T_{M+1}$ with accrual periods $\delta_i := T_{i+1} - T_i$

- **Zero coupon bonds**: $B_k(t)$, $t \in [0, T_k]$ with $B_k(T_k) = 1$

- **Forward Libor rates**: $L_1(t), \ldots, L_M(t)$

\[
L_k(t) = \frac{1}{\delta_k} \left( \frac{B_k(t)}{B_{k+1}(t)} - 1 \right), \quad t \in [0, T_k], \quad k = 1, \ldots, M
\]

**Remark**

$L_1, \ldots, L_M$ are based on simple compounding that is an investor receives 1$ at $T_k$ and pays $1 + \delta_k L_k(t)$ at $T_{k+1}$
Forward Libor Models

- **Tenor structure:** $0 = T_0 < T_1 < \ldots < T_M < T_{M+1}$ with accrual periods $\delta_i := T_{i+1} - T_i$

- **Zero coupon bonds:** $B_k(t), \ t \in [0, T_k]$ with $B_k(T_k) = 1$

- **Forward Libor rates:** $L_1(t), \ldots, L_M(t)$

$$L_k(t) = \frac{1}{\delta_k} \left( \frac{B_k(t)}{B_{k+1}(t)} - 1 \right), \quad t \in [0, T_k], \quad k = 1, \ldots, M$$

**Remark**

$L_1, \ldots, L_M$ are based on simple compounding that is an investor receives $1$ at $T_k$ and pays $1 + \delta_k L_k(t)$ at $T_{k+1}$
The BGM/Jamshidian LIBOR market model

\[ dL_i = - \sum_{j=i+1}^{M} \frac{\delta_j L_i L_j \gamma_i \cdot \gamma_j}{1 + \delta_j L_j} \, dt + L_i \gamma_i \cdot dW^{(M+1)}, \]

where \((W^{(M+1)}(t) \mid 0 \leq t \leq T_M)\) is a \(D\)-dimensional Wiener process under \(P_{M+1}\) and

\[ \gamma_i = (\gamma_{i,1}, \ldots, \gamma_{i,D}) \in \mathbb{R}^D, \quad i = 1, \ldots, M \]

are deterministic volatility vector functions, called factor loadings, to be determined via calibration.
Scalar volatility and correlation functions

A scalar volatility functions

\[ t \to \sigma_i(t) := |\gamma_i(t)| = \sqrt{\sum_{k=1}^{d} \gamma_{i,k}^2(t)}, \quad 0 \leq t \leq T_i, \ 1 \leq i \leq M, \]

and a (local) correlation functions

\[ t \to \rho_{ij}(t) := \frac{\gamma_i(t) \cdot \gamma_j(t)}{|\gamma_i(t)||\gamma_j(t)|}, \quad 0 \leq t \leq \min(T_i, T_j), \ 1 \leq i, j \leq M, \]

are the only economically relevant objects!
Scalar volatility and correlation functions

For any deterministic orthogonal matrix valued map

\[ Q : t \rightarrow Q(t) \in \mathbb{R}^{D \times D} \]

the process

\[ \tilde{W}_t = \int_0^t Q(u) dW_u \]

is again a standard Brownian motion in \( \mathbb{R}^D \). Hence, taking

\[ dW_t := Q^T d\tilde{W}_t, \]

yields the same Libor process with

\[ \tilde{\gamma} := \gamma Q^T. \]
The Hull-White Parametrization

For a fixed correlation matrix \( \varrho^{(0)} := [\varrho^{(0)}_{ij}]_{0 \leq i, j < M} \) of rank \( D \) and a fixed non-negative vector \( \Lambda := [\Lambda_i]_{0 \leq i < M} \) take

\[
\sigma_i(t) := \Lambda_{i - m(t)}, \quad 0 \leq t \leq T_i, \\
\varrho_{ij}(t) := \varrho^{(0)}_{i - m(t), j - m(t)}, \quad 0 \leq t \leq \min(T_i, T_j), \quad 1 \leq i, j \leq M
\]

with \( m(t) := \min\{m : T_m \geq t\} \).
The Hull-White Parametrization

Find unit vectors $e_i^{(0)} \in \mathbb{R}^D$ satisfying

$$\varrho_{ij}^{(0)} = e_i^{(0)} \cdot e_j^{(0)}$$

and set $e_i(t) := e_i^{(0)}$.

The dynamics of the Hull-White LIBOR model is determined by the volatility structure

$$\gamma_i(t) := \sigma_i(t) e_i(t) = \Lambda_{i-m(t)} e_i^{(0)}_{i-m(t)}, \quad 1 \leq i \leq M.$$ 

Observation

Good feature: A kind of time-shift homogeneity which is economically sensible.
The Hull-White Parametrization

Find unit vectors $e_i^{(0)} \in \mathbb{R}^D$ satisfying

$$
\varrho_{ij}^{(0)} = e_i^{(0)} \cdot e_j^{(0)}
$$

and set $e_i(t) := e_i^{(0)}$.

The dynamics of the Hull-White LIBOR model is determined by the volatility structure

$$
\gamma_i(t) := \sigma_i(t)e_i(t) = \Lambda_{i-m(t)}e_{i-m(t)}^{(0)}, \quad 1 \leq i \leq M.
$$

**Observation**

*Good feature: A kind of time-shift homogeneity which is economically sensible.*
More flexible volatility structure

Define

\[ \gamma_i(t) = c_i g(T_i - t) e^{(0)}_{i-m(t)}, \quad 0 \leq t \leq \min(T_i, T_j), \quad 1 \leq i, j \leq M, \]

\[ \varrho_{kl}^{(0)} = e^{(0)}_k \cdot e^{(0)}_l, \quad 0 \leq k, l < M \]

with suitably parameterized function \( g \) and correlation matrix \( \varrho^{(0)} \).

