Persistent inequality when learning requires a minimal standard of living*

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Abstract

This paper studies the persistence of wealth and utility inequality in a dynamic model of skill acquisition when efficient learning requires a minimal standard of living. We assume that credit markets are complete and individuals are rational, perfectly altruistic, and maximize dynastic utility. The main result is that if the minimal standard of living is sufficiently large, at any stationary equilibrium without intergenerational mobility (which always exists) there are 'poor', unskilled and 'rich', skilled dynasties. Members of rich dynasties inherit more from their parents than members of poor dynasties. The former acquire skill, while the latter remain unskilled, and—most importantly—members of rich dynasties also enjoy strictly higher utility than members of poor dynasties. When allowing for intergenerational mobility, we show that in steady state equilibrium with unequal lifetime labour earnings and sufficiently large discount rates the set of dynasties inhabiting the economy can be partitioned into three stable classes. Dynasties never switch classes and class boundaries do not converge.

Keywords: Inequality, Skill Acquisition, Standard of Living, Bequests.

JEL Classification: D33, D91, J24, J31, J62.

1 Introduction

This paper studies the persistence of wealth and utility inequality in a dynamic model of skill acquisition. There are two professions, both essential for final goods production, but only one requires a period of exclusive education. Individuals are assumed to live for two periods, to be perfectly altruistic, and to maximize dynastic utility. Credit markets

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are complete; in particular, independent of their initial wealth all agents can borrow the resources necessary to cover training costs at identical rates of interest. The central assumption of the present paper is that learning is only effective if the standard of living enjoyed during the period of learning is sufficiently high. This consumption requirement may force some individuals to deviate from their first-best consumption path, i.e. the path that would be chosen in the absence of any minimal standard of living constraint. Since poor individuals with little inherited wealth are more likely to be affected by the consumption constraint than rich individuals, the loss in lifetime utility due to the restriction (which add to the opportunity cost of foregone earnings), is higher for the poor than for the rich. Lifetime labour earnings are therefore never negatively, but may well be positively correlated with total consumption.

A main result of this paper is that for sufficiently large minimal standard of living requirements there are 'poor', unskilled and 'rich', skilled dynasties in any stationary equilibrium without intergenerational mobility. Moreover, members of rich families also enjoy strictly higher lifetime utility than members of poor dynasties. Thus, at stationary equilibrium with both types of labour being active, the economy inevitably exhibits persistent inequality in lifetime utility.

The intuition for this result is as follows: Assume the extreme case that the consumption constraint is arbitrarily small and that all agents inherit an equal amount of assets. Then in stationary equilibrium lifetime labour earnings are identical. We refer to that kind of equilibrium as an egalitarian equilibrium. In an egalitarian equilibrium obviously all agents enjoy identical first-period consumption. Now consider a minimal standard of living requirement that is larger than this first-period first-best consumption. All else equal, individuals attending school are not compensated for their loss in lifetime utility that is due to the imperfect consumption smoothing and therefore nobody attends school. Thus, there is no egalitarian equilibrium for sufficiently large minimal standard of living requirements.

It is instructive to compare this result with the Ramsey-Cass-Koopmans framework in which persistent inequality of wealth and utility may prevail as well. As in the present paper, in this model capital markets are perfect, agents are perfectly altruistic and there are no decreasing returns on investments at the individual level. Then, independent of their inherited wealth, all individuals face the same Euler equation. Even though the aggregate steady state wealth level is fully specified in this model, the personal wealth distribution is indeterminate (see, e.g., Chatterjee 1994, Caselli and Ventura 2000). In particular, it can but does not have to be unequal.

Most articles on long-run wealth and earnings inequality attribute persistent inequality to incomplete credit markets (e.g., Galor and Zeira 1993, Aghion and Bolton 1997, Freeman 1996, Ljungqvist 1993). In fact, Mookherjee and Ray (2003) show in a framework in which 'professions' are the only assets that can be inherited, that the absence of credit markets makes steady state inequality inevitable if in every steady state at least two professions with differing training costs are active. In such a setting, decreasing marginal utility implies that the willingness to invest in the education of the offspring depends positively on total income and therefore in every steady state within every dynasty all generations work within the same profession. Moreover, since all dynasties earning higher wages have always the opportunity to leave their child a profession with lower training costs, utility inequality is inevitable in long-run equilibrium.

For this strong result to hold the assumption of completely missing capital markets is necessary. However, as Mookherjee and Ray (2002) show, this assumption can be somewhat relaxed and the inevitability of long-run equilibrium is still obtained if one assumes a sufficiently large variety of training costs. In this model, even though alternative forms of assets are introduced, the inability of parents to borrow against their children's earnings implies the necessity of steady state inequality as long as in steady state total assets in the economy are too small to be able to offset the differing labour earnings.

