Stochastic Volatility in Real General Equilibrium[*]

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Abstract

In this paper I examine the propagation mechanism of stochastic volatility in a neoclassical growth model that incorporates labor market search, adjustment cost to investment, variable capital utilization and a weak short-run wealth effect, but no nominal frictions such as price stickiness. In this general equilibrium environment, stochastic volatility generates business cycle fluctuations in major macroeconomic aggregates due to the precautionary motive of risk-averse agents, yet it has no significant effects on these major aggregates as suggested by the numerical analysis of the model.

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1 Introduction

The propagation mechanism and quantitative importance of stochastic volatility in general equilibrium is still an ongoing discussion. I derive a DSGE model with stochastic volatility embedded, incorporating no nominal frictions but only adjustment cost to investment, and thereby provide a general equilibrium environment that contains real frictions only to evaluate the qualitative and quantitative implications of stochastic volatility.

Often modeled as volatility shock, stochastic volatility generates business cycle fluctuations in macroeconomic aggregates by triggering off the precautionary reactions of risk-averse households as it alters the distribution of future risk. In the baseline model where labor market search and matching à la Mortensen and Pissarides is embedded, a positive shock in the volatility of productivity increases the uncertainty in the realization of future productivity. In response to this increased risk, households lower current consumption owing to the precautionary motive, leading to an increase in the marginal utility of consumption. This increase causes the marginal cost of vacancy creation (marginal welfare loss due to vacancy creation from planner's perspective) in consumption terms to rise, and accordingly firms (planner) create less vacancies. The reduction in current vacancy then leads to a fall in future employment and output under conventional calibration. In the extended model that includes investment adjustment cost and variable capital utilization in addition to labor market search, the increased marginal utility of consumption also causes the value of current installed capital in consumption terms to rise, giving an incentive to households to slow down the depreciation of current capital stock by lowering the utilization rate, resulting in a fall in current effective capital and investment. In sum and with a weak, short-run wealth effect introduced by using the Jaimovich and Rebelo's (2009) preferences, output, consumption, investment, employment and capital in service in the extended model fall together in response to a positive shock in the volatility of productivity. The systematic reaction and positive comovemnt among these aggregates in responses to a positive shock in the volatility of investment specific technology shock, preferences shock and government spending shock can be likewise explained by the precautionary motive and the chain reaction it will initialize.

Alternative to the propagation mechanism proposed by Basu and Bundick (2012) which is based on sticky price and wage setting, neither the baseline nor the extended model includes such nominal rigidities. Moreover, as the Hosios's (1990) condition holds by construction, labor market search and matching frictions as a special type of labor adjustment cost can be internalized. The extended model therefore only contains as frictions adjustment cost to investment. In this general equilibrium environment, I find that quantitatively, the impact of stochastic volatility on macroeconomic aggregates is minimal. Even if the size of volatility shocks are reasonably large, the responses to these shocks are very small. In addition, while stochastic volatility significantly enlarges the conditional standard deviation of the aggregates, its contribution to the unconditional standard deviation is small. This result is in agreement with those reported by Bachmann and Bayer (2013), Bachmann et al. (2013) and Born and Pfeifer (2014). Furthermore, it provides a potential explanation to the sizable impact of stochastic volatility reported by Fernández-Villaverde et al. (2011a) — as they study stochastic volatility in a monetary, general equilibrium model in which a certain amount of nominal rigidities are embedded, it is reasonable to argue that the substantial effect of stochastic volatility they have observed may depend on the presence of the nominal rigidities in their model economy.

The paper is organized as follows. The baseline model is presented in section 2. In section 3, I briefly introduce the nonlinear moving average perturbation that used to solve the model, and lay out the baseline calibration for numerical analysis of the model. I present the impulse responses and moments of the baseline model and explain the propagation mechanism in section 4. In section 5, the baseline model is extended to incorporate investment adjustment cost and variable capital utilization, together with some other features that are frequently modeled in the study of stochastic volatility. The propagation mechanism is reexamined with the presence of those new ingredients. Section 6 concludes.

2 The Baseline Model

In this section, I derive a neoclassical growth model with stochastic volatility in productivity. The labor market in this model is characterized by search and matching frictions à la Mortensen and Pissarides, implemented as in Merz (1995) and Andofatto (1996).

2.1 The Baseline Model

The economy is populated by infinitely lived, identical households with Jaimovich and Rebelo's (2009) preferences (thereinafter JR preferences)

$$U_t = \frac{\left(c_t - \kappa_N \frac{n_t^{1+\gamma}}{1+\gamma} S_t\right)^{1-\kappa_F} - 1}{1-\kappa_F} \tag{1}$$

with

$$S_t = c_t^{\kappa_W} S_{t-1}^{1-\kappa_W} \tag{2}$$

where c_t is consumption, n_t the fraction of employed family members. κ_N is a strictly positive constant that scales the size of disutility rising from work and κ_F the risk aversion parameter. γ is the inverse of the Frisch elasticity of labor supply and $\kappa_W \in [0,\ 1]$ governs the size of wealth effect. When $\kappa_W = 1$, the JR preferences (1) turn into the preferences discussed in King et al. (1988) (thereinafter KPR preferences), and when $\kappa_W = 0$, it amounts to the class of preferences proposed by Greenwood et al. (1988) (thereinafter GHH preferences).

Households own the capital in the economy, and maximize the present discounted value of their life-time utility by choosing capital investment

$$\max_{i_t} \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t U_t \tag{3}$$

subject to (2) and the following budget constraint

$$c_t + i_t = w_t n_t + r_t k_t \tag{4}$$

where i_t is investment, $\beta \in (0,1)$ the discount rate, w_t wage and r_t the rental rate of capital. Households accumulate capital according the following law of motion

$$k_{t+1} = (1-\delta)k_t + i_t \tag{5}$$

where $\delta \in (0,1)$ is the capital depreciation rate. Similarly, the aggregate employment evolves according to the following

$$n_{t+1} = (1-\chi)n_t + m_t \tag{6}$$

where $\chi \in (0,1)$ denotes the exogenous constant job separation rate. m_t represents the number of job matches that are created in period t. Following Merz (1995), Andofatto (1996), Pissarides (2000), Shimer (2005), Pissarides (2009) and many others, job matches are assumed to be generated by a Cobb-Douglas matching function

$$m_t = m_0 v_t^{1-\eta}(1-n_t)^{\eta} \tag{7}$$

where m_0 is a constant scaling factor and $\eta \in (0,1)$ the elasticity of the matching function with respect to unemployment. The aggregate employment in the next period therefore is the sum of current employment that has not been destroyed, and the new employment generated by the matching function.