Observation

For \( c_i \equiv c \), model is basically time-shift homogeneous.
More flexible volatility structure

Define

\[ \gamma_i(t) = c_i g(T_i - t) e^{(0)}_{i-m(t)}, \quad 0 \leq t \leq \min(T_i, T_j), \quad 1 \leq i, j \leq M, \]
\[ \varrho_{kl}^{(0)} = e_k^{(0)} \cdot e_l^{(0)}, \quad 0 \leq k, l < M \]

with suitably parameterized function \( g \) and correlation matrix \( \varrho^{(0)} \).

**Observation**

*For \( c_i \equiv c \), model is basically time-shift homogeneous.*
Simple parametrization of scalar volatilities

\[ g(x) := g_{a,b,g_\infty}(x) := g_\infty + (1 - g_\infty + ax)e^{-bx}, \quad a, b, g_\infty > 0 \]

A typical shape of \( g : x \to g_{0.5,0.4,0.6}(x) \)
Semi-parametric full rank correlation structures

Consider initial correlation structures of the following form

\[ \varrho_{ij}^{(0)} := \frac{\min(b_i, b_j)}{\max(b_i, b_j)}, \quad 1 \leq i, j < M, \]

where the sequences \( i \rightarrow b_i \) and \( i \rightarrow b_i/b_{i+1} \) are positive and strictly increasing.

**Observation**

*Implicit consequence: \( D = M, \) w.l.g. \( b_1 = 1. \)**
Semi-parametric full rank correlation structures

The assumption that the sequence

$$i \rightarrow \frac{b_i}{b_{i+1}}$$

is increasing forces that for fixed $p$

$$i \rightarrow \rho_{i,i+p} = \text{Cor}(\Delta L_i, \Delta L_{i+p})$$

is increasing.

**Example**

*Correlation between a seven and a nine year forward is higher than the correlation between a three and a five year forward.*
Semi-parametric full rank correlation structures

The assumption that the sequence

\[ i \longrightarrow \frac{b_i}{b_{i+1}} \]

is increasing forces that for fixed \( p \)

\[ i \longrightarrow \rho_{i,i+p} = \text{Cor}(\Delta L_i, \Delta L_{i+p}) \]

is increasing.

Example

Correlation between a seven and a nine year forward is higher than the correlation between a three and a five year forward.
Semi-parametric full rank correlation structures

**Question**

Is \( \min(b_i, b_j) / \max(b_i, b_j) \) a correlation structure at all?

Take \( i \to b_i \) increasing with \( b_0 = 0, \ b_1 = 1 \) and set
\[
a_i := \sqrt{b_i^2 - b_{i-1}^2}.
\]
Consider standard normal i.i.d r.v. \( Z_k \) and set
\[
Y_i := \sum_{k=1}^{i} a_k Z_k.
\]

Then, for \( i \leq j \)
\[
\text{Cov}(Y_i, Y_j) = \sum_{k=1}^{i} a_k^2 = b_i^2
\]
and the correlations are
\[
\rho_{Y_i, Y_j} = b_i / b_j.
\]
Semi-parametric full rank correlation structures

**Question**

Is $\min(b_i, b_j)/\max(b_i, b_j)$ a correlation structure at all?

Take $i \rightarrow b_i$ increasing with $b_0 = 0$, $b_1 = 1$ and set $a_i := \sqrt{b_i^2 - b_{i-1}^2}$. Consider standard normal i.i.d r.v. $Z_k$ and set

$$Y_i := \sum_{k=1}^{i} a_k Z_k.$$

Then, for $i \leq j$

$$\text{Cov}(Y_i, Y_j) = \sum_{k=1}^{i} a_k^2 = b_i^2$$

and the correlations are $\rho_{Y_i, Y_j} = b_i/b_j$. 

Denis Belomestny (WIAS)  
Extended Libor Models and Their Calibration  
Haindorf, 7 Februar 2008  
13 / 43
Semi-parametric full rank correlation structures

For every correlation structure $b_i/b_j$, $i < j$, with increasing $b_i$ and $b_i/b_{i+1}$ it holds

$$b_i = \exp\left[\sum_{l=2}^{M} \min(l - 1, i - 1) \Delta_l\right]$$

for a nonnegative numbers $\Delta_i$, $\Delta_i \geq 0$, $2 \leq i \leq M$. In particular,

$$\Delta_i := -(\ln b_{i-1} + \ln b_{i+1} - 2 \ln b_i), \quad 2 \leq i \leq M - 1,$$

$$\Delta_M := \ln b_m - \ln b_{m-1}.$$. 
Semi-parametric full rank correlation structures

The following representation thus holds

\[ \varrho_{ij} = \frac{\min(b_i, b_j)}{\max(b_i, b_j)} = \exp \left[ - \sum_{l=i+1}^{M} \min(l - i, |j - i|) \Delta_l \right], \]

\[ \Delta_l \geq 0, \quad 2 \leq l \leq M. \]

From this representation we may derive conveniently various low parametric structures consistent with the structure

\[ \varrho_{ij} = \frac{\min(b_i, b_j)}{\max(b_i, b_j)}. \]
Example: a two parametric structure

The choice $\Delta_2 = \ldots = \Delta_{M-1} =: \alpha > 0$ and $\Delta_M =: \beta > 0$ gives the correlation structure

$$\rho_{ij} = e^{-|i-j| \left( \beta + \alpha \left( \frac{m - i + j + 1}{2} \right) \right)}, \quad i, j = 1, \ldots, m.$$  