With respect to its central assumption, the present paper is closely related to the efficiency wage literature (Baland and Ray 1991, Bliss and Stern 1978, Dasgupta and Ray 1986, Dasgupta and Ray 1987, Ray and Streufert 1993). These articles assume a minimal nutritional requirement for the productivity of physical labour and show how this may lead to efficiency-wage type of unemployment. Here, we assume a minimal standard of living for the 'productivity' of learning.

While the bulk of the paper deals with stationary equilibria without intergenerational mobility, in Section 5 we weaken this requirement of stationarity by allowing for intergenerational mobility without however giving up the assumption of constant aggregate factor intensities. We show that in every steady state equilibrium of this more general type in which lifetime labour earnings are unequally large and the discount factor exceeds one-half, society is partitioned in up to three stable classes of dynasties. "Upper class" dynasties are rich, educated, and enjoy high utility in each period, "middle class" dynasties are less rich, less happy, and permanently switch between levels of educational attainment, and "lower class" dynasties are even poorer, even less happy, and never attend school. Thus mobility is limited in two ways: First, no dynasty ever switches classes and intergenerational mobility only occurs in the middle class. Second, the general result remains valid that in each period there is necessarily inequality in income and utility.

The paper is structured as follows: Section 2 presents the model. The individual maximization problem is solved in Section 3. Section 4 characterizes stationary equilibria without intergenerational mobility and shows that such equilibria indeed always exist. Section 5 deals with more general steady state equilibria that in particular allow for intergenerational mobility. Section 6 concludes.

2 The model

2.1 Dynasties

An individual lives for two periods. When it enters the second period, one offspring enters the economy. An infinite sequence of offsprings constitute a dynasty. There is a mass of L dynasties at any date. In the sequel we index individuals by the period of their entrance into the economy. When entering the economy, every individual owns zero assets. In period $t+1$ generation t inherits a bequest of b_t and in period $t+2$ it leaves a bequest to generation $t+1$ of b_{t+1}. The capital market is perfect such that all individuals are free to borrow or lend capital at the market interest rate r_t. There is no restriction on the sign of b_t and b_{t+1}, but lenders do not give credit to a household whose descendants will never be able to pay off their dynasty's debt (no-Ponzi condition).

Generation t enjoys consumption of c_{1t} and c_{2t} in its first and, respectively, second period of life from which it derives utility $U(c_{1t}, c_{2t}) = u(c_{1t}) + \delta u(c_{2t})$, where $0 < \delta < 1$, $u'(c) > 0$, and $u''(c) < 0$. Moreover, we assume that $u(c)$ strongly penalizes zero consumption (such as, e.g., $u(c) = \ln c$), that is to say, households prefer positive consumption in both periods to any consumption pattern with zero consumption in one period. Every individual cares about the well-being of its offspring and expects its descendants to share this preference. In particular, generation t seeks to maximize its perfectly altruistic dynastic utility function

$$\sum_{\tau=t}^{\infty} \delta^{(\tau-t)} U(c_{1\tau}, c_{2\tau}). \tag{1}$$

When young, every individual has to decide whether to join the unskilled labour force immediately or to attend school in the first period and to work as a skilled worker (manager) in the second. School is free, but no labour income is earned while the young individual is attending school. Every worker supplies inelastically one unit of labour. All newborns share an equal ability to finish school. Schooling is however only effective if a minimum amount of consumption, $\widetilde{c}$, is maintained during the schooling period.

2.2 Production

There are three factors of production: skilled labour S, unskilled labour U and physical capital K. U_t denotes the mass of unskilled workers and S_t the mass of skilled workers at period t. Total savings of the household sector are used to increase the stock of capital. For simplicity we abstract from capital depreciation. The production function $F(S,U,K)$ of the economy is Cobb-Douglas:[1]

$$F(S,U,K) = S^{\alpha}U^{\beta}K^{1-\alpha-\beta},\ \alpha,\beta > 0,\ \alpha+\beta < 1. \tag{2}$$

Define $u \overset{\text{def}}{=} U/K$ and $s \overset{\text{def}}{=} S/K$. The economy is in short-run equilibrium in every period, i.e., at all dates all markets clear and factor prices are determined by the respective first-order conditions.

2.3 Stationary equilibrium

The main part of the present paper studies the existence and the properties of stationary equilibria with rational expectations and zero intergenerational mobility in educational attainment. A stationary equilibrium is defined as a sequence of short-run equilibria with constant factor intensities (u_t, s_t) at which all generations of all dynasties have perfect foresight and choose the same profession as their ancestors did. In stationary equilibrium therefore all factor prices are constant and are denoted by w_S, w_U, and r.

