Diploma Thesis

Reference-Dependent Consumer Preferences and Price Setting

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Reference-Dependent
Consumer Preferences and Price Setting

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Abstract

In this thesis, a model is presented which assumes reference-dependent preferences in the style of Tversky and Kahneman (1991). It attempts to answer the question whether consumers' loss aversion leads to price stickiness, assuming that firms maximize profits. This is done by analyzing a monopolistic firm with constant marginal cost which faces a loss-averse representative customer.

It is shown that the demand functions of loss-averse consumers are kinked; thus, so is the firm's profit function. Moreover, the kink is shown to be so pronounced that the maximum profit is attained at the associated price, as long as the producer's unit cost lies inside a certain interval. This is interpreted as "price stickiness", since the profit-maximizing price does not respond to shocks on the firm's unit cost when the shocks are small enough.

Two different specifications of consumers' reference points are proposed: in the basic model, the reference point is set exogenously and not allowed to vary over time. In an extended version of the model, the reference point depends on past consumption. The extended version is found to mimic real market prices better than the basic version. My way of modeling the determination of the reference points is defended against two competing hypotheses about their nature, proposed by Sibly (2002) and by Heidhues and Kőszegi (2004).

The extended version is modified in a second extended version to allow for consumer heterogeneity, i.e. consumers have idiosyncratic reference points. This way, it is shown that the results of the first extended version survive aggregation. Because of this result, I argue that customer heterogeneity can be neglected and the model can be used in its representative-agent version, which is much easier to solve than the version incorporating consumer heterogeneity.

The main result of my model is that it combines one kind of non-responsiveness of the profit-maximizing price—to cost shocks—with another kind of immediate responsiveness—to changes in demand. Such changes in demand occur, e.g., when the agents' nominal income changes. Thus, expansionary monetary policy has in most cases no real, but only inflationary effects, despite the observation that prices are constant over longer periods of time.

Finally, topics for further research are suggested: most importantly, upcoming versions of the model should incorporate forward-looking behavior of both firms and consumers; the analysis should be extended from partial to general equilibrium; and a calibrated version of the model should be compared with actual data on prices and costs thereafter. In addition to this theoretical work, further empirical research on the origins and effects of reference points is considered necessary.

1 Introduction

This thesis deals with two longstanding controversies of economic theory. The first is the extent to which nominal prices and wages can adjust quickly to a changing environment; the second is the controversy about the constancy of individuals' preferences.

Whichever side one chooses in these controversies has far-reaching consequences on what monetary and fiscal policy one considers appropriate. It is therefore easy to understand that the debate on sticky vs. flexible prices has been on the macroeconomic research agenda ever since the publication of the opus that founded macroeconomics—Keynes's (1936) *General Theory of Employment, Interest and Money*.

In the beginning—throughout two decades—inflexibility of the prices of goods was bluntly *assumed* by "Keynesian" theorists without providing a rationale for such inflexibilities. Since the 1950s, this view was attacked vehemently on different grounds by competing schools of thought. The assumption of rigid prices was replaced by the assumption that firms' pricing can be explained by mathematical models of profit-maximization. Some of the modern so-called "micro-founded" macro models still generate price stickiness, while others predict prices to adjust instantaneously to any change in the market environment. The existence or non-existence of price stickiness is crucial for the conduct of monetary policy, because if prices adjust quickly to any new information, expansionary monetary policy will likely only increase inflation and have no effect on employment whatsoever.

Regarding peoples' preferences, economists almost exclusively assume that

1. individuals' preferences are *constant* and
2. it is only the *absolute* quantities of the consumed goods that matter.

My thesis presents a model in which this convenient assumption is replaced by reference-dependent preferences in the style of the theory by Tversky and Kahneman (1991): people are assumed to be loss-averse. This way, preferences no longer depend only on the absolute amounts consumed, but also on *relative* quantities—vis-à-vis the reference point. Furthermore, by allowing the reference point to change over time, responding to past consumption, changes in tastes become endogenous.

My thesis tries to answer the question whether the dependence of preferences on reference points in connection with loss aversion leads to price stickiness, assuming that firms maximize profits.

Why should there at all exist such a connection between price rigidities and reference-dependent preferences? The answer is: loss aversion changes the shape of consumers' demand curves and, thus, of suppliers' profit functions. More specifically, as I will show in this thesis, the demand curves of loss-averse consumers are kinked; and I will also show that these kinks are so pronounced that the maximum profit is attained at the associated price, as long as the producer's marginal cost varies only inside a certain interval.

Hence, as long as the kinks do not shift, prices can be indeed constant over several periods. However, whenever the kinks do shift—e.g., due to an increase in nominal income—a change in the firm's profit-maximizing price can be the consequence. This is an interesting feature of my model: it combines one kind of non-responsiveness of the profit-maximizing price (to cost shocks) with another kind of immediate responsiveness (to changes in demand). Thus, expansionary monetary policy has hardly any real, but strong inflationary effects, despite the observation that prices are constant over longer periods of time. In addition, just like empirical findings on real markets, price series generated by my model exhibit short-run price stickiness and long-run price flexibility.

Remarkably, the characteristics of demand and price setting under consumer loss aversion have been studied rather superficially up to now. While there were several attempts to verify empirically the existence of loss aversion over prices (e.g., Putler, 1992; Hardie et al., 1993; Bell and Lattin, 2000), only a few publications dealt with developing theoretical models of firms' price setting when facing loss-averse customers: McDonald and Sibly (2001), Sibly (2002, 2004), and Heidhues and Kőszegi (2004).

Among the latter four models, in my opinion, only the one by Heidhues and Kőszegi takes a satisfactory approach to deriving demand and firms' price setting under loss aversion. Therefore, my own model will use certain elements of the Heidhues–Kőszegi model, but differ considerably in other aspects.

The remainder of this thesis is organized as follows: I start in *Section 2.1* by presenting a brief review of the major macroeconomic approaches that try to explain firms' price setting. *Section 2.2* provides an introduction to the theory of reference-dependent preferences. In *Section 2.3*, I present and criticize the findings of the abovementioned empirical studies and theoretical models which connect reference-dependent preferences and price setting.

Section 3 consists of an extensive list of empirical findings on the movement of individual prices as well as on the aggregate price level.

In *Section 4*, I then introduce my own model of a monopolistic firm facing loss-averse customers. The model is solved in *Section 5*, and simulation results are presented in *Section 6*. *Section 7* adds two variations to the basic model by making the reference points endogenous and allowing for consumer heterogeneity. *Section 8* discusses the predictions of all three variants.

Section 9 concludes.

2 Literature

2.1 Macroeconomic theory and price rigidities[1]

The flexibility of prices is an extremely important issue in macroeconomics, even in economics in general, since it is the system of relative prices which indicates relative scarceness of resources and, thus, channels production and investment decisions in such directions that they can contribute best to satisfying consumers' needs. Disturbances in the adjustment of prices to changes in the economic environment can, therefore, lead to an inefficient allocation of resources—i.e., suboptimal investment decisions, underemployed production facilities and an inadequate supply of goods to consumers.

While Keynes (1936) mentioned inflexibility of nominal wages in his *General Theory of Employment, Interest and Money*, it was not his central concern. According to Snowdon et al. (1994, see p. 63), in Keynes's theory indeed output instead of price adjustments in response to economic shocks are emphasized; however, major influence on the aggregate level of economic activity is seen to be caused by the money market through people's demand for liquidity and the influence of this demand on the interest rate (see pp. 66–67), which brings about the failure of Say's Law. While admitting that unemployment of workers can be caused by money wage inflexibility, Keynes did not consider this a major problem. In his view, expansive monetary policy could re-establish the equilibrium amount of output and employment (see p. 72).

It was only the interpretation of his work by Modigliani (1944) which led to the fact that "Keynesian economics was seen to be the economics of wage and price rigidities" (Snowdon et al., p. 74). However, this sort of "Keynesianism" suffered from a major weakness: "the lack of a convincing reason for wage and price rigidities" (p. 74). Still, it gained widespread acceptance, with its main tool being the so-called "neoclassical synthesis", which materialized in the IS–LM model. In this model, with the price level held constant, it is possible for the monetary authority to influence the level of output and, thus, employment.

Of course, the abovementioned "lack of a convincing reason for wage and price rigidities" drew severe criticism by competing schools of thought, such as monetarism in the late 1960s and a little later by the new classical "real business cycle" (RBC) approach. According to the monetarist model by Friedman (1968), in which he introduced the so-called "expectations-augmented Phillips curve", expansive monetary policy may have real consequences in the short run, but only inflationary results in the long run. His analysis abandoned the mere as-

1 This section draws heavily on the presentation in Snowdon et al. (1994).

sumption of rigid prices and provided a rationale for *temporary* non-adjustment of prices to monetary shocks by positing that expectations of future inflation are incorporated in current wages and prices, but that these expectations lag behind actual inflation. Thus, there is space for monetary policy to influence output and employment in the short run—but monetary policy is "neutral" with respect to real variables in the long run.

The new classical school as well assumed flexible prices and continuous market clearing. In addition to this, unlike the monetarist model described above, new classical models like the one by LUCAS (1972) did not assume adaptive expectations any further, but "rational expectations". Despite these ingredients, which favor neutrality of monetary policy regarding the real sphere, the LUCAS model still generates temporary nominal non-adjustment in response to monetary shocks. This is due to the assumed properties of the information set available to producers: "while a firm knows the current price of its own goods, the general price level for other markets only becomes known with a time lag" (SNOWDON ET AL., p. 194). This poses a "'signal extraction' problem [to the firms], in that they have to distinguish between relative and absolute price changes" (p. 195). In this context, unanticipated monetary shocks do have an influence on the level of output and employment, but *foreseeable* monetary policy does not—similar to the predictions of the monetarist approach.

While all aforementioned theories have in common that fluctuations in the economy's output are seen as reactions to movements in the money supply, the Real Business Cycle (RBC) revolution, started by KYDLAND AND PRESCOTT (1982), maintained that business cycles are not caused by monetary shocks at all. Instead, in RBC models they are explained by changes in productivity ("technology shocks") alone, with all nominal prices being totally flexible. In RBC models, money is not only neutral in the long run, but has no real effects even in the shortest of short runs.

However, the RBC models fail to explain certain important features of actual economic data: BURDA AND WYPLOSZ (1997) mention, e.g., that real wages are *acyclical* in reality, while RBC theory predicts *procyclical* real wages. Furthermore, according to BURDA AND WYPLOSZ, "the real money stock is procyclical and in particular a leading indicator" (p. 374). Since money plays no role whatsoever in RBC models, if technology shocks were indeed the driving force behind business cycles, we should not expect the real money stock to be leading, but rather lagging or coincident, since it simply reacts passively to the expansions and contractions of real output.

For this reason, among others, two new strands of micro-founded macro models have become popular: time-dependent and state-dependent ("New Keynesian") sticky-price models. The former assume that, in general, firms set prices to maximize profits; however, it is assumed either that prices must be set a fixed number of periods in advance or that in each period, only an exogenously given fraction of firms is allowed to adjust prices. The first models with the former feature were developed by PHELPS AND TAYLOR (1977) and FISCHER (1977), and with the latter feature by CALVO (1983).

According to SNOWDON ET AL. (see p. 74), the seminal papers incorporating *state*-dependent sticky-price models were AKERLOF AND YELLEN (1985), MANKIW (1985), and PARKIN (1986). In these models, profit-maximizing firms *choose* depending on their state and using rational expectations, not to update prices at every instant, because it is assumed to be costly to changes prices (so-called "menu costs", e.g. due to the need of printing new catalogs and price tags). This price stickiness paves the way for monetary policy to exert an influence on real variables.

Of course, New Keynesian economics is not uncontested either. The explanatory power of time-dependent and state-dependent sticky-price models will be discussed in Sections 3.1 and 3.2, respectively.

2.2 The case for reference-dependent preferences

2.2.1 *The point of departure: reference-independent preferences in riskless choice and in choice under risk*

Standard (by this I mean neoclassical) economic theory of choice postulates that human behavior can be described—at least on average—sufficiently well by an certain concept of rationality. A concise account of the axioms on which this theory rests can be found in VARIAN (1992, Ch. 7). Since I do not focus on criticizing this rationality concept as a whole, but rather on the *actual use* of the theory, it suffices to mention the concept's main implication: that all choices $\mathbf{x}$ and $\mathbf{y}$ out of the set of admissible choices $\mathbf{X}$ can be mapped via a function $u: \mathbf{X} \to \mathbb{R}$ ("utility function") to real numbers such that $\mathbf{x}$ is preferred over $\mathbf{y}$ if and only if $u(\mathbf{x}) > u(\mathbf{y})$. The agent's decisions are then modeled by maximizing the utility function subject to the agent's budget restriction.

While this theory of decision making is already subject to severe criticism[2], it has without further specification of the shape of the utility function virtually no empirical content. Most models gain their predictive power—e.g., on the existence, uniqueness or multiplicity, and characteristics of equilibria—from assumptions which are made—often only implicitly—in addition to the basic rationality postulates, for instance by assuming that the utility function is strict-

2 For a survey of objections to modeling human behavior this way, see CONLISK (1996).

ly concave. It is such commonly assumed specifications of the utility function to which my thesis suggests an alternative:

- In dynamic models, it is almost exclusively assumed that the utility function (thus, the preferences) does not change over time. However, some economists argue that people's preferences are strongly influenced by their economic environment[3], or that preferences are inherently unstable[4] and constructed on-the-fly, whenever a decision has to be made, because one can never have the entire choice set in mind.
- The utility function is normally defined only over the quantities of the goods consumed. Information and deliberation costs (to be able to decide at all)[5], as well as social components of preferences (e.g. fairness, altruism) are usually disregarded.
- Usually, the construction of the utility function lets the quantities of goods enter only in absolute terms. This way, potential dependence of preferences on reference states (anchoring effects[6], available alternative products, past consumption, the consumption of one's neighbor, the wage of fellow workers, and so on) is excluded.
- The neoclassical rationality concept makes no explicit statement on the use of information available to the economic agent. It only posits that, given an available set of information, the agent should behave consistently—no matter how (in)correct the information may be on which the agent bases her decisions. However, in contrast to this general possibility of misinterpretation that the theory allows for, virtually all neoclassical models make the *implicit* assumption that the economic agent interprets her information environment correctly (making preferences "description-invariant" which excludes, for example, so-called "framing" of situations).[7]

This concept of rationality and the many implicit assumptions that usually come along with its actual use is not only applied to riskless choice but also to choice under risk[8], in the form of so-called "expected utility theory" (EUT)[9]. It was decision under risk for which reference-dependent preferences and loss

3 For arguments in favor of analyzing endogenous preference changes, see Bowles (1998).
4 This is done for example by Ariely et al. (2003).
5 For the relevance of introducing such costs into economic models see, e.g., Conlisk (1996).
6 See Ariely et al. (2003).
7 This point is mentioned by Munro (2004, p. 2-2) and Starmer (2000, p. 338).
8 In line with Starmer (2000, p. 334), I define as a state of "risk" a situation in which the outcome of an agent's choice is random, but all potential consequences are known to the choice maker and can be attributed (at least subjective) probabilities. In contrast, a state of "uncertainty" is a situation in which at least some of the potential outcomes or probabilities are unknown.
9 Starmer (2000) provides a concise overview of EUT, including its achievements, shortcomings, and potential alternatives.

aversion were first introduced (by Kahneman and Tversky, 1979) as an alternative to the commonly assumed specifications of the utility function.

The standard reference-independent theory of choice not only makes it relatively easy to derive functions describing the demand and supply of goods, but it also has important implications for welfare economics and the assessment of the efficiency of markets. As soon as reference dependence is introduced in combination with an assumption that the reference point changes in a way connected to the market's behavior, the question of efficiency may become unanswerable by purely economic analysis, because the answer would involve a judgment about which reference state is the best. Tversky and Kahneman (1991, p. 1039) make a similar point:

> The standard models of decision making assume that preferences do not depend on current assets. This assumption greatly simplifies the analysis of individual choice and the prediction of trades: indifference curves are drawn without reference to current holdings, and the Coase theorem asserts that, except for transaction costs, initial entitlements do not affect final allocations. The facts of the matter are more complex.

2.2.2 *Evidence from experiments and market data on deviations from the neoclassical theory of choice*

Surveys on the deviations of actual behavior from the predictions of the neoclassical theory of choice—often called "anomalies"—that were documented in experimental or field studies, can be found in Munro (2004), Starmer (2000), and Tversky and Kahneman (1991). Although these anomalies are very diverse, the theory of reference-dependent preferences has been used to explain a larger number of them. In the following I will present only the cases relevant for my object of analysis.

Four classical findings on the deviation of actual choice from the predictions of standard theory are the following phenomena: the "endowment effect" first described by Thaler (1980), the "status quo bias" documented by Samuelson and Zeckhauser (1988), "preference reversal" as it was first observed by Lichtenstein and Slovic (1971) and Lindman (1971), as well as the "disposition effect" first examined by Shefrin and Statman (1985).

The *endowment effect* is called that way because it describes the following phenomenon frequently found in laboratory experiments: subjects are split into two subgroups. Subjects in one group are given an item (e.g., a chocolate bar), and in an incentive-compatible procedure their willingness to accept compensation (WTA) for giving up the item is elicited. Subjects in the second group are not given the item, and their willingness to pay (WTP) for the item is elicited.

Mean WTA is frequently found to be higher than mean WTP. In the absence of transaction costs, this disparity is at odds with standard theory according to which for small stakes, WTA and WTP should be equal.

A very similar phenomenon is the *status quo bias*: this term describes people's aversion to give up the status quo. SAMUELSON AND ZECKHAUSER (1988) showed it to be present in the decisions of Harvard University employees regarding enrollment in different available medical plans.

The experiment to reveal *preference reversal* is as follows: Subjects are asked to choose between two prospects: the "$-bet" which offers the chance to win a high prize at a low probability, and the "P-bet" which offers the chance to win a smaller prize at a higher probability. Later, subjects' minimum prices at which they are willing to sell the two lotteries are elicited. Studies have repeatedly shown that subjects tend to choose the P-bet, while placing a higher value on the $-bet (see STARMER, 2000, p. 338). From the perspective of EUT, this behavior is inconsistent—thus, the term "preference reversal".

The *disposition effect* is more accurately the "disposition to sell winners too early and ride losers too long", as the title of SHEFRIN AND STATMAN's (1985) groundbreaking article defines. The disposition effect has not only been verified by WEBER AND CAMERER (1998) in a simple experimental setting, but it has also been shown by ODEAN (1999) to be present in professional asset traders' decisions. In WEBER AND CAMERER's experimental setting, the artificial assets were constructed in such a way that it was clearly a mistake to hold on to assets which had suffered losses in the past—yet, subjects did so.

Reference-dependent preferences in combination with loss aversion are the most frequently used way to explain these phenomena. The next subsection provides an overview of the two seminal contributions by KAHNEMAN and TVERSKY.

2.2.3 *The theory of reference-dependent preferences in decision under risk and in riskless choice*

The concepts of preferences which depend on reference states and of loss aversion, as it was proposed in KAHNEMAN AND TVERSKY (1979) and TVERSKY AND KAHNEMAN (1991) have become known as "prospect theory".

More narrowly defined, prospect theory refers only to decision making under risk. A prospect is defined as a lottery with known potential outcomes x_i and known associated probabilities p_i, $i = 1, \ldots, n$; thus, it can be denoted by the vector $\mathbf{q} \equiv (x_1, p_1; x_2, p_2; \ldots; x_n, p_n)$. Expected-utility theory posits that an agent who possesses an endowment y (measured in the same units as the components of the prospect) values the prospect as follows:

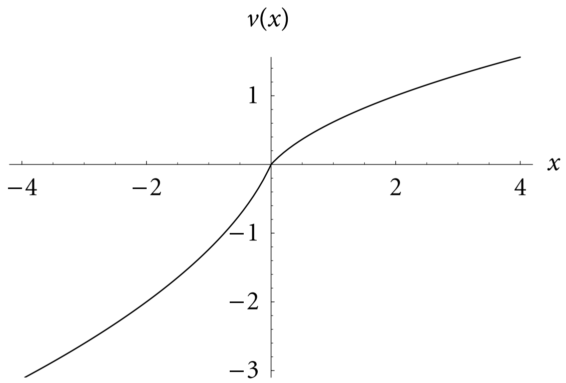


Figure 2.1: Example of a function $v(x)$ that fulfills the assumptions made in Kahneman and Tversky (1979, pp. 277–280).

$$(2.1) \qquad U(\mathbf{q}) = \sum_{i=1}^{n} p_i u(y + x_i),$$

where $u(\bullet)$ is usually assumed to be strictly increasing, differentiable and strictly concave. In contrast, prospect theory posits that people's evaluation of a prospect's components can be described by the following value function:

$$(2.2) \qquad V(\mathbf{q}) = \sum_{i=1}^{n} \pi(p_i) v(x_i),$$

where $v(x)$ has the following properties (see Kahneman and Tversky, 1979, pp. 277–280):

1 For all $x \geq 0$ ("gain"): $v''(x) \leq 0$; i.e., $v(x)$ is concave.
2 For all $x \leq 0$ ("loss"): $v''(x) \geq 0$; i.e., $v(x)$ is convex.
3 For $x > y \geq 0$: $v(y) + v(-y) > v(x) + v(-x)$, which implies:
4 For all x: $v(x) < -v(-x)$ and $v'(x) < -v'(-x)$.
5 $v(0) = 0$ (normalization).

A function with these properties is depicted in Figure 2.1.

Through these conditions prospect theory posits that an agent's evaluation of a prospect is not based on its components' contribution to *absolute* wealth, but only on whether the components increase or decrease wealth compared to the status quo. It is the change in wealth that matters. Furthermore, prospect theory posits that a loss of size x is valued more negatively than a gain of equal size, and that agents are risk-averse in the domain of gains, while being risk-seeking in the domain of losses. In addition, it is important to note that the value function $v(x)$ is not differentiable at $x = 0$.

$\pi(p)$ is a "decision weight" function. While it can be the identity function, it is usually assumed that $\pi(p) > p$ for p close to zero and that $\pi(p) < p$ for p close to one. This way, the empirically found tendency of people to overestimate the probabilities of unlikely events and to underestimate the probabilities of very likely events shall be captured.

In their 1991 article, Tversky and Kahneman extended this original theory to the domain of goods: In their "theory of reference-dependent preferences", loss aversion is suggested to exert an influence even on decisions which involve no uncertainty at all.

In contrast to prospect theory, in the theory of reference-dependent preferences the state to which a level of wealth or consumption x_i is compared is not necessarily the status quo (0) any more, but can be any real number r_i. In analogue to prospect theory, the difference $x_i - r_i$ is called a "gain" when larger than zero and a "loss" when below zero. Tversky and Kahneman provide accurate definitions of "loss aversion" and of "diminishing sensitivity to gains and losses" (pp. 1046–1050). For our purposes, however, the details of these definitions are not too important, because all attributes of the preferences of a loss-averse agent can be summarized in a value function $V_{\mathbf{r}} : \mathbf{X} \to \mathbb{R}$ which maps all choices $\mathbf{x}$ and $\mathbf{y}$ out of the set of admissible choices $\mathbf{X}$ to real numbers such that $\mathbf{x}$ is preferred over $\mathbf{y}$ if and only if $V_{\mathbf{r}}(\mathbf{x}) > V_{\mathbf{r}}(\mathbf{y})$. The subscript $\mathbf{r}$ denotes in this context that the alternatives $\mathbf{x}$ and $\mathbf{y}$ are evaluated from the reference point $\mathbf{r}$, which is also an element of $\mathbf{X}$. The existence of such a value function substantially simplifies the modeling of a loss-averse agent's behavior.

As Tversky and Kahneman show (p. 1048), the value function has to assign a more negative impact to losses than it assigns a positive impact to gains of equal size. When the agent's preferences are decomposable, $V_{\mathbf{r}}(\mathbf{x})$ has the property that the value of a bundle $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ to the consumer can be described by $V_{\mathbf{r}}(\mathbf{x}) = V(v_{\mathbf{r},1}(x_1), v_{\mathbf{r},2}(x_2), \ldots, v_{\mathbf{r},n}(x_n))$. In the special case of "constant loss aversion", the functions $v_{\mathbf{r},i}(x_i)$ are defined as follows:

$$v_{\mathbf{r},i}(x_i) = \begin{cases} u_i(x_i) - u_i(r_i) & \text{if } x_i \geq r_i \\ (u_i(x_i) - u_i(r_i))/\lambda_i & \text{if } x_i < r_i \end{cases}, \tag{2.3}$$

with $u_i : X_i \to \mathbb{R}$, $\lambda_i > 0$, $i = 1, \ldots, n$. Although not mentioned by Tversky and Kahneman, I suppose that, in addition, $u_i(x_i)$ has to be concave and strictly increasing and that $\lambda_i < 1$. For only then the function $v_{\mathbf{r},i}(x_i)$ in the case of reference-dependent preferences has the same properties as the function $v(x)$ suggested in prospect theory. (In fact, Tversky and Kahneman provide a plot of example indifference curves, allegedly with $\lambda_1 = 2$ and $\lambda_2 = 3$. I think this is a mistake, and it should be $\lambda_1 = ½$ and $\lambda_2 = ⅓$ instead. With $\lambda_i > 1$, one would have to speak of "loss proneness" rather than of "loss aversion".)

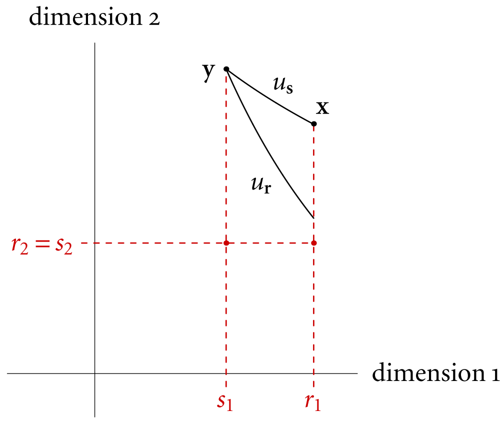


Figure 2.2: A graphic illustration of loss aversion: The agent is indifferent between the bundles **x** and **y**, given that her reference point is $\mathbf{s} = (s_1, s_2)$. When evaluating **x** and **y** from the reference point $\mathbf{r} = (r_1, r_2)$, she prefers **x** over **y**, because $y_1 < r_1$. (Source: Tversky and Kahneman, 1991, p. 1047, Figure III.)

Due to these properties, the theory of reference-dependent preferences can explain the endowment effect, the status quo bias, and preference reversal, given that the reference point is endowment: If the reference point is the endowment at the time of the decision, then purchasing a new item/lottery is asymmetric from being endowed with it and selling it. For under loss aversion giving up something is perceived as a greater loss than acquiring is perceived as a gain. The disposition effect can be explained by the theory of reference dependence, too, when last period's price of the stock is used as the reference point. However, to explain the disposition effect, not only loss aversion has to be invoked, but also the property of risk-proneness in the domain of losses and risk-aversion in the domain of gains (see Weber and Camerer, 1998, p. 170).

2.2.4 *The relevance of loss aversion for real-world markets*

So far, the focus of the evidence mentioned has been *individual* decision making. However, it is at least equally important to understand the implications of loss aversion for the *collective* outcome. As an example[10] may serve the common justification for assuming continuous and differentiable market demand functions, while most goods in fact cannot be divided or are at least sold only in fixed quantities. Therefore, most consumption choices are over discrete, and not continuous quantities, or they are even binary choices. Still, for the supermarket which serves thousands of customers simultaneously, a larger number of individual binary choices who all have different reservation prices translates

10 This example is inspired by a similar, and more formally presented, one in Schlicht (1985), p. 81.

to a monotonically decreasing demand curve. The larger the number of customers, the better average demand per customer can be approximated by a smooth, i.e. differentiable, function. Similarly, it could be that anomalies, while being present on the individual level, are invisible on the aggregate level—for example to producers.

Yet there is substantial evidence that in many situations departures of actual behavior from neoclassical choice theory do shape market outcomes to a non-negligible extent. The findings by Samuelson and Zeckhauser (1988) illustrate how flawed forecasts on the success of new products may be when neglecting the status quo bias; they were already presented in Section 2.2.2. Another study using real-market data—Odean (1999), which was already mentioned earlier as well—shows that, at least in the case of discount brokerage employees, competition between different firms does not make the disposition effect disappear.

For an impression of the wide range of economic areas in which evidence on the importance of reference-dependent preferences for explaining people's behavior has been found *in market data*, please refer to Table 2.1.

Of particular interest for this thesis is research on asymmetric reactions to price cuts and price increases. If demand functions stem from well-behaved preferences, they are monotonic and smooth. Of course, they are allowed to be non-linear so that the response to an equal-sized price increase is in general not simply the negative of the response to a price decrease. However, for *minor* price changes, responses to increases and cuts should be approximately of the same size. In contrast, if a demand curve is kinked at a certain price—which can be due to reference-dependent preferences—, around this price, reactions to increases and cuts are *asymmetric*. Such asymmetric response to price changes has been found by a number of studies, e.g. by Putler (1992), Hardie et al. (1993), and Bidwell et al. (1995). However, the most important study in this context is Bell and Lattin (2000), because unlike the aforementioned studies it allows for customer heterogeneity, which *de*creases the obtained estimate of loss aversion. Still, despite this improved accuracy Bell and Lattin obtained statistically significant estimates of asymmetric responses to price cuts and increases: for margarine, paper towels, detergents, hot dogs, bathroom tissue, and soft drinks. In other cases the reduction in the estimated coefficient was so large that the coefficient became insignificant; this happened for bacon, butter, crackers, sugar, and ice cream (see pp. 194–195).

Another important question is whether individual loss aversion disappears with increased market experience. Two experimental studies—carried out at a real market, however—by List (2003 a, 2003 b) deal with this issue, and the answer is quite definitely "yes". In both studies, List examines the WTP/WTA disparity

Study	*Commodity*	*Method*	*Result*
Samuelson and Zeckhauser (1988)	Annual health plan enrolments	Choice by existing staff versus new staff	43.1% of existing staff opted for status quo versus 22.7% of new staff (numbers should be equal according to EUT)
Johnson et al. (1993)	Car insurance	Choice of right to sue or not, with variation in default option	Given restricted rights, 20% opted for extended rights; given extended rights, 73% opted for extended rights
Hardie et al. (1993)	Supermarket goods (scanner data)	Compare demand responses to price rises and cuts	Demand elasticity for price rise > demand elasticity for price fall (standard theory: approximately identical)
Benartzi and Thaler (1995)	Investment decisions: bonds versus equities	Compare investment choices to EUT predictions	Level of risk aversion implied by bond–equity split incompatible with that implied by other risky decisions; compatible with prospect theory
Bowman et al. (1999)	Teachers' consumption	Responses of consumption to bad income news	Consumption does not fall, supporting endowment effect
List (2003 a)	Sports cards	148 sporst card traders randomly endowed with cards and invited to swap	45% (approx.) of dealers willing to swap; 6.8% of inexperienced non-dealers willing to swap (standard theory: percentage should be equal)
	Disney label pins	80 Disney pin traders randomly endowed with pins and invited to swap	25% of inexperienced traders willing to swap 40% of experienced traders willing to swap
	Sports cards	120 dealers and non-dealers WTP or WTA for sports cards	For dealers, mean WTA/mean WTP = 1.3; for non-dealers, ratio = 5.58 (standard theory: WTA/WTP ≈ 1)
Bell and Lattin (2000)	Supermarket goods (scanner data)	Compare demand response to price rises and cuts, allowing for consumer heterogeneity	Significant evidence of loss aversion for detergents, hot dogs, tissues, paper towels, margarine, and soft drinks; insignificant though positive effects for bacon, butter, ice cream, crackers, and sugar
Simonsohn and Loewenstein (2003)	Housing demand in the USA	Uses panel study of income dynamics to relate price paid by movers in new city to price paid in previous city	Movers from expensive cities pay more for same housing services than do movers from cheaper cities (standard theory: no difference should be observable)

Table 2.1: Empirical evidence from field experiments and market data on reference-dependent preferences. (Based on Munro, 2004, p. 2-10–12, excerpt from Table 2.1.)

called the "endowment effect" which I described in Section 2.2.2. In order to determine whether market participants behave rather in line with neoclassical or with prospect theory, List (2003 b) uses three distinct measures on which the two theories predict exactly the opposite of each other (see p. 2):

1 (in)dependence of trading behavior from current entitlements;
2 the size of Hicksian equivalent and Hicksian compensating surplus;
3 the curvature (convexity vs. concavity) of Hicksian equivalent surplus for relinquishing a certain number of units of a good.

The results that List obtained are consistent over all three measures and statistically highly significant: Overall, people's behavior is much better explained by prospect theory. However, there are clearly observable discrepancies between inexperienced and experienced market participants as well as between consumers and professional dealers. While the behavior of inexperienced consumers is described well by prospect theory, trading patterns converge with increased market experience towards the prediction of neoclassical theory. Also was the dealers' trading behavior observed in the experiment was closer to the predictions of neoclassical theory than to those of prospect theory.

This points in the direction that market experience (measured in years or in trading intensity) plays an important role in shaping people's preferences. It is hard to judge from this study, however, whether this process should be called "learning", or whether we should see in it not more than an endogenously triggered transition from one form of preferences to another one, without ascribing this process a positive or negative connotation. Yet, one should also keep in mind that List does not answer the question whether the causality might actually work the other way round: Maybe it is not market experience that changes people's preferences, but it is their preferences that let some participate in the market and others not.

Moreover, List does not deal with the question which markets are dominated by experienced and which by inexperienced consumers. It has to be remembered that for such an important market as the asset market, Odean (1999) found the disposition effect to be present even—maybe especially (see p. 1280) —in professional traders' decisions.

2.3 Theoretical models linking reference-dependent preferences and pricing

Apart from the aforementioned empirical models, a handful of theoretical models have been devised to explore the relation between consumer loss aversion and firms' price setting. Out of these, three were (co-)authored by Sibly:

Sibly (2002), McDonald and Sibly (2001), and Sibly (2004), while a fourth one is Heidhues and Kőszegi (2004; cited as "H&K" hereinafter).

The Sibly (2002) model also serves as the basis of McDonald and Sibly (2001) and Sibly (2004). Consequently, the two latter models share their main result with the Sibly (2002) model, and I will, therefore, restrict myself to presenting Sibly (2002) and H&K.

In his paper entitled "Loss Averse Customers and Price Inflexibility", Sibly (2002) presents a model that assumes a representative consumer's demand for a single good, produced by a monopolist, to be kinked at a reference price, p^{ref}. The kink is such that a positive deviation of the actual price from the reference price, $p > p^{ref}$, causes a *de*crease in demand which is larger than the *in*crease in demand created by a negative deviation, $p > p^{ref}$, of equal size. Sibly motivates this by referring to empirically found asymmetries in the response of demand to price cuts and increases (as presented in the previous subsection). He incorporates in his model a term q which expresses the customer's attitude towards the firm. Hence, the demand function d depends on two arguments: the product's price p and q, called the "disenchantment level". As usual, demand falls when the price increases: $d_p(p,q) < 0$. As well, demand falls when "disenchantment" (a *negative* attitude towards the firm) grows: $d_q(p,q) < 0$.

Reference dependence enters the model through the assumption that the agent's "disenchantment" towards the firm is connected to deviations of the actual price from the reference price. Disenchantment is proposed to be as follows:

$$(2.4) \qquad q = q_0 + \rho\left(\frac{p - p^{ref}}{p^{ref}}\right),$$

where q_0 is "inherited disenchantment", determined by the consumer's past experiences with the same producer. ρ has the following properties: $\rho'(\bullet) > 0$, $\rho(0) = 0$ and $\rho'(0^+) > \rho'(0^-)$. The latter property of ρ is the one that creates the demand curve's kink at the reference price.

The firm in Sibly's model does not exhibit loss aversion, but maximizes its profits. It has constant marginal cost w, so that the profit function is as follows:

$$(2.5) \qquad \pi(p,q) = (p - w)\, d(p,q).$$

Sibly assumes that π is strictly concave in p for any reference price p^{ref} and for any disenchantment level q,[11] so that there exists a unique maximizer p_m for

11 He does not elaborate on the question which restrictions this concavity imposes on the shape of d and hence, on the shape of ρ and ultimately, on the shape of the underlying utility function. In my eyes, this is a serious omission.

each p^{ref}. Sibly is interested in the circumstances that generate a $p_m = p^{ref}$. This is the case when the following conditions hold:

$$\left.\frac{\mathrm{d}\pi(p,q)}{\mathrm{d}p^-}\right|_{p=p^{ref}} > 0 \text{ and} \tag{2.6}$$

$$\left.\frac{\mathrm{d}\pi(p,q)}{\mathrm{d}p^+}\right|_{p=p^{ref}} < 0. \tag{2.7}$$

Evaluating conditions (2.6) and (2.7), one finally arrives at

$$\left.\frac{1}{E_p + E_q^+}\right|_{p=p^{ref}} < \frac{p^{ref} - w}{p^{ref}} < \left.\frac{1}{E_p + E_q^-}\right|_{p=p^{ref}}, \tag{2.8}$$

where E_p is the elasticity of demand w.r.t. p, and E_q^+ and E_q^- is the elasticity of demand w.r.t. q, evaluated for $p > p^{ref}$ and $p < p^{ref}$, respectively. E_q^+ and E_q^- incorporate $\rho'(0^-)$ and $\rho'(0^+)$, respectively. These latter two expressions are the only difference between the upper and the lower bound of (2.8).

Interval (2.8) illustrates that changes of the marginal cost w within a certain range do not alter the profit-maximizing price, which is $p_m = p^{ref}$.

The fact that the upper and lower bound of interval (2.8) differ only in the expression $\rho'(0^-)$ and $\rho'(0^+)$, respectively, makes clear that the kink in the demand function is necessary to obtain a non-degenerate interval for which it is profit-maximizing to set the actual price equal to the reference price. This property is also stressed by Sibly (pp. 326–327).

In contrast to Sibly, H&K assume consumers not to be loss-averse over prices, but over the quantities they consume. This is much closer to the original formulation of the theory of reference-dependent preferences, which is over quantities, and not prices, and it reflects the evidence obtained in experiments much better.