To gain parameter stability we set,

$$\eta := \alpha (m-1)(m-2)/2,$$

$$\rho_{\infty} := \rho_{1m} = \exp[-\alpha (m-1)(m-2)/2 - (m-1)\beta]$$

and get a two parametric structure:

$$\rho_{ij} = e^{-\frac{|i-j|}{M-1} \left( - \ln \rho_{\infty} + \eta \frac{M-i-j+1}{M-2} \right)}, \quad 0 < \eta < -\ln \rho_{\infty}$$

with $i, j = 1, \ldots, M$. 
General Libor Model under $P_{M+1}$

For $i = 1, \ldots, M$

$$\frac{dL_i(t)}{L_i(t)} = A_i^{(M+1)}(dt) + \Gamma_i^T d\mathcal{W}^{(M+1)}(t) + \int_E \psi_i(t, u) \left( \mu - \nu^{(M+1)} \right) (dt, du)$$

- $A_i$ are predictable drift processes
- $\mathcal{W}^{(M+1)}$ is a $D$-dimensional Brownian motion under $P_{M+1}$
- $\Gamma_i$ are predictable $D$-dimensional volatility processes
- $\omega \to \mu(dt, du, \omega)$ is a random point measure on $\mathbb{R}_+ \times E$ with $P_{M+1}$-compensator $\nu^{(M+1)}(dt, du)$
General Libor Model under $P_{M+1}$

For $i = 1, \ldots, M$

\[
\frac{dL_i(t)}{L_i(t)} = A_{i}^{(M+1)}(dt) + \Gamma_i^T d\mathcal{W}^{(M+1)}(t) + \int_E \psi_i(t, u) \left( \mu - \nu^{(M+1)} \right) (dt, du)
\]

- $A_i$ are predictable drift processes
- $\mathcal{W}^{(M+1)}$ is a $D$-dimensional Brownian motion under $P_{M+1}$
- $\Gamma_i$ are predictable $D$-dimensional volatility processes
- $\omega \rightarrow \mu(dt, du, \omega)$ is a random point measure on $\mathbb{R}_+ \times E$ with $P_{M+1}$-compensator $\nu^{(M+1)}(dt, du)$
General Libor Model under $P_{M+1}$

For $i = 1, \ldots, M$

\[
\frac{dL_i(t)}{L_i(t)} = A_i^{(M+1)}(dt) + \Gamma_i^T d\mathcal{W}^{(M+1)}(t) + \int_{E} \psi_i(t, u) \left( \mu - \nu^{(M+1)}(dt, du) \right)
\]

- $A_i$ are predictable drift processes
- $\mathcal{W}^{(M+1)}$ is a $D$-dimensional Brownian motion under $P_{M+1}$
- $\Gamma_i$ are predictable $D$-dimensional volatility processes
- $\omega \rightarrow \mu(dt, du, \omega)$ is a random point measure on $\mathbb{R}_+ \times E$ with $P_{M+1}$-compensator $\nu^{(M+1)}(dt, du)$
General Libor Model under $P_{M+1}$

For $i = 1, \ldots, M$

\[
\frac{dL_i(t)}{L_i(t)} = A_i^{(M+1)}(dt) + \Gamma_i^T d\mathcal{W}^{(M+1)}(t) + \int_E \psi_i(t, u) \left( \mu - \nu^{(M+1)}(u) \right) (dt, du)
\]

- $A_i$ are predictable drift processes
- $\mathcal{W}^{(M+1)}$ is a $D$-dimensional Brownian motion under $P_{M+1}$
- $\Gamma_i$ are predictable $D$-dimensional volatility processes
- $\omega \rightarrow \mu(dt, du, \omega)$ is a random point measure on $\mathbb{R}_+ \times E$ with $P_{M+1}$-compensator $\nu^{(M+1)}(dt, du)$
General Libor Model under $P_{M+1}$

For $i = 1, \ldots, M$

$$\frac{dL_i(t)}{L_i(t)} = A_i^{(M+1)}(dt) +$$

$$\Gamma_i^T d\mathcal{W}^{(M+1)}(t) + \int_{E} \psi_i(t, u) \left( \mu - \nu^{(M+1)}(dt, du) \right)$$

- $A_i$ are predictable drift processes
- $\mathcal{W}^{(M+1)}$ is a $D$-dimensional Brownian motion under $P_{M+1}$
- $\Gamma_i$ are predictable $D$-dimensional volatility processes
- $\omega \rightarrow \mu(dt, du, \omega)$ is a random point measure on $\mathbb{R}_+ \times E$ with $P_{M+1}$-compensator $\nu^{(M+1)}(dt, du)$
Libor Model under $P_{M+1}$

We consider random point measures $\mu$ of finite activity satisfying

$$\int_{E} \psi_i(t, u) \left( \mu - \nu^{(M+1)} \right) (dt, du) = d \left[ \sum_{l=1}^{N_t} \psi_i(s_l, u_l) \right],$$

where $(s_l, u_l) \in \mathbb{R}^+ \times E$ are jumps of $\mu$ and

- $N_t$ is a Poisson process with intensity $\lambda$
- $E = \mathbb{R} \times \ldots \times \mathbb{R}$
- $u_l \in \mathbb{R}^m$ is distributed with $p_1(dx_1) \cdot \ldots \cdot p_m(dx_m)$
Libor Model under $P_{M+1}$