Let $\overline{w}_S$ and $\overline{w}_U$ denote lifetime labour incomes earned by skilled and unskilled labour, respectively, i.e. $\overline{w}_S \overset{\text{def}}{=} w_S$ and $\overline{w}_U \overset{\text{def}}{=} (2+r)\,w_U$. Two important properties of stationary equilibria follow immediately from the fact that at equilibrium both types of labour are necessary to produce final output. First, $\overline{w}_U$ cannot exceed $\overline{w}_S$ for otherwise no individual would attend school. Second, it is profitable for *all* individuals—independent of their bequeathed wealth—to attend school if the wage gap $\overline{w}_S - \overline{w}_U$ is sufficiently large such that the minimal consumption $\widetilde{c}$ during the schooling period can be financed by this difference in lifetime labour earnings alone, i.e., if $\overline{w}_S - \overline{w}_U \geq (1+r)\,\widetilde{c}$. This proves the first lemma:

Lemma 1 *In stationary equilibrium* $0 \leq \overline{w}_S - \overline{w}_U < (1+r)\,\widetilde{c}$.

3 The individual problem

We solve the problem of generation t in two steps. First, we transfer the problem to maximize (1) into a more conventional problem of maximizing discounted lifetime utility over a sequence of net transfers. Let T_t denote net transfers generation t receives from its dynasty if it inherits a bequest b_t and leaves a bequest b_{t+1}:

$$T_t = (1+r)\,b_t - b_{t+1}. \tag{3}$$

[1] We use the Cobb-Douglas case simply to guarantee that both skilled and unskilled labour are employed all the time.

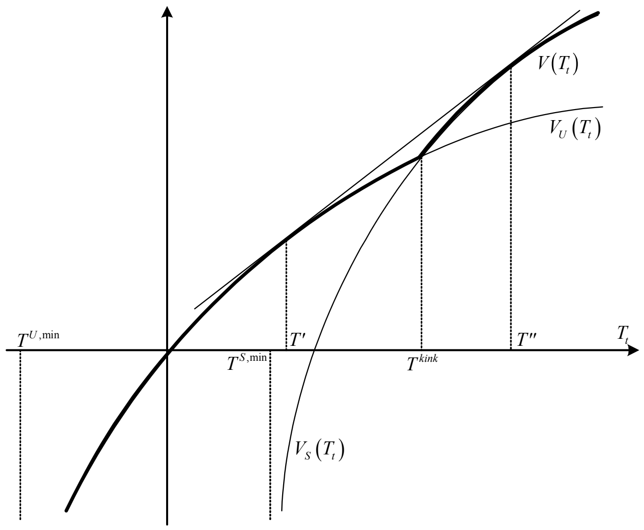

Figure 1:

Then for every T_t we determine the optimal consumption pattern of this agent in the case that it spends its life working as an unskilled worker and, separately, in the case that it decides to attend school. In a second step we characterize the optimal sequence of net transfers and, hence, bequests that maximizes dynastic utility.

3.1 The schooling problem for given bequests

Let $V_U(T_t)$ denote the solution to the maximization problem of an unskilled worker of generation t receiving net transfers T_t:

$$\max_{c_{1t},c_{2t}\in\mathbb{R}_+} U(c_{1t},c_{2t}) \text{ subject to } c_{1t}+(1+r)\,c_{2t}=T_t+\overline{w}_U.$$

This problem has a solution if and only if non-negative consumption can be established in both periods, that is, if and only if $T_t \geq T^{U,\min} \overset{\text{def}}{=} -\overline{w}_U$. Similarly, transfers T_t must not be smaller than $T^{S,\min} \overset{\text{def}}{=} (1+r)\,\widetilde{c}-\overline{w}_S$ in order to enable generation t to attend school. For any given $T_t \geq T^{S,\min}$ let $V_S(T_t)$ denote the solution to the student-manager problem:

$$\max_{c_{1t},c_{2t}\in\mathbb{R}_+} U(c_{1t},c_{2t}) \text{ subject to } c_{1t}+(1+r)\,c_{2t}=T_t+\overline{w}_S,\ c_{1t}\geq\widetilde{c}.$$

Both $V_U(\cdot)$ and $V_S(\cdot)$ are continuously differentiable and, importantly, strictly concave because $U(\cdot,\cdot)$ is strictly concave (see Figure 1). Let $V(\cdot)$ denote the upper envelope of $V_U(\cdot)$ and $V_S(\cdot)$. Notice that $V(\cdot)$ is not necessarily strictly concave, even though both $V_U(\cdot)$ and $V_S(\cdot)$ are. For later reference we denote by T'' and T' the upper and lower bound of the interval for which $V(\cdot)$ is not identical with its convex hull.