The model by H&K is an application of a general model proposed by Kőszegi and Rabin (2004; cited as “K&R” hereinafter). K&R (p. 1) claim that

> researchers have begun to apply these ideas [about reference dependent-preferences] in a handful of economic situations. Yet existing models are better suited to explaining experimental data, or to applying them in a specific context, rather than to systematically integrating them into economic theory. … In this paper we … flesh out, extend, and modify these models to build a realistic and more general theory of reference-dependent preferences that can be systematically applied to a wide array of economic settings.

The following assumptions are common to H&K and K&R: A household’s utility in riskless choice is given by $u(\mathbf{c} \mid \mathbf{r})$, where $\mathbf{c} = (c_1, c_2, \ldots, c_K) \in \mathbb{R}^K$ and

$\mathbf{r} = (r_1, r_2, ..., r_K) \in \mathbb{R}^K$. $\mathbf{c}$ is consumption and $\mathbf{r}$ is a reference point of consumption. Through combining the theory of reference-dependent preferences and prospect theory (with the probability-weighting function assumed to be the identity function), preferences over risky outcomes are modeled as follows:

$$(2.9) \quad U(F \mid \mathbf{r}) = \int_{\mathbf{c}} u(\mathbf{c} \mid \mathbf{r})\, \mathrm{d}F(\mathbf{c}),$$

where $F(\mathbf{c})$ is a (discrete or continuous) probability measure according to which consumption is drawn. In addition, the reference level itself may be a probability measure. Then, overall utility is given by:

$$(2.10) \quad U(F \mid G) = \int_{\mathbf{r}} \int_{\mathbf{c}} u(\mathbf{c} \mid \mathbf{r})\, \mathrm{d}F(\mathbf{c})\, \mathrm{d}G(\mathbf{r}).$$

This "formulation of reference-dependent utility captures … the notion that the sense of gain or loss from a given consumption outcome derives from comparing it to all outcomes in the support of the reference lottery" (K&R, p. 7).

Overall utility is split up into two components:

$$(2.11) \quad u(\mathbf{c} \mid \mathbf{r}) \equiv m(\mathbf{c}) + n(\mathbf{c} \mid \mathbf{r})$$

$m(\mathbf{c})$ is "regular" utility gained from consumption itself (therefore, I will call it "consumption utility"), whereas $n(\mathbf{c} \mid \mathbf{r})$ derives from comparing actual consumption $\mathbf{c}$ to the reference point $\mathbf{r}$. Hence, $n(\mathbf{c} \mid \mathbf{r})$ is called "gain/loss utility".

For reasons of simplicity, but also for conceptual reasons (see K&R, Section 5), m is assumed to be additive-separable:

$$(2.12) \quad m(\mathbf{c}) \equiv \textstyle\sum_{k=1}^{K} m_k(c_k),$$

where the functions m_k are differentiable and strictly increasing.

One of the main features of K&R's model is the definition of gain/loss utility n: $n(\mathbf{c} \mid \mathbf{r}) \equiv \sum_{k=1}^{K} n_k(c_k \mid r_k)$, where $n_k(c_k \mid r_k)$ is intimately connected to $m_k(c_k)$ in the following way:

$$(2.13) \quad n_k(c_k \mid r_k) \equiv \mu(m_k(c_k) - m_k(r_k)).$$

Here, $\mu(\cdot)$ is the "universal gain/loss function" and satisfies the properties imposed by prospect theory (where this function was called v, see Section 2.2.3):

A0. $\mu(x)$ is continuous for all x, twice differentiable for $x \neq 0$, and $\mu(0) = 0$.
A1. $\mu(x)$ is strictly increasing.
A2. If $y > x > 0$, then $\mu(y) - \mu(x) < \mu(-x) - \mu(-y)$.
A3. $\mu''(x) \leq 0$ for $x > 0$ and $\mu''(x) \geq 0$ for $x < 0$.
A4. $\frac{\lim_{x \to 0} \mu'(-|x|)}{\lim_{x \to 0} \mu'(|x|)} \equiv \lambda > 1$.

K&R (p. 2) remark that basing gain/loss utility on consumption utility is "an important novel restriction for a reference-dependent model to make sensible and strong predictions in many economic contexts", even though "tying the two together so tightly is likely to lead to incorrect predictions in some situation". A special property of this linkage is the prediction it makes on attitudes towards risk:

> By tying gain-loss utility in different dimensions to the consumption utility in those dimensions, our model predicts for instance that people are less bothered by risk in goods of lower consumption value than by risk in dimensions of greater consumption value. And by adding consumption utility to the decision-maker's utility function, it both replicates the predictions of Kahneman and Tversky's prospect theory value function under typical situations, where the consumption values of gains and losses are likely to be similar, and improves predictions in cases where the value function over consumption levels clearly does not apply.

An important special case is $\mu(\cdot)$ being linear, which makes the model much better tractable. This case is the one dealt with in both K&R and H&K, which is justified by K&R as follows:

> While the inequalities in A3 are most realistically considered strict to capture diminishing sensitivity, we shall often be interested in characterizing the implications of reference dependence where diminishing sensitivity does not play a big role. For doing so, we define an alternative to A3 that isolates loss aversion in our model by eliminating the diminishing sensitivity.
>
> A3′. For all $x \neq 0$, $\mu''(x) = 0$.

H&K complement this general framework outlined in K&R by a monopolistic producer with constant marginal cost c. c is a random variable to the firm with probability distribution function $\theta(c)$ and support $[\underline{c}, \overline{c}]$. The consumer is assumed to carry out a *binary* decision only: to buy a single item of the single good produced by the firm—or not to buy it. If she buys, she has to pay a price p. Therefore, $\mathbf{c} = (c_1, c_2)$ denotes the agent's consumption, with c_1 being her consumption in goods and c_2 being her consumption in money. Initial wealth is normalized to zero; hence, with the binary decision to make, either $\mathbf{c} = (1, -p)$ or $\mathbf{c} = (0, 0)$. The reference levels are $r_1 \in \{0, 1\}$ and $r_2 \in \mathbb{R}$.

Another novel feature of K&R's and H&K's model is the determination of the reference levels. Stating that a "major challenge" in developing a "fully specified model of consumer behavior with reference-dependent preferences" (H&K, p. 1) is the specification of the reference point, they call for not making "arbitrary exogenous assumptions", but assert that "a parsimonious theory of pricing and loss aversion should ideally build on a sufficiently general and precise specification of the reference point" (p. 2). To achieve this aim, H&K (p. 2; emphasis in original) assume that the

> decision-maker's reference point is determined by her recent expectations (i.e. probabilistic beliefs) about the outcomes she is going to get. … Based on this perspective, a person's reference point … depends on market conditions and her own anticipated behavior….
> … We assume that the reference point is determined endogenously, in a *personal equilibrium*, by the requirement that the stochastic outcome implied by optimal behavior conditional on expectations be consistent with expectations.[12]

The firm has to map the costs it faces to prices it charges, with the cost distribution being unknown to the consumer. Using the above described kind of rational expectations, the consumer decides upon the probability with which she will buy the item for any possible price.

H&K derive results for two variants of their model: one in which the firm commits to a pricing distribution before the consumer forms her expectations and one in which the firm does not announce its pricing distribution in advance. The results are qualitatively the same in the two cases, albeit that they require a denser cost distribution in the case without commitment (pp. 3–4):

> Our first major result is that even if marginal costs are continuously distributed, if the distribution has sufficiently high density, the firm charges finitely many prices. We interpret this as price stickiness.

H&K explain the partial non-responsiveness of the selling price to cost shocks as follows:

> Intuitively, random prices induce uncertainty for the consumer as to how much she has to pay for the good in case she finds it worthwhile to buy. If she is confronted with a relatively high buying price, she compares it to lower possible prices she could have gotten, and thus faces a monetary loss if she buys. The anticipation of this loss reduces her willingness to pay for the good. By insuring the consumer against small price shocks, therefore, the firm increases her overall willingness to pay and thus its sales revenues. If the cost distribution is sufficiently thick, this gain dominates the loss from being unable to differentiate production levels according to marginal cost.

It is important to note that the "price stickiness" generated by H&K's model is not triggered by the assumption of loss aversion alone, but by the very combination of loss aversion and the determination of the reference point they choose.

12 To use *expected* consumption in order to determined the reference point was already suggested by Bidwell et al. (1995, p. 291). However, they did not solve a full-fledged model.

3 Microeconomic and macroeconomic evidence on firms' price setting

3.1 Microeconomic evidence on firms' price setting I: "Regular prices" and sales periods

Visual inspection of time series of prices for different consumer goods reveals striking similarities, see Figure 3.1: first of all, there seems to exist something like a "regular price" for all three retail goods, which is constant for periods ranging from a couple of weeks to more than 150 weeks. Plus, all depicted price series are characterized by recurrent temporary *sales activities*: downward deviations from the "regular price" followed by a quick return to it. The duration of these deviations is usually one week or less.

The message of the price series is ambiguous: the existence of sales periods is crucial, since it can serve as an argument counter the relevance of so-called "menu costs". Models incorporating menu costs assume that it is (considerably) costly to change prices (e.g., through printing and distributing new "menus", i.e. catalogs, advertisements and price tabs); as a result, firms only change prices when their expected increase in revenue from the price change exceeds the cost of changing. Thus, prices are predicted to stay constant for much longer periods than when "menu costs" are absent.

Given the volatility of the actual price, it is hard to accept the idea that menu costs are high enough to prevent price changes. This is even more so when taking into account that sales activities are usually accompanied by putting the respective item on special display; hence, they are an example of activities that go along with comparably large menu costs.

Yet, the behavior of the "regular price", which is changed only infrequently, is in line with the predictions of "menu cost" models. The constancy of the "regular price" might be explainable by menu costs if we define them in a broader way and allow them to incorporate costs of re-optimization, the story might change. For instance, it could be the case that firms and retailers apply a rule of thumb for their short-term sales activities ("Let's decrease the product's price by around 20% vis-à-vis the 'regular price' twice a month"), while they reconsider their long-term pricing strategy (based on sector growth, expected inflation, competitors' strategies, developments on the markets for raw materials etc.) only once in a couple of months, because it is costly to acquire all the necessary information.

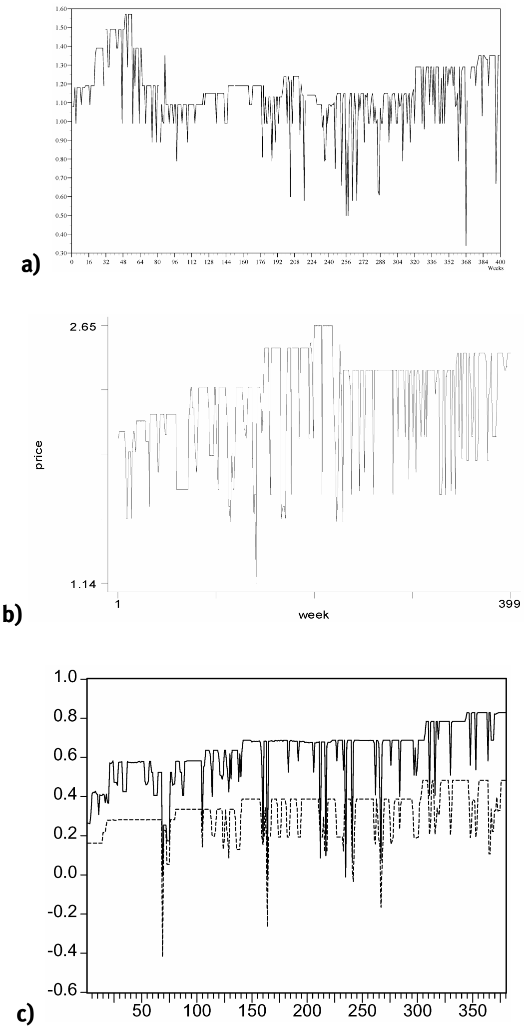


Figure 3.1:
a) Price of Frozen Concentrate Orange Juice, Heritage House, 12 oz., September 14, 1989–May 8, 1997. Discontinuities in the line indicate missing observations. (Source: LEVY ET AL., 2004, Figure 1.)
b) Price of *Triscuit*, 9.5 oz., in Dominick's Finer Foods Supermarket in Chicago. (Source: GOLOSOV AND LUCAS, 2003, p. 44, Figure 3 a.)
c) Price of *Nabisco Premium Saltines*, 16 oz., in Dominick's Finer Foods Supermarket. (Source: ROTEMBERG, 2003, p. 30, Figure 1.)

3.2 Microeconomic evidence on firms' price setting II: Frequencies of price changes

Two studies on the frequency, size and direction of price changes appeared recently this year: BILS AND KLENOW (2004) and LEVY ET AL. (2004).

Let me first summarize the findings by BILS AND KLENOW: The study is based on data from the US-American Bureau of Labor Statistics (BLS) for the

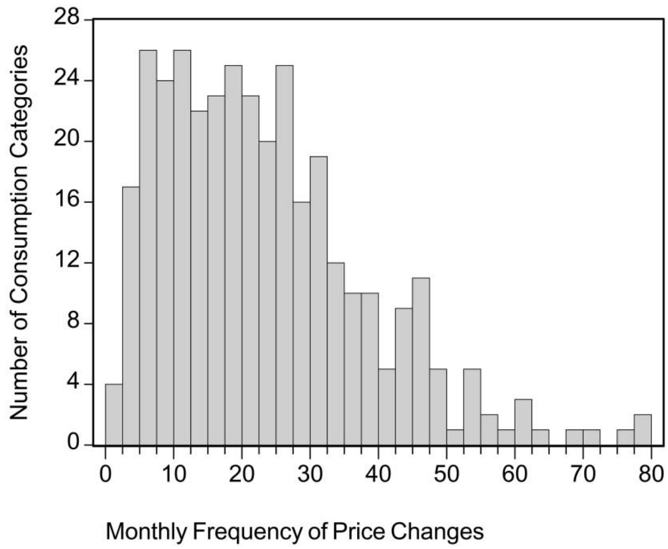


Figure 3.2: How often do how many consumption items change their prices?
(Source: Bils and Klenow, 2004, p. 952, Figure 1.)

years 1995–1997. The data consist of prices for between 70,000 and 80,000 goods and services, divided into 350 categories, that cover about 70% of consumer spending. They are collected by the BLS from around 22,000 outlets across 88 geographic areas. In densely populated areas, the data are gathered every month, in more sparsely populated areas only bimonthly.

The annual BLS *Commodities and Services Substitution Rate Table* provides, for each recorded good, how many times during that year the good's price changed with regard to the previous survey. Since—as already mentioned—the surveys take place monthly in some places and bimonthly in others, Bils and Klenow had to estimate the frequencies of monthly changes from the mixture of monthly and bimonthly frequencies reported by the BLS.[13]

It is important to note that Bils and Klenow's inference from the mixed reported frequencies to monthly frequencies *under*estimates the true frequencies of price changes. In addition, also the monthly surveys underestimate the true number of price changes during each month. This is due to the fact that (according to Chevalier et al., 2003, as cited in Bils and Klenow, 2004, p. 951) "temporary sales are … quite common" and "typically last less than one month". In this case, many price reductions at the beginning of a sales activity and the re-increases at the sales' end, will be unnoticed by monthly surveys—and even more so by bimonthly ones.

Turning to their results (see p. 951), we should therefore keep in mind that Bils and Klenow's estimates systematically underestimate the true values:

13 For details, see Bils and Klenow (2004), p. 950–951.

1 The weighted mean frequency of monthly price changes equals 26.1%, and the weighted median frequency is 20.9% (see Figure 3.2), where the weights were taken from the 1995 Consumer Expenditure Survey. That is, for half the products, the probability of a price change vis-à-vis the previous month is 20.9% or higher.
2 From this, Bils and Klenow calculate that half of the prices are constant for a period of 4.3 months or less (i.e., the median duration of prices is 4.3 months). The mean duration of prices equals 7 months.
3 Large variation across products can be observed: while some prices change only in 5% of all months—e.g., taxi fares, newspaper prices—others change in 70% of all months—e.g., those of gasoline, and fresh food. Prices of raw goods change more frequently than those of processed goods, and the prices of low-priced and high-priced goods vary more than those of medium-priced goods. Notably, the prices of durable goods are reset frequently.
4 Bils and Klenow hypothesize that the price of a good changes more often, the less value is added at one stage of the production process. The competitiveness of a market seems to increase the frequency of price changes. However, this effect disappears if one controls for the effect of "raw" vs. "processed" good.

Finally, Bils and Klenow provide an argument for not having filtered out sales activities from their data: They consider sales periods as a major consequence and indicator of price flexibility. Analogously, they have opted not to drop observations from their data when a manufacturer replaced an item by a *slightly altered* new product. This is because if the new good's price deviates from the old one's, the moment of the new good's introduction could as well have been used for changing the old one's price (see p. 957).

The aim of Bils and Klenow's paper is to check the predictions of calibrated time-dependent sticky-price models of the Calvo and Taylor type. A main features of simulated time series generated by a calibrated Calvo model is that prices' responses are relatively minor in the immediate aftermath of a shock. At the same time, once a price movement has started, it becomes relatively persistent. These features are just two sides of the same story: Both result from the assumption that in any period, any firm can adjust its price only with a certain probability below 1—no matter how huge its desire for a price change. Therefore, only a fraction of the firms can react to an observed shock and seizes this opportunity, leading only to a minor change in the aggregate price level. In the subsequent periods, again each time a fraction of the firms gets the chance to adjust prices to the new aggregate price level and in response to past shocks —leading to persistence of the inflationary movement.

Bils and Klenow state very clearly that such inertia does not correspond to their empirical findings from the BLS data (p. 949):

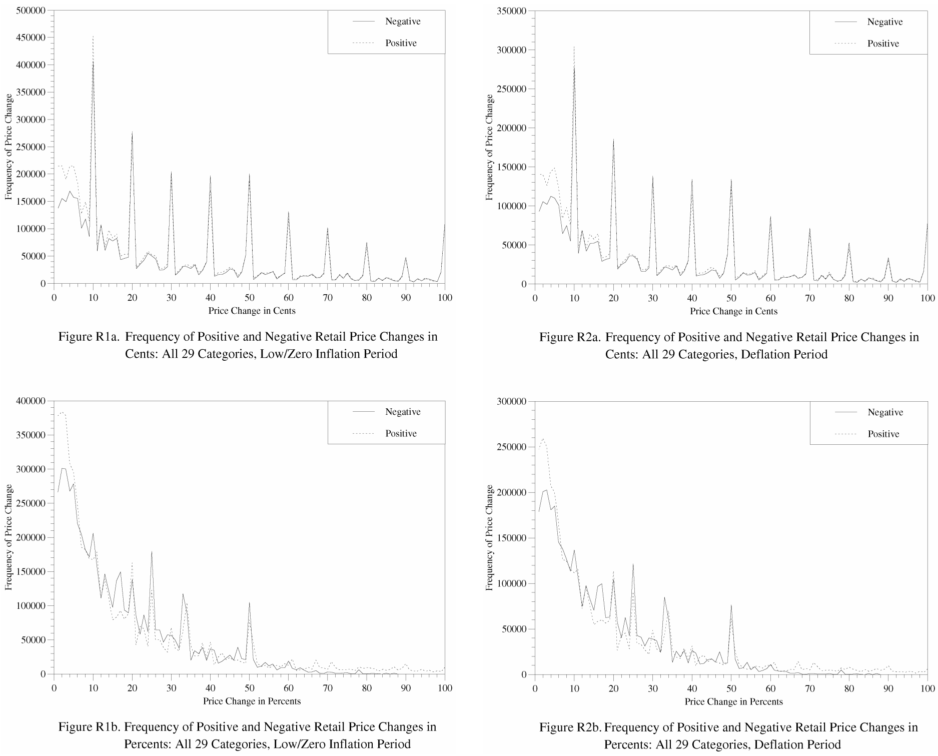


Figure 3.3: Frequencies of positive and negative price changes (in US cents and %) combined for hundreds of products that were sold at Dominick's between September 1989 and May 1997. (Source: LEVY ET AL., 2004, Figures R1a, R1b, R2a and R2b.)

We do not see this in the data. For nearly all 123 categories, inflation movements are far more volatile and transient than implied by the Calvo and Taylor models given the frequency of individual price changes in the BLS data.

3.3 Microeconomic evidence on firms' price setting III: Magnitude and direction of price changes

In contrast to BILS AND KLENOW, LEVY ET AL. (2004) focuse on the size and direction of price changes, while not taking their frequency into account. The dataset employed by LEVY ET AL. are weekly collected scanner data of retail prices from 94 stores of the "Dominick's" supermarket chain in the Chicago area. The data cover as many as 29 different product categories, with up to 400 products per category. They were collected from September 14th, 1989 to May 8th, 1997. In all, this means that "the data set contains more than 98 million weekly price observations" (p. 7). Since the largest part (86.3%) of retails grocery sales in the US takes place in supermarkets of chains of the "Dominick's" type, LEVY ET AL. consider their dataset "representative of a major class of the retail grocery trade" (p. 6). Since "retail sales account for about 9.3% of the [US] GDP", their dataset can, furthermore, be considered "representative of as much as 1.28 percent of the GDP, which seems substantial. Thus the market we are studying has a quantitative economic significance as well" (p. 6).

The main observation by LEVY ET AL. is that of "asymmetric price adjustment in the small": Across virtually all (!) product groups—and, thus, also in the combined data—small price increases (up to ca. 10 US cent) occur more often than small price decreases. Of course, this finding would be hardly surprising with constantly positive inflation, but it turns out to hold even during the low/zero inflation periods and deflation periods identified by LEVY ET AL. (see p. 3 of the main text and p. 1 of the referee appendix). In Figure 3.3, this result becomes obvious from the fact that the dotted line is significantly above the continuous line for price changes up to 18 US cent. Something noteworthy about this fact is that when retail prices rise frequently even during low/no inflation periods, prices in other categories of consumer goods must fall to obtain an overall CPI inflation of (close to) zero.

It can also be observed from Figure 3.3 (upper left and upper right panel) that the size of price decreases and increases is, in the vast majority of cases, a multiple of 10 ¢—and for these multiples of 10 ¢, the number of price decreases and increases is virtually the same: the dotted and the solid line are extremely close to each other. These two observations demonstrate that sales activities are a common characteristic of retail goods, because, as already stated, sales start with a price decrease and typically terminate with a price increase of exactly the same size.

LEVY ET AL. discuss several models that could *potentially* explain the discovered disparity in the number of small price increases and decreases (pp. 12–13). Among the models they review are ones that incorporate capacity adjustment costs on the side of the firm, imperfect competition, menu costs in combination with inflation and customer anger. However, they find that these models all predict asymmetry in the very opposite direction of that observed in the data.

3.4 Macroeconomic evidence on the movement of aggregate price indices

BURDA AND WYPLOSZ (1997, p. 359) mention that inflation (as measured by the consumer price index, CPI) is procyclical with regard to movements in GDP, but that it "systematically lag[s] behind" GDP. To illustrate this, they show a Burns–Mitchell diagram of CPI inflation, averaged over five countries (France, Germany, Italy, UK, and USA), which reveals that inflation peaks 1½ quarters after GDP. They interpret this as indicating "that prices—measured as aggregate indices—do appear to be rigid" (p. 376).

4 A model of price setting with reference-dependent preferences

4.1 Assumptions on the composition of the agent's utility

The agent is assumed to live T periods. For simplicity it is assumed that saving is impossible; hence, shifting consumption between different periods is impossible, and the agent maximizes utility separately in each period.

Following K&R, I assume the following specification of the household's utility function: A person's utility is given by $u(\mathbf{x}\,|\,\mathbf{r})$, where $\mathbf{x} = (x_1, x_2, ..., x_K) \in \mathbb{R}^K$ and $\mathbf{r} = (r_1, r_2, ..., r_K) \in \mathbb{R}^K$. $\mathbf{x}$ is consumption, and $\mathbf{r}$ is the vector of reference levels for the K goods considered; $\mathbf{r}$ is called "reference point" and assumed to be given exogenously. Overall utility is a combination of two additive components:

$$(4.1) \qquad u(\mathbf{x}\,|\,\mathbf{r}) \equiv m(\mathbf{x}) + n(\mathbf{x}\,|\,\mathbf{r}).$$

This formula can be interpreted as the household's utility consisting of two parts: The first one, $m(\mathbf{x})$, is "regular" utility gained from consumption. The second part, $n(\mathbf{x}\,|\,\mathbf{r})$, is called "gain/loss utility".[14] This utility is not derived from consumption itself, but from comparing consumption to the reference points.

To keep things simple, I assume that m and n are additive-separable as well: $m(\mathbf{x}) \equiv \sum_{k=1}^{K} m_k(x_k)$, and $n(\mathbf{x}) \equiv \sum_{k=1}^{K} n_k(x_k)$.

The functions m_k are continuously differentiable. They fulfill the standard assumptions made on utility functions, i.e. positive but decreasing marginal utility: $m'_k(x_k) \geq 0$ and $m''_k(x_k) \leq 0$ for all x_i. To exclude zero consumption of goods from being an optimal solution of the constrained maximization problem, I will furthermore assume that $\lim_{x_k \searrow 0} m'_k(x_k) = \infty$.

The functions $n_k(x_k\,|\,r_k)$ are closely connected to $m_k(x_k)$, in the following way (for a justification, see the argumentation in Section 2.3, citing K&R, p. 2):

$$(4.2) \qquad n_k(x_k\,|\,r_k) \equiv \mu(m_k(x_k) - m_k(r_k)),$$

where μ is given by

$$(4.3) \qquad \mu(x) = \begin{cases} \omega x & \text{if } x \geq 0 \\ (\lambda + \omega) x & \text{if } x < 0 \end{cases},$$

with $\omega \geq 0$ and $\lambda > 0$. This function μ fulfills the properties A1, A2, A3′, and A4, as required by K&R (p. 10) and as they were introduced in Section 2.3. Only A0 is not fulfilled in the case of $\omega = 0$. Neglecting A0 does not change the qualitative properties of the model, as is argued in Section 4.3.

14 The specification that overall utility is the sum of consumption utility and gain/loss utility is not only used by K&R, but also by Barberis et al. (2001).

4.2 A specific consumption utility function

I assume the consumption utility function to take on a specific form of CES (constant elasticity of substitution) utility. Furthermore, I restrict myself to considering two goods only. An extension of the model to include more than two goods is possible, but it will severely complicate the calculations, while important insights can already be gained from the two-good case.

I assume the utility function to have the following additive form:

$$(4.4) \quad m(\mathbf{x}) \equiv m_1(x_1) + m_2(x_2) = \alpha_1 x_1^{\rho} + \alpha_2 x_2^{\rho},$$

where $\alpha_1 \geq \alpha_2 > 0$ and $\rho \equiv (\sigma - 1)/\sigma$. $\sigma > 1$ is the elasticity of substitution between the two goods considered.[15] For m to be concave, ρ has to be smaller than unity, which holds for $\sigma > 1$.

Let M denote the nominal income that is exogenously given to the household at the beginning of each period. Then the single-period budget constraint is given by

$$(4.5) \quad p_1 x_1 + p_2 x_2 \leq M.$$

Both goods are assumed to have a strictly positive price, $p_1 > 0$ and $p_2 > 0$, so that there exists a trade-off between consuming the two goods.

4.3 Assumptions on the parameter values

Regarding the composition of consumption utility, I will assume from now on that, $½ \leq \alpha < 1$ (the lower bound is due to the condition that $\alpha_1 \geq \alpha_2$, see above),

$$(4.6) \quad m_1(x_1) = \alpha x_1^{\rho} \text{ and}$$

$$(4.7) \quad m_2(x_2) = (1 - \alpha) x_2^{\rho}.$$

Furthermore, I will assume μ to take on the simplest possible form, i.e. $\omega = 0$, $\lambda > 0$. This, of course, contradicts the original notion of "gain utility". However, in the original KAHNEMANN–TVERSKY model, gains and losses are the *only* source of utility, and no value derived from the pleasure of consumption itself is attributed to the goods. Since we do incorporate consumption utility into overall utility via $m(\mathbf{c})$, which increases in $\mathbf{c}$, a gain compared to the reference

15 This utility function is the additive-separable representation of the CES (constant elasticity of substitution) function $\tilde{m}(\mathbf{x}) \equiv A(\alpha_1 x_1^{\rho} + \alpha_2 x_2^{\rho})^{1/\rho}$. $m(\mathbf{x})$ represents the same preferences as $\tilde{m}(x)$ and can be obtained from $\tilde{m}(x)$ by applying two monotone transformations: $m(\mathbf{x}) \equiv [(1/A)\tilde{m}(\mathbf{x})]^{\rho}$, under the condition that A and ρ are greater than zero, which I assume.

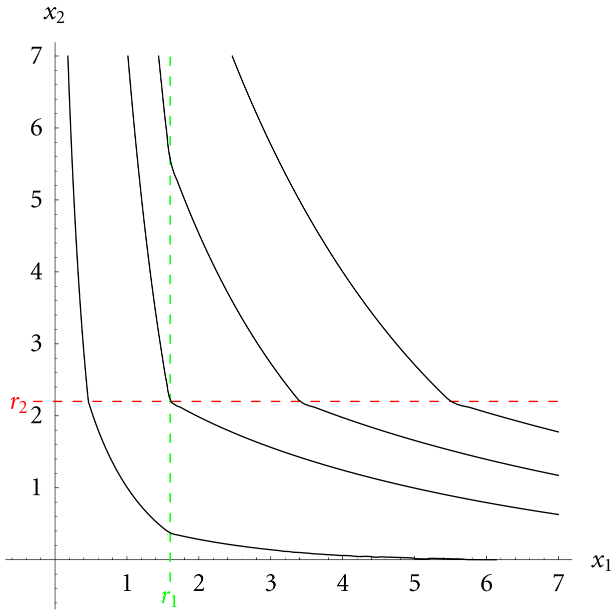


Figure 4.1: Indifference curves for loss-averse agent.[16]

level is still valued positively. This way, gain utility is still present, albeit less strongly than in the case of $\omega > 0$.

What is important about gain–loss utility is the kink in the overall utility function at the reference level; this kink is present as long as $\lambda > 0$. Therefore, setting ω to zero, while it does change the *numerical* properties of the utility function, does not alter the *qualitative* properties of the utility function. Thus, the qualitative results of the model stay the same, no matter whether $\omega > 0$ or $\omega = 0$.

This is illustrated by the indifference curves depicted in Figure 4.1: For $\lambda > 0$, they have the same properties as the indifference curves depicted in figure v of Tversky and Kahneman (1991, p. 1051): convexity (in our case even strict convexity) and kinks at the reference levels, with the effect that the marginal rate of substitution is discontinuous at the reference levels.

Remembering that

$$\begin{aligned} u(\mathbf{x} \mid \mathbf{r}) &\equiv m(\mathbf{x}) + n(\mathbf{x} \mid \mathbf{r}) \\ &= m_1(x_1) + m_2(x_2) + n_1(x_1 \mid r_1) + n_2(x_2 \mid r_2) \\ &= m_1(x_1) + m_2(x_2) + \mu(m_1(x_1) - m_1(r_1)) + \mu(m_2(x_2) - m_2(r_2)), \end{aligned}$$

16 The plot was created from $u(\mathbf{x} \mid \mathbf{r})$ as given by formula (4.8), with $\mathbf{r} = (1.6, 2.2)$ and the parameters taking on the following values: $\alpha = ⅔$, $\rho = ½$, $\lambda = 2$. λ was chosen so—probably unrealistically—high in order to make the kinks in the indifference curves pronounced enough to be visible clearly in the plot.

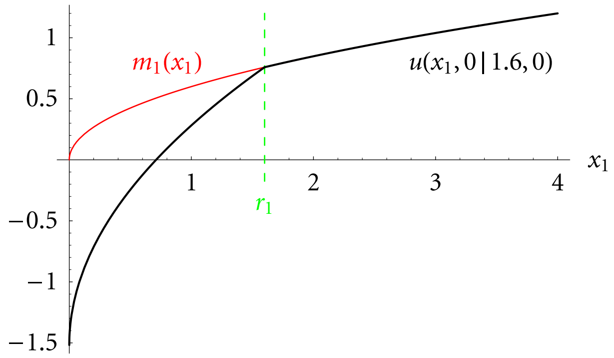


Figure 4.2: Comparison between utility function without loss aversion [$m_1(x_1)$] and with loss aversion [$u(x_1, 0\,|\,1.6, 0)$].[17]

we can plug in the definition of μ and the assumptions made in the previous paragraphs to get a utility function that has four separate branches:

$$u(\mathbf{x}\,|\,\mathbf{r}) = \tag{4.8}$$
$$\begin{cases} \alpha x_1^\rho + (1-\alpha)x_2^\rho & \text{if } x_1 \geq r_1, x_2 \geq r_2 \\ \alpha x_1^\rho + (1-\alpha)x_2^\rho + \lambda\alpha(x_1^\rho - r_1^\rho) & \text{if } x_1 < r_1, x_2 \geq r_2 \\ \alpha x_1^\rho + (1-\alpha)x_2^\rho + \lambda(1-\alpha)(x_2^\rho - r_2^\rho) & \text{if } x_1 \geq r_1, x_2 < r_2 \\ \alpha x_1^\rho + (1-\alpha)x_2^\rho + \lambda\alpha(x_1^\rho - r_1^\rho) + \lambda(1-\alpha)(x_2^\rho - r_2^\rho) & \text{if } x_1 < r_1, x_2 < r_2 \end{cases}.$$

A two-dimensional illustration of the utility function—varying one argument, with the second argument fixed to its reference level, which is assumed to be zero—is given by Figure 4.2.

I will call the four branches "branch I", "branch II", "branch III", and "branch IV", according to their order in the above utility function (see also Figure 5.2).

4.4 The maximization problem of firm 2

I will model only the behavior of the producer of good 2 ("firm 2") explicitly. This can be justified by conceiving of good 1 as an aggregate of all other products on the market. The price p_1 of this aggregate is taken as given by firm 2.

Like the consumer, the firm is assumed to exist T periods.

Firm 2 is assumed to be the only producer of good 2. For simplicity, I will assume the firm to have complete information about consumers' demand func-

17 The plot was created from $m_1(x_1)$ as given by formula (4.6) and $u(\mathbf{x}\,|\,\mathbf{r})$ as given by formula (4.8) with $x_2 = r_2 = 0$ and $r_1 = 1.6$ and the parameters taking on the following values: $\alpha = ⅔$, $\rho = ½$, $\lambda = 2$.

tions; that the production technology creates constant unit cost c_2; and that demand equals supply.

The unit cost c_2 is assumed to be random. It seems reasonable that the correlation between the cost in one period and the subsequent period is positive. This could, for instance, be due to climatic influences: When the harvest of a certain fruit was destroyed yesterday by bad weather, this will increase today's price of the fruit, and tomorrow's price will likely be higher than average, too. Therefore, I assume that the unit cost follows an AR(1) process, which is specified as follows:

$$\log c_{2,t} = \theta \log c_{2,t-1} + (1-\theta) \log \varepsilon_t \quad \text{for } 2 \le t \le T, \tag{4.9}$$

with $\log \varepsilon_t \sim N(\mu_c, \sigma_c)$, $\log c_{2,1}$ being drawn from the distribution $N(\mu_c, \sigma_c)$, and $0 \le \theta \le 1$.

At the beginning of each period, a realization of the unit cost is drawn which is valid throughout that period. Thus, the realization of the cost variable for one period is known to the firm when it solves the profit maximization problem for the respective period.

Since the household is not allowed to borrow or save and, therefore, has to consume each period's income in the respective period, the profit function of firm 2 is of identical form in every period and given by

$$\pi_2(p_1, p_2, M, c_2) \equiv (p_2 - c_2)\, x_2(p_2/p_1, M/p_1) \tag{4.10}$$

or

$$\pi_2(p_1, \tilde{p}_2, \tilde{M}, c_2) \equiv (p_1 \tilde{p}_2 - c_2)\, x_2(\tilde{p}_2, \tilde{M}). \tag{4.11}$$

In addition, let us define unit cost and profit of firm 2 relative to price 1, i.e. $\tilde{c}_2 \equiv c_2 / p_1$ and

$$\tilde{\pi}_2(\tilde{p}_2, \tilde{M}, \tilde{c}_2) \equiv \frac{\pi_2(p_1, p_2, M)}{p_1} = (\tilde{p}_2 - \tilde{c}_2)\, x_2(\tilde{p}_2, \tilde{M}). \tag{4.12}$$

The producer of good 2 maximizes profits by choosing p_2. The maximization can as well be modeled over choosing $\tilde{p}_2$, which is for a given p_1 equivalent to maximizing over p_2.

5 Solving the model

5.1 A preliminary step: Derivation of the demand functions when no loss aversion is present

5.1.1 *Derivation of the demand curves*

It will be very helpful to first solve the agent's maximization problem assuming there is *no* reference dependence. Based on the results obtained in Sections 5.1.1 and 5.1.2, Section 5.1.3 will then introduce loss aversion.