Competitive firms choose the amount of capital to rent from households and the number of vacancies to create in order to maximize the sum of their discounted, expected profit

$$\max_{k_t,v_t} \mathbb{E}_t \sum_{t=0}^{\infty} \left(\beta \frac{\lambda_{1,t+1}}{\lambda_{1,t}}\right)^t (y_t - w_t n_t - r_t k_t - \kappa_V v_t) \tag{8}$$

subject to

$$n_{t+1} = (1-\chi)n_t + q_t v_t \tag{9}$$

where $\lambda_{1,t}$ denotes the marginal utility of consumption defined in section 2.2, κ_V the constant vacancy posting cost and q_t vacancy filling rate that measures the rate at which vacancies become filled. Firms employ a labor-augmenting production function in Cobb-Douglas form to produce output y_t in period t

$$y_t = k_t^{\alpha}(e^{z_t} n_t)^{1-\alpha} \tag{10}$$

where $\alpha \in (0,1)$ is the capital share in production and z_t the productivity level that follows

$$z_t = \rho_z z_{t-1} + e^{\sigma_{z,t}} \varepsilon_{z,t}, \;\; \varepsilon_{z,t} \sim \mathcal{N}(0,1) \tag{11}$$

where ρ_z is the persistence parameter and $\varepsilon_{z,t}$ the productivity shock. As in Fernández-Villaverde et al. (2011b) and Caldara et al. (2012), $\varepsilon_{z,t}$ is scaled by a stochastic volatility level $\sigma_{z,t}$, which evolves as follows

$$\sigma_{z,t} = (1-\rho_{\sigma_z})\overline{\sigma_z} + \rho_{\sigma_z}\sigma_{z,t-1} + \tau_z \omega_{z,t}, \;\; \omega_{z,t} \sim \mathcal{N}(0,1) \tag{12}$$

where $\overline{\sigma_z}$ is the unconditional mean level of $\sigma_{z,t}$, ρ_{σ_z} the persistence parameter and $\omega_{z,t}$ the innovation in $\sigma_{z,t}$ that is scaled by a constant τ_z. The model is closed by the following resource constraint

$$c_t = y_t - i_t - \kappa_V v_t \tag{13}$$

By assuming that households and firms share the job match surplus according to firms' recruiting effort $1-\eta$,the externality induced by labor market search activities can be internalized. The model is thereby frictionless and can be presented as a social planner's problem[1]

$$V(k_t, n_t, z_t, \sigma_{z,t}) = \max_{c_t, v_t} \left\{ U_t + \beta \mathbb{E}_t V(k_{t+1}, n_{t+1}, z_{t+1}, \sigma_{z,t+1}) \right\} \tag{14}$$

subject to (1), (2), (6), (7), (10), (11), (12) and the following resource constraint

$$k_{t+1} = (1-\delta)k_t + y_t - c_t - \kappa_v v_t \tag{15}$$

which states the capital stock in the next period as the sum of current capital after depreciation and current output, net of consumption and the total cost of vacancy posting. Moreover, since the model assumes only two states for a family member, employed or unemployed, the fraction of the unemployed family members writes

$$u_t = 1 - n_t \tag{16}$$

As is usual in labor market search and matching literature, the vacancy filling rate q_t, job finding rate f_t and labor market tightness θ_t are defined as follows

$$q_t \equiv \frac{m_t}{v_t} \tag{17}$$

$$f_t \equiv \frac{m_t}{1-n_t} = \frac{m_t}{u_t} \tag{18}$$

$$\theta_t \equiv \frac{v_t}{1-n_t} = \frac{v_t}{u_t} \tag{19}$$

Both the job finding and vacancy filling rate are probabilities, and should lie between zero and one. The vacancy filling rate, however, can potentially exceed unity in simulation when the matching function takes the Cobb-Douglas form (see den Haan et al. (2000, p. 485)). To avoid introducing nonsmoothness into the policy function since in that case the perturbation methods cannot be applied, I do not restrict q_t to be less than one. The realization of q_t that exceeds unity is interpreted as that firms hire more than one worker on each posted vacancy, see Den Haan and De Wind (2012).

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[1] Except for the time-varying volatility of productivity and the JR preferences, the baseline model is a special case of the stationary version of Merz (1995) with zero search cost, and therefore her proof of the equivalence between the market model and the planner's problem directly applies.

2.2 Characterization

Apart from the constraints, the social planner's optimization problem is characterized by the following set of first order necessary conditions

$$\lambda_{1,t} = \left(c_t - \kappa_N \frac{n_t^{1+\gamma}}{1+\gamma} S_t\right)^{-\kappa_F} + \lambda_{2,t} \kappa_W c_t^{\kappa_W - 1} S_{t-1}^{1-\kappa_W} \tag{20}$$

$$\lambda_{2,t} = -\kappa_N \frac{n_t^{1+\gamma}}{1+\gamma} \left(c_t - \kappa_N \frac{n_t^{1+\gamma}}{1+\gamma} S_t\right)^{-\kappa_F} + \beta(1-\kappa_W)\mathbb{E}_t \left(\lambda_{2,t+1} c_{t+1}^{\kappa_W} S_t^{-\kappa_W}\right) \tag{21}$$

$$\lambda_{3,t} = \lambda_{1,t} \frac{\kappa_V}{m_{v,t}} \tag{22}$$

$$\lambda_{1,t} = \beta \mathbb{E}_t \left[\lambda_{1,t+1} \left(1 - \delta + y_{k,t+1}\right)\right] \tag{23}$$

$$\lambda_{3,t} = \beta \mathbb{E}_t \left[\lambda_{1,t+1} y_{n,t+1} + U_{n,t+1} + \lambda_{3,t+1} \left(1 - \chi + m_{n,t+1}\right)\right] \tag{24}$$

where $\lambda_{1,t}$, $\lambda_{2,t}$ and $\lambda_{3,t}$ are the Lagrange multipliers associated with (15), (2) and (6) respectively. Given the production function (10), the marginal productivity of capital and labor writes

$$y_{k,t} = \alpha k_t^{\alpha-1} \left(e^{z_t} n_t\right)^{1-\alpha} \tag{25}$$

$$y_{n,t} = (1-\alpha) k_t^{\alpha} \left(e^{z_t}\right)^{1-\alpha} n_t^{-\alpha} \tag{26}$$

Given the utility function (1) and the matching function (7), the disutiliy from work and the marginal contribution from vacancy and employment to job matches writes

$$U_{n,t} = -\kappa_N s_t n_t^{\gamma} \left(c_t - \kappa_N \frac{n_t^{1+\gamma}}{1+\gamma} S_t\right)^{-\kappa_F} \tag{27}$$

$$m_{v,t} = (1-\eta) m_0 v_t^{-\eta} (1-n_t)^{\eta} \tag{28}$$

$$m_{n,t} = -\eta m_0 v_t^{1-\eta} (1-n_t)^{\eta-1} \tag{29}$$

In this set of first order conditions, (20) denotes the marginal utility of consumption. (21) characterizes the dynamics of S_t in the JR preferences. From the planner's perspective, the Lagrange multiplier $\lambda_{3,t}$ in (22) represents the marginal welfare loss due to current vacancy creation, measured in consumption terms. Euler equation for consumption (23) equalizes the expected present-discounted utility value of postponing consumption of one period to its utility value today. Euler equation for employment (24) equalizes the marginal welfare loss induced by vacancy creation to its expected present-discounted marginal welfare gain. This gain is the sum of the marginal labor productivity, net of the disutility from work, and the its potential continuation. $m_{n,t+1}$ corrects the continuation as the future (un)employment stock has been changed by current vacancy creation.

3 Solution Method and Baseline Calibration

The baseline model described in section 2 does not have a known closed form solution and needs to be approximated with numerical methods. This section first introduces the method that will be used to approximate the solution, and then presents the baseline calibration for the numerical analysis of the model.

3.1 Perturbation Solution

As shown by Caldara et al. (2012) and Lan (2014), perturbation methods can solve such a model quickly with a degree of accuracy comparable to global methods. I use the nonlinear moving average perturbation derived in Lan and Meyer-Gohde (2013b) as it delivers stable nonlinear impulse responses and simulations and, as shown in Lan and Meyer-Gohde (2013a), enables analytical calculation of moments. The model is solved to third order as at least a third order approximation is necessary for the analysis of the effect of stochastic volatility.