We consider random point measures $\mu$ of finite activity satisfying

$$\int_E \psi_i(t, u) \left( \mu - \nu^{(M+1)} \right) (dt, du) = d \left[ \sum_{l=1}^{N_t} \psi_i(s_l, u_l) \right],$$

where $(s_l, u_l) \in \mathbb{R}_+ \times E$ are jumps of $\mu$ and

- $N_t$ is a Poisson process with intensity $\lambda$

- $E = \mathbb{R} \times \ldots \times \mathbb{R}$

- $u_l \in \mathbb{R}^m$ is distributed with $p_1(dx_1) \cdot \ldots \cdot p_m(dx_m)$
We consider random point measures $\mu$ of finite activity satisfying

$$
\int_E \psi_i(t, u) \left( \mu - \nu^{(M+1)} \right) (dt, du) = d \left[ \sum_{l=1}^{N_t} \psi_i(s_l, u_l) \right],
$$

where $(s_l, u_l) \in \mathbb{R}_+ \times E$ are jumps of $\mu$ and

- $N_t$ is a Poisson process with intensity $\lambda$
- $E = \mathbb{R} \times \ldots \times \mathbb{R}$
- $u_l \in \mathbb{R}^m$ is distributed with $p_1(dx_1) \cdots p_m(dx_m)$
Libor Model under $P_{M+1}$

We consider random point measures $\mu$ of finite activity satisfying

$$
\int_E \psi_i(t, u) \left( \mu - \nu^{(M+1)} \right) (dt, du) = d \left[ \sum_{l=1}^{N_t} \psi_i(s_l, u_l) \right],
$$

where $(s_l, u_l) \in \mathbb{R}_+ \times E$ are jumps of $\mu$ and

- $N_t$ is a Poisson process with intensity $\lambda$
- $E = \mathbb{R} \times \ldots \times \mathbb{R}$
- $u_l \in \mathbb{R}^m$ is distributed with $p_1(dx_1) \cdot \ldots \cdot p_m(dx_m)$
Drift term under $P_{M+1}$

The requirement that $L_i$ is a martingale under $P^{(M+1)}$ implies

$$A_i^{(M+1)}(dt) = -\sum_{j=i+1}^{M} \frac{\delta_j L_j}{1 + \delta_j L_j} \Gamma_i^T \Gamma_j dt + \lambda(t) dt \int_{\mathbb{R}^m} \psi_i(u, t) \rho(du) \left[ \prod_{j=i+1}^{M} \left( 1 + \frac{\delta_j L_j - \psi_i(t, u)}{1 + \delta_j L_j} \right) \right].$$
Dynamic of $L_i$ under $P_{i+1}$

Since $L_i$ is a martingale under $P_{i+1}$

$$\frac{dL_i}{L_i} = \Gamma_i^T dW^{(i+1)} + \int_E \psi_i(t, u)(\mu - \nu^{(i+1)})(dt, du),$$

where

$$dW^{(i+1)} = - \sum_{j=i+1}^{M} \frac{\delta_j L_j^-}{1 + \delta_j L_j^-} \Gamma_i dt + dW^{(M+1)}$$

is a standard Brownian motion under $P_{i+1}$ and

$$\nu^{(i+1)}(dt, du) = \nu^{(M+1)}(dt, du) \left[ \prod_{j=i+1}^{M} \left( 1 + \frac{\delta_j L_j^- \psi_j(t, u)}{1 + \delta_j L_j^-} \right) \right].$$
Dynamic of $L_i$ under $P_{i+1}$

The logarithmic version reads as

$$d \ln(L_i) = A^{(i+1)}(dt) + \Gamma_i^T dW^{(i+1)} + d\left[\sum_{i=1}^{N_t} \phi_i(s_l, u_l)\right]$$

with $\phi_i = \ln(1 + \psi_i)$ and

$$A^{(i+1)}(dt) = -\frac{1}{2} |\Gamma_i|^2 dt - \int_{\mathbb{R}^m} \psi_i(t, u)\nu^{(i+1)}(dt, du)$$

Observation

For $i < M$ the new compensator $\nu^{(i+1)}$ is not deterministic and $\ln(L_i)$ is generally not affine under $P_{i+1}$. 
Dynamic of $L_i$ under $P_{i+1}$

The logarithmic version reads as

$$d \ln(L_i) = A^{(i+1)}(dt) + \Gamma_i^T dW^{(i+1)} + d \left[ \sum_{i=1}^{N_t} \phi_i(s_l, u_l) \right]$$

with $\phi_i = \ln(1 + \psi_i)$ and

$$A^{(i+1)}(dt) = -\frac{1}{2} |\Gamma_i|^2 dt - \int_{\mathbb{R}^m} \psi_i(t, u) \nu^{(i+1)}(dt, du)$$

Observation

For $i < M$ the new compensator $\nu^{(i+1)}$ is not deterministic and $\ln(L_i)$ is generally not affine under $P_{i+1}$. 
Caplets

The price of $j$-th caplet at time zero is given by

$$C_j(K) = \delta_j B_{j+1}(0) E_{P_{j+1}} [(L_j(T_j) - K)^+]$$
Pricing Caplets under $P_{i+1}$

In terms of \textbf{log-forward moneyness} $v = \ln(K/L_j(0))$

$$C_j(v) := \delta_j B_{j+1}(0) L_j(0) E_{P_{j+1}}[e^{X_j(t)} - e^v]^+],$$

with $X_j(t) = \log(L_j(t)) - \log(L_j(0))$.

Define

$$O_j(v) = E_{P_{j+1}}[e^{X_j(t)} - e^v]^+] - (1 - e^v)^+,$$

then

$$F\{O_j\}(z) := \int_{\mathbb{R}} O_j(v) e^{ivz} \, dz = \frac{1 - \Phi_{P_{j+1}}(z - i; T_j)}{z(z - i)}.$$
Pricing and Calibration

Pricing of Caplets

Pricing Caplets under $P_{i+1}$

In terms of log-forward moneyness $\nu = \ln(K/L_j(0))$

$$C_j(\nu) := \delta_j B_{j+1}(0) L_j(0) E_{P_{j+1}}[(e^{X_j(t)} - e^{\nu})^+]$$

with $X_j(t) = \log(L_j(t)) - \log(L_j(0))$.