Making good 1 the numeraire good, I define the relative price between the two goods as $\tilde{p}_2 \equiv p_2 / p_1$ and the income converted into units of good 1 as $\tilde{M} \equiv M / p_1$. With these, the budget constraint (4.5) can be written as

$$x_1 + \tilde{p}_2 x_2 \le \tilde{M}. \tag{5.1}$$

When reference dependence is absent, $u(\mathbf{x} \mid \mathbf{r}) = m(\mathbf{x})$. Since both prices have been assumed to be positive, maximizing $m(\mathbf{x})$ subject to the budget constraint yields the following first-order necessary conditions:

$$\frac{m_1'(x_1)}{m_2'(x_2)} = \frac{\alpha_1 \rho x_1^{\rho-1}}{\alpha_2 \rho x_2^{\rho-1}} = \frac{p_1}{p_2} \text{ and} \tag{5.2}$$

$$p_1 x_1 + p_2 x_2 = M. \tag{5.3}$$

The budget constraint (4.5) is exhausted in the optimum (i.e., fulfilled by equality), which yields (5.3), because m is strictly increasing in both arguments. Since m is strictly concave, the first-order conditions are also sufficient and have a unique solution. Therefore, combining (5.2) and (5.3) yields the following demand functions for the goods 1 and 2:

$$x_1(p_1, p_2, M) = \frac{M}{p_1^\sigma} \cdot \frac{\alpha_1^\sigma}{\alpha_1^\sigma p_1^{1-\sigma} + \alpha_2^\sigma p_2^{1-\sigma}} \text{ and} \tag{5.4}$$

$$x_2(p_1, p_2, M) = \frac{M}{p_2^\sigma} \cdot \frac{\alpha_2^\sigma}{\alpha_1^\sigma p_1^{1-\sigma} + \alpha_2^\sigma p_2^{1-\sigma}}. \tag{5.5}$$

With the notation introduced above, equations (5.4) and (5.5) become

$$x_1(p_1, p_2, M) = \tilde{M} \frac{\alpha_1^\sigma}{\alpha_1^\sigma + \alpha_2^\sigma \tilde{p}_2^{1-\sigma}} \text{ and}$$

$$x_2(p_1, p_2, M) = \frac{\tilde{M}}{\tilde{p}_2^\sigma} \cdot \frac{\alpha_2^\sigma}{\alpha_1^\sigma + \alpha_2^\sigma \tilde{p}_2^{1-\sigma}}.$$

This shows that the demand functions can be expressed in terms of the relative price $\tilde{p}_2$ and the relative income $\tilde{M}$ alone. I therefore re-define them as follows:

$$(5.6) \quad x_1(\tilde{p}_2, \tilde{M}) \equiv \tilde{M} \cdot \frac{\alpha_1^\sigma}{\alpha_1^\sigma + \alpha_2^\sigma \tilde{p}_2^{1-\sigma}} \quad \text{and}$$

$$(5.7) \quad x_2(\tilde{p}_2, \tilde{M}) \equiv \frac{\tilde{M}}{\tilde{p}_2^\sigma} \cdot \frac{\alpha_2^\sigma}{\alpha_1^\sigma + \alpha_2^\sigma \tilde{p}_2^{1-\sigma}}.$$

5.1.2 *Comparison of demand with the reference levels*

Since under loss aversion, we will have to compare demand given by functions like the above one to the reference levels r_1 and r_2, it makes sense to do this once in advance with the general forms:

$$\begin{aligned} & x_1(\tilde{p}_2, \tilde{M}) \geq r_1 \\ \Leftrightarrow \quad & \tilde{M} \cdot \frac{\alpha_1^\sigma}{\alpha_1^\sigma + \alpha_2^\sigma \tilde{p}_2^{1-\sigma}} \geq r_1. \end{aligned}$$

Note that for non-negative $\tilde{p}_2$, this inequality can be only fulfilled if $\tilde{M} > r_1$, since $\alpha_1^\sigma / (\alpha_1^\sigma + \alpha_2^\sigma \tilde{p}_2^{1-\sigma}) < 1$. Only if $\tilde{M} > r_1 > 0$, the following rearrangements yield a meaningful condition (if $r_1 = 0$, $x_1(\tilde{p}_2, \tilde{M}) \geq r_1$ is always fulfilled):

$$\begin{aligned} & \tilde{M} \cdot \frac{\alpha_1^\sigma}{\alpha_1^\sigma + \alpha_2^\sigma \tilde{p}_2^{1-\sigma}} \geq r_1 \\ \Leftrightarrow \quad & \frac{1}{\alpha_1^\sigma + \alpha_2^\sigma \tilde{p}_2^{1-\sigma}} \geq \frac{r_1}{\alpha_1^\sigma \tilde{M}} \\ \Leftrightarrow \quad & \alpha_1^\sigma + \alpha_2^\sigma \tilde{p}_2^{1-\sigma} \leq \frac{\alpha_1^\sigma \tilde{M}}{r_1} \\ \Leftrightarrow \quad & \tilde{p}_2^{1-\sigma} \leq \frac{\alpha_1^\sigma \tilde{M} / r_1 - \alpha_1^\sigma}{\alpha_2^\sigma} \\ \Leftrightarrow \quad & \frac{1}{\tilde{p}_2^{\sigma-1}} \leq \frac{\alpha_1^\sigma \tilde{M} / r_1 - \alpha_1^\sigma}{\alpha_2^\sigma} \\ \Leftrightarrow \quad & \tilde{p}_2^{\sigma-1} \geq \frac{\alpha_2^\sigma}{\alpha_1^\sigma \tilde{M} / r_1 - \alpha_1^\sigma}. \end{aligned}$$

Thus, for the optimal consumption of good 1 to exceed its reference level r_1, we arrive at the conditions

$$(5.8) \quad \tilde{M} > r_1 \quad \text{and}$$

$$(5.9) \quad \tilde{p}_2 \geq \left(\frac{\alpha_2^\sigma}{\alpha_1^\sigma \tilde{M} / r_1 - \alpha_1^\sigma} \right)^{1/(\sigma-1)}.$$

For now, let us assume that $\tilde{M} > r_1$. Then the denominator of the r.h.s. of (5.9) is larger than zero and the r.h.s. is a real number. I denote by $\tilde{p}_2^*(\alpha_1, \alpha_2, \tilde{M})$ the value of $\tilde{p}_2$ for which (5.9) is fulfilled by equality:

$$(5.10) \quad \tilde{p}_2^*(\alpha_1, \alpha_2, \tilde{M}) \equiv \left(\frac{\alpha_2^\sigma}{\alpha_1^\sigma \tilde{M} / r_1 - \alpha_1^\sigma} \right)^{1/(\sigma-1)}.$$

Hence, if $\tilde{p}_2 = \tilde{p}_2^*$, then x_1 is equal to its reference level r_1; if $\tilde{p}_2 > \tilde{p}_2^*$, then $x_1 > r_1$; and if $\tilde{M} \le r_1$ or $\tilde{p}_2 < \tilde{p}_2^*$, then $x_1 < r_1$.

For consumption of the second good to exceed its reference level r_2, the following condition has to be satisfied:

$$(5.11) \quad x_2(\tilde{p}_2, \tilde{M}) = \frac{\tilde{M}}{\tilde{p}_2^\sigma} \cdot \frac{\alpha_2^\sigma}{\alpha_1^\sigma + \alpha_2^\sigma \tilde{p}_2^{1-\sigma}} \ge r_2.$$

Again, for non-negative $\tilde{p}_2$, $\alpha_1^\sigma / (\alpha_1^\sigma + \alpha_2^\sigma \tilde{p}_2^{1-\sigma})$ is strictly less than unity. Hence, a necessary condition for x_2 to be at least as large as r_2 ($r_2 > 0$) is that

$$(5.12) \quad \tilde{M} / \tilde{p}_2^\sigma > r_2 \Leftrightarrow \tilde{p}_2 < (\tilde{M} / r_2)^{1/\sigma}.$$

(If $r_2 = 0$, $x_2(\tilde{p}_2, \tilde{M}) \ge r_2$ is always fulfilled.)

Provided that (5.12) holds, we can rearrange the inequality as follows:

$$\begin{aligned} & \frac{\tilde{M}}{\tilde{p}_2^\sigma} \cdot \frac{\alpha_2^\sigma}{\alpha_1^\sigma + \alpha_2^\sigma \tilde{p}_2^{1-\sigma}} && \ge r_2 \\ \Leftrightarrow \quad & \frac{1}{\alpha_1^\sigma \tilde{p}_2^\sigma + \alpha_2^\sigma \tilde{p}_2} && \ge \frac{r_2}{\alpha_2^\sigma \tilde{M}}, \end{aligned}$$

which translates to

$$(5.13) \quad \alpha_1^\sigma \tilde{p}_2^\sigma + \alpha_2^\sigma \tilde{p}_2 \le \frac{\alpha_2^\sigma \tilde{M}}{r_2}.$$

Note that for non-negative $\tilde{p}_2$, this inequality implies (5.12), because it was assumed that $\alpha_1 \ge \alpha_2$. By rearranging (5.13) we can see (keeping in mind that $\sigma > 1$) that (5.12) holds, if (5.13) is fulfilled:

$$\begin{aligned} & \alpha_1^\sigma \tilde{p}_2^\sigma + \alpha_2^\sigma \tilde{p}_2 && \le \frac{\alpha_2^\sigma \tilde{M}}{r_2} \\ \Leftrightarrow \quad & \frac{\alpha_1^\sigma}{\alpha_2^\sigma} \tilde{p}_2^\sigma + \tilde{p}_2 && \le \frac{\tilde{M}}{r_2} \\ \Leftrightarrow \quad & \left[\left(\frac{\alpha_1}{\alpha_2} \right)^\sigma + \tilde{p}_2^{1-\sigma} \right] r_2 && \le \frac{\tilde{M}}{\tilde{p}_2^\sigma} \\ \Rightarrow \quad & r_2 && < \frac{\tilde{M}}{\tilde{p}_2^\sigma}. \end{aligned}$$

	$\tilde{M} > r_1$	$\tilde{M} \le r_1$
x_1	$\ge r_1$	$< r_1$
	if $\tilde{p}_2 \ge \left(\dfrac{\alpha_2^\sigma}{\alpha_1^\sigma \tilde{M} / r_1 - \alpha_1^\sigma} \right)^{1/(\sigma-1)}$	
	$< r_1$	
	if $\tilde{p}_2 < \left(\dfrac{\alpha_2^\sigma}{\alpha_1^\sigma \tilde{M} / r_1 - \alpha_1^\sigma} \right)^{1/(\sigma-1)}$	
x_2	$\ge r_2$	$\ge r_2$
	if $\alpha_1^\sigma \tilde{p}_2^\sigma + \alpha_2^\sigma \tilde{p}_2 \le \dfrac{\alpha_2^\sigma \tilde{M}}{r_2}$	if $\alpha_1^\sigma \tilde{p}_2^\sigma + \alpha_2^\sigma \tilde{p}_2 \le \dfrac{\alpha_2^\sigma \tilde{M}}{r_2}$
	$< r_2$	$< r_2$
	if $\alpha_1^\sigma \tilde{p}_2^\sigma + \alpha_2^\sigma \tilde{p}_2 > \dfrac{\alpha_2^\sigma \tilde{M}}{r_2}$	if $\alpha_1^\sigma \tilde{p}_2^\sigma + \alpha_2^\sigma \tilde{p}_2 > \dfrac{\alpha_2^\sigma \tilde{M}}{r_2}$

Table 5.1: Overview of the conditions under which x_1 and x_2 are above, equal to, or below the respective reference levels r_1 and r_2.

Thus, for x_2 to be at least as large as r_2, condition (5.13) is both necessary and sufficient.

Unfortunately, (5.13) cannot be solved for $\tilde{p}_2$ directly. However, since the l.h.s. is strictly increasing in $\tilde{p}_2$, there exists a unique value $\tilde{p}_2^{**}(\alpha_1, \alpha_2, \tilde{M})$ for which the condition is fulfilled by equality:

$$\tilde{p}_2^{**}(\alpha_1, \alpha_2, \tilde{M}) \equiv \tilde{p}_2 : \alpha_1^\sigma \tilde{p}_2^\sigma + \alpha_2^\sigma \tilde{p}_2 = \frac{\alpha_2^\sigma \tilde{M}}{r_2}. \tag{5.14}$$

$\tilde{p}_2^{**}$ can be solved for numerically when the parameter values are given.

If $\tilde{p}_2 = \tilde{p}_2^{**}$, then x_2 is equal to its reference level r_2; if $\tilde{p}_2 < \tilde{p}_2^{**}$, then $x_2 > r_2$. And if $\tilde{p}_2 > \tilde{p}_2^{**}$, then $x_2 < r_2$.

Combining these results, we get:

1. If $\tilde{M} > r_1$ and if $\tilde{p}_2^* < \tilde{p}_2^{**}$, the inequalities (5.9) and (5.13) define a unique interval $[\tilde{p}_2^*, \tilde{p}_2^{**}]$ of relative prices for which *both* consumed quantities exceed or are equal to their reference levels.
2. If $\tilde{M} > r_1$ and $\tilde{p}_2^* > \tilde{p}_2^{**}$, the inequalities (5.9) and (5.13) define a unique interval $[\tilde{p}_2^{**}, \tilde{p}_2^*]$ of relative prices for which *both* consumed quantities are below or equal to their reference levels.
3. If $\tilde{M} \le r_1$ and $\tilde{p}_2 < \tilde{p}_2^{**}$, it holds that $x_1 < r_1$ and $x_2 > r_2$.
4. If $\tilde{M} \le r_1$ and $\tilde{p}_2 > \tilde{p}_2^{**}$, it holds that $x_1 < r_1$ and $x_2 < r_2$.

All possible combinations are summarized in Table 5.1.

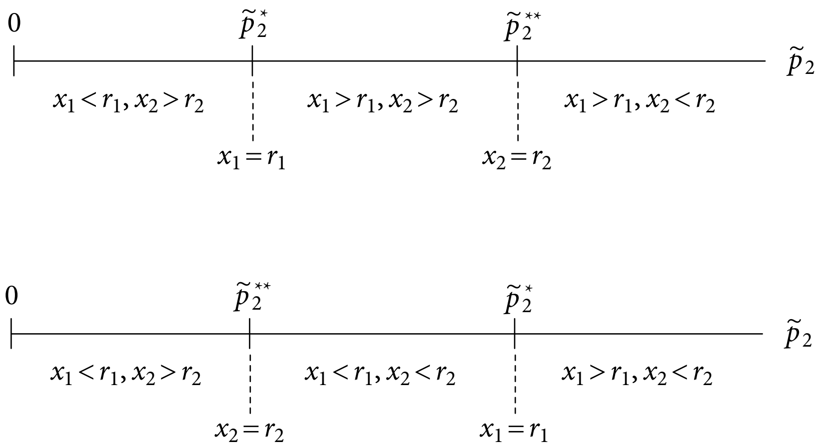


Figure 5.1: Intervals of the relative price $\tilde{p}_2$ for which x_1 and x_2 are above, below or equal to the respective reference level r_1 or r_2, provided that $\tilde{M} > r_1$.

Given that $\tilde{M} > r_1$, outside the interval $[\tilde{p}_2^*, \tilde{p}_2^{**}]$ or $[\tilde{p}_2^{**}, \tilde{p}_2^*]$, respectively, either one of the two quantities consumed is below its reference level, while the other one is above. Which is above and which below, depends on which of the inequalities (5.9) and (5.13) is fulfilled. Figure 5.1 provides an overview of these results.

Note that, as one would expect, $\tilde{p}_2^*$ is strictly *de*creasing in $\tilde{M}$, while $\tilde{p}_2^{**}$ is strictly *in*creasing in $\tilde{M}$: The higher the income, the larger is the interval $[\tilde{p}_2^*, \tilde{p}_2^{**}]$ inside which both $x_1 > r_1$ and $x_2 > r_2$. This leads us to the following considerations: As we know, it can be the case that $\tilde{p}_2^* < \tilde{p}_2^{**}$ as well as that $\tilde{p}_2^* > \tilde{p}_2^{**}$. It will be crucial later which of the two cases holds. Therefore, let us determine the income $\tilde{M}$ for which $\tilde{p}_2^* = \tilde{p}_2^{**}$ and denote this value by $\tilde{M}^*$. $\tilde{M}^*$ is given by

$$(5.15) \quad \tilde{M}^*(\alpha_1, \alpha_2) \equiv r_1 \left(1 + \frac{\alpha_2 r_2^\rho}{\alpha_1 r_1^\rho} \right),$$

as is derived in Appendix A. Due to the monotonicity of $\tilde{p}_2^*$ and $\tilde{p}_2^{**}$ in $\tilde{M}$ we can conclude that if $\tilde{M} > \tilde{M}^*$, then $\tilde{p}_2^* < \tilde{p}_2^{**}$, and if $\tilde{M} < \tilde{M}^*$, then $\tilde{p}_2^* > \tilde{p}_2^{**}$.

5.1.3 *The profit-maximizing price of firm 2*

Using the demand function derived above, the profit of firm 2, converted into units of good 1, is given by

$$\begin{aligned}\tilde{\pi}_2(\tilde{p}_2, \tilde{M}, \tilde{c}_2) &= (\tilde{p}_2 - \tilde{c}_2)\, x_2(\tilde{p}_2, \tilde{M}) \\ &= (\tilde{p}_2 - \tilde{c}_2) \frac{\tilde{M}}{\tilde{p}_2^\sigma} \cdot \frac{\alpha_2^\sigma}{\alpha_1^\sigma + \alpha_2^\sigma \tilde{p}_2^{1-\sigma}}.\end{aligned}$$

Since $x_2(\tilde{p}_2, \tilde{M}) > 0$ for all admissible $\tilde{p}_2$, $\tilde{p}_2 = \tilde{c}_2$ is the only null of $\tilde{\pi}_2$. It holds that $\tilde{\pi}_2 < 0$ for $\tilde{p}_2 < \tilde{c}_2$ and $\tilde{\pi}_2 > 0$ for $\tilde{p}_2 > \tilde{c}_2$. Thus, the profit-maximizing price will always be larger than $\tilde{c}_2$, with the consequence that the firm makes positive profits.

Unfortunately, it is impossible to obtain an analytical solution to firm 2's profit-maximization problem. However, it is shown in the following that the profit-maximizing price $\tilde{p}_2^{max}$ is unique and can be calculated numerically when the values of all parameters are given. Firm 2 increases $\tilde{p}_2$ as long as the derivative of $\tilde{\pi}_2$ w.r.t. $\tilde{p}_2$ is non-negative. Marginal profit is given by

$$\begin{aligned} &\frac{\partial \tilde{\pi}_2(\tilde{p}_2, \tilde{M}, \tilde{c}_2)}{\partial \tilde{p}_2} = \\ &\frac{\alpha_2^\sigma \tilde{M}}{\tilde{p}_2 (\alpha_1^\sigma \tilde{p}_2^\sigma + \alpha_2^\sigma \tilde{p}_2)^2} [\tilde{c}_2 \alpha_2^\sigma \tilde{p}_2 - \alpha_1^\sigma (\sigma - 1) \tilde{p}_2^{\sigma+1} + \tilde{c}_2 \alpha_2^\sigma \sigma \tilde{p}_2^\sigma]. \end{aligned} \tag{5.16}$$

Requiring this derivative to be non-negative yields:

$$\begin{aligned} & \frac{\alpha_2^\sigma \tilde{M}}{\tilde{p}_2 (\alpha_1^\sigma \tilde{p}_2^\sigma + \alpha_2^\sigma \tilde{p}_2)^2} [\tilde{c}_2 \alpha_2^\sigma \tilde{p}_2 - \alpha_1^\sigma (\sigma - 1) \tilde{p}_2^{\sigma+1} + \tilde{c}_2 \alpha_1^\sigma \sigma \tilde{p}_2^\sigma] \geq 0 \\ \Rightarrow \quad & \tilde{c}_2 \alpha_2^\sigma \tilde{p}_2 - \alpha_1^\sigma (\sigma - 1) \tilde{p}_2^{\sigma+1} + \tilde{c}_2 \alpha_1^\sigma \sigma \tilde{p}_2^\sigma \geq 0 \\ \Leftrightarrow \quad & \tilde{c}_2 \alpha_2^\sigma \tilde{p}_2 + \tilde{c}_2 \alpha_1^\sigma \sigma \tilde{p}_2^\sigma \geq \alpha_1^\sigma (\sigma - 1) \tilde{p}_2^{\sigma+1} \\ \Leftrightarrow \quad & \tilde{c}_2 \geq \frac{\alpha_1^\sigma (\sigma - 1) \tilde{p}_2^{\sigma+1}}{\alpha_2^\sigma \tilde{p}_2 + \alpha_1^\sigma \sigma \tilde{p}_2^\sigma}. \end{aligned}$$

Through division by $\tilde{p}_2^\sigma$, this can finally be written as

$$\tilde{c}_2 \geq \frac{\alpha_1^\sigma (\sigma - 1) \tilde{p}_2}{\alpha_2^\sigma \tilde{p}_2^{1-\sigma} + \alpha_1^\sigma \sigma}. \tag{5.17}$$

The latter form of the inequality shows that there exists a unique value $\tilde{p}_2$ for which the condition is fulfilled by equality: The r.h.s. of inequality (5.17) is strictly increasing in $\tilde{p}_2$, because the numerator is positive and increasing in $\tilde{p}_2$, and the denominator is positive and decreasing in $\tilde{p}_2$; both is due to $\sigma > 1$. Hence, the fraction in total is increasing in $\tilde{p}_2$. I call the value of $\tilde{p}_2$ for which (5.17) is fulfilled by equality $\tilde{p}_2^{max}$:

$$\tilde{p}_2^{max}(\alpha_1, \alpha_2, \sigma, \tilde{c}_2) \equiv \tilde{p}_2 : \tilde{c}_2 = \frac{\alpha_1^\sigma (\sigma - 1) \tilde{p}_2}{\alpha_2^\sigma \tilde{p}_2^{1-\sigma} + \alpha_1^\sigma \sigma}. \tag{5.18}$$

Note that $\tilde{p}_2^{max}$ is strictly increasing in $\tilde{c}_2$ and independent of $\tilde{M}$. Furthermore, $\lim_{\tilde{c}_2 \to \infty} \tilde{p}_2^{max} = \infty$ and $\lim_{\tilde{c}_2 \to 0} \tilde{p}_2^{max} = 0$.

5.2 Deriving the demand functions when loss aversion is present

5.2.1 *Outline of the derivation of demand curves under loss aversion*

We can now apply the general findings from Section 5.1 to our special model in which gain/loss utility is added to consumption utility. We are interested in answering the question which branch is optimally chosen by the household for all possible combinations of income and prices, i.e. which of the branches defined in Section 4.3 yields the highest utility. To do this, we have to proceed in three steps:

1 We will have to find out for all possible price–income combinations $(\tilde{p}_2, \tilde{M}) \in \mathbb{R}_+^2$, which of the branches are *feasible*. I call a branch "feasible" if the budget restriction, fulfilled by equality, allows for being on that branch.
2 For each feasible branch, we will have to calculate the optimal consumption bundle, given that the agent is on that branch and given that she behaves according to an *interior solution* of the optimization problem. This will yield conditions on $(\tilde{p}_2, \tilde{M})$ for which the respective branch is not only feasible, but also optimal.
3 In a third step, we will have to determine the optimal behavior for all price–income combinations $(\tilde{p}_2, \tilde{M})$ for which *no interior solution* exists.

5.2.2 *Step 1: The feasible branches*

For carrying out *step 1*, it is helpful to look at Figure 5.2. The reference point **r** divides the non-negative quadrant of the real plain into four quadrants:

1	quadrant I,	where $x_1 \geq r_1$ and $x_2 \geq r_2$;
2	quadrant II,	where $x_1 < r_1$ and $x_2 \geq r_2$;
3	quadrant III,	where $x_1 \geq r_1$ and $x_2 < r_2$;
4	quadrant IV,	where $x_1 < r_1$ and $x_2 < r_2$.

Of course, the agent cannot spend more on consumption than her income. On the other hand, since u is strictly increasing in both arguments, at the optimum the entire income will be exhausted. Thus, five cases arise (see Figure 5.2):

1	If $\tilde{M} \geq r_1 + \tilde{p}_2 r_2$,	branches I, II, and III are feasible.
2	If $\tilde{M} \geq r_1$ and $\tilde{p}_2 r_2 < \tilde{M} < r_1 + \tilde{p}_2 r_2$,	branches II, III, and IV are feasible.
3	If $\tilde{M} \geq r_1$ and $\tilde{M} < \tilde{p}_2 r_2$,	branches III and IV are feasible.
4	If $\tilde{M} < r_1$ and $\tilde{M} \geq \tilde{p}_2 r_2$,	branches II and IV are feasible.
5	If $\tilde{M} < r_1$ and $\tilde{M} < \tilde{p}_2 r_2$,	only branch IV is feasible.

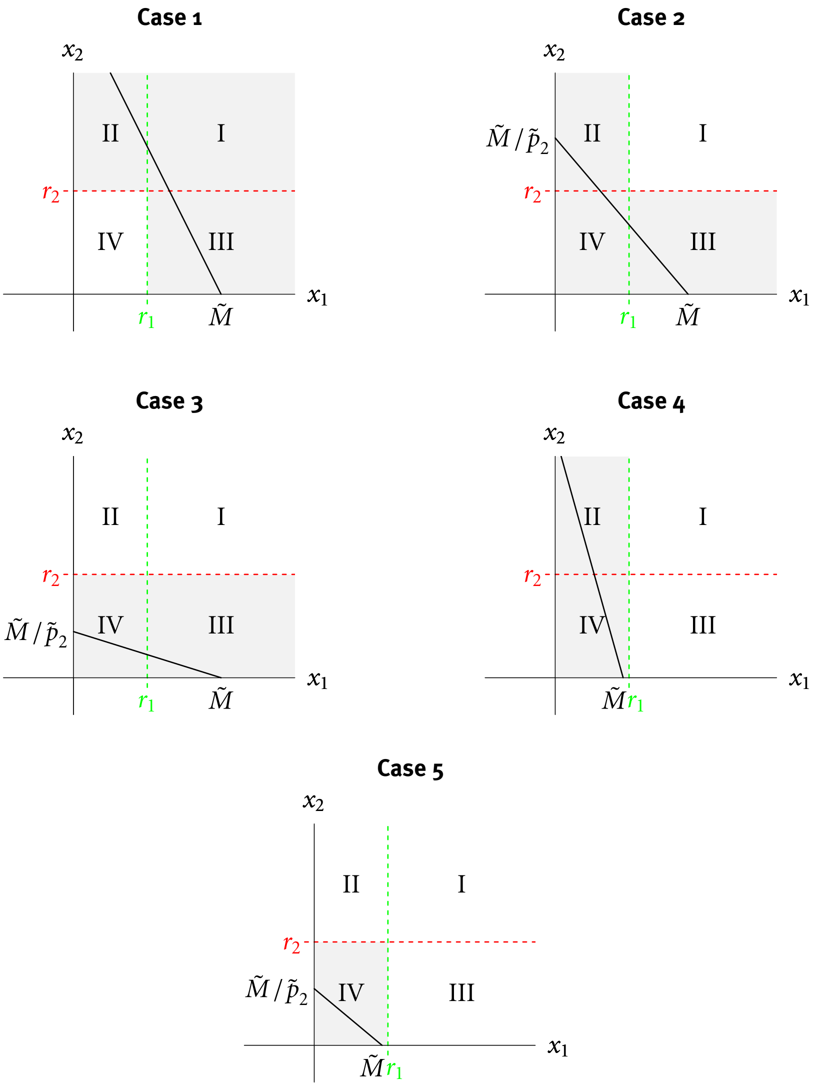


Figure 5.2: The four quadrants created by the reference point. Shaded are those quadrants in which the optimal consumption bundle can lie for the respective combination of $\tilde{M}$ and $\tilde{p}_2$.

5.2.3 *Step 2 (a): The interior solution when being on a specific branch*

As already stated, in *step 2* we have to find out the optimal branch among the feasible ones for each of the five cases. In preparation of carrying out step 2, let us rearrange (4.8) slightly:

$$
u(\mathbf{x}\,|\,\mathbf{r}) =
$$

$$
(5.19)\quad \begin{cases} \alpha x_1^{\rho} + (1-\alpha)\, x_2^{\rho} & \text{if } x_1 \geq r_1,\ x_2 \geq r_2 \\ (1+\lambda)\alpha\, x_1^{\rho} + (1-\alpha)\, x_2^{\rho} - \lambda\, \alpha\, r_1^{\rho} & \text{if } x_1 < r_1,\ x_2 \geq r_2 \\ \alpha x_1^{\rho} + (1+\lambda)(1-\alpha)\, x_2^{\rho} - \lambda\,(1-\alpha)\, r_2^{\rho} & \text{if } x_1 \geq r_1,\ x_2 < r_2 \\ (1+\lambda)\alpha\, x_1^{\rho} + (1+\lambda)(1-\alpha)\, x_2^{\rho} - \lambda\, \alpha\, r_1^{\rho} - \lambda\,(1-\alpha)\, r_2^{\rho} & \text{if } x_1 < r_1,\ x_2 < r_2 \end{cases}.
$$

Now, every branch has the form

$$
\alpha_1\, x_1^{\rho} + \alpha_2\, x_2^{\rho} - \text{const},
$$

where "const" denotes a term that depends on (r_1,r_2) but is independent of (x_1,x_2). This enables us to apply the results derived in Section 5.1.1 [i.e., formulae (5.6) and (5.7)] directly. Under the condition that we are on the respective branch I, II, III, or IV, the demand functions in the case of an interior solution are

$$
(5.20)\quad x_1^{\mathrm{I}}(\tilde{p}_2,\tilde{M}) = \tilde{M}\cdot\frac{\alpha^{\sigma}}{\alpha^{\sigma}+(1-\alpha)^{\sigma}\,\tilde{p}_2^{1-\sigma}};
$$

$$
(5.21)\quad x_2^{\mathrm{I}}(\tilde{p}_2,\tilde{M}) = \frac{\tilde{M}}{\tilde{p}_2^{\sigma}}\cdot\frac{(1-\alpha)^{\sigma}}{\alpha^{\sigma}+(1-\alpha)^{\sigma}\,\tilde{p}_2^{1-\sigma}};
$$

$$
(5.22)\quad x_1^{\mathrm{II}}(\tilde{p}_2,\tilde{M}) = \tilde{M}\cdot\frac{[(1+\lambda)\alpha]^{\sigma}}{[(1+\lambda)\alpha]^{\sigma}+(1-\alpha)^{\sigma}\,\tilde{p}_2^{1-\sigma}};
$$

$$
(5.23)\quad x_2^{\mathrm{II}}(\tilde{p}_2,\tilde{M}) = \frac{\tilde{M}}{\tilde{p}_2^{\sigma}}\cdot\frac{(1-\alpha)^{\sigma}}{[(1+\lambda)\alpha]^{\sigma}+(1-\alpha)^{\sigma}\,\tilde{p}_2^{1-\sigma}};
$$

$$
(5.24)\quad x_1^{\mathrm{III}}(\tilde{p}_2,\tilde{M}) = \tilde{M}\cdot\frac{\alpha^{\sigma}}{\alpha^{\sigma}+[(1+\lambda)(1-\alpha)]^{\sigma}\,\tilde{p}_2^{1-\sigma}};
$$

$$
(5.25)\quad x_2^{\mathrm{III}}(\tilde{p}_2,\tilde{M}) = \frac{\tilde{M}}{\tilde{p}_2^{\sigma}}\cdot\frac{[(1+\lambda)(1-\alpha)]^{\sigma}}{\alpha^{\sigma}+[(1+\lambda)(1-\alpha)]^{\sigma}\,\tilde{p}_2^{1-\sigma}};
$$

$$
(5.26)\quad x_1^{\mathrm{IV}}(\tilde{p}_2,\tilde{M}) = \tilde{M}\cdot\frac{\alpha^{\sigma}}{\alpha^{\sigma}+(1-\alpha)^{\sigma}\,\tilde{p}_2^{1-\sigma}};
$$

$$
(5.27)\quad x_2^{\mathrm{IV}}(\tilde{p}_2,\tilde{M}) = \frac{\tilde{M}}{\tilde{p}_2^{\sigma}}\cdot\frac{(1-\alpha)^{\sigma}}{\alpha^{\sigma}+(1-\alpha)^{\sigma}\,\tilde{p}_2^{1-\sigma}}.
$$

5.2.4 *Step 2 (b): Consistency*

It has to be ensured that the above listed interior solutions (5.20)–(5.27) indeed fall into the range of x_1 and x_2 for which they were derived. This can be done by application of formulae (5.8), (5.9) and (5.13) and using the definitions of $\tilde{p}_2^*$ and $\tilde{p}_2^*$ in formulae (5.10) and (5.14), respectively:

- Branch I ($x_1 \geq r_1$ and $x_2 \geq r_2$):

(5.28) $x_1 \geq r_1$: $\tilde{M} > r_1$ and $\tilde{p}_2 \geq \tilde{p}_2^{*}(\alpha, 1-\alpha, \tilde{M})$ must hold;
$x_2 \geq r_2$: $\tilde{p}_2 \leq \tilde{p}_2^{**}(\alpha, 1-\alpha, \tilde{M})$ must hold.

- Branch II (where $x_1 < r_1$ and $x_2 \geq r_2$):

(5.29) $x_1 < r_1$: it must hold that $\tilde{M} \leq r_1$
or $\tilde{M} > r_1$ and $\tilde{p}_2 < \tilde{p}_2^{*}((1+\lambda)\alpha, 1-\alpha, \tilde{M})$;
$x_2 \geq r_2$: $\tilde{p}_2 \leq \tilde{p}_2^{**}((1+\lambda)\alpha, 1-\alpha, \tilde{M})$ must hold.

- Branch III (where $x_1 \geq r_1$ and $x_2 < r_2$):

(5.30) $x_1 \geq r_1$: $\tilde{M} > r_1$ and $\tilde{p}_2 \geq \tilde{p}_2^{*}(\alpha, (1+\lambda)(1-\alpha), \tilde{M})$ must hold;
$x_2 < r_2$: $\tilde{p}_2 > \tilde{p}_2^{**}(\alpha, (1+\lambda)(1-\alpha), \tilde{M})$ must hold.

- Branch IV (where $x_1 < r_1$ and $x_2 < r_2$):

(5.31) $x_1 < r_1$: it must hold that $\tilde{M} \leq r_1$
or $\tilde{M} > r_1$ and $\tilde{p}_2 < \tilde{p}_2^{*}(\alpha, 1-\alpha, \tilde{M})$;
$x_2 < r_2$: $\tilde{p}_2 > \tilde{p}_2^{**}(\alpha, 1-\alpha, \tilde{M})$ must hold.

Condition (5.31) is a reduced version of these original inequalities:

$\tilde{p}_2 < \tilde{p}_2^{*}((1+\lambda)\alpha, (1+\lambda)(1-\alpha), \tilde{M})$; and
$\tilde{p}_2 > \tilde{p}_2^{**}((1+\lambda)\alpha, (1+\lambda)(1-\alpha), \tilde{M})$.

Since I will have to refer frequently to the various lower and upper bounds imposed by these conditions on $\tilde{p}_2$, I define

(5.32) $\tilde{p}_2^{\mathrm{I},lb} \equiv \tilde{p}_2^{*}(\alpha, 1-\alpha, \tilde{M})$;
(5.33) $\tilde{p}_2^{\mathrm{I},ub} \equiv \tilde{p}_2^{**}(\alpha, 1-\alpha, \tilde{M})$;
(5.34) $\tilde{p}_2^{\mathrm{II},ub} \equiv \min\{\tilde{p}_2^{*}((1+\lambda)\alpha, 1-\alpha, \tilde{M}), \tilde{p}_2^{**}((1+\lambda)\alpha, 1-\alpha, \tilde{M})\}$;
(5.35) $\tilde{p}_2^{\mathrm{III},lb} \equiv \max\{\tilde{p}_2^{*}(\alpha, (1+\lambda)(1-\alpha), \tilde{M}), \tilde{p}_2^{**}(\alpha, (1+\lambda)(1-\alpha), \tilde{M})\}$;
(5.36) $\tilde{p}_2^{\mathrm{IV},lb} \equiv \tilde{p}_2^{**}(\alpha, 1-\alpha, \tilde{M})$;
(5.37) $\tilde{p}_2^{\mathrm{IV},ub} \equiv \tilde{p}_2^{*}(\alpha, 1-\alpha, \tilde{M})$.

5.2.5 *Step 3: Corner solutions*

It can be shown that the above conditions do not overlap—which is done in Appendix A. On the contrary, combinations $(\tilde{p}_2, \tilde{M})$ exist for which no interior solution on the various branches is attained. For these combinations $(\tilde{p}_2, \tilde{M})$, the agent chooses a corner solution on one of the branches. Remembering that the branches are distinguished via the criteria $x_1 \gtreqless r_1$ and $x_2 \gtreqless r_2$, it becomes clear that a corner solution is characterized by $x_1 = r_1$ or $x_2 = r_2$. Hence, I define two consumption bundles

$$\text{(5.38)} \quad \mathbf{a} \equiv (x_1^{\mathbf{a}}, x_2^{\mathbf{a}}) \equiv (r_1, (\tilde{M} - r_1)/\tilde{p}_2) \text{ and}$$

$$\text{(5.39)} \quad \mathbf{b} \equiv (x_1^{\mathbf{b}}, x_2^{\mathbf{b}}) \equiv (\tilde{M} - \tilde{p}_2 r_2, r_2).$$

The following proposition describes the conditions under which bundle **a** or **b** is a consumed as an optimal solution.

Proposition 1.

Claim 1. For all $(\tilde{p}_2, \tilde{M})$ that fulfill

$$\text{(5.40)} \quad \tilde{M} \geq r_1 + \tilde{p}_2 r_2 \text{ and}$$

$$\left(\frac{(1-\alpha)^\sigma}{[\alpha(1+\lambda)]^\sigma \tilde{M}/r_1 - [\alpha(1+\lambda)]^\sigma} \right)^{1/(\sigma-1)} \leq \tilde{p}_2 \leq \left(\frac{(1-\alpha)^\sigma}{\alpha^\sigma \tilde{M}/r_1 - \alpha^\sigma} \right)^{1/(\sigma-1)},$$

consuming bundle $\mathbf{a} \equiv (x_1^{\mathbf{a}}, x_2^{\mathbf{a}}) \equiv (r_1, (\tilde{M} - r_1)/\tilde{p}_2)$ is the optimal choice.
Proof. See Appendix A.

Claim 2. For all $(\tilde{p}_2, \tilde{M})$ that fulfill

$$\text{(5.41)} \quad \tilde{M} \geq r_1 + \tilde{p}_2 r_2 \text{ as well as}$$

$$\alpha^\sigma \tilde{p}_2^\sigma + (1-\alpha)^\sigma \tilde{p}_2 \geq \frac{(1-\alpha)^\sigma \tilde{M}}{r_2} \text{ and}$$

$$\alpha^\sigma \tilde{p}_2^\sigma + [(1-\alpha)(1+\lambda)]^\sigma \tilde{p}_2 \leq \frac{[(1-\alpha)(1+\lambda)]^\sigma \tilde{M}}{r_2},$$

consuming bundle $\mathbf{b} \equiv (x_1^{\mathbf{b}}, x_2^{\mathbf{b}}) \equiv (\tilde{M} - \tilde{p}_2 r_2, r_2)$ is the optimal choice.
Proof. See Appendix A.