For the implementation of the nonlinear moving average perturbation, I collect the equilibrium conditions, i.e., the constraints of the social planner's problem with the two Euler equations, into a vector of functions

$$0 = \mathbb{E}_t[f(\mathcal{Y}_{t+1}, \mathcal{Y}_t, \mathcal{Y}_{t-1}, \varepsilon_t)] \tag{30}$$

where $\mathcal{Y}_t$ is the vector of the endogenous variables, and ε_t the vector of the exogenous shocks, assuming the function f in (30) is sufficiently smooth and all the moments of ε_t exist and finite.

The solution to (30) is a time-invariant function $\mathcal{Y}$, taking as its state variable basis the infinite sequence of realized shocks, past and present, and indexed by the perturbation parameter $\sigma \in [0,1]$ scaling the distribution of future shocks

$$\mathcal{Y}_t = \mathcal{Y}(\sigma, \varepsilon_t, \varepsilon_{t-1}, \ldots) \tag{31}$$

Assuming normality of all the shocks and setting $\sigma = 1$ as I am interested in the stochastic model, the third order approximation—a Volterra expansion, see Lan and Meyer-Gohde (2013b)—

of (31), takes the form

$$\mathcal{Y}_t^{(3)} = \overline{\mathcal{Y}} + \frac{1}{2}\mathcal{Y}_{\sigma^2} + \frac{1}{2}\sum_{i=0}^{\infty}\left(\mathcal{Y}_i + \mathcal{Y}_{\sigma^2,i}\right)\varepsilon_{t-i} + \frac{1}{2}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\mathcal{Y}_{j,i}(\varepsilon_{t-j}\otimes\varepsilon_{t-i}) + \frac{1}{6}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\mathcal{Y}_{k,j,i}(\varepsilon_{t-k}\otimes\varepsilon_{t-j}\otimes\varepsilon_{t-i}) \tag{32}$$

where $\overline{\mathcal{Y}}$ denotes the deterministic steady state of the model, at which all the partial derivatives $\mathcal{Y}_{\sigma^2}, \mathcal{Y}_{\sigma^2,i}, \mathcal{Y}_i, \mathcal{Y}_{j,i}$ and $\mathcal{Y}_{k,j,i}$ are evaluated. (32) is naturally decomposed into order of nonlinearity and risk adjustment—$\mathcal{Y}_i$, $\mathcal{Y}_{j,i}$ and $\mathcal{Y}_{k,j,i}$ capture the amplification effects of the realized shocks $(\varepsilon_t, \varepsilon_{t-1}, \ldots)$ in the policy function (31) at first, second and third order respectively. The two partial derivatives with respect to σ, $\mathcal{Y}_{\sigma^2}$ and $\mathcal{Y}_{\sigma^2,i}$ adjust the approximation for future risk.[2] While $\mathcal{Y}_{\sigma^2}$ is a constant adjustment for risk and a linear function of the variance of future shocks[3], $\mathcal{Y}_{\sigma^2,i}$ varies over time, interacting the linear response to realized shocks with the variance of future shocks essentially adjusting the model for time variation in the conditional volatility of future risk.

3.2 Baseline Calibration

The baseline model is quarterly calibrated. Table 1 summarizes the parameter values

[Table 1 about here.]

For the value of the Frisch elasticity, Ríos-Rull et al. (2012) argue that 0.72 and 1 are the most credible ones, whereas higher value can also be found in the literature, e.g., 1.25 from Merz (1995). I use 0.72 as the benchmark and will examine the quantitative implications of the model with higher Frisch elasticity. Likewise, I set the risk aversion parameter κ_F to 2 and will evaluate the effect of higher/lower risk aversion on the numerical performance of the model.

For the parameters of the stochastic volatility process, I follow Caldara et al. (2012) and set $\rho_{\sigma_z} = 0.90$ and $\tau_z = 0.06$ respectively, to match "the persistence and standard deviation of heteroskedastic component of the Solow residual during the last five decades."

In particular, I set the size of wealth effect $\kappa_W = 0.001$ as in Jaimovich and Rebelo (2009), effectively enforcing the GHH preferences. As preferences play a key role in shaping the dynamics of the baseline model, I will then analyze in detail the model with the KPR preferences.

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[2] More generally, a constant term, $\mathcal{Y}_{\sigma^3}$, at third order adjusts (32) for the skewness of the shocks. See Andreasen (2012). As I assume all the shocks are normally distributed, $\mathcal{Y}_{\sigma^3}$ is zero and not included in (32) and the rest of the analysis.

[3] See, Lan and Meyer-Gohde (2013b) for the derivation of this term.

Finally, I set the vacancy posting cost κ_V to 0.256, to match the empirical volatility of labor market tightness relative to that of labor productivity which is 7.56 as reported by Pissarides (2009).

4 Analysis of the Baseline Model

This section presents the impulse responses and theoretical moments of the baseline model. Analyzing these numerical implications leads to two observations. First, labor market search and matching, when combined with the class of preferences with little wealth effect, can generate positive comovement among consumption, output and employment in response to a shock in the volatility of productivity. Second, the impact of such a shock on major macroeconomic aggregates is quantitatively insignificant. Under the baseline calibration, output deviates from its third order accurate stochastic steady state by about -1.2×10^{-6} in response to a positive, one standard deviation shock in the volatility of productivity. Moments analysis also supports this observation by showing that the contribution from stochastic volatility to the unconditional volatility of macroeconomic aggregates is minimal.

4.1 Impulse Response

This section presents the impulse responses of major macroeconomic aggregates to a positive shock in the volatility of productivity, i.e., in $\omega_{z,t}$, then analyzes the role of several parameters and the preferences in shaping the responses.

[Figure 1 about here.]

Figure 1 depicts the impulse response and its contributing components for capital to a positive, one standard deviation shock in volatility of productivity. In both Figure 1 and 2, the upper panel displays the impulse responses at first, second and third order as deviations from their respective (non)stochastic steady states (themselves in the middle right panel). In the the middle left panel and the middle column of panels in the lower half of the figure, the contributions to the total impulse responses from the first, second and third order amplification channels, that is, $\mathcal{Y}_i$, $\mathcal{Y}_{i,i}$ and $\mathcal{Y}_{i,i,i}$ in the third order approximation (32), are displayed. Notice that there is no response in these amplification channels. All responses to this volatility shock come from the lower left panel of the figure where the time-varying risk adjustment channel $\mathcal{Y}_{\sigma^2,i}$ is displayed. In other words, for capital, a volatility shock by itself propagates solely through the time-varying risk adjustment channel.

Capital responds positively to this positive volatility shock. This captures the planner's precautionary reaction to the widening of the distribution of future productivity shocks.[4] The risk-averse planner accumulates a buffer stock in capital to insure itself against the increased risk in future productivity as it will be drawn from a more dispersed distribution.

[Figure 2 about here.]

Figure 2 details the impulse response and its contributing components for employment to a positive, one standard deviation volatility shock to productivity. Like for capital, all responses of employment to this volatility shock comes from the time-varying risk adjustment channel and there is no response in any amplification channels. In the baseline model where employment is created by matching unemployed workers with vacancies, the negative response of employment is a direct consequence of the negative response of vacancy to a positive volatility shock to productivity, see Figure 3 below

[Figure 3 about here.]