We want to determine whether, for given net transfers T_t and labour earnings $\overline{w}_U$ and $\overline{w}_S$, it is profitable for generation t of a given dynasty to attend school $(V_S(T_t) > V_U(T_t))$, to work as an unskilled worker ($V_S(T_t) < V_U(T_t)$), or whether it is indifferent over both types of labour ($V_S(T_t) = V_U(T_t)$). According to the incentive constraints of Lemma 1, we have to distinguish two cases:

First, consider the case $\overline{w}_S = \overline{w}_U$. Denote the minimal net transfers that just suffice for an unconstraint first-best consumption pattern of a student-manager as $\widetilde{T}$. Then generation t clearly chooses to provide unskilled labour in both periods of its life whenever net transfers are smaller than $\widetilde{T}$ because with equal lifetime earnings there is no compensation for the distortion caused by the consumption constraint. Whenever the consumption constraint does not bind, however, which is the case for all net transfers $T_t \geq \widetilde{T}$, individuals are indifferent over working as skilled or unskilled workers.

Second, consider the case $0 < \overline{w}_S - \overline{w}_U < (1+r)\widetilde{c}$. On the one hand, for $T_t = T^{S,\min}$ consumption of generation t is positive in both periods when working as an unskilled, but zero in the first period when choosing to attend school. Thus, for small net transfers agents choose not to attend school. On the other hand, for all $T_t \geq \widetilde{T}$ generation t clearly attends school because $\overline{w}_S > \overline{w}_U$ and the consumption constraint does not bind. Continuity of $V_U(\cdot)$ and $V_S(\cdot)$ therefore implies the existence of a (unique) level of transfers, say T^{kink}, such that $V_U\left(T^{\text{kink}}\right) = V_S\left(T^{\text{kink}}\right)$ (see Figure 1).[2] Thus for all $T_t \in \left[T^{U,\min}, T^{\text{kink}}\right)$ generation t supplies unskilled labour, for all $T_t > T^{\text{kink}}$ it attends school.

3.2 Optimal bequests

Optimal net transfers of generation t are obtained by determining a feasible sequence of net transfers that solves for

$$\max_{\langle T_\tau \rangle_{\tau=t}^{\infty}} \Sigma_{\tau=t}^{\infty} \delta^{\tau-t} V(T_\tau) \tag{4}$$

while respecting the no-Ponzi condition. This sequence necessarily satisfies the Euler equation

$$V'(T_\tau) = \delta(1+r)V'(T_{\tau+1}) \text{ for all } \tau \geq t.$$

The Euler equation reveals that in stationary equilibrium $\delta(1+r)$ must be equal to one. Otherwise in the long run one type of labour would not be supplied. So in stationary equilibrium the optimal sequences of marginal indirect utility are always stationary.

Stationarity of marginal indirect utility does however not imply that in stationary equilibrium the sequence of net transfers is stationary as well, for $V(\cdot)$ is in general non-concave. But since a non-stationary sequence of net transfers implies that the sequence of

[2] Uniqueness of T^{kink} is implied by the fact that whenever both indirect utility functions intersect, $V_S(\cdot)$ cuts $V_U(\cdot)$ from below. To see this notice that $\partial V_i/\partial T = \partial U(c_1, c_2)/\partial c_2$, $i = S, U$, and that at T^{kink} both the total value of consumption of the educated is greater than the respective value of the unskilled workers and the consumption constraint still binds.

schooling decisions is not stationary either, we can state the next lemma:

Lemma 2 *Suppose the economy is in stationary equilibrium. Then the optimal sequence of net transfers is stationary for all generations of all dynasties. In particular, all agents leave a bequest of the same amount as they inherited themselves from their parent.*

The last statement is a consequence of the no-Ponzi and a transversality condition (see, e.g., Ekeland and Scheinkman 1986).

4 Stationary equilibrium

4.1 Characterization

As outlined in the introduction, with complete credit markets and no minimal standard of living required for learning, in long-run equilibrium lifetime labour earnings are the same in both professions. Moreover, *all* individuals, regardless of the level of inherited wealth, are indifferent over whether to become an unskilled worker or a student-manager. As one might expect, these conclusions are robust to small increases of the minimal standard of living. But how large can $\widetilde{c}$ maximal be for an egalitarian equilibrium to exist? For perfect utility equality to prevail in stationary equilibrium, all dynasties must receive identical interest payments since the costs (in utility terms) of obtaining education are never higher for the rich than for the poor. In a stationary equilibrium with equal lifetime labour earnings the consumption constraint therefore does not bind anybody.

Define $\widetilde{c}^{\min}$ as the upper bound on the minimal standard of living such that egalitarian equilibria exist. The assumption of perfect altruism implies that with constant factor prices first- and second-period consumption are identical whenever the consumption constraint does not bind. In egalitarian equilibrium therefore per capita consumption equals per capita production.[3] This implies that an egalitarian equilibrium exists if and only if $\widetilde{c} \leq \widetilde{c}^{\min} = F\left(S^{*}, U^{*}, K^{*}\right)/2L = F\left(s^{*}, u^{*}, 1\right)/\left(2s^{*} + u^{*}\right)$, where the skilled and unskilled intensities (s^{*}, u^{*}) are appropriately chosen such as to satisfy the equilibrium conditions $\overline{w}_S = \overline{w}_U$ and $(1 + r)\,\delta = 1$.