Claim 3. For all $(\tilde{p}_2, \tilde{M})$ that fulfill

$$\text{(5.42)} \quad r_1 < \tilde{M} < r_1 + \tilde{p}_2 r_2 \text{ and}$$

$$\left(\frac{(1-\alpha)^\sigma}{\alpha^\sigma \tilde{M}/r_1 - \alpha^\sigma} \right)^{1/(\sigma-1)} \leq \tilde{p}_2 \leq \left(\frac{[(1+\lambda)(1-\alpha)]^\sigma}{\alpha^\sigma \tilde{M}/r_1 - \alpha^\sigma} \right)^{1/(\sigma-1)},$$

consuming bundle **a** is the optimal choice.
Proof. See Appendix A.

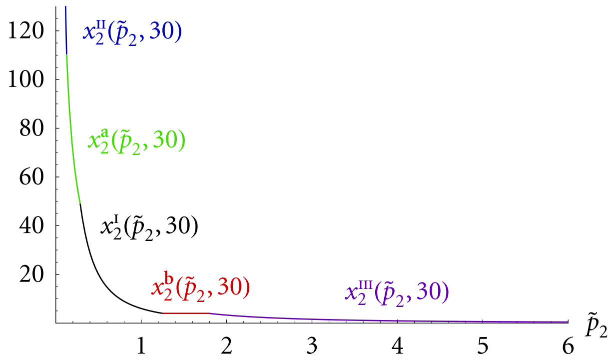


Figure 5.3: An example of the demand function's composition from various branches and the bundles **a** and **b**.[18]

Claim 4. For all $(\tilde{p}_2, \tilde{M})$ that fulfill

(5.43) $\tilde{p}_2 r_2 < \tilde{M} < r_1 + \tilde{p}_2 r_2$ as well as

$$[\alpha(1+\lambda)]^\sigma \tilde{p}_2^\sigma + (1-\alpha)^\sigma \tilde{p}_2 \geq \frac{(1-\alpha)^\sigma \tilde{M}}{r_2} \quad \text{and}$$

$$\alpha^\sigma \tilde{p}_2^\sigma + (1-\alpha)^\sigma \tilde{p}_2 \leq \frac{(1-\alpha)^\sigma \tilde{M}}{r_2},$$

consuming bundle **b** is the optimal choice.

Proof. See Appendix A.

The intuition behind these corner solutions is as follows: At $x_1 = r_1$ and $x_2 = r_2$, $n_1(x_1 \mid r_1)$ and $n_2(x_2 \mid r_2)$, respectively, are kinked. That is, the marginal utility from increasing the consumption of a good is lower when the respective reference level is exceeded than when the consumption of that good falls short of its reference level. This has the consequence that $x_1 = r_1$ as long as the marginal utility of consuming good 2 is lower than the marginal *dis*utility from x_1 falling below r_1 and at the same time higher than the marginal utility that would be gained from increasing x_1 above r_1.

Formally, $x_1 = r_1$ (bundle **a** is consumed) if $x_1 + \tilde{p}_2 x_2 = \tilde{M}$ and

$$\tilde{p}_2 m_1'(x_1) \leq \frac{\partial u(\mathbf{x} \mid \mathbf{r})}{\partial x_2} \leq \tilde{p}_2 (1+\lambda) m_1'(x_1),$$

Vice versa, $x_2 = r_2$ (bundle **b** is consumed) if $x_1 + \tilde{p}_2 x_2 = \tilde{M}$ and

$$\frac{1}{\tilde{p}_2} m_2'(x_2) \leq \frac{\partial u(\mathbf{x} \mid \mathbf{r})}{\partial x_1} \leq \frac{1}{\tilde{p}_2}(1+\lambda) m_2'(x_2).$$

18 The plot was created using $r_1 = 16$, $r_2 = 4$, $\alpha = ⅔$, $\rho = ½$, $\lambda = ½$.

5.2.6 *Summary: The complete demand functions for both goods*

To be able to write the demand functions in a compact way, define the vector of demand for the two goods:

$$(5.44)\quad \mathbf{x}^j(\tilde{p}_2,\tilde{M}) \equiv (x_1^j(\tilde{p}_2,\tilde{M}),\, x_2^j(\tilde{p}_2,\tilde{M})) \text{ for all } j \in \{\text{I, II, III, IV}\}.$$

Combining the results from the entire Section 5.1.3 up to here, we can write the demand functions for the two goods as follows:

$$(5.45)\quad \mathbf{x}(\tilde{p}_2,\tilde{M}) = \begin{cases} \mathbf{x}^{\text{I}}(\tilde{p}_2,\tilde{M}) & \text{if condition (5.28) holds} \\ \mathbf{x}^{\text{II}}(\tilde{p}_2,\tilde{M}) & \text{if condition (5.29) holds} \\ \mathbf{x}^{\text{III}}(\tilde{p}_2,\tilde{M}) & \text{if condition (5.30) holds} \\ \mathbf{x}^{\text{IV}}(\tilde{p}_2,\tilde{M}) & \text{if condition (5.31) holds} \\ (r_1,(\tilde{M}-r_1)/\tilde{p}_2) & \text{if condition (5.40) or (5.42) holds} \\ (\tilde{M}-\tilde{p}_2 r_2,\, r_2) & \text{if condition (5.41) or (5.43) holds} \end{cases};$$

This formula serves its purpose in summarizing all necessary information in a compact way—e.g., it can serve as the basis for the implementation of numerical examples with a mathematical software.[19] It is, however, difficult to interpret, since they do not reveal the sequence in which the branches will be chosen, when holding $\tilde{M}$ constant and increasing $\tilde{p}_2$ continuously, starting from a value close to zero.

This information is only revealed after a couple of additional deliberations that are summarized in the following proposition.

Proposition 2.

In formula (5.15) the value $\tilde{M}^*(\alpha_1,\alpha_2)$ was defined. Using this formula, I define

$$(5.46)\quad \tilde{M}_{\text{I}}^* \equiv \tilde{M}^*(\alpha,1-\alpha) \equiv r_1\left(1+\frac{(1-\alpha)r_2^\rho}{\alpha r_1^\rho}\right);$$

$$(5.47)\quad \tilde{M}_{\text{II}}^* \equiv \tilde{M}^*((1+\lambda)\alpha,1-\alpha) = r_1\left(1+\frac{(1-\alpha)r_2^\rho}{(1+\lambda)\alpha r_1^\rho}\right);$$

$$(5.48)\quad \tilde{M}_{\text{III}}^* \equiv \tilde{M}^*(\alpha,(1+\lambda)(1-\alpha)) = r_1\left(1+\frac{(1+\lambda)(1-\alpha)r_2^\rho}{\alpha r_1^\rho}\right);$$

$$(5.49)\quad \tilde{M}_{\text{IV}}^* \equiv \tilde{M}^*((1+\lambda)\alpha,(1+\lambda)(1-\alpha)) = r_1\left(1+\frac{(1+\lambda)(1-\alpha)r_2^\rho}{(1+\lambda)\alpha r_1^\rho}\right) = \tilde{M}_{\text{I}}^*.$$

19 For instance, Figure 5.3 has been created using formula (5.45) (more accurately, the formula's Mathematica equivalent).

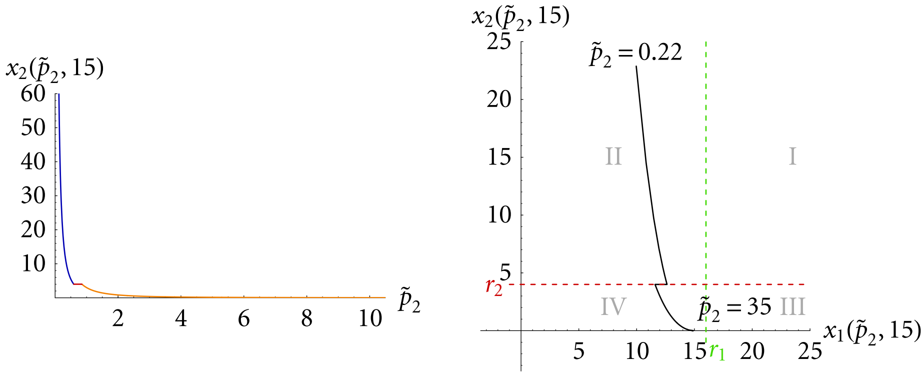


Figure 5.4: Example of a demand function of good 2 (blue indicates the segment stemming from branch II, red from bundle **b**, and orange from branch IV) and of combined demand of goods 1 and 2 in case A.

Since $\lambda > 0$, the four values r_1, $\tilde{M}^*_{\mathrm{I}}$ $(= \tilde{M}^*_{\mathrm{IV}})$, $\tilde{M}^*_{\mathrm{II}}$, $\tilde{M}^*_{\mathrm{III}}$ can be ordered as follows:

(5.50) $r_1 < \tilde{M}^*_{\mathrm{II}} < \tilde{M}^*_{\mathrm{I}} < \tilde{M}^*_{\mathrm{III}}$.

This gives rise to a set of five intervals, for each of which a separate claim follows below[20]. For each interval, the branches and bundles are chosen in a different sequence in the course of increasing $\tilde{p}_2$ continuously, starting from a value slightly above zero:

Claim 1. If $\tilde{M} \leq r_1$ ("case A"), then the agent chooses sequentially the branches/bundles II – **b** – IV in the course of increasing $\tilde{p}_2$ continuously on the interval $(0, \infty)$. This translates to the following demand function:

$$\mathbf{x}^{\mathrm{A}}(\tilde{p}_2, \tilde{M}) =$$

$$(5.51)\quad \begin{cases} \mathbf{x}^{\mathrm{II}}(\tilde{p}_2, \tilde{M}) & \text{if } [(1+\lambda)\alpha]^\sigma \tilde{p}_2^\sigma + (1-\alpha)^\sigma \tilde{p}_2 \leq \dfrac{(1-\alpha)^\sigma \tilde{M}}{r_2} \\ ((\tilde{M} - \tilde{p}_2 r_2), r_2) & \begin{array}{l} \text{if } [(1+\lambda)\alpha]^\sigma \tilde{p}_2^\sigma + (1-\alpha)^\sigma \tilde{p}_2 > \dfrac{(1-\alpha)^\sigma \tilde{M}}{r_2} \\ \text{and } \alpha^\sigma \tilde{p}_2^\sigma + (1-\alpha)^\sigma \tilde{p}_2 \leq \dfrac{(1-\alpha)^\sigma \tilde{M}}{r_2} \end{array} \\ \mathbf{x}^{\mathrm{IV}}(\tilde{p}_2, \tilde{M}) & \text{if } \alpha^\sigma \tilde{p}_2^\sigma + (1-\alpha)^\sigma \tilde{p}_2 > \dfrac{(1-\alpha)^\sigma \tilde{M}}{r_2} \end{cases};$$

Proof. See Appendix A.

20 On the page of each claim, an example demand function for the respective case is depicted.

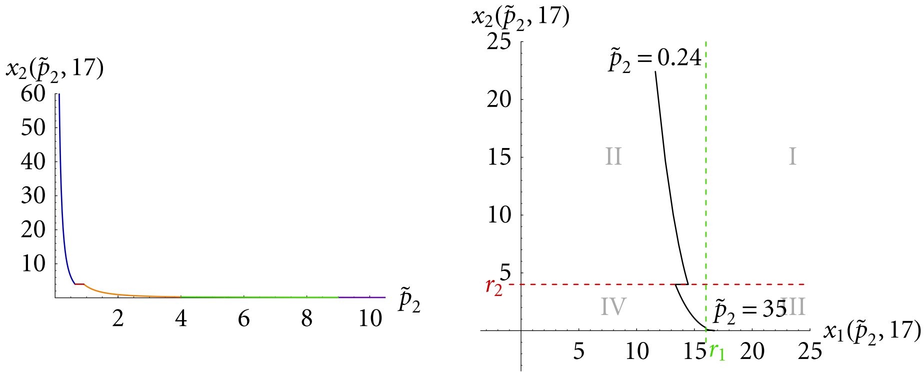


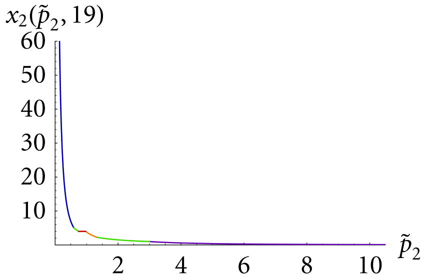
Figure 5.5: Example of a demand function of good 2 (blue indicates the segment stemming from branch II, red from bundle **b**, orange from branch IV, green from bundle **a**, and purple from branch III) and combined demand of goods 1 and 2 in case B.

Claim 2. If $r_1 < \tilde{M} \leq \tilde{M}^*_{\text{II}}$ ("case B"), then the agent chooses sequentially the branches/bundles II – **b** – IV – **a** – III in the course of increasing $\tilde{p}_2$ continuously on the interval $(0, \infty)$. This translates to the following demand function:

$$\mathbf{x}^{\text{B}}(\tilde{p}_2, \tilde{M}) =$$

$$\begin{cases} \mathbf{x}^{\text{II}}(\tilde{p}_2, \tilde{M}) & \text{if } [(1+\lambda)\alpha]^\sigma \tilde{p}_2^\sigma + (1-\alpha)^\sigma \tilde{p}_2 \leq \dfrac{(1-\alpha)^\sigma \tilde{M}}{r_2} \\ (r_2, \tilde{M} - \tilde{p}_2 r_2) & \text{if } [(1+\lambda)\alpha]^\sigma \tilde{p}_2^\sigma + (1-\alpha)^\sigma \tilde{p}_2 > \dfrac{(1-\alpha)^\sigma \tilde{M}}{r_2} \\ & \text{and } \alpha^\sigma \tilde{p}_2^\sigma + (1-\alpha)^\sigma \tilde{p}_2 \leq \dfrac{(1-\alpha)^\sigma \tilde{M}}{r_2} \\ \mathbf{x}^{\text{IV}}(\tilde{p}_2, \tilde{M}) & \text{if } \alpha^\sigma \tilde{p}_2^\sigma + (1-\alpha)^\sigma \tilde{p}_2 > \dfrac{(1-\alpha)^\sigma \tilde{M}}{r_2} \\ & \text{and } \tilde{p}_2 < \left(\dfrac{(1-\alpha)^\sigma}{\alpha^\sigma \tilde{M}/r_1 - \alpha^\sigma}\right)^{1/(\sigma-1)} \\ (r_1, (\tilde{M} - r_1)/\tilde{p}_2) & \text{if } \left(\dfrac{(1-\alpha)^\sigma}{\alpha^\sigma \tilde{M}/r_1 - \alpha^\sigma}\right)^{1/(\sigma-1)} \leq \tilde{p}_2 < \left(\dfrac{[(1+\lambda)(1-\alpha)]^\sigma}{\alpha^\sigma \tilde{M}/r_1 - \alpha^\sigma}\right)^{1/(\sigma-1)} \\ \mathbf{x}^{\text{III}}(\tilde{p}_2, \tilde{M}) & \text{if } \tilde{p}_2 \geq \left(\dfrac{[(1+\lambda)(1-\alpha)]^\sigma}{\alpha^\sigma \tilde{M}/r_1 - \alpha^\sigma}\right)^{1/(\sigma-1)} \end{cases} ; \tag{5.52}$$

Proof. See Appendix A.



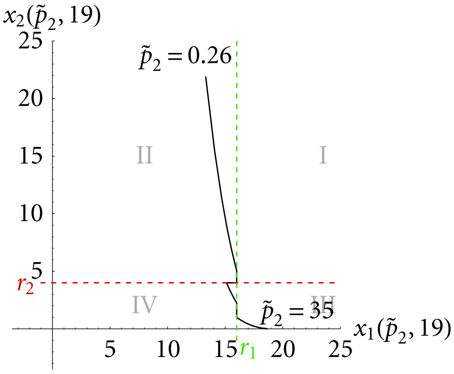


Figure 5.6: Example of a demand function of good 2 (blue indicates the segment stemming from branch II, green from bundle **a**, red from bundle **b**, orange from branch IV, green from bundle **a** again, purple from branch III) and of combined demand of goods 1 and 2 in case C.

Claim 3. If $\tilde{M}^*_{\mathrm{II}} < \tilde{M} \leq \tilde{M}^*_{\mathrm{I}}$ ("case C"), then the agent chooses sequentially the branches/bundles II – **a** – **b** – IV – **a** – III in the course of increasing $\tilde{p}_2$ continuously on the interval $(0, \infty)$. This translates to the demand function

$$\mathbf{x}^{\mathrm{C}}(\tilde{p}_2, \tilde{M}) =$$

$$\begin{cases} \mathbf{x}^{\mathrm{II}}(\tilde{p}_2, \tilde{M}) & \text{if } \tilde{p}_2 < \left(\dfrac{(1-\alpha)^\sigma}{[(1+\lambda)\alpha]^\sigma \tilde{M}/r_1 - [(1+\lambda)\alpha]^\sigma} \right)^{1/(\sigma-1)} \\ (r_1, (\tilde{M}-r_1)/\tilde{p}_2) & \text{if } \left(\dfrac{(1-\alpha)^\sigma}{[(1+\lambda)\alpha]^\sigma \tilde{M}/r_1 - [(1+\lambda)\alpha]^\sigma} \right)^{1/(\sigma-1)} \leq \tilde{p}_2 < \dfrac{\tilde{M}-r_1}{r_2} \\ (\tilde{M}-\tilde{p}_2 r_2, r_2) & \text{if } \dfrac{\tilde{M}-r_1}{r_2} < \tilde{p}_2 \\ & \text{and } \alpha^\sigma \tilde{p}_2^\sigma + (1-\alpha)^\sigma \tilde{p}_2 \leq \dfrac{(1-\alpha)^\sigma \tilde{M}}{r_2} \\ \mathbf{x}^{\mathrm{IV}}(\tilde{p}_2, \tilde{M}) & \text{if } \alpha^\sigma \tilde{p}_2^\sigma + (1-\alpha)^\sigma \tilde{p}_2 > \dfrac{(1-\alpha)^\sigma \tilde{M}}{r_2} \\ & \text{and } \tilde{p}_2 < \left(\dfrac{(1-\alpha)^\sigma}{\alpha^\sigma \tilde{M}/r_1 - \alpha^\sigma} \right)^{1/(\sigma-1)} \\ (r_1, (\tilde{M}-r_1)/\tilde{p}_2) & \text{if } \left(\dfrac{(1-\alpha)^\sigma}{\alpha^\sigma \tilde{M}/r_1 - \alpha^\sigma} \right)^{1/(\sigma-1)} \leq \tilde{p}_2 < \left(\dfrac{[(1+\lambda)(1-\alpha)]^\sigma}{\alpha^\sigma \tilde{M}/r_1 - \alpha^\sigma} \right)^{1/(\sigma-1)} \\ \mathbf{x}^{\mathrm{III}}(\tilde{p}_2, \tilde{M}) & \text{if } \tilde{p}_2 \geq \left(\dfrac{[(1+\lambda)(1-\alpha)]^\sigma}{\alpha^\sigma \tilde{M}/r_1 - \alpha^\sigma} \right)^{1/(\sigma-1)} \end{cases} ; \tag{5.53}$$

Proof. See Appendix A.

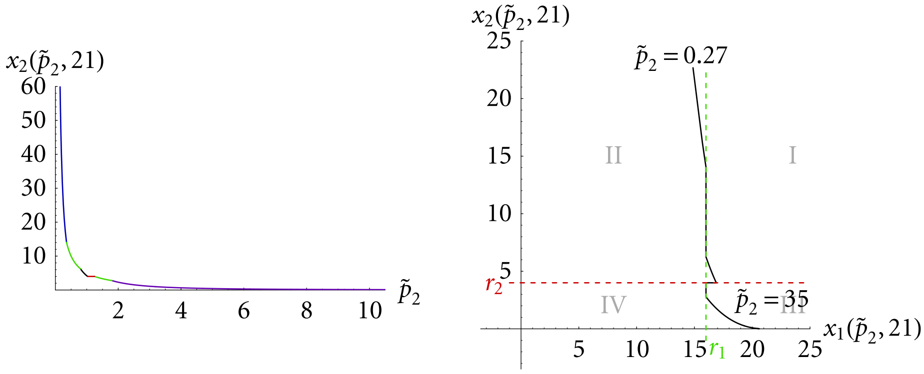


Figure 5.7: Example of a demand function of good 2 (blue indicates the segment stemming from branch II, green from bundle **a**, black from branch I, red from bundle **b**, green from bundle **a** again, and purple from branch III) and combined demand of goods 1 and 2 in case D.

Claim 4. If $\tilde{M}_{\mathrm{I}}^* < \tilde{M} \leq \tilde{M}_{\mathrm{III}}^*$ ("case D"), then the agent chooses sequentially the branches/bundles II – **a** – I – **b** – **a** – III in the course of increasing $\tilde{p}_2$ continuously on the interval $(0, \infty)$. This translates to the demand function

$$\mathbf{x}^{\mathrm{D}}(\tilde{p}_2, \tilde{M}) =$$

$$(5.54)\quad \begin{cases} \mathbf{x}^{\mathrm{II}}(\tilde{p}_2, \tilde{M}) & \text{if } \tilde{p}_2 < \left(\dfrac{(1-\alpha)^\sigma}{[(1+\lambda)\alpha]^\sigma \tilde{M}/r_1 - [(1+\lambda)\alpha]^\sigma} \right)^{1/(\sigma-1)} \\ (r_1, (\tilde{M}-r_1)/\tilde{p}_2) & \text{if } \left(\dfrac{(1-\alpha)^\sigma}{[(1+\lambda)\alpha]^\sigma \tilde{M}/r_1 - [(1+\lambda)\alpha]^\sigma} \right)^{1/(\sigma-1)} \leq \tilde{p}_2 < \left(\dfrac{(1-\alpha)^\sigma}{\alpha^\sigma \tilde{M}/r_1 - \alpha^\sigma} \right)^{1/(\sigma-1)} \\ \mathbf{x}^{\mathrm{I}}(\tilde{p}_2, \tilde{M}) & \text{if } \tilde{p}_2 \geq \left(\dfrac{(1-\alpha)^\sigma}{\alpha^\sigma \tilde{M}/r_1 - \alpha^\sigma} \right)^{1/(\sigma-1)} \\ & \text{and } \alpha^\sigma \tilde{p}_2^\sigma + (1-\alpha)^\sigma \tilde{p}_2 \leq \dfrac{(1-\alpha)^\sigma \tilde{M}}{r_2} \\ (\tilde{M} - \tilde{p}_2 r_2, r_2) & \text{if } \alpha^\sigma \tilde{p}_2^\sigma + (1-\alpha)^\sigma \tilde{p}_2 > \dfrac{(1-\alpha)^\sigma \tilde{M}}{r_2} \\ & \text{and } \tilde{p}_2 < \dfrac{\tilde{M}-r_1}{r_2} \\ (r_1, (\tilde{M}-r_1)/\tilde{p}_2) & \text{if } \dfrac{\tilde{M}-r_1}{r_2} \leq \tilde{p}_2 < \left(\dfrac{[(1+\lambda)(1-\alpha)]^\sigma}{\alpha^\sigma \tilde{M}/r_1 - \alpha^\sigma} \right)^{1/(\sigma-1)} \\ \mathbf{x}^{\mathrm{III}}(\tilde{p}_2, \tilde{M}) & \text{if } \tilde{p}_2 \geq \left(\dfrac{[(1+\lambda)(1-\alpha)]^\sigma}{\alpha^\sigma \tilde{M}/r_1 - \alpha^\sigma} \right)^{1/(\sigma-1)} \end{cases} ;$$

Proof. See Appendix A.

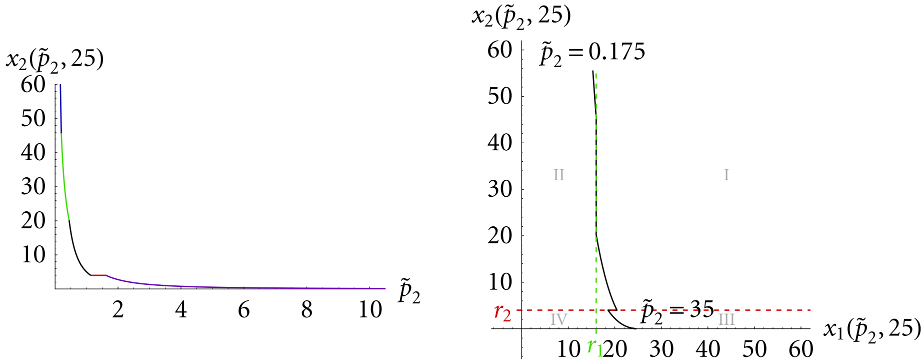


Figure 5.8: Example of a demand function of good 2 (blue indicates the segment stemming from branch II, green from bundle **a**, black from branch I, red from bundle **b**, and purple from branch III) and combined demand of goods 1 and 2 in case E.

Claim 5. If $\tilde{M}^{*}_{\mathrm{III}} < \tilde{M}$ ("case E"), then the agent chooses sequentially the branches/bundles II – **a** – I – **b** – III in the course of increasing $\tilde{p}_2$ continuously on the interval $(0, \infty)$. This translates to the following demand function:

$$\mathbf{x}^{\mathrm{E}}(\tilde{p}_2, \tilde{M}) = \tag{5.55}$$
$$\begin{cases} \mathbf{x}^{\mathrm{II}}(\tilde{p}_2, \tilde{M}) & \text{if } \tilde{p}_2 < \left(\dfrac{(1-\alpha)^{\sigma}}{[(1+\lambda)\alpha]^{\sigma}\tilde{M}/r_1 - [(1+\lambda)\alpha]^{\sigma}} \right)^{1/(\sigma-1)} \\ (r_1, (\tilde{M}-r_1)/\tilde{p}_2) & \text{if } \left(\dfrac{(1-\alpha)^{\sigma}}{[(1+\lambda)\alpha]^{\sigma}\tilde{M}/r_1 - [(1+\lambda)\alpha]^{\sigma}} \right)^{1/(\sigma-1)} \leq \tilde{p}_2 < \left(\dfrac{(1-\alpha)^{\sigma}}{\alpha^{\sigma}\tilde{M}/r_1 - \alpha^{\sigma}} \right)^{1/(\sigma-1)} \\ \mathbf{x}^{\mathrm{I}}(\tilde{p}_2, \tilde{M}) & \text{if } \tilde{p}_2 \geq \left(\dfrac{(1-\alpha)^{\sigma}}{\alpha^{\sigma}\tilde{M}/r_1 - \alpha^{\sigma}} \right)^{1/(\sigma-1)} \\ & \text{and } \alpha^{\sigma}\tilde{p}_2^{\sigma} + (1-\alpha)^{\sigma}\tilde{p}_2 \leq \dfrac{(1-\alpha)^{\sigma}\tilde{M}}{r_2} \\ (\tilde{M} - \tilde{p}_2 r_2, r_2) & \text{if } \alpha^{\sigma}\tilde{p}_2^{\sigma} + (1-\alpha)^{\sigma}\tilde{p}_2 > \dfrac{(1-\alpha)^{\sigma}\tilde{M}}{r_2} \\ & \text{and } \alpha^{\sigma}\tilde{p}_2^{\sigma} + [(1+\lambda)(1-\alpha)]^{\sigma}\tilde{p}_2 \leq \dfrac{[(1+\lambda)(1-\alpha)]^{\sigma}\tilde{M}}{r_2} \\ \mathbf{x}^{\mathrm{III}}(\tilde{p}_2, \tilde{M}) & \text{if } \alpha^{\sigma}\tilde{p}_2^{\sigma} + [(1+\lambda)(1-\alpha)]^{\sigma}\tilde{p}_2 > \dfrac{[(1+\lambda)(1-\alpha)]^{\sigma}\tilde{M}}{r_2} \end{cases};$$

Proof. See Appendix A.

The reader might wonder why we obtain such different demand functions for different income levels. This phenomenon is caused by the steepness of the agent's indifference cuves in the quadrants I–IV (as they were defined in Figure 5.2) in combination with the steepness of the budget line. Let me illustrate this with the help of an example:

Recall that the highest value of $\tilde{p}_2$ for which an interior solution on branch II is attained, is

$$\tilde{p}_2^{\text{II},ub} \equiv \min\{\tilde{p}_2^*((1+\lambda)\alpha, 1-\alpha, \tilde{M}), \tilde{p}_2^{**}((1+\lambda)\alpha, 1-\alpha, \tilde{M})\};$$

see equation (5.34). Recall that $\tilde{p}_2^*$ decreases and $\tilde{p}_2^{**}$ increases in $\tilde{M}$, and that the two values coincide when $\tilde{M} = \tilde{M}_{\text{II}}^*$. Thus, if $\tilde{M} < \tilde{M}_{\text{II}}^*$, then $\tilde{p}_2^{**} < \tilde{p}_2^*$, and the interior solution on branch II connects to bundle **b**. Conversely, if $\tilde{M} > \tilde{M}_{\text{II}}^*$, the interior solution on branch II connects to bundle **a**.

In quadrant II (where $x_1 < r_1$ and $x_2 \geq r_2$), the agent's indifference curves are relatively steep, see Figure 4.1. For a relatively low income, $\tilde{M} < \tilde{M}_{\text{II}}^*$, the budget restriction has to be relatively steep as well (i.e., $\tilde{p}_2$ is small) in order to cut through quadrant II at all. In combination, this leads to the fact that the two curves are tangent as long as long as $\tilde{p}_2 < \tilde{p}_2^{**}$. Quite the reverse, for a relatively high income, $\tilde{M} > \tilde{M}_{\text{II}}^*$, the budget restriction is comparatively flat when cutting through quadrant II. In combination with the steep indifference cuve, this leads to the fact that the two curves are tangent as long as long as $\tilde{p}_2 < \tilde{p}_2^*$.

Analogous reasoning can be applied for the other branches. The results were collected in Proposition 2.

5.2.7 *Continuity of the demand functions*

The results from the two previous subsections combined show that for any price–income combination $(\tilde{p}_2, \tilde{M})$, a unique demand vector exists. Furthermore, the demand functions for the two goods are continuous.

> **Proposition 3.** The demand functions for good 1 and for good 2 are continuous in $\tilde{p}_2$ and in $\tilde{M}$.
>
> **Proof.** See Appendix A.

Knowing that the demand functions are continuous will facilitate solving the profit-maximization problem of the producer of good 2.

5.3 Solving the profit maximization problem of firm 2

Since firm 2 is assumed to be the only producer of good 2, it possesses market power and can choose the price of good 2 to maximize its profits. What we are interested in is whether the profit-maximizing price exhibits stickiness, i.e. non-responsiveness to a changing environment. As a benchmark, consider the standard monopoly model without information or menu costs: here, any change in the monopoly's marginal cost changes the profit-maximizing price. Thus, any non-responsiveness of the profit-maximizing price to cost changes in my model is caused by the introduction of loss-averse consumers.

This is exactly the kind of stickiness that occurs in the models by Sibly (2002) and by H&K: Changes of the marginal cost inside certain intervals do not translate to changes of the profit-maximizing price. Since the consumers' demand curves generated by my model are kinked, my model has the potential of generating price stickiness, too.

To determine its profit-maximizing price, firm 2 has to know which of the five cases A–E outlined in Proposition 2 prevails. It is important for the monopolist to distinguish between these cases, since a consumer's demand function changes its shape quite drastically across the different situations (see Figures 5.4–5.8).

In every case A–E, the price $\tilde{p}_2^{max,\,global}(\tilde{c}_2, \tilde{M}, \mathbf{r})$ which maximizes profits globally can be determined unanimously, however only numerically. Therefore, I cannot provide a functional form for $\tilde{p}_2^{max,\,global}(\tilde{c}_2, \tilde{M}, \mathbf{r})$, but will have to describe the way to solve the profit maximization problem with the help of an example. In the example, the values are chosen such that case D prevails, i.e. $\tilde{M}_{\mathrm{I}}^* < \tilde{M} \leq \tilde{M}_{\mathrm{III}}^*$. Doing the analysis for this case is instructive, since it can be applied to all other cases analogously.

Assume that the parameters take on the following values: $\alpha = 2/3$, $\rho = 1/2$ (i.e. $\sigma = 2$), $\lambda = 1/2$. Furthermore, the agent has reference levels $r_1 = 16$, $r_2 = 4$, and income $\tilde{M} = 21$.

For these parameter values, we get $\tilde{M}_{\mathrm{II}}^* = 18\frac{2}{3}$, $\tilde{M}_{\mathrm{I}}^* = 20$, $\tilde{M}_{\mathrm{III}}^* = 22$. As stated above, the agent's demand function in this example is that of case D, which is given by formula (5.54). In case D, the agent chooses sequentially the branches/bundles II – **a** – I – **b** – **a** – III in the course of increasing $\tilde{p}_2$ continuously on the interval $(0, \infty)$. Profit could potentially be maximized at any of these branches/bundles, because $\lim_{\tilde{c}_2 \to 0} \tilde{p}_2^{max} = 0$ and $\lim_{\tilde{c}_2 \to \infty} \tilde{p}_2^{max} = \infty$. Hence, for each branch/bundle, the locally highest attainable profit has to be determined. Afterwards, to find the global maximum, all local maxima have to be compared.

Several profit functions, associated with different cost levels, are drawn in Figure 5.9. From the figure it can be seen that for low unit costs (0.015 and 0.05) the profit is maximized for prices $\tilde{p}_2$ somewhere between 0 and 0.2. For intermediate unit costs (0.15, 0.25, 0.5, and 0.75), the profit-maximizing price equals 1.8; and for $\tilde{c}_2 = 1$, profit is maximized at a price above 2.

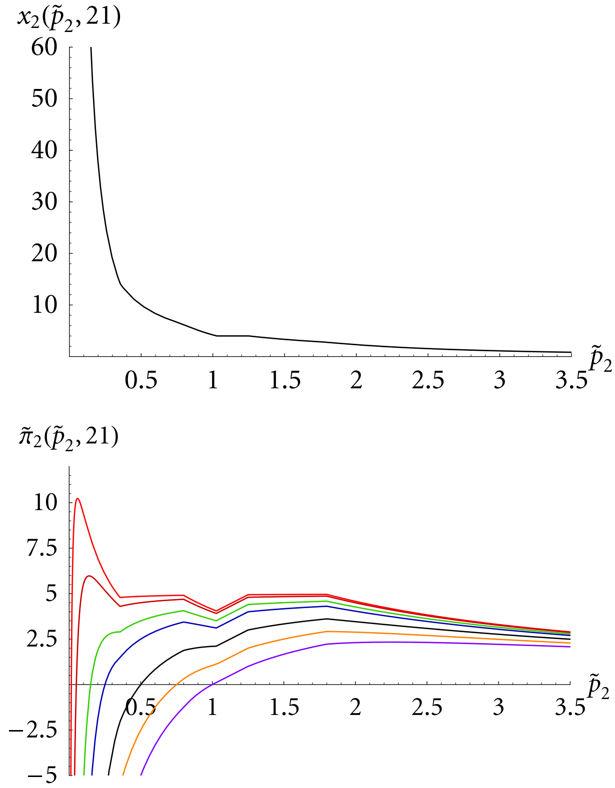


Figure 5.9: Demand function and profit functions of firm 2 for the hypothetical parameter values assumed throughout Section 5.3. The cost functions are drawn for $\tilde{c}_2$ equal to 0.015, 0.05, 0.15, 0.25, 0.5, 0.75 and 1 (ordered from top to bottom).

Recall the definition of $\tilde{p}_2^{\max}(\alpha_1, \alpha_2, \sigma, \tilde{c}_2)$ in formula (5.18):

$$\tilde{p}_2^{\max}(\alpha_1, \alpha_2, \sigma, \tilde{c}_2) \equiv \tilde{p}_2 : \tilde{c}_2 = \frac{\alpha_1^{\sigma}(\sigma - 1)\tilde{p}_2}{\alpha_2^{\sigma}\tilde{p}_2^{1-\sigma} + \alpha_1^{\sigma}\sigma}.$$

Since α_1 and α_2 vary across the different branches in the way determined in Section 5.2.3, the formula for $\tilde{p}_2^{max}$ differs across the four branches. I define

$$(5.56) \quad \tilde{p}_2^{\max,\mathrm{I}} \equiv \tilde{p}_2^{max}(\alpha, 1-\alpha, \sigma, \tilde{c}_2);$$

$$(5.57) \quad \tilde{p}_2^{\max,\mathrm{II}} \equiv \tilde{p}_2^{max}((1+\lambda)\alpha, 1-\alpha, \sigma, \tilde{c}_2);$$

$$(5.58) \quad \tilde{p}_2^{\max,\mathrm{III}} \equiv \tilde{p}_2^{max}(\alpha, (1+\lambda)(1-\alpha), \sigma, \tilde{c}_2);$$

$$(5.59) \quad \tilde{p}_2^{\max,\mathrm{IV}} \equiv \tilde{p}_2^{max}((1+\lambda)\alpha, (1+\lambda)(1-\alpha), \sigma, \tilde{c}_2) = \tilde{p}_2^{max}(\alpha, 1-\alpha, \sigma, \tilde{c}_2).$$

Of course, we have to check for consistency again: It can happen that the profit-maximizing price defined on a certain branch lies outside the condition which has to be met for an interior solution to be attained on the respective branch. Based on the reasoning in Section 5.1.3, the following holds:

- If $\tilde{p}_2^{\max,\mathrm{I}} < \tilde{p}_2^{\mathrm{I},lb}$, then firm 2's profit, given that the consumer is on branch I, is maximized by setting $\tilde{p}_2$ equal to $\tilde{p}_2^{\mathrm{I},lb}$.
If $\tilde{p}_2^{\max,\mathrm{I}} > \tilde{p}_2^{\mathrm{I},ub}$, then firm 2's profit, given that the consumer is on branch I, is maximized by setting $\tilde{p}_2$ equal to $\tilde{p}_2^{\mathrm{I},ub}$.
- If $\tilde{p}_2^{\max,\mathrm{II}} > \tilde{p}_2^{\mathrm{II},ub}$, then firm 2's profit, given that the consumer is on branch II, is maximized by setting $\tilde{p}_2$ equal to $\tilde{p}_2^{\mathrm{II},ub}$.
- If $\tilde{p}_2^{\max,\mathrm{III}} < \tilde{p}_2^{\mathrm{III},lb}$, then firm 2's profit, given that the consumer is on branch III, is maximized by setting $\tilde{p}_2$ equal to $\tilde{p}_2^{\mathrm{III},lb}$.
- For $\tilde{p}_2^{\max,\mathrm{IV}}$ and branch IV the same as for $\tilde{p}_2^{\max,\mathrm{I}}$ and branch I holds.