Figure 3 displays the responses of consumption, investment, vacancy and output as deviations from their third order accurate stochastic steady states to a positive, one standard shock in productivity. The social planner accumulates a buffer stock of capital by increasing current investment on impact of the shock. As the allocation has not changed, it finances this investment through a decrease in current consumption. With the capital stock being fixed on impact as it is a state variable and with the productivity having not changed,[5] current output does not change on impact. The instantaneous increase in investment translate into an increase in capital stock in the next period according to the law of motion of capital (5). Furthermore, the decrease in current consumption results in an increase in the marginal utility of consumption, which in turn increases the marginal loss, in consumption terms, in welfare due to vacancy creation, see (22). Given the

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[4] See also Fernández-Villaverde and Rubio-Ramírez (2010) and van Binsbergen et al. (2012) for precautionary savings behavior in DSGE perturbation.

[5] Note that, it is the distribution governing future productivity shocks that is being shocked here, not the level of productivity itself.

matching function (7) and that employment is a state variable that can not be adjusted on impact, the planner chooses to decrease its vacancy creation effort to counteract such additional welfare loss. As a result, less job match is created in current period (not pictured), translating into a drop in employment in the subsequent period according to the law of motion of employment (6).

Under the baseline calibration in section 3.2, the boosting effect from this increased capital stock on output is outweighted by the adverse effect from the decreased employment in the next period, resulting in a fall in output immediately after impact. Thus, the baseline model predicts a recession following an increase in risk of future productivity.[6] The volatility shock is persistent but not permanent. As the shock dies out and productivity shocks fail to materialize from their widened distribution, the planner winds down its buffer stock of capital by increasing consumption and vacancy creation, leading to a fall in investment, an increase in employment and a quick rebound followed by an overshoot in output.[7]

4.1.1 Role of Risk Aversion, Frisch Elasticity and Job Separation Rate

To examine the role of risk aversion κ_F, the Frisch elasticity $1/\gamma$ and the job separation rate χ in shaping impulse responses, it is convenient to consider the baseline model with the exact GHH preferences, i.e., $\kappa_W = 0$. In this case S_t becomes a constant and can be normalized to one[8], and the marginal utility of consumptions writes

$$\lambda_{1,t}^{GHH} = \frac{1}{\left(c_t - \kappa_N \frac{n_t^{1+\gamma}}{1+\gamma}\right)^{\kappa_F}} \tag{33}$$

As shown in the preceding analysis of impulse responses, a positive shock to the volatility of productivity leads to an increase in the marginal utility of consumption. Note that $\frac{n_t^{1+\gamma}}{1+\gamma}$ is increasing in $1/\gamma$ given that $n_t \in (0,\ 1)$. Holding everything else constant, a fixed amount of drop in current consumption translates into a larger increase in $\lambda_{1,t}$ when the Frisch elasticity is high, and therefore a deeper cutback in vacancy than that with a lower Frisch elasticity. Consequentially, employment in the next period is lower, leading to a deeper contraction in output. See Figure 4 for the responses of consumption, the marginal utility of consumption and output to a positive,

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[6] While the impulse responses for the macroeconomic variables are not pictured with their contributing components, responses of these variables to a volatility shock come solely from the time-varying risk adjustment channel.

[7] This pattern of response of output to a positive volatility shock is consistent with that found by Bloom (2009).

[8] See Jaimovich and Rebelo (2009) for more details.

one standard deviation shock in volatility of productivity with $1/\gamma$ equals to 0.5 and 0.72 and 1.25 respectively.

[Figure 4 about here.]

The risk aversion parameter κ_F determines the magnitude of planner's precautionary motive. A highly risk-averse planner is motivated to build up a buffer stock of capital larger than that a less risk-averse planner would build though increasing current investment in response to an increase in future risk of productivity. When κ_F is extremely high, increasing current investment and cutting down current consumption is not enough to support the construction of the desirable amount of capital buffer stock. The planner then chooses to further decrease vacancy creation so that more resource can be used for investment. This leads to a deeper drop in employment, and therefore in output in the next period. See Figure 5 for the responses of consumption, investment, vacancy and output to a positive, one standard deviation shock in volatility of productivity with κ_F equals to 2, 10 and 20.

[Figure 5 about here.]

The job separation rate does not play a significant role in determining the response of the marginal utility of consumption to a volatility shock. Yet it can alter the size of the response of vacancy — as the law of motion of employment (6) implies, to reach the same amount of employment stock in the next period, the planner facing a high job separation rate needs to create more vacancies in current period to produce a larger employment inflow than that with a low job separation rate. Therefore, in response to a positive shock in volatility of productivity, the planner facing a low job separation rate needs to decrease vacancy further than that with a high job separation rate, leading to a lower employment stock in the next period and therefore a deeper drop in output.

[Figure 6 about here.]

Figure 6 depicts the responses of vacancy, employment and output to a positive, one standard deviation volatility shock in productivity with χ equals to 0.1, 0.07 and 0.036 respectively. $\chi = 0.07$ is used in the baseline calibration and is taken from Merz (1995). $\chi = 0.036$ is the monthly separation rate reported by Shimer (2005) and Pissarides (2009) uses this value for quarterly calibration, assuming separation rate is constant within the quarter. Otherwise it aggregates to a quarterly separation rate of 0.1, see Shimer (2005).

4.1.2 The KPR Preferences and Volatility of Wage

When the baseline model is equipped with the KPR preferences, i.e., $\kappa_W = 1$, a positive shock in the volatility of productivity might lead to an increase in output. To understand the reason for such a counterintuitive result, it is useful to analyze the propagation mechanism of such a volatility shock in the market setup of the baseline model. In the market setup, firms' recruiting effort is characterized by the following first order necessary conditions

$$\lambda_{3,t} = \lambda_{1,t}\frac{\kappa_V}{q_t} \tag{34}$$

$$\lambda_{3,t} = \beta\mathbb{E}_t\left[\lambda_{1,t+1}\left(y_{n,t+1} - w_{t+1} + \frac{\kappa_V}{q_{t+1}}(1-\chi)\right)\right] \tag{35}$$

where $\lambda_{3,t}$ in (34) is the marginal vacancy posting cost measured in consumption terms, and conditional on the current vacancy filling rate q_t. (35) equalizes that cost to its expected, discounted benefit. w_t in (35) denotes the market wage. Under the assumption that households and firms split match surplus according to firms' recruiting effort, the market wage takes the following form

$$w_t = \eta\left(y_{n,t} + \kappa_V\frac{v_t}{1-n_t}\right) + (1-\eta)\left(\frac{-U_{n,t}}{\lambda_{1,t}}\right) \tag{36}$$

With GHH preferences, the disutility of work writes

$$U_{n,t}^{GHH} = -\kappa_N n_t^{\gamma}\left(c_t - \kappa_N\frac{n_t^{1+\gamma}}{1+\gamma}\right)^{-\kappa_F} \tag{37}$$

Inserting the previous equation and the marginal utility of consumption (33) in (36) yields the market wage with the GHH preferences

$$w_t^{GHH} = \eta\left(y_{n,t} + \kappa_V\frac{v_t}{1-n_t}\right) + (1-\eta)\kappa_N n_t^{\gamma} \tag{38}$$

With KPR preferences, the marginal utility of consumption and disutility of work writes

$$\lambda_{1,t}^{KPR} = c_t^{-\kappa_F}\left(1 - \kappa_N\frac{n_t^{1+\gamma}}{1+\gamma}\right)^{1-\kappa_F} \tag{39}$$

$$U_{n,t}^{KPR} = -\kappa_N n_t^{\gamma} c_t^{1-\kappa_F}\left(1 - \kappa_N\frac{n_t^{1+\gamma}}{1+\gamma}\right)^{-\kappa_F} \tag{40}$$