Define

$$O_j(\nu) = E_{P_{j+1}}[(e^{X_j(t)} - e^{\nu})^+] - (1 - e^{\nu})^+,$$

then

$$F\{O_j\}(z) := \int_{\mathbb{R}} O_j(\nu) e^{i\nu z} \, dz = \frac{1 - \Phi_{P_{j+1}}(z - i; T_j)}{z(z - i)}.$$
Characteristic Function of $L_M$ under $P_{M+1}$

Since

$$
\frac{d \ln(L_M)}{dt} = -\frac{1}{2} |\Gamma_i|^2 dt + \Gamma_i^T d\mathcal{N}^{(M+1)}(t) + d\left[ \sum_{l=1}^{N_t} \phi_i(s_l, u_l) \right]
$$

and $N_t$, $\mathcal{N}^{(M+1)}$ and $u_l$ are mutually independent

$$
\Phi_{P_{M+1}}(z; T) = \Phi_{P_{M+1}}^C(z; T) \Phi_{P_{M+1}}^J(z; T),
$$

where $\Phi_{P_{M+1}}^C(z; T)$ ( $\Phi_{P_{M+1}}^J(z; T)$) is the c.f. of continuous (jump) part.
Specification Analysis: Continuous Part

For some predictable vector volatility process \((v_1(t), \ldots, v_d(t))\) define

\[
\Gamma_i = \begin{pmatrix}
\sqrt{1 - r_{SV}^2 \gamma_i 1} \\
\sqrt{1 - r_{SV}^2 \gamma_i 2} \\
\ddots \\
r_{SV} \beta_i 1 \sqrt{v_1(t)} \\
r_{SV} \beta_i d \sqrt{v_d(t)}
\end{pmatrix}, \quad \mathcal{W}^{(M+1)} = \begin{pmatrix}
W_1^{(M+1)} \\
W_2^{(M+1)} \\
\ddots \\
W_d^{(M+1)} \\
\overline{W}_1^{(M+1)} \\
\overline{W}_2^{(M+1)} \\
\ddots \\
\overline{W}_d^{(M+1)}
\end{pmatrix}
\]

with mutually independent \(d\)-dimensional Brownian motions \(W^{(M+1)}\) and \(\overline{W}^{(M+1)}\).
Specification Analysis: Continuous Part

Let

\[ \gamma_i(t) = c_i g(T_i - t)e_i, \quad e_i \in \mathbb{R}^d, \]

where

- \( c_i > 0 \) are loading factors
- \( g_i(\cdot) \) is a scalar volatility function
- \( e_i \) are unit vectors coming from the decomposition of the correlation matrix \( \zeta \)

\[ \zeta_{ij} = e_i^\top e_j, \quad 1 \leq i, j \leq M, \]

be a deterministic volatility structure of the input Libor market model calibrated to ATM caps and ATM swaptions.
Specification Analysis: Continuous Part

Define a new \textbf{time independent} volatility structure via

\[
\beta_i \beta_j = \frac{1}{\min(i, j)} \sum_{k=1}^{\min(i, j)} \frac{1}{T_k} \int_0^{T_k} \gamma_i^T(t) \gamma_j(t) \, dt.
\]

Remark

The covariance of the process \( \xi_i(t) := \int_0^t \Gamma_i^T(t) d\mathcal{W}^{(M+1)} \) satisfies

\[
\text{cov}(\xi_i(t), \xi_j(t)) \approx \int_0^t \gamma_i^T(t) \gamma_j(t) \, ds
\]

and is approximately the same as in the input LMM.
Specifcation Analysis: Continuous Part

Define a new time independent volatility structure via

$$\beta_i^T \beta_j = \frac{1}{\min(i, j)} \sum_{k=1}^{\min(i,j)} \frac{1}{T_k} \int_0^{T_k} \gamma_i^T(t) \gamma_j(t) \, dt.$$ 

Remark

The covariance of the process $\xi_i(t) := \int_0^t \Gamma_i^T(t) d\mathcal{W}^{(M+1)}$ satisfies

$$\text{cov}(\xi_i(t), \xi_j(t)) \approx \int_0^t \gamma_i^T(t) \gamma_j(t) \, ds$$

and is approximately the same as in the input LMM.
Two possible specifications for the volatility process $v$

- **Stochastic Volatility Heston Model**
  
  $$d\nu_k = \kappa_k (1 - \nu_k)dt + \sigma_k \rho_k \sqrt{\nu_k} dW_k^{(M+1)} + \sigma_k \sqrt{(1 - \rho_k^2)} \nu_k dV_k^{(M+1)},$$

- **Stochastic Volatility BN Model**
  
  $$d\nu_k = \kappa_k \nu_k dt + \sigma_k \rho_k dW_k^{(M+1)} + \sigma_k \sqrt{(1 - \rho_k^2)} dV_k^{(M+1)}.$$
Specification Analysis: Continuous Part

Two possible specifications for the volatility process $\nu$

- **Stochastic Volatility Heston Model**

\[
d\nu_k = \kappa_k (1 - \nu_k) dt + \sigma_k \rho_k \sqrt{\nu_k} d\mathcal{W}_k^{(M+1)} + \sigma_k \sqrt{1 - \rho_k^2} \sqrt{\nu_k} dV_k^{(M+1)},
\]

- **Stochastic Volatility BN Model**

\[
d\nu_k = \kappa_k \nu_k dt + \sigma_k \rho_k d\mathcal{W}_k^{(M+1)} + \sigma_k \sqrt{1 - \rho_k^2} dV_k^{(M+1)}.
\]
Specification Analysis: Continuous Part