In the next—and final—step, the profit-maximizing price, given that the agent consumes bundle **a** or **b**, is calculated. If the agent consumes bundle **a**, firm 2's profit is

$$(\tilde{p}_2 - \tilde{c}_2)\frac{\tilde{M} - r_1}{\tilde{p}_2} = \tilde{M} - r_1 - \tilde{c}_2\frac{\tilde{M} - r_1}{\tilde{p}_2},$$

which is obviously strictly increasing in $\tilde{p}_2$. If the agent consumes bundle **b**, firm 2's profit is

$$(\tilde{p}_2 - \tilde{c}_2)r_2,$$

which is strictly increasing in $\tilde{p}_2$ as well. Therefore, firm 2 sets the price equal to the upper bound of the interval on which it is optimal for the agent to consume bundle **a** or **b**, respectively. Now, the upper bound of an interval on which it is optimal to consume **a** or **b** is always either $\tilde{p}_2^{\mathrm{I},lb}$, $\tilde{p}_2^{\mathrm{III},lb}$, or $\tilde{p}_2^{\mathrm{IV},lb}$. Hence, the profit-maximizing price, given that the agent consumes **a** or **b**, is already among the profit-maximizing prices determined for the branches I, II, III, and IV.

Furthermore, from the above list of potentially profit-maximizing prices, the values $\tilde{p}_2^{\mathrm{I},ub}$, $\tilde{p}_2^{\mathrm{II},ub}$, and $\tilde{p}_2^{\mathrm{IV},ub}$ can be excluded: At these upper bounds, either bundle **a** or bundle **b** is consumed. For the two bundles, we have just found out that the price is pushed upward until the lower bound of the neighboring branch is reached. Hence, the profit-maximizing price can never be attained at the upper bound of a branch—is attained either at an interior solution on one of the four branches or at the *lower* bound of branch I, III, or IV.

For the given parameter values, the branches I, II, and III are feasible, whose upper and lower bounds as follows [given by formula (5.54)]:

- upper bound of the interval on which an interior solution on branch II is attained: $\tilde{p}_2^{\mathrm{II},ub} \doteq 0.355556$;
- lower bound of the interval on which an interior solution on branch I is attained: $\tilde{p}_2^{\mathrm{I},lb} \doteq 0.8$; respective upper bound: $\tilde{p}_2^{\mathrm{I},ub} \doteq 1.02744$;
- lower bound of the interval on which an interior solution on branch III is attained: $\tilde{p}_2^{\mathrm{III},lb} \doteq 1.8$.

By plugging these four values into (5.18), we obtain the unit costs at which exactly these four prices are the profit-maximizing ones, given that the agent is at the respective interior solution:

- $\tilde{c}_2^{\text{II},ub} \doteq 0.153754$;
- $\tilde{c}_2^{\text{I},lb} \doteq 0.345946$; $\tilde{c}_2^{\text{I},ub} \doteq 0.458001$;
- $\tilde{c}_2^{\text{III},lb} \doteq 0.778378$.

Hence, to obtain the price that maximizes firm 2's profits globally, the following comparisons have to be carried out:

- If $\tilde{c}_2 \leq \tilde{c}_2^{\text{II},ub}$,
  the profits made at $\tilde{p}_2^{\max,\text{II}}$, at $\tilde{p}_2^{\text{I},lb}$, and at $\tilde{p}_2^{\text{III},lb}$ have to be compared.
- If $\tilde{c}_2^{\text{II},ub} \leq \tilde{c}_2 \leq \tilde{c}_2^{\text{I},lb}$,
  the profits made at $\tilde{p}_2^{\text{I},lb}$ and at $\tilde{p}_2^{\text{III},lb}$ have to be compared.
- If $\tilde{c}_2^{\text{I},lb} \leq \tilde{c}_2 \leq \tilde{c}_2^{\text{I},ub}$,
  the profits made at $\tilde{p}_2^{\max,\text{I}}$ and at $\tilde{p}_2^{\text{III},lb}$ have to be compared.
- If $\tilde{c}_2^{\text{I},ub} \leq \tilde{c}_2 \leq \tilde{c}_2^{\text{III},lb}$,
  the profit made at $\tilde{p}_2^{\text{III},lb}$ has to be calculated.
- If $\tilde{c}_2^{\text{III},lb} \leq \tilde{c}_2$,
  the profit made at $\tilde{p}_2^{\max,\text{III}}$ has to be calculated.

Out of the potential maximizers for a given $\tilde{c}_2$, the one that yields the highest profit is $\tilde{p}_2^{max,global}(\tilde{c}_2, \tilde{M}, \mathbf{r})$.

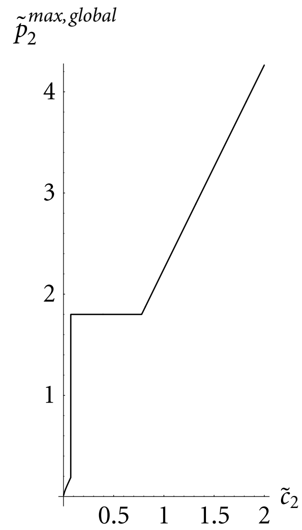


Figure 6.1: The profit-maximizing price as a function of marginal cost in case D ($\tilde{M}$ = 21) .

6 Model results

6.1 Non-responsiveness of the profit-maximizing price to cost changes

At the end of the previous section (Section 5.3), it was demonstrated with the help of an example case how to find the price which maximizes firm 2's profits. For the parameter constellation already assumed in Section 5.3, Section 6 presents results regarding non-responsiveness of $\tilde{p}_2^{max,global}(\tilde{c}_2, \tilde{M}, \mathbf{r})$ with respect to cost changes.

Figure 6.1 depicts $\tilde{p}_2^{max,global}(\tilde{c}_2, 21, (16,4))$. The most important thing about this plot is that it is horizontal for $\tilde{c}_2 \in [0.07235, \tilde{c}_2^{\text{III},lb}] = [0.07235, 0.77838]$, i.e., it indeed reveals non-responsiveness of the profit-maximizing price with respect to changes in firm 2's unit costs, as long as the original and the changed cost lie inside a certain interval. The profit-maximizing price throughout this interval of unit costs is 1.8.

This corresponds to the results we obtained earlier from visual inspection of Figure 5.9: There, for $\tilde{c}_2$ equal to 0.15, 0.25, 0.5, and 0.75, the profit-maximizing price was found to be 1.8.

The results for other hypothetical incomes (such that the cases A, B, C, and E, respectively, hold) are presented in Figure 6.2. It is obvious that the profit-maximizing price is non-responsive with respect to cost changes inside a certain interval for the cases A, B, C, and E, too.

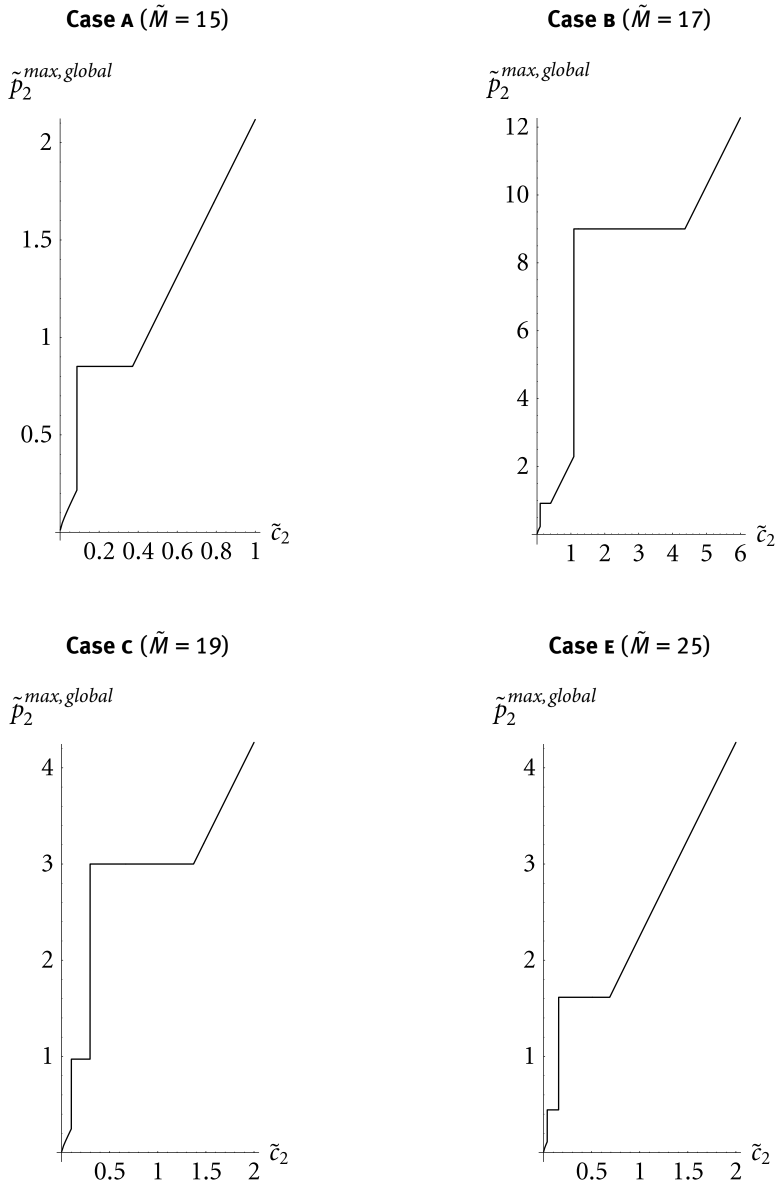


Figure 6.2: The profit-maximizing price as a function of marginal cost in cases A, B, C, and E.

Since the $\tilde{p}_2^{max}$ of each branch and bundle is non-decreasing in $\tilde{c}_2$ (see Section 5.1.3), it is possible to state that $\tilde{p}_2^{max, global}$ is non-decreasing in $\tilde{c}_2$.

In contrast, no such monotonicity exists in the connection between $\tilde{p}_2^{max, global}$ and the household's income $\tilde{M}$. This is illustrated by the upper panel of Figure 6.3: there are income levels for which $\tilde{p}_2^{max, global}(0.15, \tilde{M}, (16, 4))$ does not respond to changes in $\tilde{M}$. This non-responsiveness is due to the fact that (5.18) is independent of $\tilde{M}$, which is an artifact of the assumed utility function

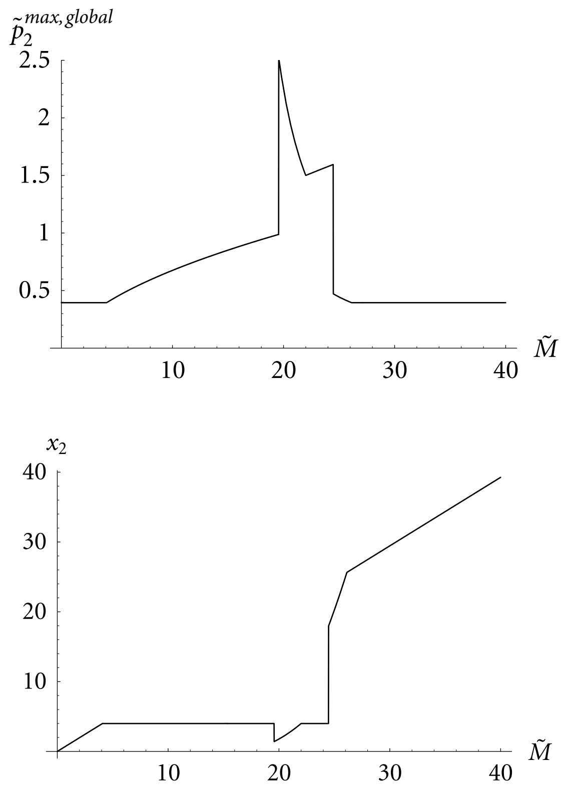


Figure 6.3: Non-monotonic dependence of the profit-maximizing price and of the associated output x_2 on income $\tilde{M}$. [21]

$m(\mathbf{x})$. Consequently, without loss aversion, the plot in the upper panel of Figure 6.3 would be entirely flat. This means that loss aversion, while reducing one kind of responsiveness—to cost changes—creates another kind of responsiveness—to changes in income.

The lower panel of Figure 6.3 shows a plot of $x_2(\tilde{p}_2^{max, global}(0.15, \tilde{M}, (16,4)))$. For the horizontal parts in the plot of $\tilde{p}_2^{max, global}(0.15, \tilde{M}, (16,4))$, any increase in $\tilde{M}$ has a positive impact on the output level x_2. However, for the ranges of income inside which $\tilde{p}_2^{max, global}$ reacts to changes in $\tilde{M}$, the positive impact that a higher $\tilde{M}$ could have on output is totally offset by price increases.

This finding has an important implication for the role of monetary policy: Over a wide range of incomes, an increase in income only increases $\tilde{p}_2$, but not production and consumption x_2. That is, in my model, expansionary monetary policy—increasing nominal income $\tilde{M}$—has in many cases no real, but only inflationary effects.

21 The plot was created using $r_1 = 16$, $r_2 = 4$, $\alpha = 2/3$, $\rho = 1/2$, $\lambda = 1/2$, and $\tilde{c}_2 = 0.15$.

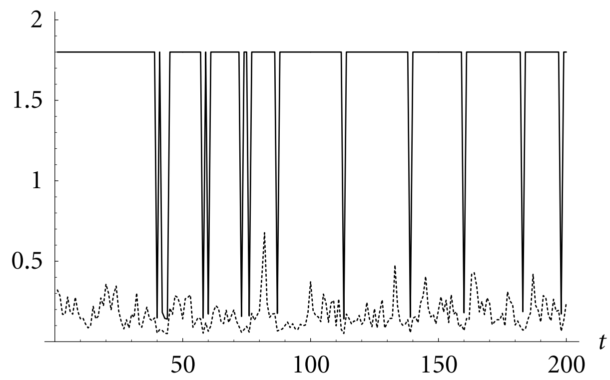


Figure 6.4: Simulated time series of unit costs $\tilde{c}_2$ (dashed line) and of the profit-maximizing price $\tilde{p}_2$ (solid line); $\mu_c = \log 0.15$ and $\sigma_c = 0.9$.

6.2 Simulation of a price series

In order to illustrate the elaboration in Sections 5.3 and 6.1, a numerical example is analyzed. In this example the parameters and reference levels take on the same hypothetical values that were already used in Section 5: $\alpha = ⅔$, $\rho = ½$ (i.e., $\sigma = 2$), $\lambda = ½$, $r_1 = 16$, and $r_2 = 4$. As in Section 5.3, I furthermore assume the agent's income to be $\tilde{M} = 21$. Regarding the distribution of c_2 the following assumptions are made: $\sigma_c = 0.9$ and $\mu_c = \log 0.15$ so that $\mathrm{E}[c_2] = \exp(\mu_c) = 0.15$. The number of simulated periods is set to $T = 200$.

A simulated time series of unit costs and of the profit-maximizing prices associated with these cost realizations is shown in Figure 6.4.[22] To examine the robustness of the model's results, a second simulation was done, in which the mean and the variance of the AR(1) process were increased: $\mu_c = \log 0.25$ and $\sigma_c = 1.1$. The result of this simulation is presented in Figure 6.5.

6.3 Discussion of the model's properties

Comparison of Figure 6.4 with the actual price series depicted in Figure 3.1 reveals that the actual and the simulated price series are quite similar. Both exhibit constancy of the good's price over a large number of periods. This constancy is in both cases interrupted by short periods of downward deviations, with quick return to the price that prevailed before.

22 The simulation of the two time series was programmed in the mathematical software Mathematica (version 5.0). The Mathematica code of the simulation is not provided as a printout, but it is contained in the CD-ROM to be found at the back cover of this thesis.

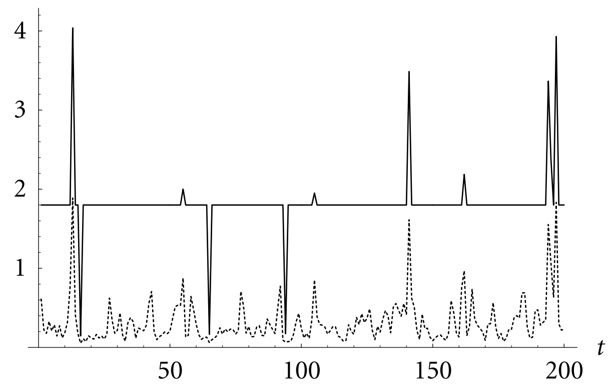


Figure 6.5: Simulated time series of unit costs $\tilde{c}_2$ (dashed line) and of the profit-maximizing price $\tilde{p}_2$ (solid line); $\mu_c = \log 0.25$ and $\sigma_c = 1.1$.

The fact that we do not observe upward deviations of the price from its "regular" level in Figure 6.4 is due to the choice of the mean and standard deviation of the cost distribution. This is demonstrated by Figure 6.5: Here, upward deviations of the price from the "regular" price can be observed as well as downward deviations.

This leads us to the shortcomings of the model as it is specified so far. Four major weaknesses are important to mention:

1 While the model is able to create constancy in the profit-maximizing price of a consumer good, for certain values of μ_c and σ_c, it creates too many upward deviations of the actual price from the "regular price" in comparison with empirical data. In empirical data, such upward deviations are observed rather rarely [see Figure 3.1: once in panel a (week 85), twice in panel b (weeks 190 and 220) only].
2 If the mean of the cost distribution did not fall into the range [0.07235, 0.77838] much less price rigidity would be observable, because in this case the assumed AR(1) process would create many cost realizations outside this interval.
3 In addition, Figure 3.1 reveals that the "regular price" also changes from time to time. Since in my model so far, constancy of the reference levels is assumed, neither the interval of costs for which the "regular price" is chosen nor the "regular price" itself varies over time, contradicting the empirical data. This is accompanied by another contradiction to the empirical data: As presented in Section 3.3, LEVY ET AL. (2004) found that small price increases are more frequent than small price decreases. In my simulations, the price always returns to the invariable "regular price", thereby leading to an *identical* number of price increases and price decreases of the same size.

4 Yet another shortcoming of the model so far, could be that the demand side is represented by a single consumer. The modeled consumer was interpreted as a representative agent. However, it is doubtful whether such a representative agent should be equipped with preferences incorporating loss aversion, since loss aversion describes a characteristic of *individual* behavior. For example, if incomes or reference levels vary across agents, all individual demand curves are kinked, but at different consumption levels, thus making aggregate demand look rather smooth.

To overcome the first three shortcomings, in Section 7.1 the reference points are no longer given exogenously, but made dependent on the consumer's past consumption. Shortcoming no. 4 is tackled in Section 7.2. There, it is examined how *individual* loss aversion is reflected in the collective demand curve resulting from aggregation over a large number agents with idiosyncratic income and reference levels.

7 Variations

7.1 Extended model I: Making the reference points endogenous

7.1.1 *Motivation of the extended model I*

Up to now, the reference point was given exogenously and not allowed to change throughout the $T = 200$ simulated periods. Several authors (Bell and Lattin, 2000; Bidwell et al., 1995; Kőszegi and Rabin, 2004; Munro and Sugden, 2003; Slonim and Garbarino, 1999) stressed, however, that reference points change over time, and arious suggestions on the evolution of reference points have been made.

In their seminal paper, Tversky and Kahneman (1991, p. 1046) admit explicitly that "[t]he question of the origin and the determinants of the reference state lies beyond the scope of the present article". Still, referring to the findings of other researchers, they state that "the reference state usually corresponds to the decision maker's current position", notwithstanding that "it can also be influenced by aspirations, expectations, norms, and social comparisons". Indeed, Bowman et al. (1999) solve a theoretical model in which the current reference point is equal to past consumption, and they present evidence from five countries that confirms the predictions of their model.

Bidwell et al. (1995, p. 290–291) and Kőszegi and Rabin (2004, pp. 2 and 14–17), in contrast, suggest that an agent's reference point is given by her expectations of future prices and, thus, future consumption. While being very appealing due to its forward-looking character, this, of course, entails new conceptual problems, because the individual formation of expectations has to be modeled appropriately. Furthermore, it is unclear which role the *current* reference point should play in the calculation of expected future consumption; this issue is to be discussed in detail in Section 8.

Due to the empirical evidence and the problems of the alternative proposed by Kőszegi and Rabin, in the modification of my model which I present below, I therefore assume that the representative consumer's reference levels at a given period depend on past consumption. The way I model this dependence rests on the hypothesis that the reference levels are rather stable, i.e. they do not adjust immediately each time that actual consumption deviates from the reference levels.

My assumption on the stability of reference levels is motivated by the consideration that people are normally able to distinguish between *temporary* price changes—e.g., a sales activity—and persistent *trends* in the price. Imagine, for instance, that a product goes on temporary sale—which is usually indicated

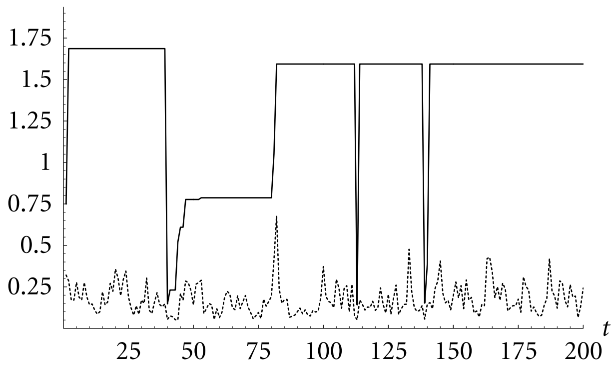


Figure 7.1: Simulated time series of unit costs $\tilde{c}_2$ (dashed line) and of the profit-maximizing price $\tilde{p}_2$ (solid line) when the reference levels adjust to past consumption; $\mu_c = \log 0.15$ and $\sigma_c = 0.9$.

as such by price tags in eye-catching colors etc.—and the agent chooses to consume more than her average during the sales period. Her reference level for the respective product will probably be the same after the sale as it was before, because she knew from the start that the price decrease would be only temporary and that she would not be able to purchase a large amount of the good permanently.

7.1.2 *Specification of the extended model I*

On the basis of these considerations, I model the connection between past consumption and current reference levels as follows: At any period t, actual consumption in the past t_{comp} periods is compared to the reference levels during these periods. (If $t_{comp} < t$, then comparison is carried out over the previous t periods.) Only if *during all t_{comp} periods* actual consumption of a good i, $x_{i,t-\tau}$, $\tau = 1, \ldots, t_{comp}$, deviated from the respective reference level $r_{i,t-\tau}$ by more than d%, then a new reference level $r_{i,t}$ is set. A new reference level is assumed to be given by the consumption of the previous period, $r_{i,t} = x_{i,t-1}$, otherwise $r_{i,t} = r_{i,t-1}$.

In general, the new reference level might as well be average consumption over a couple of previous periods. However, if the adaptation of a new reference level is interpreted as a reaction to a suspected trend in prices, it would not be logical for the consumer to calculate average consumption over amounts which she already suspects to be lower or higher than her future consumption. Of course, the same argument applies to taking last period's consumption as the new reference level. Therefore, setting the current reference level equal to last period's consumption should be understood as a simple formalization of the heuristic, "I noticed that prices have decreased (increased). This means, I will

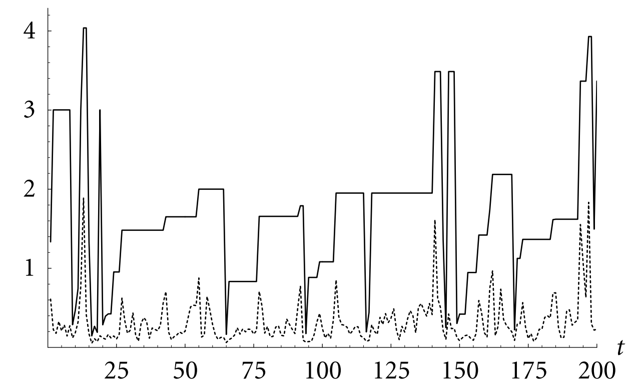


Figure 7.2: Simulated time series of unit costs $\tilde{c}_2$ (dashed line) and of the profit-maximizing price $\tilde{p}_2$ (solid line) when the reference levels adjust to past consumption; $\mu_c = \log 0.25$ and $\sigma_c = 1.1$.

be able to consume more (less) of good i during the upcoming periods, and I would like to consume in the future at least as much as I consume today."

7.1.3 *Results of the extended model I*

The results of incorporating the above described connection between past consumption and current reference levels in my model are shown in Figures 7.1 and 7.2. For the two simulations the same parameter values were used as in Section 6.2, with the difference that the initial reference point $\mathbf{r}_1 = (r_{1,1}, r_{2,1})$ was not set to (16,4) but to (0,0). This reflects the notion that reference levels are connected to a consumer's becoming accustomed to consuming a certain amount of the goods available. Hence, at the beginning there is nothing yet that the consumer could have become used to consuming. (Any other initial reference point changes the numerical results, but not the qualitative properties of the simulation.) The simulated cost series which firm 2 faces are the same ones as used for the simulations in Section 6.2, with the parameters of the cost distribution being $\mu_c = \log 0.15$ and $\sigma_c = 0.9$ for the first simulated cost series (Figure 7.1) and $\mu_c = \log 0.25$ and $\sigma_c = 1.1$ for the second one (Figure 7.2).

Like in the price series generated by the original model, there are times at which the simulated profit-maximizing price stays constant over several periods, interrupted by short deviations. In contrast to the original model, the extended model also generates shifts of the "regular price" to higher or lower levels, just as observed in empirical data (see Figure 3.1, especially panel a). Interestingly, the simulations create more small price increases than small price decreases, as found in empirical data by Levy et al. (2004).

The extended model I illustrates yet another interesting point in which the theory of reference-dependent preferences deviates from standard consumer

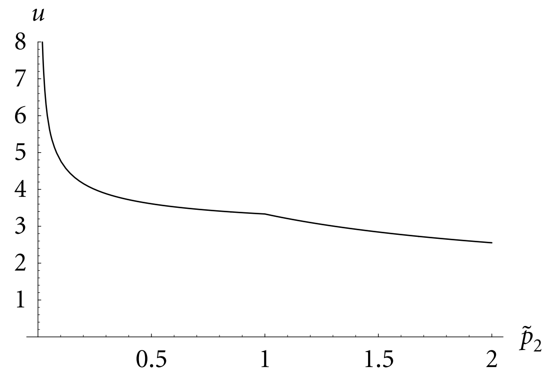


Figure 7.3: Local risk aversion over price changes around the price which generated the reference point.

theory: the attitude towards risk over price changes. Standard theory posits that agents are risk-loving w.r.t. price changes: A direct utility function that is concave in the quantities of the consumed goods gives rise to an indirect utility function which is *convex* in the goods' prices. By application of EUT, agents with standard preferences prefer a "price lottery" with mean $\overline{p}$ over paying the price $\overline{p}$ with certainty.

Under loss aversion, this is locally just the other way round: As Figure 7.3 shows, the indirect utility function is locally *concave* in prices. The figure was created as follows: For an agent with $\alpha = 2/3$, $\rho = 1/2$, $\lambda = 2$, $r_1 = r_2 = 0$, and $\tilde{M} = 20$, the optimal consumption bundle for $\tilde{p}_2 = 1$ was calculated—which yields $\mathbf{x} = (16, 4)$. The consumption bundle $\mathbf{x}$ was then taken to be the new reference level $\mathbf{r}$. With $\mathbf{r} = (16, 4)$ and all other parameters left unchanged, the agent's indirect utility for $\tilde{p}_2 \in [0, 2]$ was plotted. One can see that the kink in the indirect utility function occurs at $\tilde{p}_2 = 1$, the very price that gave rise to the reference level $\mathbf{r} = (16, 4)$.

Thus, under loss aversion, price rigidities can increase agents' average utility. They might, therefore, be desirable.

7.1.4 *Discussion of the extended model I*

Unfortunately, the majority of *small* price increases over *small* price decreases is matched by the opposite proportion of *large* price decreases and *large* price increases; this is not observed in real-world retail prices. In addition, this asymmetry feature of the simulated price series is an artifact of the assumed cost distribution—which is right-skewed—rather than of the model's core properties.

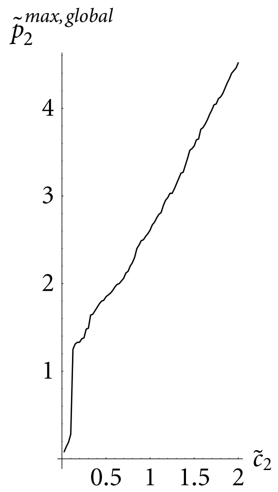


Figure 7.4: The profit-maximizing price as a function of marginal cost in the case of consumers with idiosyncratic (randomly determined) reference levels.

Despite that, we can conclude that allowing the reference levels to vary increases the similarity between the simulated price series and the empirical ones—the weaknesses no. 1 and 3 of the original model have been overcome.

This is also true for weakness no. 2 (which stated that in the original model it is a matter of chance only whether the mean of the cost distribution falls inside the interval for which the profit-maximizing price is constant or not). By incorporating a dependence of the current reference levels on past consumption, it is the last periods' prices which determine today's reference levels. Since the prices during the previous periods reflect the firm's cost distribution, the reference levels ultimately depend on the firm's cost distribution. This mechanism makes sure that in a large of periods, the reference levels are such that the costs are inside the interval for which the profit-maximizing price is constant.

7.2 Extended model II: Allowing for consumer heterogeneity

7.2.1 *The extended model II*

The phenomenon that the profit-maximizing price which a monopolist facing a loss-averse representative consumer chooses is identical for a range of unit costs, is caused by the fact that the demand of a loss-averse consumer is perfectly inelastic inside a certain interval of prices, determined by the reference levels. Therefore, if a firm faces not only one consumer but a large number of consumers with idiosyncratic reference levels, the phenomenon of price stickiness might disappear: The kinks and inelastic segments of the demand functions do not coincide, but are attained at different prices. Hence, the kink in the firm's profit function, generating the same profit-maximizing price for a range of production costs (recall Figure 5.9), disappears. This is confirmed by Figure 7.4: in contrast to Figure 6.1, there is no interval of costs any more for which the profit-maximizing price is constant—Figure 7.4 has no horizontal part. It was generated by aggregating demand over $n = 50$ consumers who differed with regard to income and reference levels. To make Figure 7.4 comparable to Figure 6.1, the income $\tilde{M}_j$ of each consumer j, $j = 1, \ldots, n$, was drawn from the following log-normal distribution: $\log \tilde{M}_j \sim N(3.045, 0.3)$; hence, the mean income was $e^{3.045} \approx 21$. (The sample mean turned out to be 21.34.) Each agent's reference levels were then calculated as random fractions of the agent's income.

However, this argumentation is not valid any more, when current reference levels likely depend on past consumption. In extended model II, the connection between past consumption and current reference levels was assumed to be the same as in extended model I. As already argued in Section 7.1.4, this makes the reference levels depend ultimately on the producer's cost distribution. This, in turn, has the consequence that a large share of the reference points is coordinated. It is still true that different households' reference levels differ from each other. However, the reference levels were the optimal consumption choices of many households for a given price in a previous period. This means that around this price, all these households stick to their reference levels. Hence, despite heterogeneity in the reference levels, there is a range of prices for which all these households consume their reference levels, and the upper bound of this range is under certain circumstances the price which globally maximizes profits. Through this channel, it is the fact that a price was once set in the past which makes this very price the profit-maximizing one in future periods.

7.2.2 *Results of the extended model II*

To make the plots comparable, the simulation shown in Figure 7.5 was created using the parameter values introduced in Section 6.2. As well, the cost series is the same that was already employed previously and the list of random incomes is the same as used for Figure 7.4. Total demand was generated by aggregating the individual demand curves of $n = 50$ consumers. Hence, as a consequence of the train of thought in Section 7.2.1, any stickiness of the profit-maximizing price must be triggered by the assumed dependence of consumers' idiosyncratic reference points on past consumption.

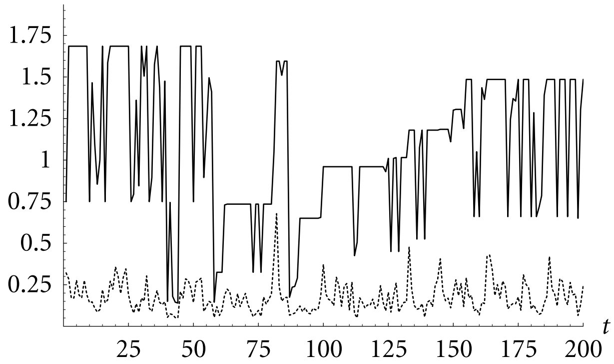


Figure 7.5: Simulated time series of unit costs $\tilde{c}_2$ (dashed line) and of the profit-maximizing price $\tilde{p}_2$ (solid line) when the reference levels adjust to past consumption and consumers have idiosyncratic reference levels; $\mu_c = \log 0.15$ and $\sigma_c = 0.9$.

The simulated series of profit-maximizing prices resulting from these assumptions indeed exhibits the familiar "regular price" phenomenon. Furthermore, temporary downward deviations from and return to the "regular price" can be observed.

For this model, it is practically impossible to obtain analytical conditions determining the profit-maximizing price in the form of those derived in Section 5.3. Therefore, in every period, I calculated aggregate demand and the resulting profit for all $\tilde{p}_2 \in \{k/200 \,|\, k \in \mathbb{N} \wedge \tilde{c}_{2,t} \leq k/200 \leq 10\,\tilde{c}_{2,t}\}$; from this set of prices, I chose the one with the highest profit. While it is theoretically possible that the profit-maximizing price lies above the upper bound $10\,\tilde{c}_2$, this is virtually excluded for the assumed values of α and σ. Admittedly, a finer grid of prices at which profit is evaluated yields more accurate results. To check whether such refinement changes anything significantly, I decreased the increment between subsequent prices at which profit is evaluated, to 1/1000. Neither did the qualitative properties change nor did the profit-maximizing price on the finer grid deviate from that on the coarser grid by more than the increment of the coarser grid, indicating that indeed the same optimal price is approximated. On top, in practice there is also a lower bound (1 cent) on price differentiation; thus, the numerical search for the profit-maximizing price might even increase rather than decrease the model's realism.

7.2.3 *Discussion of the extended model II*

Just like the two previous versions, extended model II, in which reference levels differ across consumers but change over time in a similar manner for all consumers, creates a profit-maximizing price which is non-responsive to changes in production costs within a certain interval. However, compared to the series depicted in Figure 7.1, the series with heterogeneous consumers is much more volatile.

This means that for purely qualitative analysis, it is possible to use extended model I instead of extended model II. Using extended model I has the advantage that a representative agent is analyzed instead of a large number of heterogeneous agents. This enables us to calculate the range of costs for which the profit-maximizing price is non-responsive, without having to rely on numerical simulation. Analyzing model I is, thus, much easier than analyzing model II.

This is especially important because the model still lacks a lot of features that should be incorporated in upcoming versions, such as forward-looking behavior of both consumers and producers (expectations; saving; maximization of the discounted sum of profits over all periods instead of maximizing in each period separately). Incorporating forward-looking behavior is mandatory, since it is obvious that in the extended versions, the price setting of today is likely to influence demand and, thus, profits tomorrow via changing in the reference levels. More specifically, there exists a trade-off between current profits and future profits, because low prices today mean high exploitable reference levels tomorrow, and vice versa. For empirical evidence on this relationship, see Slonim and Garbarino (2000, pp. 3 and 12).

8 Discussion

This discussion focuses on three topics related to the model I presented and solved in the previous sections:

1 its behavioral basis, especially the question of reference quantities vs. reference prices and the determinants of the reference point;
2 the model's time series properties and its ability to explain the empirically observed price setting of firms, contrasted by the predictions of competing theories;
3 shortcomings of the model and possible extensions in the future.

Regarding *topic 1*, it can be said that the behavioral basis of the way I model consumer behavior has been the subject of various studies which, without exception, answered the question whether people exhibit loss aversion in the affirmative. This evidence on the relevance of loss aversion was presented in Sections 2.2.2 and 2.2.4. Admittedly, none of these studies examined reference dependence in the very context that I model—repeat purchase of consumer goods. Instead, its relevance for financial markets, the choice of insurances, and enrollment in health plans was studied.

There is, however, one study which lends support to the hypotheses on which my model is built, i.e. to the assumptions that people are also loss-averse over everyday consumption, and that the current reference point depends on past consumption: it is Bowman et al.'s (1999) model of loss-averse consumers who have to decide on how to react to changes in their permanent income. In the model, the current reference point is assumed to be equal to past consumption. Bowman et al.'s result is that agents are reluctant to decrease current consumption in response to bad news about future income, given that there is sufficient uncertainty about future income (see p. 956). Bowman et al. find empirical evidence supporting this prediction from five different countries. Hence, their joint hypothesis of loss aversion over income and the reference point being equal to past consumption seems to hold.

Two possible alternatives to my specification suggested by other researchers were presented in Section 2.3: one is to conceive of loss aversion not over the quantities consumed of a certain good, but over the price of that good. This is done by Sibly (2002) and by several empirical studies cited in Section 2.2.4. The second alternative is to stay in the framework of loss aversion over quantities, but assume the reference levels to be determined through expected consumption in a rational-expectations equilibrium. This is the approach taken by Heidhues and Kőszegi (2004).