Inserting the previous two equations in (36) yields the market wage with the KPR preferences

$$w_t^{KPR} = \eta\left(y_{n,t} + \kappa_V \frac{v_t}{1-n_t}\right) + (1-\eta)\kappa_N n_t^{\gamma}\left(\frac{c_t}{1-\kappa_N \frac{n_t^{1+\gamma}}{1+\gamma}}\right) \tag{41}$$

The crucial difference between the two wages above is that w_t^{KPR} includes current consumption whereas w_t^{GHH} does not. In the light of Greenwood et al.'s (1988) interpretation, w_t^{GHH} is determined independently of households' intertemporal consumption decision, though such a wage can be considered as the result from a two-sided (households and firms) bargaining process. This property of w_t^{GHH} also enables the following interpretation of the propagation mechanism of a positive volatility shock in productivity — with the GHH preferences, firms can lower down current wage by creating less vacancies to insure themselves against the potential decrease in current profit in response to a positive volatility shock in productivity. On the other hand, households reduce current consumption to build up a buffer stock of capital. While this would increase the marginal utility of consumption and therefore the marginal, conditional cost of vacancy posting in consumption terms, such an increase has been offset by the decrease in firms' vacancy creation behavior which leads to a lower vacancy filling rate. Finally, the decrease in vacancy creation leads to a drop in employment in the next period, and a consequential fall in output.

With the KPR preferences, firms do not necessarily reduce vacancy in order to cut down current wage and thereby counteract the potential profit loss — the drop in current consumption driven by households' precautionary motive already decreases current wage, i.e., w_t^{KPR} is also decreasing in c_t. In fact, under the baseline calibration with the KPR preferences, firms choose to increase vacancy to partly compensate the excessive drop in current wage resulting from the fall of current consumption in response to a positive volatility shock in productivity, leading to a rise of employment in the next period, and eventually an increase in output, see Figure 7.

[Figure 7 about here.]

It is still possible, however, for the baseline model with the KPR preferences to generate a decrease in output in response to increased future risk in productivity. One option is to assume a low level of risk aversion. As is discussed in section 4.1.1, current consumption decreases less with a low κ_F than it would with a high κ_F, and therefore firms still need to cut down current vacancies to ensure a sufficiently large drop in current wage. Then employment in the next period drops and output decreases. See Figure 8 for the impulse responses of macro quantities with κ_W setting to one and κ_F to 1 instead of 2 in the baseline calibration.[9]

[Figure 8 about here.]

Note that, when the baseline model is equipped with the GHH preferences, the market wage is determined independently of consumption and therefore becomes less volatile as one source of its volatility has been removed. In other words, the GHH preferences implicitly posit a wage which is less volatile than that associated with the KPR preferences. This observation provides an alternative perspective to understand the propagation mechanism of volatility shock proposed by Basu and Bundick (2012) in a monetary model with sticky wage, stick price and the KPR preferences.[10]

4.2 Moment Comparison

This section examines the contribution from stochastic volatility to the conditional and unconditional volatility of major macroeconomic aggregates respectively. While stochastic volatility can induce a significant amount of additional variations in conditional volatility, its contribution to the unconditional volatility is minimal.

4.2.1 Conditional Variance

The conditional variance of endogenous variables can be expressed as follows

$$var_t\left(\mathcal{Y}_{t+1}\right) = \mathbb{E}_t\left[\left(\mathcal{Y}_{t+1} - \mathbb{E}_t\mathcal{Y}_{t+1}\right)\left(\mathcal{Y}_{t+1} - \mathbb{E}_t\mathcal{Y}_{t+1}\right)'\right] \tag{42}$$

where $\mathbb{E}_t\mathcal{Y}_{t+1}$ denotes the conditional mean. Adding this conditional variance as an additional variable to the vector of endogenous variables and solving the model to third order delivers the third order accurate conditional variance.

[Figure 9 about here.]

Figure 9 depicts the simulated time paths of the third order accurate conditional variance of the endogenous variables with and without stochastic volatility (blue and red line respectively). When

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[9] $\kappa_W = 1$ and $\kappa_F = 1$ effectively enforce a special case of the KPR preferences: $U_t = \log c_t - \kappa_N \frac{n_t^{1+\gamma}}{1+\gamma}$.

[10] They further send the KPR preferences to the recursive utility framework à la Epstein and Zin, in order to calibrate their model with asset pricing data.

there is no volatility shock, the conditional variance of all variables exhibit minimal fluctuations along the simulation path. Adding stochastic volatility, however, induces a substantial amount of variations in the conditional variances. This is consistent with the interpretation that volatility shocks are a source of conditional heteroskedasticity, see Andreasen (2012).

4.2.2 Unconditional Standard Deviation

As noted by Andreasen (2012), the presence of stochastic volatility may induce additional variation in endogenous variables when a DSGE model is solved to third order. While it is difficult to isolate the effect of volatility shock in a nonlinear environment as noted by Fernández-Villaverde and Rubio-Ramírez (2007), the contribution from volatility shock and from its interaction with level shock to the total unconditional volatility of macroeconomic aggregates can be measured by computing the unconditional standard deviation with and without volatility shock respectively, and then examining the difference.

[Table 2 about here.]

Table 2 documents the unconditional standard deviation of endogenous variables in the absence and presence of volatility shock in productivity (column 2 and 3 respectively), and reports the difference in percentage (last column). Note that the presence of stochastic volatility indeed leads to an increase in the unconditional standard deviation of all endogenous variables, confirming Andreasen's (2012) simulation-based observation. Such increase, however, is very small across all the variables.

[Table 3 about here.]

Table 3 repeats the above unconditional volatility comparison. Yet all the unconditional standard deviations are computed with a higher risk aversion ($\kappa_F = 5$), a higher Frisch elasticity ($1/\gamma = 1.25$) and a lower job separation rate ($\chi = 0.036$), as the preceding discussion has shown that such set of parameter values will enlarge the impact of a volatility shock. Under this risk sensitive calibration, stochastic volatility contributes more to the unconditional volatility of variables than under the baseline calibration (percentage difference in the last column is uniformly larger than that in the last column of Table 2). Nevertheless, such contribution is still very small in levels.

5 The Extended Model

In this section, I extend the baseline model in section 2 to include adjustment cost to investment and variable capital utilization. Jaimovich and Rebelo (2009) show that a general equilibrium model with these two features and the class of preferences with little wealth effect can generate the positive comovement among major macroeconomic aggregates, such as output, consumption, investment and employment in response to a news shock. In the light of their analysis, I show that the extend model also restores the positive comovement particularly between investment and consumption in response to a volatility shock, as argued for in Basu and Bundick (2012).

To facilitate comparison to the results in the literature, I also add consumption habit formation,[11] noting that this is not required for the extended model to predict a recession in major macroeconomic aggregates in response to a positive shock in the volatility of productivity. Moreover, I add preferences shock, investment technology shock and government spending shock to the extended model.[12] The volatility of all these three shocks are allowed to change over time.

5.1 The Extended Model

With consumption habit formation, variable capital utilization and investment adjustment cost incorporated, the planner faces the following maximization problem[13]

$$\max_{c_t, i_t, v_t, x_t} \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t \left[\frac{e^{b_t} \left(c_t - \kappa_C c_{t-1} - \kappa_N \frac{n_t^{1+\gamma}}{1+\gamma} S_t \right)^{1-\kappa_F} - 1}{1-\kappa_F} \right] \tag{43}$$

with

$$S_t = (c_t - \kappa_C c_{t-1})^{\kappa_W} S_{t-1}^{1-\kappa_W} \tag{44}$$

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[11] See Bidder and Smith (2012), Christiano et al. (2013) and Born and Pfeifer (2014) for incorporating consumption habit formation in their analysis of volatility shocks in general equilibrium models.