Two possible specifications for the volatility process $\nu$

- **Stochastic Volatility Heston Model**

  \[ dv_k = \kappa_k (1 - \nu_k) dt + \sigma_k \varrho_k \sqrt{\nu_k} dW_k^{(M+1)} + \sigma_k \sqrt{1 - \varrho_k^2} \nu_k dV_k^{(M+1)}, \]

- **Stochastic Volatility BN Model**

  \[ dv_k = \kappa_k \nu_k dt + \sigma_k \varrho_k dW_k^{(M+1)} + \sigma_k \sqrt{1 - \varrho_k^2} dV_k^{(M+1)}. \]
Specification Analysis

It holds

$$
\Phi_{P_{M+1}}^C(z; T) = \Phi_{D,P_{M+1}}^C(z; T) \Phi_{SV,P_{M+1}}^C(z; T),
$$

where

$$
\Phi_{D,P_{M+1}}^C(z; T) = \exp \left( -\frac{1}{2} \theta_M^2(T) \left( z^2 + i z \right) \right), \quad \theta_M^2(T) = \int_0^T |\gamma_M|^2 dt
$$

and

$$
\Phi_{SV,P_{M+1}}^C(z; T) = \exp \left( A_M(z; T) + B_M(z; T) \right)
$$
In particular

\[ A_M(z; T) = \frac{\kappa_M}{\sigma^2_M} \left\{ (a_M + d_M) T - 2 \ln \left[ \frac{1 - g_M e^{d_M T}}{1 - g_M} \right] \right\} \]

\[ B_M(z; T) = \frac{(a_M + d_M)(1 - e^{d_M T})}{\sigma^2_M(1 - g_M e^{d_M T})} \]

and

\[ a_M = \kappa_M - i \varrho_M \omega_M z \]
\[ d_M = \sqrt{a_M^2 + \omega_M^2 (z^2 + iz)} \]
\[ g_M = \frac{a_M + d_M}{a_M - d_M}, \quad \omega_M = r_{SV} \beta_{MM} \sigma_M \]
As can be easily seen

\[ \lim_{z \to \infty} \frac{A_M(z; T)}{z} = -\alpha_M \omega_M \left( i \varrho_M + \sqrt{1 - \varrho_M^2} \right) T \]

and

\[ \lim_{z \to \infty} \frac{B_M(z; T)}{z} = -\frac{\sqrt{1 - \varrho_M^2} + i \varrho_M}{\sigma_M} \]

with

\[ \alpha_M := \frac{\kappa_M}{\sigma_M^2} \]
Specification Analysis

Let us take $\phi_i(u, t) = u^\top \beta_i$, then the characteristic function of the jump part is given by

$$
\Phi_{P_{M+1}}^J(z; T) = \exp \left( \lambda T \int_\mathbb{R} (e^{izv} - 1) \mu_M(v) \, dv \right),
$$

where $\mu_M$ is the density of $u^\top \beta_M(t)$.

Observation

*Due to the Riemann-Lebesgue theorem*

$$
\Phi_{P_{M+1}}^J(z; T) \to \exp(-\lambda T), \quad |z| \to \infty.
$$
Specification Analysis

Let us take $\phi_i(u, t) = u^\top \beta_i$, then the characteristic function of the jump part is given by

$$\Phi_{P_{M+1}}^J(z; T) = \exp \left( \lambda T \int_{\mathbb{R}} (e^{izv} - 1) \mu_M(v) \, dv \right),$$

where $\mu_M$ is the density of $u^\top \beta_M(t)$.

Observation

Due to the Riemann-Lebesgue theorem

$$\Phi_{P_{M+1}}^J(z; T) \rightarrow \exp(-\lambda T), \quad |z| \rightarrow \infty.$$
Specification Analysis: Asymptotic Properties

Computing sequentially

\[ \mathcal{L}_2 := \lim_{z \to \infty} \log(\Phi_{P_{M+1}}(z; T))/z^2, \]

\[ \mathcal{L}_1 := \lim_{z \to \infty} \left[ \log(\Phi_{P_{M+1}}(z; T))/z - (z + i)\mathcal{L}_2 \right], \]

\[ \mathcal{L}_0 := \lim_{z \to \infty} \left[ \log(\Phi_{P_{M+1}}(z; T)) - (z^2 + iz)\mathcal{L}_2 - z\mathcal{L}_1 \right], \]

we get

\[ \mathcal{L}_0 = -\lambda, \quad \mathcal{L}_2 = -\frac{1}{2}\theta_M^2(T) \]

and

\[ \text{Re } \mathcal{L}_1 = -\frac{\sqrt{1 - \rho_M^2}}{\sigma_M} - \alpha_M\omega_M\sqrt{1 - \rho_M^2} T, \]

\[ \text{Im } \mathcal{L}_1 = -\frac{\theta_M}{\sigma_M} - \alpha_M\omega_M\rho_M T \]
Pricing and Calibration
Specification Analysis

**Specification Analysis: Asymptotic Properties**

Computing sequentially

\[ \mathcal{L}_2 := \lim_{z \to \infty} \frac{\log(\Phi_{P_{M+1}}(z; T))}{z^2}, \]

\[ \mathcal{L}_1 := \lim_{z \to \infty} \left[ \frac{\log(\Phi_{P_{M+1}}(z; T))}{z} - (z + i)\mathcal{L}_2 \right], \]

\[ \mathcal{L}_0 := \lim_{z \to \infty} \left[ \log(\Phi_{P_{M+1}}(z; T)) - (z^2 + iz)\mathcal{L}_2 - z\mathcal{L}_1 \right], \]

we get

\[ \mathcal{L}_0 = -\lambda, \quad \mathcal{L}_2 = -\frac{1}{2} \theta_M^2(T) \]

and

\[ \text{Re} \mathcal{L}_1 = -\sqrt{1 - \rho_M^2} \frac{\alpha_M\omega_M}{\sigma_M} \sqrt{1 - \rho_M^2} T, \quad \text{Im} \mathcal{L}_1 = -\frac{\rho_M}{\sigma_M} - \alpha_M\omega_M\rho_M T \]
Specification Analysis: Asymptotic Properties