Both these approaches, however, suffer from weaknesses, as I will argue in the following. The most important argument against modeling loss aversion over prices is that this is in stark contrast to the original meaning of loss aversion. Actually, for certain prices, Sibly's model predicts the outright opposite of my model with loss aversion over quantities: Loss aversion in the sense of Tversky and Kahneman (1991) implies that for a certain range of prices of a good, the demand for the respective good is *totally inelastic*. Sibly's demand function, in contrast, is elastic everywhere, and there exists a price above which demand becomes even more elastic. Hence, the only thing that "loss aversion" over prices and loss aversion over quantities have in common is the assumption that the human decision-making process comprises two steps: First, forming a reference point and, second, comparing the actual numbers (items consumed or price, respectively) to that reference point.

In order to avoid such confusion of the two distinct concepts of reference dependence, it is in order not to speak of "loss aversion" in the case of reference prices. Instead, the term "customer anger at price increases" should be used, as it is done by Rotemberg (2003). Rotemberg (p. 12, footnote 11) even explicitly mentions the proximity between his model and Sibly's:

> The optimization problem of a firm setting a new price is then very similar to the optimization considered in Sibly (2002) for the case when consumers are "loss averse."

Of course, it sounds comprehensible that customers are reluctant to buy a good they discover to cost more than the reference price. However, it is also comprehensible that, when being accustomed to a certain level of consumption, this reluctance to buy after a price increase is transient. That is, after a couple of days, consumers do not care about the fact that the price has increased any further, because they do not want consumption to fall short of their reference levels.

Indeed, any large aversion to price increases is hard to reconcile with the finding by Levy et al. (2004) that I presented in Section 3.3: small price increases are more frequent than price decreases. Now, a price increase is unlikely to boost profits if demand decreases substantially in response to it. Thus, if it were true that consumers react strongly to price increases, it would be surprising that price increases are as common as Levy et al. discover.

However, one possibility remains to reconcile large demand decreases in response to price increases with Levy et al.'s finding: Firms might actually be forced to carry out desired price increases in small steps *just because* of the presence of customer anger: producers hope that many *small* increases go through with their customers unnoticed, thus preventing the harsh reaction that might follow a single large increase. This idea is the heart of Rotemberg's "customer anger" model and of Reis's (2004) model of rationally inattentive consumers. To explain their empirical findings, Levy et al. present a model of "rationally inattentive" consumers of their own. However, their model is so simplistic and their solution so flawed[23] that it has to be refuted out of hand.

Put aptly, in my opinion the simple models by Sibly and Levy et al. have to be rejected. The first one cannot be reconciled with the facts, and both have serious theoretical shortcomings. In contrast, the more sophisticated models by Rotemberg and Reis make predictions similar to those of my own model. However, they do so on the basis of entirely different assumptions about agents' behavior—full knowledge of price changes and loss aversion in my case, unnoticed price changes and no loss aversion in the case of Reis (2004) and Rotemberg (2003). Hence, the three distinct approaches should be seen as competitors, with each of them yielding useful insights.

I will now turn to discussing the differences between my model and that by H&K. In line with K&R (pp. 2 and 14–17), H&K suggest that the reference point is formed by the agent's expectations of future consumption. This is very appealing due to its forward-looking character, but it also entails new conceptual problems, since the individual formation of expectations has to be modeled appropriately.

Another at least equally important criticism is that the way h&k model the determination of the reference point seems inconsistent with the very notion of loss aversion. In their determination of a new reference point, the current reference point plays no role at all. The new reference point is determined in "personal equilibrium": what the agent expects to do, given the reference point, has to equal the reference point. Of course, this is a reasonable property, since—in Bidwell et al.'s (1995, p. 291) words—it "precludes behavior that implies self-deception".

My point of criticism is another: If people are loss-averse over the quantities they consume, why should they readily abandon reference levels they have entertained in the past? H&K's hypothesis that the reference point is purely forward-looking implies that the reference point for the next period changes *instantaneously* when new information arrives. For example, when the relative price changes in the two-goods case, *ceteris paribus*, one reference level will probably go up, and the second one will probably go down. But shouldn't the agent suffer loss aversion in this case, too? Loss aversion means that an agent is reluctant to let a good's consumption fall short of its reference level. It seems reasonable that such an agent is also reluctant to let consumption *in the near future, as she expects it today*, fall short of today's reference level. Granting that agents may be able to foresee—and sometimes even bring about—changes in their preferences (i.e., reference points), one should at least require that these changes be not be too drastic within a short period of time.

23 For instance, they call a situation (the fact that the consumer does not change its consumption despite a price increase and already knows this in advance, p. 20) an equilibrium which clearly is none.

Introducing loss aversion to the determination of expectations-based reference levels may have important consequences: Imagine, for instance, the situation that an agent expects her future income to drop. *Ceteris paribus*, according to H&K's model, she would adjust the reference levels for all the goods she consumes downward. Now, assume instead that the agent is reluctant to let future reference levels decrease below current reference levels. With the value function for each good being convex in the domain of losses, she might decide to decrease a few reference levels drastically in order to keep all remaining ones at their previous levels—instead of lowering all levels by a small amount.

Hence, it is reasonable to assume that the reference levels valid in one period influence the subsequent ones—as captured by my model, while being ignored by H&K. Since the aim is to construct a formal model of human behavior, this dispute cannot, of course, be resolved by purely theoretical deliberations. In the end, it is necessary to find out empirically which hypothesis regarding the foundation of reference levels better describes human decision making.

Turning to *topic 2*—the properties of the time series of profit-maximizing prices generated by my model—it can be stated that the simulated price series resemble time series of actual market prices quite accurately. At least, this is true with respect to the *qualitative* properties: a "regular price" is observable which is constant over a couple of periods, interrupted by short downward deviations and immediate returns ("sales periods"). I have to admit, of course, that in this early stage of research I have not yet examined the *quantitative* properties of these time series, e.g. minimum and maximum price, variance/mean ratio, serial correlations. It would be important to calibrate the model such that these values equal those of real price series. Once this is done, one can find out if the numerical value of σ, which determines the curvature of the consumption utility function, has a reasonable size and if the calibration's coefficient of loss aversion λ takes on a reasonable value as well. Simultaneously, size and variance of the unit costs creating sufficient variability in the profit-maximizing price have to be examined as well.

Nevertheless, it can be noted that my model has an edge over the contending models employing time-dependent or state-dependent sticky prices. State-dependent price setting has difficulties explaining the frequent occurrence of sales periods, which my model can explain. Time-dependent sticky price models were found by Bils and Klenow (2004) to create too low volatility and too much persistence in inflation. My model does not create such inertia: Firms can reset prices whenever this increases their profits. It is important to note that what appears to be "stickiness" of prices in my model is *non-responsiveness to cost shocks only*. As soon as, e.g., a change in the price of the other good or in the agent's wage changes her income $\tilde{M}$, the profit-maximizing price will likely adjust to this change. This is illustrated by the upper panel of Figure 6.3:

While the profit-maximizing price is independent of $\tilde{M}$ for very low and very high incomes, in a quite large intermediate range it responds to any change in income, however small the change is.

That is, my model combines one kind of non-responsiveness of the profit-maximizing price (to cost shocks) with another kind of immediate responsiveness (to changes in demand). Through this combination, price series generated by my model exhibit short-run price stickiness and long-run price flexibility, just like empirical findings on real markets. This has crucial implications for monetary policy: a change in nominal income due to a monetary contraction or expansion has over a wide range of $\tilde{M}$ no real, but only inflationary effects (see the lower panel of Figure 6.3), despite the observation that prices are constant over several periods.

Let us finally turn to *topic 3* of this discussion: shortcomings of the model and suggestions for further research. The most obvious shortcoming is the fact that both firms and consumers are assumed to suffer from myopia—imposed in my model by not allowing consumers to save: neither firms nor consumers take into account the consequences of their current behavior for consumers' future preferences. However, these consequences exist through the dependence of reference levels on past consumption. Thus, any firm that maximizes profits and expects to exist more than a single period should not maximize each period's profit separately, but the sum of all periods' discounted profits jointly. Analogously, the same holds true for the consumers and their utility maximization.

In upcoming versions of the model, such forward-looking behavior of both producers and consumers should be incorporated. Furthermore, based on the analysis of the partial equilibrium in this thesis, a general-equilibrium model should be developed. After this has been done, a calibrated version of the model can be compared with actual data on prices and costs, as I already outlined in this discussion when dealing with topic 2.

Apart from this theoretical work, a lot of empirical research—in experiments and field studies—on the origins and effects of reference points has to be conducted, so that it can be decided what role past consumption and expected future consumption play in the determination of the reference point.

9 Summary and conclusion

This thesis dealt with two controversies that have been subject of debate in economic research for a long time: first, the source and extent of rigidity of nominal prices and second, the constancy of agents' preferences. A model was developed which assumes reference-dependent preferences and loss aversion in the style of Tversky and Kahneman (1991). The question it tried to answer is whether the dependence of preferences on reference points in connection with loss aversion leads to price stickiness, assuming that firms maximize profits.

Therefore, consumers' demand functions under loss aversion were derived and shown to be kinked at the reference levels. In the next step, a monopolistic firm with constant marginal cost serving the loss-averse consumers was introduced. It was shown that the firm's profit-maximizing price does not respond to changes in marginal cost as long as the cost lies inside a certain interval. This was interpreted as a form of "price stickiness", i.e. non-responsiveness of the price to a shock on another variable.

Using pseudo-randomly drawn unit costs, a time series of profit-maximizing prices was simulated and compared to time series of actual market prices. The simulated and the real-world price series were found to have very similar qualitative properties: Both are characterized by the emergence of a "regular price", where a price is called "regular price" when it is attained a couple of periods in a row, and/or if deviations from it occur, they last for only a few periods.

The model was presented first in a basic version, where a representative agent was assumed whose reference point was given exogenously and not allowed to vary over time. Since studies on reference-dependent preferences frequently emphasize that the reference levels change over time, in an extended version, the reference point was made dependent on past consumption. It was shown via a simulated time series of profit-maximizing prices that the quality of the model's predictions on price setting improves in comparison with the basic model. The way I model the determination of the reference points was defended against the two competing hypotheses about their nature put forth by Sibly (2002) and by Heidhues and Kőszegi (2004).

Using the same dependence of the reference point on past consumption as in the first extended version, in a second extended version, the representative consumer was replaced by a number of heterogeneous consumers with idiosyncratic reference levels. This way, it was shown that the results of the first extended version survive aggregation, because their qualitative properties do not change. Due to this result, I argued that customer heterogeneity can be neglected and the model can be used in its representative-agent version, which is much easier to solve than the version incorporating consumer heterogeneity.

The main result of my model is that it combines one kind of non-responsiveness of the profit-maximizing price—to cost shocks—with another kind of immediate responsiveness—to changes in demand. Such changes in demand occur, e.g., when the agents' nominal income changes. I showed that expansionary monetary policy has over a wide range of nominal incomes no real, but only inflationary effects, despite the observation that prices are constant over longer periods of time. In addition, just like empirical findings on real market prices, price series generated by my model exhibit short-run price stickiness and long-run price flexibility.

Finally, I discussed shortcomings of my model and suggested topics for further research: The present model suffers from not allowing for forward-looking behavior, neither on the side of the consumers nor on the side of the firm. In upcoming versions of the model, forward-looking behavior of both firms and consumers should be incorporated, and the analysis should be extended from partial equilibrium to general equilibrium. When this has been achieved, a calibrated version of the model should be compared with actual data on prices and costs in order to check that price rigidity also occurs when plugging in realistic values for the coefficient of loss aversion and the variance of unit costs. In addition to this theoretical work, further empirical research on the origins and effects of reference points is deemed necessary.

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Appendix A: Derivations and proofs

Derivation of $\tilde{M}^*(\alpha_1,\alpha_2)$

$\tilde{M}^*(\alpha_1,\alpha_2)$ is the value at which conditions (5.9) and (5.13) collapse into a single admissible value of $\tilde{p}_2$. Hence, for $\tilde{M}=\tilde{M}^*$, both mentioned conditions have to hold by equality. Hence, in combination they require that

$$\alpha_1^\sigma\left(\left(\frac{\alpha_2^\sigma}{\alpha_1^\sigma\tilde{M}/r_1-\alpha_1^\sigma}\right)^{1/(\sigma-1)}\right)^\sigma+\alpha_2^\sigma\left(\frac{\alpha_2^\sigma}{\alpha_1^\sigma\tilde{M}/r_1-\alpha_1^\sigma}\right)^{1/(\sigma-1)}=\frac{\alpha_2^\sigma\tilde{M}}{r_2}$$

$$\Leftrightarrow\quad \frac{\alpha_1^\sigma}{\alpha_2^\sigma}\left(\frac{\alpha_2^\sigma}{\alpha_1^\sigma\tilde{M}/r_1-\alpha_1^\sigma}\right)^{\sigma/(\sigma-1)}+\left(\frac{\alpha_2^\sigma}{\alpha_1^\sigma\tilde{M}/r_1-\alpha_1^\sigma}\right)^{1/(\sigma-1)}=\frac{\tilde{M}}{r_2}.$$

Since $\rho\equiv(\sigma-1)/\sigma$, this translates to

$$\Leftrightarrow\quad \frac{\alpha_1^\sigma}{\alpha_2^\sigma}\left(\frac{\alpha_2^\sigma}{\alpha_1^\sigma\tilde{M}/r_1-\alpha_1^\sigma}\right)^{1/\rho}+\left(\frac{\alpha_2^\sigma}{\alpha_1^\sigma\tilde{M}/r_1-\alpha_1^\sigma}\right)^{(1/\rho)-1}=\frac{\tilde{M}}{r_2}$$

$$\Leftrightarrow\quad \left(\frac{\alpha_2^\sigma}{\alpha_1^\sigma\tilde{M}/r_1-\alpha_1^\sigma}\right)^{1/\rho}\left[\frac{\alpha_1^\sigma}{\alpha_2^\sigma}+\left(\frac{\alpha_2^\sigma}{\alpha_1^\sigma\tilde{M}/r_1-\alpha_1^\sigma}\right)^{-1}\right]=\frac{\tilde{M}}{r_2}$$

$$\Leftrightarrow\quad \left(\frac{\alpha_2^\sigma}{\alpha_1^\sigma\tilde{M}/r_1-\alpha_1^\sigma}\right)^{1/\rho}\left[\frac{\alpha_1^\sigma}{\alpha_2^\sigma}+\frac{\alpha_1^\sigma(\tilde{M}/r_1-1)}{\alpha_2^\sigma}\right]=\frac{\tilde{M}}{r_2}$$

$$\Leftrightarrow\quad \left(\frac{\alpha_2^\sigma}{\alpha_1^\sigma\tilde{M}/r_1-\alpha_1^\sigma}\right)^{1/\rho}\left[\frac{\alpha_1^\sigma}{\alpha_2^\sigma}\cdot\frac{\tilde{M}}{r_1}\right]=\frac{\tilde{M}}{r_2}$$

$$\Leftrightarrow\quad \frac{\alpha_1^\sigma}{\alpha_2^\sigma}\left(\frac{\alpha_2^\sigma}{\alpha_1^\sigma\tilde{M}/r_1-\alpha_1^\sigma}\right)^{1/\rho}=\frac{r_1}{r_2}$$

$$\Leftrightarrow\quad \frac{\alpha_1^{\sigma-(\sigma/\rho)}}{\alpha_2^{\sigma-(\sigma/\rho)}}=\frac{r_1}{r_2}\cdot\left(\frac{\tilde{M}-r_1}{r_1}\right)^{1/\rho}$$

$$\Leftrightarrow\quad \frac{\alpha_1^{\sigma\rho-\sigma}}{\alpha_2^{\sigma\rho-\sigma}}=\left(\frac{r_1}{r_2}\right)^\rho\cdot\frac{\tilde{M}-r_1}{r_1}$$

$$\Leftrightarrow\quad \frac{\alpha_1^{\sigma(\rho-1)}}{\alpha_2^{\sigma(\rho-1)}}\left(\frac{r_2}{r_1}\right)^\rho r_1=\tilde{M}-r_1$$

$$\Leftrightarrow\quad r_1+\frac{\alpha_1^{(1/(1-\rho))(\rho-1)}}{\alpha_2^{(1/(1-\rho))(\rho-1)}}\left(\frac{r_2}{r_1}\right)^\rho r_1=\tilde{M}$$

$$\Leftrightarrow \qquad r_1 + \frac{\alpha_1^{-1}}{\alpha_2^{-1}} \left(\frac{r_2}{r_1} \right)^{\rho} r_1 \;=\; \tilde{M}.$$

We finally obtain formula (5.15):

$$r_1 \left(1 + \frac{\alpha_2 r_2^{\rho}}{\alpha_1 r_1^{\rho}} \right) \;=\; \tilde{M}.$$

Combinations $(\tilde{p}_2, \tilde{M})$ for which no interior solution is attained

- *Branches I and II:* The conditions admitting for an interior solution on branch I or II exclude each other: the first part of (5.28) and the first part of (5.29) combined would require that

$$\left(\frac{(1-\alpha)^{\sigma}}{\alpha^{\sigma} \tilde{M} / r_1 - \alpha^{\sigma}} \right)^{1/(\sigma-1)} < \left(\frac{(1-\alpha)^{\sigma}}{[\alpha(1+\lambda)]^{\sigma} \tilde{M} / r_1 - [\alpha(1+\lambda)]^{\sigma}} \right)^{1/(\sigma-1)}.$$

  This cannot be fulfilled, since $\lambda > 0$. Consequently, for

$$\text{(A.1)} \qquad \left(\frac{(1-\alpha)^{\sigma}}{[\alpha(1+\lambda)]^{\sigma} \tilde{M} / r_1 - [\alpha(1+\lambda)]^{\sigma}} \right)^{1/(\sigma-1)} < \tilde{p}_2 < \left(\frac{(1-\alpha)^{\sigma}}{\alpha^{\sigma} \tilde{M} / r_1 - \alpha^{\sigma}} \right)^{1/(\sigma-1)},$$

  an interior solution is attained neither on branch I nor on branch II.

- *Branches I and III:* The conditions admitting for an interior solution on branch I or III exclude each other, since (5.28) and (5.30) cannot be fulfilled simultaneously. The second part of (5.30) can be rearranged to give

$$\frac{\alpha^{\sigma} \tilde{p}_2^{\sigma}}{(1+\lambda)^{\sigma}} + (1-\alpha)^{\sigma} \tilde{p}_2 > \frac{(1-\alpha)^{\sigma} \tilde{M}}{r_2}.$$

  Combined with the second part of (5.28) this requires

$$\frac{\alpha^{\sigma} \tilde{p}_2^{\sigma}}{(1+\lambda)^{\sigma}} + (1-\alpha)^{\sigma} \tilde{p}_2 > \alpha^{\sigma} \tilde{p}_2^{\sigma} + (1-\alpha)^{\sigma} \tilde{p}_2.$$

  This is impossible to fulfill as long as $\lambda > 0$. Consequently, for

$$\text{(A.2)} \qquad \alpha^{\sigma} \tilde{p}_2^{\sigma} + (1-\alpha)^{\sigma} \tilde{p}_2 > \frac{(1-\alpha)^{\sigma} \tilde{M}}{r_2} \quad \text{and}$$

$$\alpha^{\sigma} \tilde{p}_2^{\sigma} + [(1-\alpha)(1+\lambda)]^{\sigma} \tilde{p}_2 < \frac{[(1-\alpha)(1+\lambda)]^{\sigma} \tilde{M}}{r_2},$$

  an interior solution is attained neither on branch I nor on branch III.

- *Branches* I *and* IV: The conditions admitting for an interior solution on branch I or IV, (5.28) and (5.31), state exactly the opposite of each other, thereby excluding the possibility of being fulfilled simultaneously.

- *Branches* II *and* III: Again, the conditions admitting for an interior solution on branch II or III, (5.29) and (5.30), contradict each other. Unlike in the cases of the other branches, here the first *as well as* the second parts of the conditions exclude each other. The first parts would require that

$$\left(\frac{[(1-\alpha)(1+\lambda)]^{\sigma}}{\alpha^{\sigma}\tilde{M}/r_1-\alpha^{\sigma}}\right)^{1/(\sigma-1)} < \left(\frac{(1-\alpha)^{\sigma}}{[\alpha(1+\lambda)]^{\sigma}\tilde{M}/r_1-[\alpha(1+\lambda)]^{\sigma}}\right)^{1/(\sigma-1)},$$

which is impossible as long as $\lambda>0$. Hence, for

$$\left(\frac{(1-\alpha)^{\sigma}}{[\alpha(1+\lambda)]^{\sigma}\tilde{M}/r_1-[\alpha(1+\lambda)]^{\sigma}}\right)^{1/(\sigma-1)} < \tilde{p}_2 < \left(\frac{[(1-\alpha)(1+\lambda)]^{\sigma}}{\alpha^{\sigma}\tilde{M}/r_1-\alpha^{\sigma}}\right)^{1/(\sigma-1)}, \tag{A.3}$$

an interior solution is attained neither on branch II nor on branch III.

As far as the second parts of conditions (5.29) and (5.30) are concerned, divide the second part of condition (5.30) by $(1+\lambda)^{\sigma}$ to get

$$\frac{\alpha^{\sigma}\tilde{p}_2^{\sigma}}{(1+\lambda)^{\sigma}}+(1-\alpha)^{\sigma}\tilde{p}_2 > \frac{(1-\alpha)^{\sigma}\tilde{M}}{r_2}.$$

Combined with the second part of (5.29), this requires that

$$\frac{\alpha^{\sigma}}{(1+\lambda)^{\sigma}}\tilde{p}_2^{\sigma}+(1-\alpha)^{\sigma}\tilde{p}_2 > [\alpha(1+\lambda)]^{\sigma}\tilde{p}_2^{\sigma}+(1-\alpha)^{\sigma}\tilde{p}_2,$$

which is impossible as long as $\lambda>0$. Hence, for

$$\begin{aligned}&[\alpha(1+\lambda)]^{\sigma}\tilde{p}_2^{\sigma}+(1-\alpha)^{\sigma}\tilde{p}_2 > \frac{(1-\alpha)^{\sigma}\tilde{M}}{r_2} \quad \text{and}\\ &\alpha^{\sigma}\tilde{p}_2^{\sigma}+[(1-\alpha)(1+\lambda)]^{\sigma}\tilde{p}_2 < \frac{[(1-\alpha)(1+\lambda)]^{\sigma}\tilde{M}}{r_2},\end{aligned} \tag{A.4}$$

an interior solution is attained neither on branch II nor on branch III. Note that the intervals (A.3) and (A.4) include the intervals on which an interior solution on branch I or branch IV, respectively, exists.

- *Branches II and IV:* Conditions (5.29) and (5.31) exclude each other, because the right-hand sides of their second parts are identical and, therefore, require that

$$\alpha^{\sigma}\,\tilde{p}_2^{\sigma} + (1-\alpha)^{\sigma}\,\tilde{p}_2 > [\alpha(1+\lambda)]^{\sigma}\,\tilde{p}_2^{\sigma} + (1-\alpha)^{\sigma}\,\tilde{p}_2,$$

  which is not the case if, as assumed, $\lambda > 0$. Consequently, branches II and IV do not overlap either, and for

$$(\text{A.5})\quad [\alpha(1+\lambda)]^{\sigma}\,\tilde{p}_2^{\sigma} + (1-\alpha)^{\sigma}\,\tilde{p}_2 > \frac{(1-\alpha)^{\sigma}\,\tilde{M}}{r_2} \text{ and}$$
$$\alpha^{\sigma}\,\tilde{p}_2^{\sigma} + (1-\alpha)^{\sigma}\,\tilde{p}_2 < \frac{(1-\alpha)^{\sigma}\,\tilde{M}}{r_2},$$

  an interior solution is attained neither on branch II nor on branch IV.

- *Branches III and IV:* Conditions (5.30) and (5.31) contradict each other: Their first parts require that,

$$\left(\frac{(1-\alpha)^{\sigma}}{\alpha^{\sigma}\,\tilde{M}/r_1 - \alpha^{\sigma}}\right)^{1/(\sigma-1)} > \left(\frac{[(1-\alpha)(1+\lambda)]^{\sigma}}{\alpha^{\sigma}\,\tilde{M}/r_1 - \alpha^{\sigma}}\right)^{1/(\sigma-1)},$$

  which is not fulfilled if $\lambda > 0$, which we assumed to be the case. Consequently, branches III and IV do not overlap either, and for

$$(\text{A.6})\quad \left(\frac{(1-\alpha)^{\sigma}}{\alpha^{\sigma}\,\tilde{M}/r_1 - \alpha^{\sigma}}\right)^{1/(\sigma-1)} < \tilde{p}_2 < \left(\frac{[(1+\lambda)(1-\alpha)]^{\sigma}}{\alpha^{\sigma}\,\tilde{M}/r_1 - \alpha^{\sigma}}\right)^{1/(\sigma-1)},$$

  an interior solution is attained neither on branch III nor on branch IV.

- In summary: The conditions for obtaining an interior solution are mutually exclusive for all branches I–IV.

Proposition 1

Claim 1

For all $(\tilde{p}_2, \tilde{M})$ that fulfill

$$\text{(A.7)} \quad \tilde{M} \geq r_1 + \tilde{p}_2 r_2 \text{ and}$$

$$\left(\frac{(1-\alpha)^{\sigma}}{[\alpha(1+\lambda)]^{\sigma}\, \tilde{M}/r_1 - [\alpha(1+\lambda)]^{\sigma}} \right)^{1/(\sigma-1)} \leq \tilde{p}_2 \leq \left(\frac{(1-\alpha)^{\sigma}}{\alpha^{\sigma}\, \tilde{M}/r_1 - \alpha^{\sigma}} \right)^{1/(\sigma-1)},$$

consuming bundle $\mathbf{a} \equiv (x_1^{\mathbf{a}}, x_2^{\mathbf{a}}) \equiv (r_1, (\tilde{M} - r_1)/\tilde{p}_2)$ is the optimal choice.

Proof. In the situation that $\tilde{M} \geq r_1$ and, furthermore, that $\tilde{p}_2$ is low enough so that $\tilde{M} \geq r_1 + \tilde{p}_2 r_2$, branches I, II, and III are feasible. In this case, it is possible to consume of both goods at least or more than the reference levels. Hence, there is *no trade-off* of the sort that a gain in one good is met by a loss in the respective other good.

It has to be shown that when consuming bundle **a**, neither shifting resources from the consumption of good 1 towards good 2 nor in the opposite direction can improve the agent's utility for $\tilde{p}_2$ inside the interval defined by (A.7). For this purpose, let us calculate the utility that the agent would get from consuming $(x_1^{\mathbf{a}} + \delta, x_2^{\mathbf{a}} - \delta/\tilde{p}_2) = (r_1 + \delta, (\tilde{M} - r_1 - \delta)/\tilde{p}_2)$, $\delta > 0$: Since $\tilde{M} \geq r_1 + \tilde{p}_2 r_2$, for a sufficiently small δ, of both goods more than the respective reference level is consumed, i.e. $x_2^{\mathbf{a}} - \delta/\tilde{p}_2 \geq r_2$. Therefore, $n_1(x_1 \,|\, r_1) = n_2(x_2 \,|\, r_2) = 0$, so that the utility is given by

$$\text{(A.8)} \quad u(r_1 + \delta, (\tilde{M} - r_1 - \delta)/\tilde{p}_2 \,|\, r_1, r_2) = \alpha(r_1 + \delta)^{\rho} + (1-\alpha)[(\tilde{M} - r_1 - \delta)/\tilde{p}_2]^{\rho}.$$

Furthermore, we will need the change in this utility when δ is marginally increased:

$$\text{(A.9)} \quad \begin{aligned} &\frac{\mathrm{d}u(r_1 + \delta, (\tilde{M} - r_1 - \delta)/\tilde{p}_2 \,|\, r_1, r_2)}{\mathrm{d}\delta} \\ &= \alpha\rho(r_1 + \delta)^{\rho-1} - (1-\alpha)\frac{1}{\tilde{p}_2}\,\rho\left(\frac{\tilde{M} - r_1 - \delta}{\tilde{p}_2}\right)^{\rho-1}. \end{aligned}$$

For the bundle $(r_1 + \delta, (\tilde{M} - r_1 - \delta)/\tilde{p}_2)$ to be optimal, increasing δ must not pay off, i.e. this derivative has to be non-positive:

$$\text{(A.10)} \quad \alpha\rho(r_1 + \delta)^{\rho-1} - (1-\alpha)\frac{1}{\tilde{p}_2}\,\rho\left(\frac{\tilde{M} - r_1 - \delta}{\tilde{p}_2}\right)^{\rho-1} \leq 0.$$

This inequality can be reduced by the factor ρ, and in the special case of bundle **a**, δ equals zero, so that the condition simplifies to

$$\alpha r_1^{\rho-1} - (1-\alpha)\frac{1}{\tilde{p}_2}\left(\frac{\tilde{M}-r_1}{\tilde{p}_2}\right)^{\rho-1} \leq 0. \tag{A.11}$$

Solving this inequality for $\tilde{p}_2$:

$$\begin{aligned} \alpha r_1^{\rho-1} &\leq (1-\alpha)\ \frac{(\tilde{M}-r_1)^{\rho-1}}{\tilde{p}_2^{\rho}} \\ \Leftrightarrow \qquad \tilde{p}_2^{\rho} &\leq \frac{1-\alpha}{\alpha}\left(\frac{\tilde{M}-r_1}{r_1}\right)^{\rho-1} \\ \Leftrightarrow \qquad \tilde{p}_2 &\leq \left(\frac{1-\alpha}{\alpha}\right)^{1/\rho}\left(\frac{\tilde{M}-r_1}{r_1}\right)^{(\rho-1)/\rho}. \end{aligned}$$

With $\rho = (\sigma-1)/\sigma$, this condition can be rewritten as:

$$\begin{aligned} \tilde{p}_2 &\leq \left(\frac{1-\alpha}{\alpha}\right)^{\sigma/(\sigma-1)}\left(\frac{\tilde{M}-r_1}{r_1}\right)^{1/(1-\sigma)} \\ \Leftrightarrow \qquad \tilde{p}_2 &\leq \left(\frac{1-\alpha}{\alpha}\right)^{\sigma/(\sigma-1)}\left(\frac{\tilde{M}}{r_1}-1\right)^{1/(1-\sigma)} \\ \Leftrightarrow \qquad \tilde{p}_2 &\leq \left(\frac{1-\alpha}{\alpha}\right)^{\sigma/(\sigma-1)}\left(\frac{1}{\tilde{M}/r_1-1}\right)^{1/(\sigma-1)}. \end{aligned}$$

This finally delivers

$$\tilde{p}_2 \leq \left(\frac{(1-\alpha)^{\sigma}}{\alpha^{\sigma}\tilde{M}/r_1-\alpha^{\sigma}}\right)^{1/(\sigma-1)}. \tag{A.12}$$

Hence, for $\tilde{p}_2$ below this threshold, the utility cannot be increased by shifting consumption from good 2 in favor of good 1, starting from $\mathbf{a} \equiv (r_1, (\tilde{M}-r_1)/\tilde{p}_2)$.

In the next step, it has to be proven that when consuming bundle **a**, utility cannot be increased by shifting consumption in the opposite direction, i.e. from good 1 towards good 2, either. For this purpose, let us calculate the utility that the agent would get from consuming $(r_1-\delta, (\tilde{M}-r_1+\delta)/\tilde{p}_2)$, $\delta > 0$. Since now $x_1 < r_1$ and $x_2 > r_2$, $n_1(x_1 | r_1) = \lambda(\alpha x_1^{\rho} - \alpha r_1^{\rho})$ and $n_2(x_2 | r_2) = 0$; thus, the utility is given by

$$\begin{aligned} &u(r_1-\delta, (\tilde{M}-r_1+\delta)/\tilde{p}_2 | r_1, r_2) \\ &= (1+\lambda)\alpha(r_1-\delta)^{\rho} + (1-\alpha)[(\tilde{M}-r_1+\delta)/\tilde{p}_2]^{\rho} - \lambda\alpha r_1^{\rho}. \end{aligned} \tag{A.13}$$

The derivative of this utility level w.r.t. δ is

$$\begin{aligned}(\text{A.14})\quad & \frac{\mathrm{d}u(r_1-\delta,(\tilde{M}-r_1+\delta)/\tilde{p}_2\,|\,r_1,r_2)}{\mathrm{d}\delta}\\ &= -(1+\lambda)\alpha\rho(r_1-\delta)^{\rho-1}+(1-\alpha)\frac{1}{\tilde{p}_2}\rho\left(\frac{\tilde{M}-r_1+\delta}{\tilde{p}_2}\right)^{\rho-1}.\end{aligned}$$

For the bundle $(r_1-\delta,(\tilde{M}-r_1+\delta)/\tilde{p}_2)$ to be optimal, increasing δ must not pay off, i.e. this derivative has to be non-positive:

$$(\text{A.15})\quad -(1+\lambda)\alpha\rho(r_1-\delta)^{\rho-1}+(1-\alpha)\frac{1}{\tilde{p}_2}\rho\left(\frac{\tilde{M}-r_1+\delta}{\tilde{p}_2}\right)^{\rho-1} \le 0.$$

This inequality can be reduced by the factor ρ, and in the special case of bundle **a**, δ equals zero, so that the condition simplifies to

$$(\text{A.16})\quad -(1+\lambda)\alpha r_1^{\rho-1}+(1-\alpha)\frac{1}{\tilde{p}_2}\left(\frac{\tilde{M}-r_1}{\tilde{p}_2}\right)^{\rho-1} \le 0.$$

Solving this inequality for $\tilde{p}_2$:

$$\begin{aligned} & (1+\lambda)\alpha r_1^{\rho-1} \ge (1-\alpha)\frac{(\tilde{M}-r_1)^{\rho-1}}{\tilde{p}_2^{\rho}}\\ \Leftrightarrow\quad & \tilde{p}_2^{\rho} \ge \frac{1-\alpha}{(1+\lambda)\alpha}\left(\frac{\tilde{M}-r_1}{r_1}\right)^{\rho-1}\\ \Leftrightarrow\quad & \tilde{p}_2 \ge \left(\frac{1-\alpha}{(1+\lambda)\alpha}\right)^{1/\rho}\left(\frac{\tilde{M}-r_1}{r_1}\right)^{(\rho-1)/\rho}.\end{aligned}$$

With $\rho=(\sigma-1)/\sigma$, this condition can be rewritten as:

$$\begin{aligned} & \tilde{p}_2 \ge \left(\frac{1-\alpha}{(1+\lambda)\alpha}\right)^{\sigma/(\sigma-1)}\left(\frac{\tilde{M}-r_1}{r_1}\right)^{1/(1-\sigma)}\\ \Leftrightarrow\quad & \tilde{p}_2 \ge \left(\frac{1-\alpha}{(1+\lambda)\alpha}\right)^{\sigma/(\sigma-1)}\left(\frac{\tilde{M}}{r_1}-1\right)^{1/(1-\sigma)}\\ \Leftrightarrow\quad & \tilde{p}_2 \ge \left(\frac{1-\alpha}{(1+\lambda)\alpha}\right)^{\sigma/(\sigma-1)}\left(\frac{1}{\tilde{M}/r_1-1}\right)^{1/(\sigma-1)}.\end{aligned}$$

This finally delivers the condition that

$$(\text{A.17})\quad \tilde{p}_2 \ge \left(\frac{(1-\alpha)^{\sigma}}{[(1+\lambda)\alpha]^{\sigma}\,\tilde{M}/r_1-[(1+\lambda)\alpha]^{\sigma}}\right)^{1/(\sigma-1)}.$$

Hence, for $\tilde{p}_2$ above this threshold, the utility cannot be increased by shifting consumption from good 1 in favor of good 2, starting from $\mathbf{a} \equiv (r_1, (\tilde{M} - r_1)/\tilde{p}_2)$.

Combining conditions (A.12) and (A.17) delivers the interval of $\tilde{p}_2$ for which it pays off neither to shift consumption from good 1 towards good 2 nor the opposite direction, starting from bundle **a**. This is the interval for which bundle **a** is optimal, as asserted in Proposition 1.

Claim 2

For all $(\tilde{p}_2, \tilde{M})$ that fulfill

$$\text{(A.18)} \quad \tilde{M} \geq r_1 + \tilde{p}_2 r_2 \text{ as well as}$$

$$\alpha^\sigma \tilde{p}_2^\sigma + (1-\alpha)^\sigma \tilde{p}_2 \geq \frac{(1-\alpha)^\sigma \tilde{M}}{r_2} \text{ and}$$

$$\alpha^\sigma \tilde{p}_2^\sigma + [(1-\alpha)(1+\lambda)]^\sigma \tilde{p}_2 \leq \frac{[(1-\alpha)(1+\lambda)]^\sigma \tilde{M}}{r_2},$$

consuming bundle $\mathbf{b} \equiv (\tilde{M} - \tilde{p}_2 r_2, r_2)$ is the optimal choice.