[12] See Justiniano and Primiceri (2008), Fernández-Villaverde et al. (2011a) and Born and Pfeifer (2014) for incorporating these three shocks in analyzing the quantitative impact of volatility shocks in general equilibirium models.

[13] As the model is no longer frictionless, households and firms' problem should be presented and solved separately. Yet for notational ease, I still present the model as a planner's problem, with the same set of equilibrium conditions that would come from the corresponding market model.

where $\kappa_C \in (0,1)$ governs the persistence of consumption habit and b_t denotes the preferences shock process. The law of motion of capital and production function now take the following form

$$k_{t+1} = (1-\delta_t)k_t + e^{\mu_t}(1-\phi_t)\, i_t \tag{45}$$

$$y_t = e^{z_t}(x_t k_t)^{\alpha} n_t^{1-\alpha} \tag{46}$$

where δ_t denotes the depreciation function, x_t the capital utilization rate and ϕ_t the investment adjustment cost function. μ_t denotes the investment-specific technology shock process. The depreciation function takes the following functional form as proposed by Baxter and Farr (2005)

$$\delta_t = \frac{\delta_1}{1+\delta_2} x_t^{1+\delta_2} + \delta_0 \tag{47}$$

where δ_0 and δ_1 will be chosen such that capital is fully utilized in the deterministic steady state. δ_2 denotes the elasticity of marginal depreciation with respect to the utilization rate.

As in Justiniano and Primiceri (2008), Fernández-Villaverde et al. (2011b), Bidder and Smith (2012), Born and Pfeifer (2014) and many others, the investment adjustment cost function takes the following quadratic form

$$\phi_t = \frac{\kappa_I}{2}\left(\frac{i_t}{i_{t-1}} - 1\right)^2 \tag{48}$$

where κ_I is positive and governs the curvature of the function.

In addition, the government purchases goods and service and balances its budget in each period. This government spending is financed by a lump-sum tax and therefore the resource constraint of the extended model writes

$$y_t = c_t + i_t + \kappa_V v_t + e^{g_t} \tag{49}$$

where g_t denotes the government spending and is assumed to be an exogenous process.

Analogous to the productivity process (11), the preferences shock process b_t, investment shock process μ_t and the government spending process g_t are driven by their corresponding exogenous innovations with stochastic volatility and take the following form

$$b_t = \rho_b b_{t-1} + e^{\sigma_{b,t}}\varepsilon_{b,t}, \;\; \varepsilon_{b,t} \sim \mathcal{N}(0,1) \tag{50}$$

$$\mu_t = \rho_\mu \mu_{t-1} + e^{\sigma_{\mu,t}}\varepsilon_{\mu,t}, \;\; \varepsilon_{\mu,t} \sim \mathcal{N}(0,1) \tag{51}$$

$$g_t = (1-\rho_g)\bar{g} + \rho_g g_{t-1} + e^{\sigma_{g,t}}\varepsilon_{g,t}, \;\; \varepsilon_{g,t} \sim \mathcal{N}(0,1) \tag{52}$$

where ρ_b, ρ_μ and ρ_g are persistence parameters, and $\bar{g}$ the deterministic steady state value of government spending. Likewise, analogous to the stochastic volatility process that scales the productivity shock (12), the stochastic volatility in the above three processes, $\sigma_{b,t}$, $\sigma_{\mu,t}$ and $\sigma_{g,t}$ are all assumed to take the following form

$$\sigma_{\zeta,t} = (1-\rho_{\sigma_\zeta})\overline{\sigma_\zeta} + \rho_{\sigma_\zeta}\sigma_{\zeta,t-1} + \tau_\zeta \omega_{\zeta,t}, \ \ \omega_{\zeta,t} \ \sim \mathcal{N}(0,1) \tag{53}$$

where ρ_{σ_ζ} governs the persistence and $\zeta \in \{b, \mu, g\}$. $\overline{\sigma_\zeta}$ denotes the respective unconditional mean level of $\sigma_{b,t}$, $\sigma_{\mu,t}$ and $\sigma_{g,t}$. τ_ζ scales the volatility shock.

5.2 Characterization and Calibration

Defining $\lambda_{1,t}$, $\lambda_{2,t}$, $\lambda_{3,t}$ and $\lambda_{4,t}$ as the Lagrangian multipliers associated with the resource constraint (49), the S_t dynamic (44), the law of motion of employment (6) and capital (45) respectively, setting up the associated Lagrangian function and differentiating with respect to the corresponding control and state variables deliver the following set of first order necessary conditions that characterizes the equilibrium of the extended model

$$\begin{aligned} \lambda_{1,t} = {} & e^{b_t}\left(c_t - \kappa_C c_{t-1} - \kappa_N \frac{n_t^{1+\gamma}}{1+\gamma} S_t\right)^{-\kappa_F} + \lambda_{2,t}\kappa_W \left(c_t - \kappa_C c_{t-1}\right)^{\kappa_W - 1} S_{t-1}^{1-\kappa_W} \\ & - \beta\kappa_C \mathbb{E}_t\left[e^{b_{t+1}}\left(c_{t+1} - \kappa_C c_t - \kappa_N \frac{n_{t+1}^{1+\gamma}}{1+\gamma} S_{t+1}\right)^{-\kappa_F}\right] \\ & - \beta\kappa_C\kappa_W \mathbb{E}_t\left[\lambda_{2,t+1}\left(c_{t+1} - \kappa_C c_t\right)^{\kappa_W - 1} S_t^{1-\kappa_W}\right] \end{aligned} \tag{54}$$

$$\begin{aligned} \lambda_{2,t} = {} & -\kappa_N \frac{n_t^{1+\gamma}}{1+\gamma}\left(c_t - \kappa_C c_{t-1} - \kappa_N \frac{n_t^{1+\gamma}}{1+\gamma} S_t\right)^{-\kappa_F} \\ & + \beta(1-\kappa_W)\mathbb{E}_t\left[\lambda_{2,t+1}(c_{t+1} - \kappa_C c_t)^{\kappa_W} S_t^{-\kappa_W}\right] \end{aligned} \tag{55}$$

$$\lambda_{3,t} = \lambda_{1,t}\frac{\kappa_V}{m_{v,t}} \tag{56}$$

$$\lambda_{1,t} y_{x,t} = \lambda_{4,t} k_t \delta_{x,t} \tag{57}$$

$$\begin{aligned} \lambda_{1,t} = {} & e^{\mu_t}\lambda_{4,t}\left[1 - \kappa_i\left(\frac{i_t}{i_{t-1}} - 1\right)\frac{i_t}{i_{t-1}} - \frac{\kappa_i}{2}\left(\frac{i_t}{i_{t-1}} - 1\right)^2\right] \\ & + \beta\mathbb{E}_t\left[e^{\mu_{t+1}}\lambda_{4,t+1}\kappa_i\left(\frac{i_{t+1}}{i_t} - 1\right)\left(\frac{i_{t+1}}{i_t}\right)^2\right] \end{aligned} \tag{58}$$

$$\lambda_{4,t} = \beta\mathbb{E}_t\left[\lambda_{1,t+1} y_{k,t+1} + \lambda_{4,t+1}(1-\delta_{t+1})\right] \tag{59}$$

$$\lambda_{3,t} = \beta\mathbb{E}_t\left[U_{n,t+1} + \lambda_{1,t+1} y_{n,t+1} + \lambda_{3,t}\left(1 - \chi + m_{n,t+1}\right)\right] \tag{60}$$

with $y_{x,t} = \alpha y_t/x_t$, $\delta_{x,t} = \delta_1 x_t^{\delta_2}$, $y_{k,t} = \alpha y_t/k_t$ and $y_{n,t} = (1-\alpha)y_t/n_t$. $U_{n,t}$, $m_{v,t}$ and $m_{n,t}$ are as defined by (27), (28) and (29). The four remaining first order conditions with respect to the Lagrangian multipliers are the four constraints with which the multipliers are associated.