For every $\widetilde{c}$ that is indeed *greater* than $\widetilde{c}^{\min}$ an egalitarian equilibrium cannot exist because the consumption constraint would bind those individuals attending school who however would not be compensated for the deviation from their optimal consumption pattern. In fact, in case of $\widetilde{c} > \widetilde{c}^{\min}$ for $\overline{w}_S = \overline{w}_U$ to hold in stationary equilibrium the average bequest made by all skilled dynasties has to strictly exceed the average bequest

[3] Recall that capital does not depreciate and because of $\overline{w}_S = \overline{w}_U$, agents leave bequests as large as they receive (Lemma 2) such that total savings are zero.

amounts made by all unskilled dynasties so that because of the relatively high interest payments the skilled dynasties are left unaffected by the consumption constraint and thus are willing to attend school.

So far we have studied the (im)possibility of utility equality in stationary equilibria with *equal* lifetime labour earnings. With a positive consumption constraint, however, $\overline{w}_S > \overline{w}_U$ may hold in stationary equilibrium and one may wonder how the resulting kink in the indirect lifetime utility function $V(\cdot)$ affects our results on the inevitability of wealth and utility inequality.[4] As argued in Section 3.1, if $V(\cdot)$ has a kink, skilled and unskilled individuals enjoy identical lifetime utility if and only if every agent of every dynasty leaves assets of $b^{\text{kink}} = T^{\text{kink}}/r$ to its child. However, exactly because of the kink of $V(\cdot)$ at T^{kink} no dynasty will choose such a sequence of bequests. An individual whose inherited wealth is in a neighbourhood of b^{kink} would rather choose to leave a bequest that is either smaller or larger than b^{kink} instead of leaving a bequest of exactly b^{kink}. But this cannot occur in stationary equilibrium, as claimed in Lemma 2.

The following inequality theorem summarizes the above characterization of stationary equilibria for sufficiently large standard of living requirements.

Theorem 3 *Suppose the minimal standard of living required for schooling, $\widetilde{c}$, is larger than $\widetilde{c}^{\min}$. Then, in stationary equilibrium the average bequest made by all skilled dynasties is strictly greater than the average bequest made by all unskilled dynasties. There is a positive number of poor and unskilled dynasties and a positive number of rich and skilled dynasties. Lifetime and dynastic utility of the rich and skilled individuals is strictly larger than that of the poor and unskilled. Moreover, if the difference in lifetime labour earnings, $\overline{w}_S - \overline{w}_U$, is positive, there is a non-degenerated interval $\left(b^{kink} - \Delta, b^{kink} + \Delta\right)$, $\Delta > 0$, dividing the society into dynasties with poor uneducated members and constant bequests smaller than $b^{kink} - \Delta$ and dynasties with rich educated members with constant bequests larger than $b^{kink} + \Delta$.*

4.2 Existence

The following conditions are necessary and sufficient for the existence of stationary equilibria: factor intensities, say (s^{**}, u^{**}), imply that $(1+r)\,\delta = 1$ and that the incentive constraints of Lemma 1 are satisfied. (The production technology (2) guarantees that such factor intensities indeed exist.) Furthermore, the wealth distribution has to support a stationary bequest distribution as well as educational choices corresponding to (s^{**}, u^{**}).

[4] Notice that in order to prove the necessity of wealth and utility inequality in stationary equilibria with $\overline{w}_S > \overline{w}_U$ we do not need to assume that $\widetilde{c} > \widetilde{c}^{\min}$. See also the discussion in Section 5.

Proving existence for *any* $\widetilde{c}$ is complicated by the fact that $\overline{w}_S = \overline{w}_U$ can impossibly hold at stationary equilibrium if $\widetilde{c}$ is sufficiently large[5] and that for $\overline{w}_S > \overline{w}_U$ the indirect lifetime utility function $V(\cdot)$ is non-concave (see Figure 1). For any $\widetilde{c}$, it is rather straightforward to construct wealth distributions for which the poor uneducated dynasties choose to remain uneducated. It is somewhat more cumbersome to show that for any $\widetilde{c}$ there are factor intensities for which total wealth can be split among rich and poor such that also the rich have no incentive to switch between attending school in one generation and not attending school in a later period—a problem which does not arise in a framework with strictly concave lifetime utility. A constructive proof of the following proposition is provided in Funk and Vogel (2003):

Proposition 4 *For all parameter specifications of the model there is a non-empty set of (s,u) such that all these (s,u) are the factor intensities of infinitely many stationary equilibria. These equilibria differ with respect to the distribution of total wealth between the rich and the poor group.*