Proof. It has to be shown that when consuming bundle **b**, neither shifting resources from the consumption of good 1 towards good 2 nor in the opposite direction can improve the agent's utility for $\tilde{p}_2$ inside the interval defined by (A.18). For this purpose, let us calculate the utility that the agent would get from consuming $(x_1^{\mathbf{b}} - \tilde{p}_2 \delta, x_2^{\mathbf{b}} + \delta) = (\tilde{M} - \tilde{p}_2(r_2 + \delta), r_2 + \delta)$, $\delta > 0$: Since $\tilde{M} \geq r_1 + \tilde{p}_2 r_2$, for a sufficiently small δ, of both goods more than the respective reference level is consumed, i.e. $x_1^{\mathbf{b}} - \tilde{p}_2 \delta \geq r_1$. Therefore, $n_1(x_1 \mid r_1) = n_2(x_2 \mid r_2) = 0$, so that the utility is given by

$$\text{(A.19)} \quad u(\tilde{M} - \tilde{p}_2(r_2 + \delta), r_2 + \delta \mid r_1, r_2) = \alpha(\tilde{M} - \tilde{p}_2(r_2 + \delta))^\rho + (1-\alpha)(r_2 + \delta)^\rho.$$

Furthermore, we will need the change in this utility when δ is marginally increased:

$$\text{(A.20)} \quad \frac{\mathrm{d}u(\tilde{M} - \tilde{p}_2(r_2 + \delta), r_2 + \delta \mid r_1, r_2)}{\mathrm{d}\delta} = -\alpha \tilde{p}_2 \rho (\tilde{M} - \tilde{p}_2(r_2 + \delta))^{\rho-1} + (1-\alpha)\rho (r_2 + \delta)^{\rho-1}.$$

For the bundle $(\tilde{M} - \tilde{p}_2(r_2 + \delta), r_2 + \delta)$ to be optimal, increasing δ must not pay off, i.e. this derivative has to be non-positive:

$$\text{(A.21)} \quad -\alpha \tilde{p}_2 \rho (\tilde{M} - \tilde{p}_2(r_2 + \delta))^{\rho-1} + (1-\alpha)\rho (r_2 + \delta)^{\rho-1} \leq 0.$$

This inequality can be reduced by the factor ρ, and in the special case of bundle $\mathbf{b}$, δ equals zero, so that the condition simplifies to

$$-\alpha\,\tilde{p}_2\,(\tilde{M}-\tilde{p}_2 r_2)^{\rho-1}+(1-\alpha)\,r_2^{\rho-1}\le 0. \tag{A.22}$$

This inequality cannot be solved analytically for $\tilde{p}_2$; still, it yields a condition of which we can conclude that a unique lower bound on $\tilde{p}_2$ exists and that for $\tilde{p}_2$ not below this lower bound, the inequality holds. Rearranging gives

$$\begin{aligned}
& \alpha\,\tilde{p}_2\,(\tilde{M}-\tilde{p}_2 r_2)^{\rho-1} \ge (1-\alpha)\,r_2^{\rho-1} \\
\Leftrightarrow\quad & \alpha^{-1}\,\tilde{p}_2^{-1}\,(\tilde{M}-\tilde{p}_2 r_2)^{1-\rho} \le (1-\alpha)^{-1}\,r_2^{1-\rho} \\
\Leftrightarrow\quad & (1-\alpha)(\tilde{M}-\tilde{p}_2 r_2)^{1-\rho} \le \alpha\,\tilde{p}_2\,r_2^{1-\rho},
\end{aligned}$$

which, since $\rho=(\sigma-1)/\sigma$, translates to

$$\begin{aligned}
& (1-\alpha)(\tilde{M}-\tilde{p}_2 r_2)^{1/\sigma} \le \alpha\,\tilde{p}_2\,r_2^{1/\sigma} \\
\Leftrightarrow\quad & (1-\alpha)^{\sigma}\,(\tilde{M}-\tilde{p}_2 r_2) \le \alpha^{\sigma}\,\tilde{p}_2^{\sigma}\,r_2 \\
\Leftrightarrow\quad & \alpha^{\sigma}\,\tilde{p}_2^{\sigma} \ge (1-\alpha)^{\sigma}\,\frac{\tilde{M}-\tilde{p}_2 r_2}{r_2} \\
\Leftrightarrow\quad & \alpha^{\sigma}\,\tilde{p}_2^{\sigma} \ge (1-\alpha)^{\sigma}\left(\frac{\tilde{M}}{r_2}-\tilde{p}_2\right).
\end{aligned}$$

This finally gives rise to

$$\alpha^{\sigma}\,\tilde{p}_2^{\sigma}+(1-\alpha)^{\sigma}\,\tilde{p}_2 \ge \frac{(1-\alpha)^{\sigma}\,\tilde{M}}{r_2}. \tag{A.23}$$

Hence, for $\tilde{p}_2$ above this threshold, utility cannot be increased by shifting consumption from good 1 in favor of good 2, starting from $\mathbf{b}\equiv(\tilde{M}-\tilde{p}_2 r_2, r_2)$.

In the next step, it has to be proven that when consuming bundle $\mathbf{b}$, utility cannot be increased by shifting consumption in the opposite direction, i.e. from good 2 towards good 1, either. For this purpose, let us calculate the utility that the agent gets from consuming $(\tilde{M}-\tilde{p}_2(r_2-\delta), r_2-\delta)$, $\delta>0$. Since now $x_1>r_1$ and $x_2<r_2$, $n_1(x_1\,|\,r_1)=0$ and $n_2(x_2\,|\,r_2)=\lambda((1-\alpha)\,x_2^{\rho}-(1-\alpha)\,r_2^{\rho})$; thus, the utility is given by

$$\begin{aligned}
& u(\tilde{M}-\tilde{p}_2(r_2-\delta), r_2-\delta\,|\,r_1,r_2) \\
& = \alpha(\tilde{M}-\tilde{p}_2(r_2-\delta))^{\rho}+(1+\lambda)(1-\alpha)(r_2-\delta)^{\rho}-\lambda(1-\alpha)\,r_2^{\sigma}.
\end{aligned} \tag{A.24}$$

The derivative of this utility level w.r.t. δ is

$$\begin{aligned}(\text{A.25})\quad &\frac{\mathrm{d}u(\tilde{M}-\tilde{p}_2(r_2-\delta), r_2-\delta \mid r_1, r_2)}{\mathrm{d}\delta}\\ &= \alpha\,\tilde{p}_2\,\rho(\tilde{M}-\tilde{p}_2(r_2-\delta))^{\rho-1}-(1+\lambda)(1-\alpha)\,\rho\,(r_2-\delta)^{\rho-1}.\end{aligned}$$

For the bundle $(\tilde{M}-\tilde{p}_2(r_2-\delta), r_2-\delta)$ to be optimal, increasing δ must not pay off, i.e. this derivative has to be non-positive:

$$(\text{A.26})\quad \alpha\,\tilde{p}_2\,\rho(\tilde{M}-\tilde{p}_2(r_2-\delta))^{\rho-1}-(1+\lambda)(1-\alpha)\,\rho\,(r_2-\delta)^{\rho-1} \le 0.$$

This inequality can be reduced by the factor ρ, and in the special case of bundle **b**, δ equals zero, so that the condition simplifies to

$$(\text{A.27})\quad \alpha\,\tilde{p}_2(\tilde{M}-\tilde{p}_2 r_2)^{\rho-1} \le (1+\lambda)(1-\alpha)r_2^{\rho-1}.$$

This inequality cannot be solved analytically for $\tilde{p}_2$; still, it yields a condition of which we can conclude that a unique upper bound on $\tilde{p}_2$ exists and that for $\tilde{p}_2$ not above this upper bound, the inequality holds. Rearranging gives

$$\begin{aligned}\alpha^{-1}\tilde{p}_2^{-1}(\tilde{M}-\tilde{p}_2 r_2)^{1-\rho} &\ge (1+\lambda)^{-1}(1-\alpha)^{-1}\,r_2^{1-\rho}\\ \Leftrightarrow\quad (1+\lambda)(1-\alpha)(\tilde{M}-\tilde{p}_2 r_2)^{1-\rho} &\ge \alpha\,\tilde{p}_2 r_2^{1-\rho},\end{aligned}$$

which, since $\rho = (\sigma-1)/\sigma$, translates to

$$\begin{aligned}(1+\lambda)(1-\alpha)(\tilde{M}-\tilde{p}_2 r_2)^{1/\sigma} &\ge \alpha\,\tilde{p}_2 r_2^{1/\sigma}\\ \Leftrightarrow\quad [(1+\lambda)(1-\alpha)]^{\sigma}\,(\tilde{M}-\tilde{p}_2 r_2) &\ge \alpha^{\sigma}\,\tilde{p}_2^{\sigma}\,r_2\\ \Leftrightarrow\quad \alpha^{\sigma}\,\tilde{p}_2^{\sigma} &\le [(1+\lambda)(1-\alpha)]^{\sigma}\frac{\tilde{M}-\tilde{p}_2 r_2}{r_2}\\ \Leftrightarrow\quad \alpha^{\sigma}\,\tilde{p}_2^{\sigma} &\le [(1+\lambda)(1-\alpha)]^{\sigma}\left(\frac{\tilde{M}}{r_2}-\tilde{p}_2\right).\end{aligned}$$

This finally gives rise to

$$(\text{A.28})\quad \alpha^{\sigma}\,\tilde{p}_2^{\sigma}+[(1+\lambda)(1-\alpha)]^{\sigma}\,\tilde{p}_2 \le \frac{[(1+\lambda)(1-\alpha)]^{\sigma}\,\tilde{M}}{r_2}.$$

Hence, for $\tilde{p}_2$ below this threshold, utility cannot be increased by shifting consumption from good 2 in favor of good 1, starting from $\mathbf{b} \equiv (\tilde{M}-\tilde{p}_2 r_2, r_2)$.

Combining conditions (A.23) and (A.28) delivers the interval of $\tilde{p}_2$ for which it pays off neither to shift consumption from good 1 towards good 2 nor the opposite direction, starting from bundle **b**. This is the interval for which bundle **b** is optimal, as asserted in Proposition 2.

Claim 3

For all $(\tilde{p}_2, \tilde{M})$ that fulfill

$$r_1 < \tilde{M} < r_1 + \tilde{p}_2 r_2 \text{ and}$$
$$\left(\frac{(1-\alpha)^\sigma}{\alpha^\sigma \tilde{M}/r_1 - \alpha^\sigma}\right)^{1/(\sigma-1)} \leq \tilde{p}_2 \leq \left(\frac{[(1+\lambda)(1-\alpha)]^\sigma}{\alpha^\sigma \tilde{M}/r_1 - \alpha^\sigma}\right)^{1/(\sigma-1)}, \tag{A.29}$$

consuming bundle $\mathbf{a} \equiv (r_1, (\tilde{M}-r_1)/\tilde{p}_2)$ is the optimal choice.

Proof. The first part of condition (A.29) can be achieved by $\tilde{M} \geq r_1$ and $\tilde{p}_2$ being relatively high so that $\tilde{M} < r_1 + \tilde{p}_2 r_2$; alternatively, $\tilde{M} < r_1$, but $\tilde{p}_2$ is low enough so that $\tilde{M} \geq \tilde{p}_2 r_2$, and at the same time high enough so that $\tilde{M} < r_1 + \tilde{p}_2 r_2$. Then, branches II, III, and IV, or III and IV, or II and IV are feasible.

In this case, it is impossible to simultaneously consume of both goods more than the reference levels. It is possible, however, to consume more than the reference level of one good, when consuming less than the reference level of the respective other good. Hence, *there is a trade-off* of the sort that a gain in one good is met by a loss in the respective other good. This trade-off changes the intervals on which it is optimal to consume bundle **a** or **b** vis-à-vis the situation in claims 1 and 2, where of both goods more than the reference levels could be consumed simultaneously.

It has to be shown that when consuming bundle **a**, neither shifting resources from the consumption of good 1 towards good 2 nor in the opposite direction can improve the agent's utility for $\tilde{p}_2$ inside the interval defined by (A.29). For this purpose, let us calculate the utility that the agent would get from consuming $(x_1^{\mathbf{a}}+\delta, x_2^{\mathbf{a}}-\delta/\tilde{p}_2) = (r_1+\delta, (\tilde{M}-r_1-\delta)/\tilde{p}_2)$, $\delta > 0$. Since $r_1 < \tilde{M} < r_1 + \tilde{p}_2 r_2$, consuming $x_1 = r_1$ means that $x_2 < r_2$. Therefore, $n_1(x_1 \mid r_1) = 0$ and $n_2(x_2 \mid r_2) = \lambda((1-\alpha)x_2^\rho - (1-\alpha)r_2^\rho)$, so that the utility is given by

$$\begin{aligned} &u(r_1+\delta, (\tilde{M}-r_1-\delta)/\tilde{p}_2 \mid r_1, r_2) \\ &= \alpha(r_1+\delta)^\rho + (1-\alpha)(1+\lambda)[(\tilde{M}-r_1-\delta)/\tilde{p}_2]^\rho - \lambda(1-\alpha)r_2^\rho \end{aligned}. \tag{A.30}$$

Furthermore, we will need the change in this utility when δ is marginally increased:

$$\begin{aligned} &\frac{\mathrm{d}u(r_1+\delta, (\tilde{M}-r_1-\delta)/\tilde{p}_2 \mid r_1, r_2)}{\mathrm{d}\delta} \\ &= \alpha\rho(r_1+\delta)^{\rho-1} - (1-\alpha)(1+\lambda)\frac{1}{\tilde{p}_2}\rho\left(\frac{\tilde{M}-r_1-\delta}{\tilde{p}_2}\right)^{\rho-1}. \end{aligned} \tag{A.31}$$

For the bundle $(r_1+\delta, (\tilde{M}-r_1-\delta)/\tilde{p}_2)$ to be optimal, increasing δ must not pay off, i.e. this derivative has to be non-positive:

$$\alpha\rho(r_1+\delta)^{\rho-1}-(1-\alpha)(1+\lambda)\frac{1}{\tilde{p}_2}\rho\left(\frac{\tilde{M}-r_1-\delta}{\tilde{p}_2}\right)^{\rho-1}\leq 0. \tag{A.32}$$

This inequality can be reduced by the factor ρ, and in the special case of bundle **a**, δ equals zero, so that the condition simplifies to

$$\alpha r_1^{\rho-1}-(1-\alpha)(1+\lambda)\frac{1}{\tilde{p}_2}\left(\frac{\tilde{M}-r_1}{\tilde{p}_2}\right)^{\rho-1}\leq 0. \tag{A.33}$$

Solving this inequality for $\tilde{p}_2$ is analogous to solving the inequality involved in the proof of Proposition 1, with the factor $1-\alpha$ replaced by $(1-\alpha)(1+\lambda)$. This delivers

$$\tilde{p}_2\leq\left(\frac{[(1-\alpha)(1+\lambda)]^{\sigma}}{\alpha^{\sigma}\tilde{M}/r_1-\alpha^{\sigma}}\right)^{1/(\sigma-1)}. \tag{A.34}$$

Hence, for $\tilde{p}_2$ below this threshold, the utility cannot be increased by shifting consumption from good 2 in favor of good 1, starting from $\mathbf{a}\equiv(r_1, (\tilde{M}-r_1)/\tilde{p}_2)$.

In the next step, it has to be proven that when consuming bundle **a**, utility cannot be increased by shifting consumption in the opposite direction, i.e. from good 1 towards good 2, either. For this purpose, let us calculate the utility that the agent gets from consuming $(r_1-\delta, (\tilde{M}-r_1+\delta)/\tilde{p}_2)$, $\delta>0$. Since now $x_1<r_1$ and, for sufficiently small δ, $x_2<r_2$, gain–loss utility is $n_1(x_1\,|\,r_1)=\lambda(\alpha x_1^{\rho}-\alpha r_1^{\rho})$ and $n_2(x_2\,|\,r_2)=\lambda((1-\alpha)x_2^{\rho}-(1-\alpha)r_2^{\rho})$; thus, total utility is given by

$$\begin{aligned}&u(r_1-\delta, (\tilde{M}-r_1+\delta)/\tilde{p}_2\,|\,r_1,r_2)=\\&\alpha(1+\lambda)(r_1-\delta)^{\rho}+(1-\alpha)(1+\lambda)[(\tilde{M}-r_1+\delta)/\tilde{p}_2]^{\rho}-\lambda\alpha r_1^{\rho}-\lambda(1-\alpha)r_2^{\rho}\end{aligned} \tag{A.35}$$

The derivative of this utility level w.r.t. δ is

$$\begin{aligned}&\frac{\mathrm{d}u(r_1-\delta, (\tilde{M}-r_1+\delta)/\tilde{p}_2\,|\,r_1,r_2)}{\mathrm{d}\delta}\\&=-(1+\lambda)\alpha\rho(r_1-\delta)^{\rho-1}+(1-\alpha)(1+\lambda)\frac{1}{\tilde{p}_2}\rho\left(\frac{\tilde{M}-r_1+\delta}{\tilde{p}_2}\right)^{\rho-1}.\end{aligned} \tag{A.36}$$

For the bundle $(r_1-\delta, (\tilde{M}-r_1+\delta)/\tilde{p}_2)$ to be optimal, increasing δ must not pay off, i.e. this derivative has to be non-positive:

$$-(1+\lambda)\alpha\rho(r_1-\delta)^{\rho-1}+(1-\alpha)(1+\lambda)\frac{1}{\tilde{p}_2}\rho\left(\frac{\tilde{M}-r_1+\delta}{\tilde{p}_2}\right)^{\rho-1}\leq 0. \tag{A.37}$$

This inequality can be reduced by the factors $1+\lambda$ and ρ, and in the special case of bundle **a**, δ equals zero, so that the condition simplifies to

$$(\text{A.38}) \quad -\alpha r_1^{\rho-1} + (1-\alpha)\frac{1}{\tilde{p}_2}\left(\frac{\tilde{M}-r_1}{\tilde{p}_2}\right)^{\rho-1} \leq 0.$$

This is the same condition as (A.11), with the inequality holding in just the opposite direction. Therefore, the solution of the inequality is given by a reversed version of (A.12):

$$(\text{A.39}) \quad \tilde{p}_2 \geq \left(\frac{(1-\alpha)^\sigma}{\alpha^\sigma \tilde{M}/r_1 - \alpha^\sigma}\right)^{1/(\sigma-1)}.$$

Combining conditions (A.34) and (A.39) delivers the interval of $\tilde{p}_2$ for which it pays off neither to shift consumption from good 1 towards good 2 nor the opposite direction, starting from bundle **a**. This is the interval for which bundle **a** is optimal, as asserted in Proposition 3.

Claim 4

For all $(\tilde{p}_2, \tilde{M})$ that fulfill

$$(\text{A.40}) \quad \begin{aligned} &\tilde{p}_2 r_2 < \tilde{M} < r_1 + \tilde{p}_2 r_2 \text{ as well as} \\ &[\alpha(1+\lambda)]^\sigma \tilde{p}_2^\sigma + (1-\alpha)^\sigma \tilde{p}_2 \geq \frac{(1-\alpha)^\sigma \tilde{M}}{r_2} \text{ and} \\ &\alpha^\sigma \tilde{p}_2^\sigma + (1-\alpha)^\sigma \tilde{p}_2 \leq \frac{(1-\alpha)^\sigma \tilde{M}}{r_2}, \end{aligned}$$

consuming bundle $\mathbf{b} \equiv (\tilde{M} - \tilde{p}_2 r_2, r_2)$ is the optimal choice.

Proof. It has to be shown that when consuming bundle **b**, neither shifting resources from the consumption of good 1 in favor of good 2 nor in the opposite direction can improve the agent's utility for $\tilde{p}_2$ inside the interval defined by (A.40). For this purpose, let us calculate the utility that the agent gets from consuming $(\tilde{M} - \tilde{p}_2(r_2+\delta), r_2+\delta)$, $\delta > 0$. Since $\tilde{p}_2 r_2 < \tilde{M} < r_1 + \tilde{p}_2 r_2$, consuming $x_2 = r_2$ means that $x_1 < r_1$. Therefore, $n_1(x_1 \mid r_1) = \lambda(\alpha x_1^\rho - \alpha r_1^\rho)$ and $n_2(x_2 \mid r_2) = 0$; thus, the utility is given by

$$(\text{A.41}) \quad \begin{aligned} &u(\tilde{M} - \tilde{p}_2(r_2+\delta), r_2+\delta \mid r_1, r_2) \\ &= \alpha(1+\lambda)(\tilde{M} - \tilde{p}_2(r_2+\delta))^\rho + (1-\alpha)(r_2+\delta)^\rho - \lambda\alpha(\tilde{M} - \tilde{p}_2(r_2+\delta))^\rho. \end{aligned}$$

Furthermore, we will need the change in this utility when δ is marginally increased:

$$\begin{aligned}(\text{A.42})\quad &\frac{\mathrm{d}u(\tilde{M}-\tilde{p}_2(r_2+\delta), r_2+\delta \mid r_1, r_2)}{\mathrm{d}\delta}\\ &= -\alpha(1+\lambda)\,\tilde{p}_2\,\rho\,(\tilde{M}-\tilde{p}_2(r_2+\delta))^{\rho-1}+(1-\alpha)\,\rho\,(r_2+\delta)^{\rho-1}.\end{aligned}$$

For the bundle $(\tilde{M}-\tilde{p}_2(r_2+\delta), r_2+\delta)$ to be optimal, increasing δ must not pay off, i.e. this derivative has to be non-positive:

$$(\text{A.43})\quad -\alpha(1+\lambda)\,\tilde{p}_2\,\rho\,(\tilde{M}-\tilde{p}_2(r_2+\delta))^{\rho-1}+(1-\alpha)\,\rho\,(r_2+\delta)^{\rho-1} \le 0.$$

This inequality can be reduced by the factor ρ, and in the special case of bundle **b**, δ equals zero, so that the condition simplifies to

$$(\text{A.44})\quad -\alpha(1+\lambda)\,\tilde{p}_2\,(\tilde{M}-\tilde{p}_2 r_2)^{\rho-1}+(1-\alpha)\,r_2^{\rho-1} \le 0.$$

This inequality is the same as (A.22), with the factor $-\alpha$ replaced by $-\alpha(1+\lambda)$. Therefore, the rearrangements can be carried out analogously to the proof of Proposition 1 and finally give rise to

$$(\text{A.45})\quad [\alpha(1+\lambda)]^{\sigma}\,\tilde{p}_2^{\sigma}+(1-\alpha)^{\sigma}\,\tilde{p}_2 \ge \frac{(1-\alpha)^{\sigma}\,\tilde{M}}{r_2}.$$

Hence, for $\tilde{p}_2$ above this threshold, utility cannot be increased by shifting consumption from good 1 in favor of good 2, starting from $\mathbf{b} \equiv (\tilde{M}-\tilde{p}_2 r_2, r_2)$.

In the next step, it has to be proven that when consuming bundle **b**, utility cannot be increased by shifting consumption in the opposite direction, i.e. from good 2 towards good 1, either. For this purpose, let us calculate the utility that the agent gets from consuming $(\tilde{M}-\tilde{p}_2(r_2-\delta), r_2-\delta)$, $\delta>0$. Since now, $x_2<r_2$ and, for sufficiently small δ, $x_1<r_1$, gain–loss utility is $n_1(x_1 \mid r_1)=\lambda(\alpha x_1^{\rho}-\alpha r_1^{\rho})$ and $n_2(x_2 \mid r_2)=\lambda((1-\alpha)\,x_2^{\rho}-(1-\alpha)\,r_2^{\rho})$; thus, total utility is given by

$$\begin{aligned}(\text{A.46})\quad &u(\tilde{M}-\tilde{p}_2(r_2-\delta), r_2-\delta \mid r_1, r_2) =\\ &(1+\lambda)\alpha(\tilde{M}-\tilde{p}_2(r_2-\delta))^{\rho}+(1+\lambda)(1-\alpha)(r_2-\delta)^{\rho}-\lambda\,\alpha\,r_1^{\sigma}-\lambda(1-\alpha)\,r_2^{\sigma}.\end{aligned}$$

Furthermore, we will need the change in this utility when δ is marginally increased:

$$\begin{aligned}(\text{A.47})\quad &\frac{\mathrm{d}u(\tilde{M}-\tilde{p}_2(r_2-\delta), r_2-\delta \mid r_1, r_2)}{\mathrm{d}\delta}\\ &= (1+\lambda)\alpha\,\tilde{p}_2\,\rho\,(\tilde{M}-\tilde{p}_2(r_2-\delta))^{\rho-1}-(1+\lambda)(1-\alpha)\,\rho\,(r_2-\delta)^{\rho-1}.\end{aligned}$$

For the bundle $(\tilde{M}-\tilde{p}_2(r_2-\delta), r_2-\delta)$ to be optimal, increasing δ must not pay off, i.e. this derivative has to be non-positive:

$$(\text{A.48}) \quad (1+\lambda)\alpha\,\tilde{p}_2\,\rho(\tilde{M}-\tilde{p}_2(r_2-\delta))^{\rho-1}-(1+\lambda)(1-\alpha)\,\rho\,(r_2-\delta)^{\rho-1} \le 0.$$

This inequality can be reduced by the factors $1+\lambda$ and ρ, and in the special case of bundle **b**, δ equals zero, so that the condition simplifies to

$$(\text{A.49}) \quad \alpha\,\tilde{p}_2(\tilde{M}-\tilde{p}_2 r_2)^{\rho-1} \le (1-\alpha)\,r_2^{\rho-1}.$$

This inequality is the same as condition (A.22), with the inequality in the opposite direction. Thus, it gives rise to a reversed version of (A.23):

$$(\text{A.50}) \quad \alpha^{\sigma}\,\tilde{p}_2^{\sigma}+(1-\alpha)^{\sigma}\,\tilde{p}_2 \le \frac{(1-\alpha)^{\sigma}\,\tilde{M}}{r_2}.$$

Hence, for $\tilde{p}_2$ below this threshold, utility cannot be increased by shifting consumption from good 2 in favor of good 1, starting from $\mathbf{b} \equiv (\tilde{M}-\tilde{p}_2 r_2, r_2)$.

Combining conditions (A.45) and (A.50) delivers the interval of $\tilde{p}_2$ for which it pays off neither to shift consumption from good 1 towards good 2 nor the opposite direction, starting from bundle **b**. This is the interval for which bundle **b** is optimal, as asserted in Claim 4.

Proposition 2

Arguments common to the proofs of all five claims

In formula (5.15) the value $\tilde{M}^*(\alpha_1,\alpha_2)$ was defined by

$$\tilde{M}^*(\alpha_1,\alpha_2) \equiv r_1\left(1+\frac{\alpha_2 r_2^{\rho}}{\alpha_1 r_1^{\rho}}\right)$$

We have on branch I that $\alpha_1=\alpha$ and $\alpha_2=1-\alpha$. I will denote the value $\tilde{M}^*(\alpha,1-\alpha)$ by $\tilde{M}_{\mathrm{I}}^*$:

$$(\text{A.51}) \quad \tilde{M}_{\mathrm{I}}^* \equiv \tilde{M}^*(\alpha,1-\alpha) \equiv r_1\left(1+\frac{(1-\alpha)\,r_2^{\rho}}{\alpha\,r_1^{\rho}}\right)$$

On branch IV it holds that $\alpha_1=(1+\lambda)\,\alpha$ and $\alpha_2=(1+\lambda)\,(1-\alpha)$. Since $\tilde{M}_{\mathrm{IV}}^* \equiv \tilde{M}^*((1+\lambda)\alpha, (1+\lambda)(1-\alpha)) = \tilde{M}^*(\alpha,1-\alpha)$, $\tilde{M}_{\mathrm{I}}^*$ and $\tilde{M}_{\mathrm{IV}}^*$ are identical. Therefore, $\tilde{M}_{\mathrm{I}}^*$ separates two situations:

1 If $\tilde{M} > \tilde{M}_{\mathrm{I}}^*$, their exists a non-empty interval of prices $\tilde{p}_2$ for which of both goods more than their reference levels is consumed. That is,

if $\tilde{M} > \tilde{M}^*_{\text{I}}$, the agent attains for suitable $\tilde{p}_2$ an interior solution on branch I. At the same time, an income $\tilde{M} > \tilde{M}^*_{\text{I}}$ has the consequence that the interval of prices $\tilde{p}_2$ for which an interior solution on branch IV could be attained is empty.

2 Vice versa, if $\tilde{M} \leq \tilde{M}^*_{\text{I}}$, the interval of prices $\tilde{p}_2$ for which an interior solution on branch I is attained is empty. At the same time, if $\tilde{M} \leq \tilde{M}^*_{\text{I}}$, the interval of prices $\tilde{p}_2$ for which an interior solution on branch IV is attained is non-empty.

In analogy to this calculation for the branches I and IV, one can determine the values of $\tilde{M}^*$ for the branches II and III. In the case of the branches II and III, comparing $\tilde{M}$ and $\tilde{M}^*$ tells us something different, however, than it does in the case of branches I and IV: Here, it is not the emptiness of the branches that is checked. Rather, the question is answered which of the two bundles **a** or **b** is consumed at the *upper* bound of branch II and at the *lower* bound of branch III (see my elaboration on p. 49).

Branch II does not impose a lower bound on $\tilde{p}_2$ but only an upper bound, see condition (5.29). Recall the definition of the values $\tilde{p}^*_2(\alpha_1, \alpha_2, \tilde{M})$ and $\tilde{p}^*_2(\alpha_1, \alpha_2, \tilde{M})$ from Section 5.1.2. On branch II, $\alpha_1 = (1+\lambda)\,\alpha$ and $\alpha_2 = 1-\alpha$, so that the upper bound of branch II, provided that $\tilde{M} > r_1$, is given by

$$\min\{\tilde{p}^*_2((1+\lambda)\alpha, 1-\alpha, \tilde{M}),\, \tilde{p}^{**}_2((1+\lambda)\alpha, 1-\alpha, \tilde{M})\}.$$

Let $\tilde{M}^*_{\text{II}}$ denote the value of $\tilde{M}$ at which $\tilde{p}^*_2((1+\lambda)\alpha, 1-\alpha, \tilde{M})$ and $\tilde{p}^{**}_2((1+\lambda)\alpha, 1-\alpha, \tilde{M})$ coincide. Then the following relations hold:

- If $\tilde{M} < \tilde{M}^*_{\text{II}}$, then $\tilde{p}^{\text{II},ub}_2 = \tilde{p}^{**}_2((1+\lambda)\alpha, 1-\alpha, \tilde{M})$, so that at $\tilde{p}^{\text{II},ub}_2$, $x_2 = r_2$ and $x_1 < r_1$ are consumed (bundle **b**).
- If $\tilde{M} = \tilde{M}^*_{\text{II}}$, then $\tilde{p}^*_2((1+\lambda)\alpha, 1-\alpha, \tilde{M})$ and $\tilde{p}^{**}_2((1+\lambda)\alpha, 1-\alpha, \tilde{M})$ coincide, and it holds that $\lim_{\tilde{p}_2 \nearrow \tilde{p}^{\text{II},ub}_2} (x_1, x_2) = (r_1, r_2)$.
- If $\tilde{M} > \tilde{M}^*_{\text{II}}$, then $\tilde{p}^{\text{II},ub}_2 = \tilde{p}^*_2((1+\lambda)\alpha, 1-\alpha, \tilde{M})$, so that $\lim_{\tilde{p}_2 \nearrow \tilde{p}^{\text{II},ub}_2} x_1 = r_1$ and $\lim_{\tilde{p}_2 \nearrow \tilde{p}^{\text{II},ub}_2} x_2 > r_2$ (bundle **a**).

$\tilde{M}^*_{\text{II}}$ is given by

$$\text{(A.52)} \quad \tilde{M}^*_{\text{II}} \equiv \tilde{M}^*((1+\lambda)\alpha, 1-\alpha) = r_1 \left(1 + \frac{(1-\alpha)\, r_2^{\rho}}{(1+\lambda)\,\alpha r_1^{\rho}}\right).$$

Being at an interior solution on branch III does not impose an upper bound on $\tilde{p}_2$ but only a lower bound, see condition (5.30). On branch III, $\alpha_1 = \alpha$ and $\alpha_2 = (1+\lambda)\,(1-\alpha)$, so that this lower bound, provided that $\tilde{M} > r_1$, is given by

$$\max\{\tilde{p}^*_2(\alpha, (1+\lambda)(1-\alpha), \tilde{M}),\, \tilde{p}^{**}_2(\alpha, (1+\lambda)(1-\alpha), \tilde{M})\}.$$

Let $\tilde{M}^*_{\text{III}}$ denote the value of $\tilde{M}$ at which $\tilde{p}_2^*(\alpha, (1+\lambda)(1-\alpha), \tilde{M})$ and $\tilde{p}_2^{**}(\alpha, (1+\lambda)(1-\alpha), \tilde{M})$ coincide. Then the following relations hold:

- If $\tilde{M} < \tilde{M}^*_{\text{III}}$, then $\tilde{p}_2^{\text{III},lb} = \tilde{p}_2^*(\alpha, (1+\lambda)(1-\alpha), \tilde{M})$, so that at $\tilde{p}_2^{\text{III},lb}$, $x_1 = r_1$ and $x_2 < r_2$ are consumed (bundle **a**).
- If $\tilde{M} = \tilde{M}^*_{\text{III}}$, then $\tilde{p}_2^*(\alpha, (1+\lambda)(1-\alpha), \tilde{M})$ and $\tilde{p}_2^{**}(\alpha, (1+\lambda)(1-\alpha), \tilde{M})$ coincide and it holds that $\lim_{\tilde{p}_2 \searrow \tilde{p}_2^{\text{III},lb}} (x_1, x_2) = (r_1, r_2)$.
- If $\tilde{M} > \tilde{M}^*_{\text{III}}$, then $\tilde{p}_2^{\text{III},lb} = \tilde{p}_2^{**}(\alpha, (1+\lambda)(1-\alpha), \tilde{M})$, so that $\lim_{\tilde{p}_2 \searrow \tilde{p}_2^{\text{III},lb}} x_2 = r_2$ and $\lim_{\tilde{p}_2 \searrow \tilde{p}_2^{\text{III},lb}} x_1 > r_1$ (bundle **b**).

$\tilde{M}^*_{\text{III}}$ is given by

$$\text{(A.53)} \quad \tilde{M}^*_{\text{III}} \equiv \tilde{M}^*(\alpha, (1+\lambda)(1-\alpha)) = r_1 \left(1 + \frac{(1+\lambda)(1-\alpha) r_2^{\rho}}{\alpha r_1^{\rho}} \right).$$

Since $\lambda > 0$, the four values r_1, $\tilde{M}^*_{\text{I}}$, $\tilde{M}^*_{\text{II}}$, and $\tilde{M}^*_{\text{III}}$ can be ordered as follows:

$$\text{(A.54)} \quad r_1 < \tilde{M}^*_{\text{II}} < \tilde{M}^*_{\text{I}} < \tilde{M}^*_{\text{III}}.$$

Claim 1

If $\tilde{M} \leq r_1$ ("case A"), then the agent chooses sequentially the branches/bundles II – **b** – IV in the course of increasing $\tilde{p}_2$ continuously on the interval $(0, \infty)$. This translates to the following demand function:

$$\text{(A.55)} \quad \mathbf{x}^{\text{A}}(\tilde{p}_2, \tilde{M}) = \begin{cases} \mathbf{x}^{\text{II}}(\tilde{p}_2, \tilde{M}) & \text{if } [(1+\lambda)\alpha]^{\sigma} \tilde{p}_2^{\sigma} + (1-\alpha)^{\sigma} \tilde{p}_2 \leq \dfrac{(1-\alpha)^{\sigma} \tilde{M}}{r_2} \\ ((\tilde{M} - \tilde{p}_2 r_2), r_2) & \begin{array}{l} \text{if } [(1+\lambda)\alpha]^{\sigma} \tilde{p}_2^{\sigma} + (1-\alpha)^{\sigma} \tilde{p}_2 > \dfrac{(1-\alpha)^{\sigma} \tilde{M}}{r_2} \\ \text{and } \alpha^{\sigma} \tilde{p}_2^{\sigma} + (1-\alpha)^{\sigma} \tilde{p}_2 \leq \dfrac{(1-\alpha)^{\sigma} \tilde{M}}{r_2} \end{array} \\ \mathbf{x}^{\text{IV}}(\tilde{p}_2, \tilde{M}) & \text{if } \alpha^{\sigma} \tilde{p}_2^{\sigma} + (1-\alpha)^{\sigma} \tilde{p}_2 > \dfrac{(1-\alpha)^{\sigma} \tilde{M}}{r_2} \end{cases};$$

Proof. In case A, only branches II and IV are feasible. Branch II imposes an upper bound on $\tilde{p}_2$ only—in case A, the second part of condition (5.29)—while branch IV imposes a lower bound—in case A, the second part of condition (5.30). As has been shown above (in this appendix), none of the intervals inside which interior solutions are attained overlap. The gap between the intervals of branches II and IV is equal to condition (5.43), under which it is optimal to consume bundle **b** according to Proposition 1.

Claim 2

If $r_1 < \tilde{M} \leq \tilde{M}^*_{\text{II}}$ ("case B"), then the agent chooses sequentially the branches/bundles II – **b** – IV – **a** – III in the course of increasing $\tilde{p}_2$ continuously on the interval $(0,\infty)$. This translates to the following demand function:

$$\mathbf{x}^{\mathrm{B}}(\tilde{p}_2,\tilde{M}) =$$

$$(\text{A.56})\quad \begin{cases} \mathbf{x}^{\mathrm{II}}(\tilde{p}_2,\tilde{M}) & \text{if } [(1+\lambda)\alpha]^\sigma \tilde{p}_2^\sigma + (1-\alpha)^\sigma \tilde{p}_2 \leq \dfrac{(1-\alpha)^\sigma \tilde{M}}{r_2} \\ (r_2,\ \tilde{M}-\tilde{p}_2 r_2) & \begin{array}{l}\text{if } [(1+\lambda)\alpha]^\sigma \tilde{p}_2^\sigma + (1-\alpha)^\sigma \tilde{p}_2 > \dfrac{(1-\alpha)^\sigma \tilde{M}}{r_2} \\ \text{and } \alpha^\sigma \tilde{p}_2^\sigma + (1-\alpha)^\sigma \tilde{p}_2 \leq \dfrac{(1-\alpha)^\sigma \tilde{M}}{r_2}\end{array} \\ \mathbf{x}^{\mathrm{IV}}(\tilde{p}_2,\tilde{M}) & \begin{array}{l}\text{if } \alpha^\sigma \tilde{p}_2^\sigma + (1-\alpha)^\sigma \tilde{p}_2 > \dfrac{(1-\alpha)^\sigma \tilde{M}}{r_2} \\ \text{and } \tilde{p}_2 < \left(\dfrac{(1-\alpha)^\sigma}{\alpha^\sigma \tilde{M}/r_1 - \alpha^\sigma}\right)^{1/(\sigma-1)}\end{array} \\ (r_1,\ (\tilde{M}-r_1)/\tilde{p}_2) & \text{if } \left(\dfrac{(1-\alpha)^\sigma}{\alpha^\sigma \tilde{M}/r_1 - \alpha^\sigma}\right)^{1/(\sigma-1)} \leq \tilde{p}_2 < \left(\dfrac{[(1+\lambda)(1-\alpha)]^\sigma}{\alpha^\sigma \tilde{M}/r_1 - \alpha^\sigma}\right)^{1/(\sigma-1)} \\ \mathbf{x}^{\mathrm{III}}(\tilde{p}_2,\tilde{M}) & \text{if } \tilde{p}_2 \geq \left(\dfrac{[(1+\lambda)(1-\alpha)]^\sigma}{\alpha^\sigma \tilde{M}/r_1 - \alpha^\sigma}\right)^{1/(\sigma-1)} \end{cases} ;$$

Proof. In case B, branches II and III are feasible, and I and IV are potentially feasible. Since $\tilde{M} \leq \tilde{M}^*_{\text{II}} < \tilde{M}^*_{\text{I}}$, an interior solution on branch I is excluded, but will be attained on branch IV.