Among this set of equilibrium conditions, (54) and (55) define the marginal utility of consumption in the presence of habit formation, and when $\kappa_C = 0$, they reduce to (20) and (21) respectively. Identical to (22), (56) denotes the marginal loss in welfare due to vacancy creation in consumption terms. (57) characterizes the optimal capital utilization rate by equating the marginal benefit in consumption terms to the marginal cost in terms of additional units of capital being worn out. (58) is the Euler equation for investment in the presence of adjustment cost. As in the baseline model, (59) and (60) are the Euler equations for consumption and employment respectively.

For numerical analysis of the extended model, in addition to the baseline calibration in section 3.2, the capital utilization elasticity parameter δ_2 is set to 1, see Basu and Kimball (1997). Consumption habit persistence κ_C is set to 0.54 as reported in Born and Pfeifer (2014). Given the value of δ_2 and κ_C, the investment adjustment cost elasticity κ_I is accordingly chosen to be 10 such that in response to a positive shock to the volatility of productivity, investment decreases. At the deterministic steady state, government spending $\bar{g}$ is equal to 20% of output as reported in Born and Pfeifer (2014). As a starting point, the persistence and volatility of the preferences shock process, investment shocks process and government spending process are assumed to be the same as those of the productivity process, i.e., $\rho_\zeta = \rho_z = 0.95$, $\rho_{\sigma_\zeta} = \rho_{\sigma_z} = 0.90$, $\overline{\sigma_\zeta} = \overline{\sigma_z} = \ln(0.07)$ and $\tau_\zeta = \tau_z = 0.06$ for $\zeta \in \{b, \mu, g\}$. Owing to the presence of these additional shock processes, the endogenous variables in the extended model are in general more volatile than those in the baseline model. The vacancy posting cost κ_V is thereby set to 0.6, to keep the volatility of labor market tightness relative to that of labor productivity still equal to 7.56. Note that the baseline model is nested in the extended model — when $\kappa_I = \kappa_C = 0$, $\delta_2 \to \infty$ and all the shocks except the productivity shock shut down, the extended model reduces to the baseline model.

5.3 Impulse Responses

This section presents and analyzes the responses of macroeconomic variables to a positive shock in the volatility of productivity, investment technology, preferences and government spending. Except for the volatility of investment technology where a positive shock leads to a boom, an increase in the volatility of all the other three shocks leads to a recession, consistent with the pattern reported by Born and Pfeifer (2014).

Quantitatively, the impact of a volatility shock on the macroeconomic aggregates is very small. For example, under the extended calibration, output deviates from its third order accurate stochastic steady state by about -1.2×10^{-5} in response to a positive, one standard deviation shock in the volatility of productivity. Its responses to such a shock in the volatility of investment technology, preferences and government spending are even smaller in terms of absolute value.

5.3.1 Shock to the Volatility of Productivity

Investment adjustment cost plays an important role in shaping the impulse responses of endogenous variables of the extended model, as summarized by the capital utilization equation (57). Inserting the functional form of $y_{x,t}$ and $\delta_{x,t}$ in (57) and rearranging yields

$$1 = \frac{\lambda_{4,t}}{\lambda_{1,t}} x_t^{1-\alpha+\delta_2} \left[\delta_1 k_t^{1-\alpha} \left(e^{z_t} n_t\right)^{\alpha-1}\right] \tag{61}$$

where $\lambda_{4,t}/\lambda_{1,t}$ is the value of installed capital in terms of consumption as noted in Jaimovich and Rebelo (2009). Terms inside the bracket are constant and state variables and will not change on impact of volatility shocks. With the presence of adjustment cost to investment, building up a buffer stock of capital in response to a positive volatility shock to productivity by increasing current investment becomes riskier. Instead, manipulating the installed capital on impact is less risky (and possible since utilization rate is a control variable) as the installed capital will not respond to the changes of risk in future productivity, and hence its value in terms of consumption increases on impact. This increase in value makes the installed capital more costly to replace, giving the planner an incentive to slow down the depreciation by lowering the utilization rate and decreasing current investment. Still, driven by the precautionary motive, the planner wants to build up a buffer stock of capital in response to a positive volatility shock to productivity and now it chooses to cut down current consumption to achieve that — the saved stock of current consumption will build up the buffer stock of capital through the resource constraint (45) and (49)(not pictured) in a less risky manner relative to that through increasing investment as current consumption is not involved in the production process and therefore less sensitive to the change in the volatility of future productivity.

[Figure 10 about here.]

Figure 10 depicts the impulse responses of macroeconomic variables, expressed as deviations from their third order accurate stochastic steady states, to a positive, one standard deviation shock in the volatility of productivity, i.e., in $\omega_{z,t}$. As in the baseline model, the decrease in current consumption results in an increase in the marginal utility of consumption. Yet this increase in $\lambda_{1,t}$ is dominated by the increase in the value of installed capital $\lambda_{4,t}$ and therefore $\lambda_{4,t}/\lambda_{1,t}$ increases on impact. The fall in current utilization rate leads to a decrease in effective capital (the lower panel). With productivity having not changed (again, it is only the volatility of the distribution of future productivity shocks that is being shocked) and current employment being fixed, current output in the extended model decline on impact due to this decrease in current effective capital. The increase in the marginal utility of consumption also increases the marginal loss in consumption terms in welfare due to vacancy creation. The planner therefore cuts down current vacancy, leading to a decline in employment in next period. This fall reinforces the decrease in output in the subsequent period and therefore the extended model predicts a deeper and more prolonged recession than the baseline model in response to increased future risk in productivity.

5.3.2 Shock to the Volatility of Investment Technology

To analyze the transmission mechanism of a shock to the volatility of investment technology, it is convenient to interpret the investment level shock $\varepsilon_{\mu,t}$ as the disturbance to the process by which current investment is transformed into installed capital to be used in production, see Justiniano et al. (2010) and Justiniano et al. (2011). When a positive shock hits the volatility of $\varepsilon_{\mu,t}$, the efficiency of this transformation becomes more uncertain, and therefore the planner increases current investment to ensure a sufficient amount of investment will be converted into capital for production purpose. An increase in current investment leads to a fall in the value of installed capital in consumption terms, and as noted by Jaimovich and Rebelo (2009), this fall occurs because adjustment cost to investment implies that higher levels of current investment reduce the cost of investment in the next period. The fall in $\lambda_{4,t}/\lambda_{1,t}$ lowers the value of installed capital, making it less costly to replace, so it is efficient to increase current utilization rate to speed up depreciation.

[Figure 11 about here.]