5 Steady state equilibrium with intergenerational mobility

In the previous sections we examined properties and existence of stationary equilibria which by definition also require stationary schooling decisions within every dynasty, that is, which by definition exclude intergenerational mobility. In the present section we weaken the requirement of stationarity by allowing for intergenerational mobility and address the question how much mobility can possibly occur at a steady state equilibrium. We say that a sequence of short-run equilibria is a steady state equilibrium if factor intensities (u_t, s_t) are constant and all generations of all dynasties have perfect foresight and behave perfectly rational. Existence of such equilibria obviously follows from Proposition 4 since stationary equilibria without intergenerational mobility are a particular type of steady state equilibria. It is also easy to see that Lemma 1 generalizes to all steady state equilibria.

The present section shows that in the interesting case of a kinked lifetime utility function at steady state equilibrium mobility is limited and, as with stationary equilibria, inequality remains unavoidable. If further the discount factor δ exceeds $1/2$, at such steady state equilibrium society can be classified into up to three groups characterized by different levels of wealth, utility, and different patterns of decisions about educational

[5] As shown above, if $\overline{w}_S$ equals $\overline{w}_U$, stationary (and even egalitarian) equilibria exist for all $\widetilde{c} \leq \widetilde{c}^{\min}$. There is, however, no stationary equilibrium for $\widetilde{c}$ sufficiently large: The wealth of every dynasty providing unskilled labour cannot be smaller than $-\overline{w}_U$ and net transfers of dynasties providing skilled labour must be at least $\widetilde{T}$. The claim follows since $\widetilde{T}$ increases without bound in $\widetilde{c}$ and since total wealth is uniquely pinned down by (s^{**}, u^{**}).

attainment. As will be shown, no dynasty switches between these classes and the wealth and utility boundaries between the "upper class" and the "lower class" do not converge.

The upper and lower class are those classes already known from the previous sections. The possibility of a "middle class" is new and due to the fact that we now no longer exclude intergenerational mobility. Consider that interesting case $\overline{w}_S > \overline{w}_U$ studied in Section 3.1 in which the indirect lifetime utility function $V(\cdot)$ has a kink at T^{kink}.[6] Depending on the distribution of wealth endowments, some dynasties find it optimal to switch between different levels of net transfers and hence between different levels of educational attainment. The middle class consists of such dynasties.

Remember that the kinked shape of V results from the construction of V from two strictly concave functions V^S and V^U (see Section 3.1). Thus, whenever switching between differing amounts of net transfers maximizes an individual's dynastic utility, this dynasty will switch between *exactly* two levels of net transfers, say $\overline{T}$ and $\underline{T}$ (with $\overline{T} > \underline{T}$). Let χ_i denote an index function that indicates whether generation i chooses net transfers of $\underline{T}$ ($\chi_i = 0$) or $\overline{T}$ ($\chi_i = 1$). We may then rewrite dynastic utility of generation t of a given dynasty as

$$V\left(\underline{T}\right)\sum_{i=t}^{\infty}\delta^{i-t} + \left[V\left(\overline{T}\right) - V\left(\underline{T}\right)\right]\sum_{i=t}^{\infty}\delta^{i-t}\chi_i. \tag{5}$$

Denote the maximal feasible stationary net transfers amount this generation could choose as T^{stat} where, by the same reasoning as used to derive Lemma 2, we find that $T^{\text{stat}} = rb_t$ for generation t inheriting b_t. For a sequence of $\overline{T}$s and $\underline{T}$s to maximize dynastic utility the discounted value of such a switching sequence must be equal to the discounted value of a stationary sequence of net transfers amounts of T^{stat}. Substitution of the equilibrium condition $\delta(1+r) = 1$ into this constraint yields

$$(1-\delta)\sum_{i=t}^{\infty}\delta^{i-t}\chi_i = \frac{T^{\text{stat}} - \underline{T}}{\overline{T} - \underline{T}}. \tag{6}$$

The left-hand side of this constraint is between zero and one for all possible switching sequences of indices χ_i. So constraint (6) implies the ordering $\overline{T} > T^{\text{stat}} > \underline{T}$. Note however that the left-hand side can take on *every* number between zero and one if and only if δ is not smaller than $1/2$, that is, if in equilibrium the gross rate of interest $1+r$ does not exceed two.[7] Thus, if $1 + r \leq 2$ holds, for any given T^{stat}, $\overline{T}$, and $\underline{T}$ with $\overline{T} > T^{\text{stat}} > \underline{T}$ there always exists a sequence of $\overline{T}$s and $\underline{T}$s (and, respectively, of indices

[6] Note that factor intensities u and s can remain stationary only if both the bounds of Lemma 1 and the condition $\delta(1+r) = 1$ are satisfied for otherwise in one period one type of labour would not be supplied. Thus in steady state equilibrium V^S never dominates V^U and so in steady state equilibrium with $\overline{w}_S > \overline{w}_U$ indirect lifetime utility V is necessarily kinked.