Computing sequentially

\[ L_2 := \lim_{z \to \infty} \log(\Phi_{P_M+1}(z; T))/z^2, \]

\[ L_1 := \lim_{z \to \infty} \left[ \log(\Phi_{P_M+1}(z; T))/z - (z + i)L_2 \right], \]

\[ L_0 := \lim_{z \to \infty} \left[ \log(\Phi_{P_M+1}(z; T)) - (z^2 + iz)L_2 - zL_1 \right], \]

we get

\[ L_0 = -\lambda, \quad L_2 = -\frac{1}{2} \theta^2_M(T) \]

and

\[ \text{Re} \ L_1 = -\frac{\sqrt{1 - \phi^2_M}}{\sigma_M} - \alpha_M \omega_M \sqrt{1 - \phi^2_M} T, \quad \text{Im} \ L_1 = -\frac{\phi_M}{\sigma_M} - \alpha_M \omega_M \phi_M T \]
Parameters Estimation: Linearization

Observation

From the knowledge of $\mathcal{L}_1(T)$ for two different $T$ one can reconstruct all parameters of the SV process.

Theorem

\[ \psi_{P_{M+1}}(z; T) := \log(\Phi_{P_{M+1}}(z; T)) \]
\[ = \mathcal{L}_2(z^2 + iz) + \mathcal{L}_1z + \mathcal{L}_0 + R_0 + R_1(z), \]

where $R_0 = R_0(\alpha_M, \kappa_M, \varrho_M, \omega_M)$ is a constant not depending on $\lambda$ and

\[ R_1(z) \to 0, \quad |z| \to \infty. \]
Observation

*From the knowledge of $\mathcal{L}_1(T)$ for two different $T$ one can reconstruct all parameters of the SV process.*

Theorem

\[ \psi_{PM+1}(z; T) := \log(\Phi_{PM+1}(z; T)) = \mathcal{L}_2(z^2 + iz) + \mathcal{L}_1 z + \mathcal{L}_0 + R_0 + R_1(z), \]

where $R_0 = R_0(\alpha_M, \kappa_M, \varrho_M, \omega_M)$ is a constant not depending on $\lambda$ and $R_1(z) \to 0, \ |z| \to \infty.$
Parameters Estimation: Projection Estimators

We find estimates for $\mathcal{L}_2$, $\mathcal{L}_1$ and $\mathcal{L}_0$ in the form of weighted averages

\[
\hat{\mathcal{L}}_{2,U} := \int \text{Re}(\tilde{\Psi}_{P_{M+1}}(u))w^U_2(u) \, du,
\]
\[
\hat{\mathcal{L}}_{1,U} := \int \text{Im}(\tilde{\Psi}_{P_{M+1}}(u))w^U_1(u) \, du - i\hat{\mathcal{L}}_{2,U},
\]
\[
\hat{\mathcal{L}}_{0,U} := \int \text{Re}(\tilde{\Psi}_{P_{M+1}}(u))w^U_0(u) \, du - \hat{R}_0
\]

with

\[
\tilde{\Psi}_{P_{M+1}}(u) := \ln \left(1 - u(u + i)F\{\tilde{O}_M\}(u + i)\right).
\]
Parameters Estimation

The weights are given by

\[ w_2^U = U^{-3} w_2(u/U), \quad w_1^U = U^{-2} w_1(u/U), \quad w_0^U = U^{-1} w_0(u/U), \]

where

\[
\int_{-1}^{1} w_2(u) du = 0, \quad \int_{-1}^{1} u w_2(u) du = 0, \quad \int_{-1}^{1} u^2 w_2(u) du = 1,
\]

\[
\int_{-1}^{1} w_1(u) du = 0, \quad \int_{-1}^{1} u w_1(u) du = 1,
\]

\[
\int_{-1}^{1} w_0(u) du = 1, \quad \int_{-1}^{1} u w_0(u) du = 0, \quad \int_{-1}^{1} u^2 w_0(u) du = 0.
\]
Parameters Estimation: Jump distribution

Define

\[ F\{\hat{\mu}_M\}(z) = \tilde{\Psi}_{P_{M+1}}(z; T) - \hat{\mathcal{L}}_2(z^2 + iz) - \hat{\mathcal{L}}_1 z - \hat{\mathcal{L}}_0 - \hat{R}_0 \]

or equivalently

\[ \hat{\mu}_M := F^{-1} \left[ \left( \tilde{\Psi}_{P_{M+1}}(\cdot; T) - \hat{\mathcal{L}}_2(\cdot^2 + i\cdot) - \hat{\mathcal{L}}_1 \cdot - \hat{\mathcal{L}}_0 - \hat{R}_0 \right) 1_{[-U,U]}(\cdot) \right] \]

Remark

Due to lack of data and numerical errors \( \hat{\mu}_M \) may not be a density and needs to be corrected.
Parameters Estimation: Jump distribution