The three feasible bundles will be chosen in the sequence II – IV – III. This is due to the fact that the intervals inside which interior solutions are attained do not overlap, as has been proven above (in this appendix). Given that the intervals do not overlap, the conditions for being at an interior solution on the respective branch allow only for the sequence II – IV – III: Branch II only imposes an upper bound on $\tilde{p}_2$ —in case C, the second part of condition (5.29) —while branch IV imposes an upper as well as a lower bound—condition (5.31) —and branch III imposes only a lower bound—in case B, the first part of condition (5.30).

It now has to be shown that condition (5.43) describes exactly the gap between the intervals for which interior solutions on the branches II and IV are attained and that (5.42) describes exactly the gap between the intervals for which interior solutions on the branches IV and III are attained.

As far as the gap between branches II and IV is concerned, it is obvious that the upper bound on $\tilde{p}_2$ allowing for an interior solution on branch II,

$\tilde{p}_2^{**}(\alpha, (1+\lambda)(1-\alpha), \tilde{M})$, is the same as the lower bound above which bundle **b** is optimal. It is equally obvious that the upper bound of the interval inside which **b** is optimal, $\tilde{p}_2^{**}(\alpha, 1-\alpha, \tilde{M})$, is also the lower bound of branch IV.

It remains to be shown that inside the interval $[\tilde{p}_2^{**}((1+\lambda)\alpha, 1-\alpha, \tilde{M}), \tilde{p}_2^{**}(\alpha, 1-\alpha, \tilde{M})]$, the requirements $\tilde{p}_2 r_2 < \tilde{M}$ and $\tilde{M} < r_1 + \tilde{p}_2 r_2$ are both fulfilled so that the entire condition (5.43) holds. The first one, $\tilde{p}_2 r_2 < \tilde{M}$, is indeed fulfilled, because at the highest eligible $\tilde{p}_2$, i.e. at $\tilde{p}_2 = \tilde{p}_2^{**}(\alpha, 1-\alpha, \tilde{M})$, it holds that

$$
\begin{aligned}
& & \alpha^\sigma \tilde{p}_2^\sigma + (1-\alpha)^\sigma \tilde{p}_2 &= \frac{(1-\alpha)^\sigma \tilde{M}}{r_2} \\
&\Leftrightarrow & \tilde{p}_2 r_2 &= \tilde{M} - \frac{\alpha^\sigma}{(1-\alpha)^\sigma} \tilde{p}_2^\sigma r_2 \\
&\Rightarrow & \tilde{p}_2 r_2 &< \tilde{M}.
\end{aligned}
$$

The second one, $\tilde{M} < r_1 + \tilde{p}_2 r_2$, holds as well, since the lowest eligible $\tilde{p}_2$, i.e. $\tilde{p}_2 = \tilde{p}_2^{**}((1+\lambda)\alpha, 1-\alpha, \tilde{M})$, is simultaneously the upper bound of the interior solution on branch II. At that $\tilde{p}_2$, by the construction of condition (5.29), $x_2 = r_2$ and $x_1 < r_1$. Since $x_1 = \tilde{M} - \tilde{p}_2 r_2$, we get that $\tilde{M} - \tilde{p}_2 r_2 < r_1 \Leftrightarrow \tilde{M} < r_1 + \tilde{p}_2 r_2$.

Turning to the gap between branches IV and III, it can be stated that it is obvious that the upper bound on $\tilde{p}_2$ allowing for an interior solution on branch IV, $\tilde{p}_2^{*}(\alpha, 1-\alpha, \tilde{M})$, is the same as the lower bound above which bundle **a** is optimal according to (5.42). It is equally obvious that the upper bound of the interval inside which **a** is optimal, $\tilde{p}_2^{*}(\alpha, (1+\lambda)(1-\alpha), \tilde{M})$, is also the lower bound of branch III.

It remains to be shown that inside the interval $[\tilde{p}_2^{*}(\alpha, 1-\alpha, \tilde{M}), \tilde{p}_2^{*}(\alpha, (1+\lambda)(1-\alpha), \tilde{M})]$, the requirement $\tilde{M} < r_1 + \tilde{p}_2 r_2$ is fulfilled so that the entire condition (5.42) holds. This is indeed the case, because the lowest eligible $\tilde{p}_2$, i.e. $\tilde{p}_2 = \tilde{p}_2^{*}(\alpha, 1-\alpha, \tilde{M})$, is simultaneously the upper bound of the interior solution on branch IV. At that $\tilde{p}_2$, by the construction of condition (5.31), $x_1 = r_1$ and $x_2 < r_2$. Since $x_2 = (\tilde{M} - r_1)/\tilde{p}_2$, we get that $(\tilde{M} - r_1)/\tilde{p}_2 < r_2 \Leftrightarrow \tilde{M} < r_1 + \tilde{p}_2 r_2$.

Claim 3

If $\tilde{M}^*_{\text{II}} < \tilde{M} \le \tilde{M}^*_{\text{I}}$ ("case C"), then the agent chooses sequentially the branches/bundles II – **a** – **b** – IV – **a** – III in the course of increasing $\tilde{p}_2$ continuously on the interval $(0,\infty)$. This translates to the demand function

$$\mathbf{x}^{\text{C}}(\tilde{p}_2,\tilde{M}) =$$

$$\begin{cases} \mathbf{x}^{\text{II}}(\tilde{p}_2,\tilde{M}) & \text{if } \tilde{p}_2 < \left(\dfrac{(1-\alpha)^\sigma}{[(1+\lambda)\alpha]^\sigma \tilde{M}/r_1 - [(1+\lambda)\alpha]^\sigma}\right)^{1/(\sigma-1)} \\ (r_1,(\tilde{M}-r_1)/\tilde{p}_2) & \text{if } \left(\dfrac{(1-\alpha)^\sigma}{[(1+\lambda)\alpha]^\sigma \tilde{M}/r_1 - [(1+\lambda)\alpha]^\sigma}\right)^{1/(\sigma-1)} \le \tilde{p}_2 < \dfrac{\tilde{M}-r_1}{r_2} \\ (\tilde{M}-\tilde{p}_2 r_2, r_2) & \text{if } \dfrac{\tilde{M}-r_1}{r_2} < \tilde{p}_2 \text{ and } \alpha^\sigma \tilde{p}_2^\sigma + (1-\alpha)^\sigma \tilde{p}_2 \le \dfrac{(1-\alpha)^\sigma \tilde{M}}{r_2} \\ \mathbf{x}^{\text{IV}}(\tilde{p}_2,\tilde{M}) & \text{if } \alpha^\sigma \tilde{p}_2^\sigma + (1-\alpha)^\sigma \tilde{p}_2 > \dfrac{(1-\alpha)^\sigma \tilde{M}}{r_2} \text{ and } \tilde{p}_2 < \left(\dfrac{(1-\alpha)^\sigma}{\alpha^\sigma \tilde{M}/r_1 - \alpha^\sigma}\right)^{1/(\sigma-1)} \\ (r_1,(\tilde{M}-r_1)/\tilde{p}_2) & \text{if } \left(\dfrac{(1-\alpha)^\sigma}{\alpha^\sigma \tilde{M}/r_1 - \alpha^\sigma}\right)^{1/(\sigma-1)} \le \tilde{p}_2 < \left(\dfrac{[(1+\lambda)(1-\alpha)]^\sigma}{\alpha^\sigma \tilde{M}/r_1 - \alpha^\sigma}\right)^{1/(\sigma-1)} \\ \mathbf{x}^{\text{III}}(\tilde{p}_2,\tilde{M}) & \text{if } \tilde{p}_2 \ge \left(\dfrac{[(1+\lambda)(1-\alpha)]^\sigma}{\alpha^\sigma \tilde{M}/r_1 - \alpha^\sigma}\right)^{1/(\sigma-1)} \end{cases} ; \tag{A.57}$$

Proof. In case C, branches II and III are feasible, and I and IV are potentially feasible. Since $\tilde{M} \le \tilde{M}^*_{\text{I}}$, an interior solution on branch I is excluded, but it will be attained on branch IV.

The three feasible bundles will be chosen in the sequence II – IV – III. This is due to the fact that the intervals inside which interior solutions are attained do not overlap, as has been proven above (in this appendix). Given that the intervals do not overlap, the conditions for being at an interior solution on the respective branch allow only for the sequence II – IV – III: Branch II only imposes an upper bound on $\tilde{p}_2$ —in case C, the first part of condition (5.29)—while branch IV imposes an upper as well as a lower bound—condition (5.31)—and branch III imposes only a lower bound—in case C, the first part of condition (5.30).

Now, two things have to be shown:

1 that the combination of conditions (5.40) and (5.43) describes exactly the gap between the intervals on which interior solutions on the branches II and IV are attained;
2 that condition (5.42) describes exactly the gap between the intervals on which interior solutions on the branches IV and III are attained.

As far as the gap between branches II and IV is concerned, it is obvious that the upper bound on $\tilde{p}_2$ allowing for an interior solution on branch II, $\tilde{p}_2^*((1+\lambda)\alpha, 1-\alpha, \tilde{M})$, is the same as the lower bound of the interval in which bundle **a** is optimal. It is equally obvious that the upper bound of the interval inside which **a** is optimal, $\tilde{p}_2^{**}(\alpha, 1-\alpha, \tilde{M})$, is also the lower bound of branch IV.

It remains to be shown that the interval $[\tilde{p}_2^*((1+\lambda)\alpha, 1-\alpha, \tilde{M}), \tilde{p}_2^{**}(\alpha, 1-\alpha, \tilde{M})]$ includes the value $\tilde{p}_2$ at which $\tilde{M} = r_1 + \tilde{p}_2 r_2$. This is indeed the case, because $\tilde{p}_2^*((1+\lambda)\alpha, 1-\alpha, \tilde{M})$ is simultaneously the upper bound on the existence of an interior solution on branch II. At that $\tilde{p}_2$, by the construction of condition (5.29), $x_1 = r_1$ and $x_2 > r_2$. Since $x_2 = (\tilde{M} - r_1)/\tilde{p}_2$, we get that $(\tilde{M} - r_1)/\tilde{p}_2 > r_2 \Leftrightarrow \tilde{M} > r_1 + \tilde{p}_2 r_2$ at $\tilde{p}_2^*((1+\lambda)\alpha, 1-\alpha, \tilde{M})$. The largest eligible $\tilde{p}_2$, $\tilde{p}_2^{**}(\alpha, 1-\alpha, \tilde{M})$, is simultaneously the lower bound of branch IV. At that $\tilde{p}_2$, by the construction of condition (5.31), $x_2 = r_2$ and $x_1 < r_1$. Since $x_1 = \tilde{M} - \tilde{p}_2 r_2$, we get that $\tilde{M} - \tilde{p}_2 r_2 < r_1 \Leftrightarrow \tilde{M} < r_1 + \tilde{p}_2 r_2$ at $\tilde{p}_2^{**}(\alpha, 1-\alpha, \tilde{M})$. Hence, the interval $[\tilde{p}_2^*((1+\lambda)\alpha, 1-\alpha, \tilde{M}), \tilde{p}_2^{**}(\alpha, 1-\alpha, \tilde{M})]$ includes the value $\tilde{p}_2$ at which $\tilde{M} = r_1 + \tilde{p}_2 r_2$. The segment $[\tilde{p}_2^*((1+\lambda)\alpha, 1-\alpha, \tilde{M}), (\tilde{M} - r_1)/r_2]$ is contained in condition (5.40), hence for that segment consuming bundle **a** is optimal; the segment $[(\tilde{M} - r_1)/r_2, \tilde{p}_2^{**}(\alpha, 1-\alpha, \tilde{M})]$ is contained in condition (5.43), hence for this segment consuming bundle **b** is optimal.

Turning to the gap between branches IV and III, it becomes obvious that in case C, the conditions that have to fulfilled are the very same as in the gap between branches IV and III in case B. Therefore, the proof that in this gap consuming bundle **a** is optimal is the same as in the proof of Claim 2.

Claim 4

If $\tilde{M}_{\mathrm{I}}^{*} < \tilde{M} \le \tilde{M}_{\mathrm{III}}^{*}$ ("case D"), then the agent chooses sequentially the branches/bundles II – **a** – I – **b** – **a** – III in the course of increasing $\tilde{p}_2$ continuously on the interval $(0,\infty)$. This translates to the demand function

$$\mathbf{x}^{\mathrm{D}}(\tilde{p}_2,\tilde{M}) =$$

$$\text{(A.58)} \quad \begin{cases} \mathbf{x}^{\mathrm{II}}(\tilde{p}_2,\tilde{M}) & \text{if } \tilde{p}_2 < \left(\dfrac{(1-\alpha)^{\sigma}}{[(1+\lambda)\alpha]^{\sigma}\,\tilde{M}/r_1 - [(1+\lambda)\alpha]^{\sigma}}\right)^{1/(\sigma-1)} \\ (r_1,(\tilde{M}-r_1)/\tilde{p}_2) & \text{if } \left(\dfrac{(1-\alpha)^{\sigma}}{[(1+\lambda)\alpha]^{\sigma}\,\tilde{M}/r_1 - [(1+\lambda)\alpha]^{\sigma}}\right)^{1/(\sigma-1)} \le \tilde{p}_2 < \left(\dfrac{(1-\alpha)^{\sigma}}{\alpha^{\sigma}\tilde{M}/r_1 - \alpha^{\sigma}}\right)^{1/(\sigma-1)} \\ \mathbf{x}^{\mathrm{I}}(\tilde{p}_2,\tilde{M}) & \begin{array}{l}\text{if } \tilde{p}_2 \ge \left(\dfrac{(1-\alpha)^{\sigma}}{\alpha^{\sigma}\tilde{M}/r_1 - \alpha^{\sigma}}\right)^{1/(\sigma-1)} \\ \text{and } \alpha^{\sigma}\tilde{p}_2^{\sigma} + (1-\alpha)^{\sigma}\tilde{p}_2 \le \dfrac{(1-\alpha)^{\sigma}\tilde{M}}{r_2}\end{array} \\ (\tilde{M}-\tilde{p}_2 r_2, r_2) & \begin{array}{l}\text{if } \alpha^{\sigma}\tilde{p}_2^{\sigma} + (1-\alpha)^{\sigma}\tilde{p}_2 > \dfrac{(1-\alpha)^{\sigma}\tilde{M}}{r_2} \\ \text{and } \tilde{p}_2 < \dfrac{\tilde{M}-r_1}{r_2}\end{array} \\ (r_1,(\tilde{M}-r_1)/\tilde{p}_2) & \text{if } \dfrac{\tilde{M}-r_1}{r_2} \le \tilde{p}_2 < \left(\dfrac{[(1+\lambda)(1-\alpha)]^{\sigma}}{\alpha^{\sigma}\tilde{M}/r_1 - \alpha^{\sigma}}\right)^{1/(\sigma-1)} \\ \mathbf{x}^{\mathrm{III}}(\tilde{p}_2,\tilde{M}) & \text{if } \tilde{p}_2 \ge \left(\dfrac{[(1+\lambda)(1-\alpha)]^{\sigma}}{\alpha^{\sigma}\tilde{M}/r_1 - \alpha^{\sigma}}\right)^{1/(\sigma-1)} \end{cases} ;$$

Proof. In case D, branches II and III are feasible, and I and IV are potentially feasible. Since $\tilde{M} > \tilde{M}_{\mathrm{I}}^{*}$, an interior solution on branch IV is excluded, but it will be attained on branch I.

The three feasible bundles will be chosen in the sequence II – I – III. This is due to the fact that the intervals inside which interior solutions are attained do not overlap, as has been proven above (in this appendix). Given that the intervals do not overlap, the conditions for being at an interior solution on the respective branch allow only for the sequence II – I – III: Branch II only imposes an upper bound on $\tilde{p}_2$ —in case D, the first part of condition (5.29)—while branch I imposes an upper as well as a lower bound—condition (5.28)—and branch III imposes only a lower bound—in case D, the first part of condition (5.30).

Now, two things have to be shown:

1. that condition (5.40) describes exactly the gap between the intervals on which interior solutions on the branches II and I are attained;
2. that the combination of conditions (5.41) and (5.42) describes exactly the gap between the intervals on which interior solutions on the branches I and III are attained.

As far as the gap between branches II and I is concerned, it is obvious that the upper bound on $\tilde{p}_2$ allowing for an interior solution on branch II, $\tilde{p}_2^*((1+\lambda)\alpha, 1-\alpha, \tilde{M})$, is the same as the lower bound of the interval in which bundle **a** is optimal. It is equally obvious that the upper bound of the interval inside which **a** is optimal, $\tilde{p}_2^*(\alpha, 1-\alpha, \tilde{M})$, is also the lower bound of branch I.

It remains to be shown that for all $\tilde{p}_2$ in $[\tilde{p}_2^*((1+\lambda)\alpha, 1-\alpha, \tilde{M}), \tilde{p}_2^*(\alpha, 1-\alpha, \tilde{M})]$, it holds that $\tilde{M} \geq r_1 + \tilde{p}_2 r_2$, so that the entire condition (5.40) is fulfilled. This is indeed the case, because $\tilde{p}_2^*((1+\lambda)\alpha, 1-\alpha, \tilde{M})$ is simultaneously the upper bound on the existence of an interior solution on branch II. At that $\tilde{p}_2$, by the construction of condition (5.29), $x_1 = r_1$ and $x_2 > r_2$. Since $x_2 = (\tilde{M} - r_1)/\tilde{p}_2$, we get that $(\tilde{M} - r_1)/\tilde{p}_2 > r_2 \Leftrightarrow \tilde{M} > r_1 + \tilde{p}_2 r_2$ at $\tilde{p}_2^*((1+\lambda)\alpha, 1-\alpha, \tilde{M})$. The largest eligible $\tilde{p}_2$, $\tilde{p}_2^*(\alpha, 1-\alpha, \tilde{M})$, is simultaneously the lower bound of branch I. At that $\tilde{p}_2$, by the construction of condition (5.28), $x_1 = r_1$ and $x_2 > r_2$. Since $x_2 = (\tilde{M} - r_1)/\tilde{p}_2$, we get that $(\tilde{M} - r_1)/\tilde{p}_2 > r_2 \Leftrightarrow \tilde{M} > r_1 + \tilde{p}_2 r_2$ at $\tilde{p}_2^*(\alpha, 1-\alpha, \tilde{M})$. Hence, throughout the interval $[\tilde{p}_2^*((1+\lambda)\alpha, 1-\alpha, \tilde{M}),\ \tilde{p}_2^*(\alpha, 1-\alpha, \tilde{M})]$, it holds that $\tilde{M} > r_1 + \tilde{p}_2 r_2$.

Turning to the gap between branches I and III, it is obvious that the upper bound on $\tilde{p}_2$ allowing for an interior solution on branch I, $\tilde{p}_2^{**}(\alpha, 1-\alpha, \tilde{M})$, is the same as the lower bound of the interval in which bundle **b** is optimal. It is equally obvious that the upper bound of the interval inside which **a** is optimal, $\tilde{p}_2^*(\alpha, (1+\lambda)(1-\alpha), \tilde{M})$, is also the lower bound of branch III.

It remains to be shown that the interval $[\tilde{p}_2^{**}(\alpha, 1-\alpha, \tilde{M}), \tilde{p}_2^*(\alpha, (1+\lambda)(1-\alpha), \tilde{M})]$ includes the value $\tilde{p}_2$ at which $\tilde{M} = r_1 + \tilde{p}_2 r_2$. This is indeed the case, because $\tilde{p}_2^{**}(\alpha, 1-\alpha, \tilde{M})$ is simultaneously the upper bound on the existence of an interior solution on branch I. At that $\tilde{p}_2$, by the construction of condition (5.28), $x_2 = r_2$ and $x_1 \geq r_1$. Since $x_1 = \tilde{M} - \tilde{p}_2 r_2$, we get that $\tilde{M} - \tilde{p}_2 r_2 \geq r_1 \Leftrightarrow \tilde{M} \geq r_1 + \tilde{p}_2 r_2$ at $\tilde{p}_2^{**}(\alpha, 1-\alpha, \tilde{M})$, the lowest eligible $\tilde{p}_2$. The largest eligible $\tilde{p}_2$, $\tilde{p}_2^*(\alpha, (1+\lambda)(1-\alpha), \tilde{M})$, is simultaneously the lower bound of branch III. At that $\tilde{p}_2$, by the construction of condition (5.30), $x_1 = r_1$ and $x_2 < r_2$. Since $x_2 = (\tilde{M} - r_1)/\tilde{p}_2$, we get that $(\tilde{M} - r_1)/\tilde{p}_2 < r_2 \Leftrightarrow \tilde{M} < r_1 + \tilde{p}_2 r_2$ at $\tilde{p}_2^*(\alpha, (1+\lambda)(1-\alpha), \tilde{M})$. Hence, the interval $[\tilde{p}_2^{**}(\alpha, 1-\alpha, \tilde{M}), \tilde{p}_2^*(\alpha, (1+\lambda)(1-\alpha), \tilde{M})]$ includes the value $\tilde{p}_2$ at which $\tilde{M} = r_1 + \tilde{p}_2 r_2$. The segment $[\tilde{p}_2^{**}(\alpha, 1-\alpha, \tilde{M}), (\tilde{M} - r_1)/r_2]$ is contained in condition (5.41), hence for that segment consuming bundle **b** is optimal; the segment $[(\tilde{M} - r_1)/r_2, \tilde{p}_2^*(\alpha, (1+\lambda)(1-\alpha), \tilde{M})]$ is contained in condition (5.42), hence for this segment consuming bundle **a** is optimal.

Claim 5

If $\tilde{M}^*_{\text{III}} < \tilde{M}$ ("case E"), then the agent chooses sequentially the branches/bundles II – **a** – I – **b** – III in the course of increasing $\tilde{p}_2$ continuously on the interval $(0,\infty)$. This translates to the following demand function:

$$\mathbf{x}^{\text{E}}(\tilde{p}_2,\tilde{M}) =$$

$$\text{(A.59)}\quad \begin{cases} \mathbf{x}^{\text{II}}(\tilde{p}_2,\tilde{M}) & \text{if } \tilde{p}_2 < \left(\dfrac{(1-\alpha)^\sigma}{[(1+\lambda)\alpha]^\sigma \tilde{M}/r_1 - [(1+\lambda)\alpha]^\sigma}\right)^{1/(\sigma-1)} \\ (r_1,(\tilde{M}-r_1)/\tilde{p}_2) & \text{if } \left(\dfrac{(1-\alpha)^\sigma}{[(1+\lambda)\alpha]^\sigma \tilde{M}/r_1 - [(1+\lambda)\alpha]^\sigma}\right)^{1/(\sigma-1)} \le \tilde{p}_2 < \left(\dfrac{(1-\alpha)^\sigma}{\alpha^\sigma \tilde{M}/r_1 - \alpha^\sigma}\right)^{1/(\sigma-1)} \\ \mathbf{x}^{\text{I}}(\tilde{p}_2,\tilde{M}) & \begin{array}{l}\text{if } \tilde{p}_2 \ge \left(\dfrac{(1-\alpha)^\sigma}{\alpha^\sigma \tilde{M}/r_1 - \alpha^\sigma}\right)^{1/(\sigma-1)} \\ \text{and } \alpha^\sigma \tilde{p}_2^\sigma + (1-\alpha)^\sigma \tilde{p}_2 \le \dfrac{(1-\alpha)^\sigma \tilde{M}}{r_2}\end{array} \\ (\tilde{M}-\tilde{p}_2 r_2, r_2) & \begin{array}{l}\text{if } \alpha^\sigma \tilde{p}_2^\sigma + (1-\alpha)^\sigma \tilde{p}_2 > \dfrac{(1-\alpha)^\sigma \tilde{M}}{r_2} \\ \text{and } \alpha^\sigma \tilde{p}_2^\sigma + [(1+\lambda)(1-\alpha)]^\sigma \tilde{p}_2 \le \dfrac{[(1+\lambda)(1-\alpha)]^\sigma \tilde{M}}{r_2}\end{array} \\ \mathbf{x}^{\text{III}}(\tilde{p}_2,\tilde{M}) & \text{if } \alpha^\sigma \tilde{p}_2^\sigma + [(1+\lambda)(1-\alpha)]^\sigma \tilde{p}_2 > \dfrac{[(1+\lambda)(1-\alpha)]^\sigma \tilde{M}}{r_2} \end{cases} ;$$

Proof. In case E, branches II and III are feasible, and I and IV are potentially feasible. Since $\tilde{M} > \tilde{M}^*_{\text{III}} > \tilde{M}^*_{\text{I}}$, an interior solution on branch IV is excluded, but it will be attained on branch I.

The three feasible bundles will be chosen in the sequence II – I – III. This is due to the fact that the intervals inside which interior solutions are attained do not overlap, as has been proven above (in this appendix). Given that the intervals do not overlap, the conditions for being at an interior solution on the respective branch allow only for the sequence II – I – III: Branch II only imposes an upper bound on $\tilde{p}_2$ —in case E, the first part of condition (5.29)—while branch I imposes an upper as well as a lower bound—condition (5.28)—and branch III imposes only a lower bound—in case E, the second part of condition (5.30).

It now has to be shown that condition (5.40) describes exactly the gap between the intervals for which interior solutions on the branches II and I are attained and that (5.41) describes exactly the gap between the intervals for which interior solutions on the branches I and III are attained.

As far as the gap between branches II and I is concerned, it becomes obvious that in case E, the conditions that have to fulfilled are the very same as in the gap between branches II and I in case D. Therefore, the proof that in this gap consuming bundle **a** is optimal is the same as in the proof of Claim 4.

Turning to the gap between branches I and III, it can be stated that it is obvious that the upper bound on $\tilde{p}_2$ allowing for an interior solution on branch I, $\tilde{p}_2^{**}(\alpha, 1-\alpha, \tilde{M})$, is the same as the lower bound of the interval inside which bundle **b** is optimal according to (5.41). It is equally obvious that the upper bound of the interval inside which **b** is optimal, $\tilde{p}_2^{**}(\alpha, (1+\lambda)(1-\alpha), \tilde{M})$, is also the lower bound of branch III.

It remains to be shown that inside the interval $[\tilde{p}_2^{**}(\alpha, 1-\alpha, \tilde{M}),$ $\tilde{p}_2^{**}(\alpha, (1+\lambda)(1-\alpha), \tilde{M})]$, the requirement $\tilde{M} \geq r_1 + \tilde{p}_2 r_2$ is fulfilled so that the entire condition (5.41) holds. This is indeed the case, because the highest eligible $\tilde{p}_2$, i.e. $\tilde{p}_2 = \tilde{p}_2^{**}(\alpha, (1+\lambda)(1-\alpha), \tilde{M})$, is simultaneously the lower bound of the interior solution on branch III. At that $\tilde{p}_2$, by the construction of condition (5.30), $x_2 = r_2$ and $x_1 \geq r_1$. Since $x_1 = \tilde{M} - \tilde{p}_2 r_2$, we get that $\tilde{M} - \tilde{p}_2 r_2 \geq r_1 \Leftrightarrow \tilde{M} \geq r_1 + \tilde{p}_2 r_2$.

Proposition 3

The demand functions for good 1 and for good 2 are continuous in $\tilde{p}_2$ and in $\tilde{M}$.

Proof. The five cases A–E introduced in Section 5.2.6 define demand functions for all non-negative values of $\tilde{p}_2$ and $\tilde{M}$. Furthermore, in the framework of each case, a unique branch or bundle is assigned to each non-negative price–income combination $(\tilde{p}_2, \tilde{M})$. Therefore, the demand function combined over all cases A–E covers the entire non-negative quadrant of the real plane with axes $\tilde{p}_2$ and $\tilde{M}$.

Under these circumstances, proving that the demand functions for good 1 and good 2 are continuous in $\tilde{p}_2$ as well as in $\tilde{M}$ can be done in two steps:

1. For every case A–E, it must be shown that the demand functions are continuous in $(\tilde{p}_2, \tilde{M})$.
2. Afterwards, it has to be proven that when switching from one case to the next, i.e. when $\tilde{M}$ equals r_1, $\tilde{M}^*_{\text{II}}$, $\tilde{M}^*_{\text{I}}$, or $\tilde{M}^*_{\text{III}}$, no discontinuities arise.

Step 1: Using the proofs of the five claims of Proposition 2, it can be shown that the demand functions of goods 1 and 2 are indeed continuous in $(\tilde{p}_2, \tilde{M})$:

- In case A, the branches/bundles chosen are, ordered by increasing $\tilde{p}_2$, II – **b** – IV. It has been shown in the first subsection ("Arguments common to the proofs of all >ve claims") of the proof of Proposition 2 that in case A at the upper bound of branch II and at the lower bound of branch IV, bundle **b** is consumed. This is equal to the bundle consumed in the gap between

branches II and IV.
Therefore, the demand for both goods is continuous in $\tilde{p}_2$ and $\tilde{M}$.

- In case B, the branches/bundles chosen are, ordered by increasing $\tilde{p}_2$, II – **b** – IV – **a** – III. It has been shown in the first subsection of the proof of Proposition 2 that in case B at the upper bound of branch II and at the lower bound of branch IV, bundle **b** is consumed. This is equal to the bundle consumed in the gap between branches II and IV.
In case B at the upper bound of branch IV and at the lower bound of branch III, bundle **a** is consumed. This is equal to the bundle consumed in the gap between branches IV and III.
Therefore, the demand for both goods is continuous in $\tilde{p}_2$ and $\tilde{M}$.
- In case C, the branches/bundles chosen are, ordered by increasing $\tilde{p}_2$, II – **a** – **b** – IV – **a** – III. It has been shown in the first subsection of the proof of Proposition 2 that in case C at the upper bound of branch II, bundle **a** is consumed, while at the lower bound of branch IV, bundle **b** is consumed. In the proof of Claim 3 of Proposition 2 it is shown that from the upper bound of branch II to $\tilde{p}_2 = (\tilde{M} - r_1)/r_2$, bundle **a** is consumed. Between $\tilde{p}_2 = (\tilde{M} - r_1)/r_2$ and the lower bound of branch IV, bundle **b** is consumed. Therefore, the demand functions are continuous at the upper bound of branch II and at the lower bound of branch IV. At $\tilde{p}_2 = (\tilde{M} - r_1)/r_2$, the bundles **a** and **b** are identical, hence the demand functions are also continuous at $\tilde{p}_2 = (\tilde{M} - r_1)/r_2$.
In case C at the upper bound of branch IV and at the lower bound of branch III, bundle **a** is consumed. This is equal to the bundle consumed in the gap between branches IV and III.
Therefore, the demand for both goods is continuous in $\tilde{p}_2$ and $\tilde{M}$.
- In case D, the branches/bundles chosen are, ordered by increasing $\tilde{p}_2$, II – **a** – I – **b** – **a** – III. It has been shown in the first subsection of the proof of Proposition 2 that in case D at the upper bound of branch II and at the lower bound of branch I, bundle **a** is consumed. This is equal to the bundle consumed in the gap between branches II and I.
In case D at the upper bound of branch I, bundle **b** is consumed, while at the lower bound of branch III, bundle **a** is consumed. In the proof of Claim 4 of Proposition 2 it is shown that from the upper bound of branch I to $\tilde{p}_2 = (\tilde{M} - r_1)/r_2$, bundle **b** is consumed. Between $\tilde{p}_2 = (\tilde{M} - r_1)/r_2$ and the lower bound of branch III, bundle **a** is consumed. Therefore, the demand functions are continuous at the upper bound of branch I and at the lower bound of branch III. At $\tilde{p}_2 = (\tilde{M} - r_1)/r_2$, the bundles **b** and **a** are identical, hence the demand functions are also continuous at $\tilde{p}_2 = (\tilde{M} - r_1)/r_2$.
Therefore, the demand for both goods is continuous in $\tilde{p}_2$ and $\tilde{M}$.

- In case E, the branches/bundles chosen are, ordered by increasing $\tilde{p}_2$, II – **a** – I – **b** – III. It has been shown in the first subsection of the proof of Proposition 2 that in case E at the upper bound of branch II and at the lower bound of branch I, bundle **a** is consumed. This is equal to the bundle consumed in the gap between branches II and I.
  In case E at the upper bound of branch I and at the lower bound of branch III, bundle **b** is consumed. This is equal to the bundle consumed in the gap between branches I and III.
  Therefore, the demand for both goods is continuous in $\tilde{p}_2$ and $\tilde{M}$.

Thus, for each of the five cases A–E, the demand functions of both goods are continuously defined.

Step 2: It remains to be shown that also when switching from one case to the other, no discontinuities occur.

- Switch between cases A and B, i.e. $\tilde{M} = r_1$:
  In case A the attained branches and bundles are II – **b** – IV, while in case B they are II – **b** – IV – **a** – III. At $\tilde{M} = r_1$ the intervals for attaining branch III or bundle **a** in formula (A.56) are not defined. Therefore, for $\tilde{M} \searrow r_1$ the agent will be on branch II or IV or consume bundle **b**. These branches and the bundle are those that are attained in case A. Hence, the switch from case A to B and vice versa occurs at a constellation for which the continuity in $\tilde{M}$ has already been shown above.
- Switch between cases B and C, i.e. $\tilde{M} = \tilde{M}^*_{\text{II}}$:
  In case B the attained branches and bundles are II – **b** – IV – **a** – III, while in case C they are II – **a** – **b** – IV – **a** – III. At $\tilde{M} = \tilde{M}^*_{\text{II}}$ the two values $\tilde{p}^*_2((1+\lambda)\alpha, 1-\alpha, \tilde{M})$ and $\tilde{p}^{**}_2((1+\lambda)\alpha, 1-\alpha, \tilde{M})$ coincide, so that at the upper bound of branch II, the bundle (r_1, r_2) is consumed. That is, at the upper bound of branch II, it holds that $r_1 + \tilde{p}_2 r_2 = \tilde{M}$. This is the condition at which in case C the switch between bundles **a** and **b** occurs. Since at $r_1 + \tilde{p}_2 r_2 = \tilde{M}$ the two bundles are identical, at that price–income combination $(\tilde{p}_2, \tilde{M})$ no discontinuity arises. For all other price–income combinations, the branches and the bundle attained are identical in the two cases. Hence, the switch from case B to C and vice versa occurs at a constellation for which the continuity in $\tilde{M}$ has already been shown above.
- Switch between cases C and D, i.e. $\tilde{M} = \tilde{M}^*_{\text{I}}$:
  In case C the attained branches and bundles are II – **a** – **b** – IV – **a** – III, while in case D they are II – **a** – I – **b** – **a** – III. At $\tilde{M} = \tilde{M}^*_{\text{I}}$ the conditions for being at an interior solution on the two branches I and IV collapse into a single combination $(\tilde{p}_2, \tilde{M})$ at which the bundle (r_1, r_2) is consumed. Therefore, at this combination, it holds that $r_1 + \tilde{p}_2 r_2 = \tilde{M}$. This is the condition at which in case D the switch between bundles **a** and **b** occurs. Since at $r_1 + \tilde{p}_2 r_2 = \tilde{M}$

the two bundles are identical, at that price–income combination $(\tilde{p}_2, \tilde{M})$ no discontinuity arises. Furthermore, at that price–income combination, the only existing interior solution on branch I is attained. Hence, for $\tilde{M} \searrow \tilde{M}_{\mathrm{I}}^{*}$ branch I and bundle **b** both converge to bundle **a**, and the set of attained brunches/bundles in case D reduces to II – **a** – III. The same argumentation holds for case C: For $\tilde{M} \nearrow \tilde{M}_{\mathrm{I}}^{*}$ branch IV and bundle **b** both converge to bundle **a**, and the set of attained brunches/bundles in case C reduces to II – **a** – III as well. For each case separately the continuity in $\tilde{M}$ has already been shown above. Therefore, the switch from case C to D and vice versa causes no discontinuities.

- Switch between cases D and D, i.e. $\tilde{M} = \tilde{M}_{\mathrm{III}}^{*}$:
  In case D the attained branches and bundles are II – **a** – I – **b** – **a** – III. while in case E they are II – **a** – I – **b** – III. At $\tilde{M} = \tilde{M}_{\mathrm{III}}^{*}$ the two values $\tilde{p}_2^{*}(\alpha, (1+\lambda)(1-\alpha), \tilde{M})$ and $\tilde{p}_2^{**}(\alpha, (1+\lambda)(1-\alpha), \tilde{M})$ coincide, so that at the lower bound of branch III, the bundle (r_1, r_2) is consumed. That is, at the lower bound of branch III, it holds that $r_1 + \tilde{p}_2 r_2 = \tilde{M}$. This is the condition at which in case D the switch between bundles **b** and **a** occurs. Since at $r_1 + \tilde{p}_2 r_2 = \tilde{M}$ the two bundles are identical, at that price–income combination $(\tilde{p}_2, \tilde{M})$ no discontinuity arises. For all other price–income combinations, the branches and the bundle attained are identical in the two cases. Hence, the switch from case D to E and vice versa occurs at a constellation for which the continuity in $\tilde{M}$ has already been shown above.

Gathering all results from step 1 and step 2, it can be concluded that the demand for both goods is continuous in $\tilde{p}_2 > 0$ and $\tilde{M} \geq 0$.