[7] In the binary numerical system every real number can be represented by a sequence consisting only of zeros and ones. In particular, for every real number between zero and $1/(1-\delta)$ there is a sequence

χ) such that Equation 6 is satisfied. For that reason in the remaining we always assume that $\delta \geq 1/2$.[8]

Insertion of (6) into (5) yields an expression for dynastic utility in any given period as a continuous function of a dynasty's endowment T^{stat} and its chosen $\overline{T}$ and $\underline{T}$:

$$\frac{1}{1-\delta}\left[V\left(\underline{T}\right)+\left(T^{\text{stat}}-\underline{T}\right)\frac{V\left(\overline{T}\right)-V\left(\underline{T}\right)}{\overline{T}-\underline{T}}\right]$$

It follows from this expression for dynastic utility, the Euler equation, and Figure 1 that for all endowments T^{stat} with its level of lifetime utility $V\left(T^{\text{stat}}\right)$ *on* the convex hull of $V(\cdot)$ switching between differing net transfers amounts does never attain maximal dynastic utility. Vice versa, if $V\left(T^{\text{stat}}\right)$ is *not* on the convex hull of $V(\cdot)$, switching between $\underline{T}=T'$ and $\overline{T}=T''$ is chosen instead.[9] Of course, even though in every period all

$\{\chi_i\}_{i\geq 0}$ with $\chi_i \in \{0,1\}$ such that the tail-sum $\sum_{i=0}^{\infty}\delta^i\chi_i$ with $\delta = 1/2$ equals this given real number. By this analogy, it is obvious that for smaller values for δ a representation of this kind does not exist any more for every given number between zero and $1/(1-\delta)$ since all χ_i remain to be restricted to be either zero or one—independent of the given δ.

[8] The household behaviour in stationary equilibria with $1+r>2$ is far less tractable than in the case of small gross rate of interest. The reason is that almost no numbers between zero and one can be represented by the left-hand side of condition (6) since the index function χ can take on only the values zero and one. Thus, for any given T^{stat} in (T',T'') the choice of $\overline{T}$ and $\underline{T}$ depends on T^{stat} in a complicated matter and in the present paper we therefore refrain from a further exploration of this matter.

[9] In fact, in case of $\delta \geq 1/2$ for almost all T^{stat} in (T',T'') the corresponding optimal switching sequence is chaotic in the sense that it is non-periodic and exhibits sensitive dependence on initial conditions. Non-periodicity is simply due to the fact that the set of all eventually periodic sequences of indices χ is countable (see Kamihigashi 2000). Thus, for almost no T^{stat} in the interval (T',T'') there exists a *periodic* sequence of T's and T''s satisfying Equation 6. Sensitive dependence on initial conditions is implied by the fact that $1+r>1$: However close is inherited wealth of two dynasties at some point in time, if bequests are not identically large, in finite time their bequest amounts will be a given small $\Delta > 0$ apart (for an introduction into the study of chaotic orbits see Alligood et al. 1996, ch 3).

Further insights into the pattern of optimal switching sequences can be gained once we specify how agents choose between T' and T'' in case for either choice there exists a sequence that attains maximal dynastic utility. (As it turns out, ambiguities exist whenever $1+r<2$.) We may then resolve this ambiguity by, for example, assuming that in case of indifference between T' and T'' agents always attend school and thus choose T''. More generally, we may assume that all dynasties with their T_i^{stat} above a certain level of net transfers in (T',T''), say T^0 (where of course T^0 is within the ambiguous region), attend school and those with $T_i^{\text{stat}} \leq T^0$ do not. One implication of such a specification is that there are one or two non-trivial intervals, depending on the choice of T^0, such that in the long run the T_i^{stat} of no dynasty will be within one of these intervals. In fact, for any feasible given T^0 the T_i^{stat} of all dynasties with an initial T_t^{stat} in (T',T'') will eventually be in a subinterval of (T',T''), say $\left(T^1,T^2\right)$. For the study of orbits of T_i^{stat}, $i \geq t$, in this subinterval a suitable normalization of net transfers proves to be helpful: simply examine the dynamics of the variable $\left(T_i^{\text{stat}}-T^1\right)/\left(T^2-T^1\right)$ which will always remain between zero and one. The dynamics of this normalized system is then described by the equation

$$x_{i+1} = a + bx_i \pmod 1$$

individuals with their T^{stat} being in the interval (T', T'') choose to switch between the same net transfers amounts T' and T'', the pattern of the switching sequences of these dynasties (their sequences $\{\chi_i\}_{i\geq t}$) still differ depending on their endowment T^{stat} in a given period. Notwithstanding switching between identical net transfers amounts, higher received bequests in a given period always result in higher dynastic utility in that period. The following lemma summarizes these findings:

Lemma 5 *Suppose the economy is in steady state equilibrium with $\overline{w}_S > \overline{w}_U$ such that the indirect lifetime utility function $V(\cdot)$ has a kink. Then for every dynasty whose maximal feasible stationary net transfers T^{stat} in any given period are either not larger than T' or not smaller than T'' the optimal sequence of net transfers is stationary. Moreover, if the gross rate of interest does not exceed two, a dynasty switches forever between the two levels of net transfers T' and T'' whenever in any period its endowment T^{stat} is in the interval (T', T'').*

Lemma 5 describes how in equilibrium for a kinked lifetime utility function and a sufficiently large discount rate society can be split into three distinct classes: a "lower class" of poor and uneducated dynasties with their T^{stat} not exceeding T', a "middle class" of dynasties that switch between net transfers amounts T' and T'', and an "upper class" of rich and educated dynasties with T^{stat} of each dynasty not being smaller than T''. As the lemma shows, in spite of the social mobility of some dynasties, the *partition* of dynasties into these three classes *is* stable over time. Moreover, since both T' and T'' depend only on prices and preferences and hence are stationary in steady state equilibrium, class boundaries do not converge.

It is straightforward to show that in equilibrium lifetime utility cannot be identically large for all dynasties in any period because this would imply that either all dynasties switch between the same net transfers amounts or stick with a stationary sequence of net transfers amounts of $T^{\text{stat}} = T^{\text{kink}}$. In case of the former in every period one type of labour would not be supplied which cannot occur in equilibrium. In case of the latter, as argued in Section 4, the stationary sequence of levels of net transfers does not maximize dynastic utility.

This finding that in no period all dynasties choose the same level of net transfers, further reveals that in steady state equilibrium the distribution of bequests and hence of dynastic utility cannot remain perfectly equal over time. If it is equal in one period then it is necessarily unequal in the next, for in every period some dynasties choose higher net transfers amounts than others. Theorem 6 summarizes the findings of this section:

where $0 \leq a < 1$ depends on T', T'', and the choice of T^0 and $b = 1 + r \leq 2$. See Flatto and Lagarias (1996) and the papers cited there for an in-depth study of this equation.

Theorem 6 (Limited Intergenerational Mobility) *Suppose the discount rate δ is not smaller than $1/2$ and the economy is in steady state equilibrium with $\overline{w}_S > \overline{w}_U$. Then the set of dynasties inhabiting the economy is partitioned into (up to) three connected* stable *classes (such that a dynasty never switches classes): Each member of the "upper class" always leaves the same bequests as its ancestors and is skilled, each member of the "lower class" always leaves the same bequests as its ancestors and is unskilled, while dynasties of the "middle class" persistently switch between different levels of net transfers and so persistently exhibit intergenerational mobility in educational attainment. At every period lifetime and dynastic utility of every upper-class member is higher than that of every middle-class member, which is in turn higher than that of every lower-class member. Furthermore, in every period some individuals enjoy higher lifetime utility than others and dynastic utility is never* persistently *distributed equally across dynasties.*

6 Conclusion

This paper has examined the effect of a minimal standard of living requirement for effective schooling on the distribution of wealth and utility in stationary equilibrium. We find that in every stationary equilibrium without intergenerational mobility there are poor and rich dynasties whenever the minimal consumption requirement is sufficiently large.

If we allow for intergenerational mobility at steady state equilibrium and assume that the discount rate δ is sufficiently large, at steady state equilibrium with unequal lifetime labour earnings mobility is confined to the dynasties of a middle class. Moreover, middle-class dynasties never leave their class. Formally, at the centre of this result is the kink in the indirect lifetime utility function which in the present paper arises naturally out of our main assumption of a minimal consumption requirement in the schooling period.

In a variant of the present setting this feature of a middle class whose members persistently switch levels of educational attainment can also be used to explain unequal treatment of siblings. Assume for instance that parents come in couples with two children. Then with a kinked lifetime utility function for each child, middle-class parents may send one of their children to school, while the other one may have to work on the parental farm—even though both children are equally gifted and capital markets are perfect.

As a second variation of the basic model, assume that educated parents have—in addition to financial considerations—an intrinsic preference for education of their progeny. Incorporating such intrinsic preferences for education into our model obviously reduces the range of wealth endowments that defines the middle class. For sufficiently strong preferences for education persistent switching between education levels would indeed become rare. Such an additional assumption would however not interfere with the main claim of the paper: A sufficiently high minimal consumption makes inequality inevitable at stationary equilibrium while education-dependent preference itself are not sufficient to obtain the same result.

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