Define

\[ F\{\hat{\mu}_M\}(z) = \tilde{\Psi}_{P_{M+1}}(z; T) - \hat{\mathcal{L}}_2(z^2 + iz) - \hat{\mathcal{L}}_1 z - \hat{\mathcal{L}}_0 - \hat{R}_0 \]

or equivalently

\[ \hat{\mu}_M := F^{-1} \left[ (\tilde{\Psi}_{P_{M+1}}(\cdot; T) - \hat{\mathcal{L}}_2(\cdot^2 + i\cdot) - \hat{\mathcal{L}}_1 \cdot - \hat{\mathcal{L}}_0 - \hat{R}_0) 1_{[-U,U]}(\cdot) \right] \]

**Remark**

*Due to lack of data and numerical errors \( \hat{\mu}_M \) may not be a density and needs to be corrected.*
Parameters Estimation: Further optimization

Upon finding

\[ (\hat{\mathcal{L}}_0, U, \hat{\mathcal{L}}_1, U, \hat{\mathcal{L}}_2, U) \longrightarrow \hat{T} := (\hat{\sigma}_M, \hat{\varrho}_M, \hat{\kappa}_M, \lambda) \]

we may

- consider \( \hat{T} \) as a final set of parameters or
- consider nonlinear least-squares

\[ \mathcal{J}(T) = \sum_{i=1}^{N} w_i | C_M^T(K_i) - C_M(K_i) |^2 \]

and minimize \( \mathcal{J}(T) \) over the parametric set \( S \subset \mathbb{R}^4 \) taking as initial value \( \hat{T} \).
Approximative dynamics of $L_i$ under $P_{i+1}$

It holds approximately

$$\frac{dL_i}{L_i} \approx \Gamma_i^T dW^{(i+1)} + \int_E e^{u^T \beta_j} (\mu - \tilde{\nu}^{(i+1)}) (dt, du),$$

where $dW^{(i+1)}$ is a standard Brownian motion under $P_{i+1}$ and

$$\tilde{\nu}^{(i+1)} (dt, du) = \nu^{(M+1)} (dt, du) \left[ \prod_{j=i+1}^M \left( 1 + \frac{\delta_j L_j(0) e^{u^T \beta_j}}{1 + \delta_j L_j(0)} \right) \right].$$
Approximative dynamics of $v_k$ under $P_{i+1}$

By freezing the Libors at their initial values we obtain an approximative $v_k$ dynamics

$$dv_k \approx \kappa_k^{(i+1)} \left( \theta_k^{(i+1)} - v_k \right) dt + \sigma_k \sqrt{v_k} \left( \varrho_k d\tilde{W}_k^{(i+1)} + \sqrt{1 - \varrho_k^2} d\tilde{W}_k^{(i+1)} \right)$$

with reversion speed parameter

$$\kappa_k^{(i+1)} := \left( \kappa_k - r_{SV} \sigma_k \varrho_k \sum_{j=i+1}^{M} \frac{\delta_j L_j(0)}{1 + \delta_j L_j(0) \beta_{jk}} \right),$$

and mean reversion level

$$\theta_k^{(i+1)} := \frac{\kappa_k}{\kappa_k^{(i+1)}}.$$
Pricing Caplets under $P_{M+1}$

The price of $j$-th caplet at time zero can be alternatively written as

$$C_j(K) = \delta_j B_{M+1}(0) E_{P_{M+1}} \left[ \frac{B_{j+1}(T_j)}{B_{M+1}(T_j)} (L_j(T_j) - K)^+ \right]$$

Note that

$$\frac{B_{j+1}(T_j)}{B_{M+1}(T_j)} = \prod_{k=j+1}^{M} (1 + \delta_k L_k(T_j))$$

$$= \prod_{k=j+1}^{M} (1 + \delta_k) E_\xi \exp \left( \sum_{k=j+1}^{M} \xi_k \ln(L_k(T_j)) \right),$$

where $\{\xi_k\}_{k=j+1}^{M}$ are independent random variables and each $\xi_k$ takes two values 0 and 1 with probabilities $1/(1 + \delta_k)$ and $\delta_k/(1 + \delta_k)$. 
Pricing Caplets under $P_{M+1}$

Thus,

$$F\{\mathcal{O}_j\}(z) = \frac{1 - E_{\xi} \Phi_{M+1}(z - i, \xi_{j+1}, \ldots, \xi_M)}{z(z - i)},$$

where $\Phi_{M+1}(z_j, z_{j+1}, \ldots, z_M)$ is the joint characteristic function of $(\ln(L_j(T_j)), \ldots, \ln(L_M(T_j)))$ under $P_{M+1}$.

**Remark**

*Instead of terminal measure $P_{M+1}$ we could consider $P_{l+1}$ with $1 < l < M + 1$.***
Pricing Caplets under $P_{M+1}$

Thus,

$$F\{\mathcal{O}_j\}(z) = \frac{1 - E_\xi \Phi_{M+1}(z - i, \xi_{j+1}, \ldots, \xi_M)}{z(z - i)},$$

where $\Phi_{M+1}(z_j, z_{j+1}, \ldots, z_M)$ is the joint characteristic function of $(\ln(L_j(T_j)), \ldots, \ln(L_M(T_j)))$ under $P_{M+1}$.

**Remark**

*Instead of terminal measure $P_{M+1}$ we could consider $P_{l+1}$ with $1 < l < M + 1$.***
Calibration in work

Calibration results for 14.08.2007

Caplet volas for different caplet periods

[17.5, 18]

[15, 15.5]

[12.5, 13]

[10, 10.5]
Belomestny, D. and Spokoiny, V.  
Spatial aggregation of local likelihood estimates with applications to classification,  

Belomestny, D. and Reiβ, M.  
*Spectral calibration of exponential Lévy models*,  

Belomestny, D. and Schoenmakers, J.  
A jump-diffusion Libor model and its robust calibration,  
*SFB649 Discussion Paper*, 2006, **037**.