Figure 11 displays the impulse responses of macroeconomic quantities as deviations from their third order accurate stochastic steady state to a positive, one standard deviation shock in the volatility of investment technology, i.e., in $\omega_{i,t}$. As increasing current investment secures a sufficient amount of installed capital for production and of capital input in the next period, the planner chooses to increase current consumption (followed by a decline), leading to a fall in the marginal utility of consumption. This fall in $\lambda_{1,t}$ is dominated by the decline in $\lambda_{4,t}$ and therefore the value of installed capital $\lambda_{4,t}/\lambda_{1,t}$ falls. The increased current utilization rate results in an increase in effective capital (the lower panel), leading to an increase in output on impact. The fall in $\lambda_{1,t}$ also leads to an increase in current vacancy creation and future employment. The latter makes the increase in output even more persistent. In sum, a positive volatility shock to investment technology leads to a boom.

5.3.3 Shock to the Volatility of Preferences and Government Spending

Since both preferences and government spending shocks, i.e. $\varepsilon_{b,t}$ and $\varepsilon_{g,t}$, are demand shocks, a positive shock that hits their volatility leads to a future aggregate demand with high uncertainty. The planner thereby increases its precautionary savings by cutting down current consumption to ensure that future demand can be met.

[Figure 12 about here.]

[Figure 13 about here.]

Figure 12 and 13 depict the impulse responses of macroeconomic variables, expressed as deviations from their third order accurate stochastic steady states, to a positive, one standard deviation shock in the volatility of preferences and government spending, i.e., in $\omega_{b,t}$ and in $\omega_{g,t}$, respectively. Through a market lens, when future aggregate demand becomes more uncertain, firms choose to rent a smaller amount of effective capital for production purpose from households. As capital is a state variable and being fixed on impact, this decline in the demand of effective capital leads to a fall in current utilization rate. On the other hand, as current consumption has been cut back on impact owing to precautionary motive, a buffer stock of capital will be built using this saved consumption stock in the next period. This crowds out the need of increasing current investment in order to build up the buffer stock of capital. As a result, current investment drops. The fall in current utilization rate leads to a decline in output on impact. The decrease in current consumption leads to an increase in the marginal utility of consumption, a fall in current vacancy and future employment, which reinforces the drop in output in the subsequent period. Put it together, a positive volatility shock to preferences and government spending leads to a recession.

5.3.4 Size of Volatility Shock

In the extended calibration, the standard deviations of the four volatility shocks are all set to be 0.06, and the impulse responses reported in section 5.3.1 - 5.3.3 are generated accordingly. In the literature, the size of these standard deviations vary. For example, the standard deviation of volatility shock in productivity, i.e., τ_z, ranges from 0.01 (see Andreasen (2012) and Justiniano and Primiceri (2008)), to 0.15 (see Bidder and Smith (2012)), and to 0.312 (see Born and Pfeifer (2014)). Note that, first, a volatility shock of large size will not qualitatively alter the impulse responses of macroeconomic aggregates in the extended model, that is, a large, positive shock in the volatility of productivity still leads to a recession. Second, the responses of macroeconomic aggregates are still small even if the standard deviation of volatility shock is reasonably large. For example, output deviates from its third order accurate stochastic steady state by about -1.2×10^{-4} in response to a positive volatility shock with $\tau_z = 0.624$ in productivity. $\tau_z = 0.624$ mimics the two-standard deviation volatility shock used in Born and Pfeifer (2014) for policy risk study.

5.4 Moments

This section presents the unconditional standard deviation of the macroeconomic aggregates in the extended model. Table 4 reports the unconditional standard deviations computed in the absence and the presence of stochastic volatility from productivity shock, investment technology shock, preferences shock and government spending shock (column 2 and 3 respectively). The difference in percentage between these two set of values are shown in the last column.

[Table 4 about here.]

Like in the baseline model (see Table 2), the contribution from stochastic volatility to the unconditional volatility of the macroeconomic aggregates is very small, although the extended model includes four different sources (instead of one in the baseline model, i.e., the productivity shock) of stochastic volatility. Table 5 reports the approximate portion of the total contribution from the four sources of stochastic volatility to the unconditional standard deviation.

[Table 5 about here.]

The second column of the table repeats the unconditional standard deviations with the presence of all the four sources of stochastic volatility for reference. The third column documents the unconditional standard deviations when the volatility of the productivity shock is hold constant but that of the other three shocks are still allowed to vary over time. Column 4 reports the percentage difference between the previous two columns, and therefore can be considered as the contribution from the stochastic volatility in productivity to the unconditional standard deviation. Analogously, column 5, 7 and 9 documents the unconditional standard deviation without stochastic volatility in investment technology, government spending and preferences respectively, and column 6, 8 and 10 reports the contribution in percentage from the three sources of stochastic volatility respectively.

There are two important observations. First, stochastic volatility in productivity and investment technology contributes most to the unconditional standard deviation, yet that in government spending and preferences contributes almost nothing. This is consistent with the observation made in Justiniano and Primiceri (2008) and Fernández-Villaverde et al. (2011b). Second, the percentage contributions shown in column 4, 6, 8 and 10 only approximate the individual contribution from the four sources of stochastic volatility respectively, and thereby not necessarily add up to the percentage contribution reported in the last column of Table 4. The remaining contribution comes from the interplay among the volatility shocks and between the level and volatility shocks.

6 Conclusion

I have presented a business cycle model that includes Jaimovich and Rebelo’s (2009) preferences, search and matching frictions in labor market, investment adjustment cost and variable capital utilization as its key ingredients that are used to explain the propagation mechanism of stochastic volatility in general equilibrium. By construction, the Hosios’s (1990) condition holds in the model economy and the frictions induced by search and matching activities in labor market can be internalized. The model thereby encompasses no frictions other than adjustment cost on investment, providing an environment to observe the structural and statistical implications of stochastic volatility almost in isolation.

The model is solved to third order using the nonlinear moving average perturbation, and analyzed under conventional, quarterly calibration. The impulse responses shows that the model predicts a recession in response to a positive shock in the volatility of productivity, government spending and preferences, and envisions a boom if such a positive shock hits the volatility of investment technology, consistent with the pattern reported by Born and Pfeifer (2014) and many other studies in this literature. On the quantitative side, both the impulse responses and unconditional standard deviations suggest that the impact of stochastic volatility on the major macroeconomics aggregates is very small, though stochastic volatility largely increases the conditional volatility of those aggregates. Since the model incorporates no nominal rigidities such as sticky wage and price, the numerical analysis of the model supports the argument that the large impact of stochastic volatility found in Fernández-Villaverde et al. (2011a) and others using a general equilibrium model with the above rigidities embedded may come from the interaction between stochastic volatility and those nominal frictions.

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Table 1: Baseline Calibration

Symbol	Description	Value	Source
β	Discount rate	0.99	Standard value
α	Capital share	0.34	Ríos-Rull et al. (2012)
δ	Capital depreciation rate	0.019	Ríos-Rull et al. (2012)
γ	Inverse of Frisch elasticity	1/0.72	Ríos-Rull et al. (2012)
κ_F	Risk aversion	2	Standard value
χ	Job separation rate	0.07	Merz (1995)
η	Matching elasticity w.r.t. unemployment	0.5	Pissarides (2009)
n_{ss}	Steady state employment	0.94	Pissarides (2009)
θ_{ss}	Steady state labor market tightness	0.72	Shimer (2005)
m_0	Job match scaling factor	0.36	Deduced
ρ_z	Persistence of productivity process	0.95	Caldara et al. (2012)
$\rho_{\sigma z}$	Persistence of volatility shock	0.90	Caldara et al. (2012)
$\overline{\sigma_z}$	Unconditional mean of productivity shock	$\ln(0.007)$	Caldara et al. (2012)
τ_z	Standard deviation of volatility shock	0.06	Caldara et al. (2012)
κ_W	Wealth effect scaling factor	0.001	Enforcing GHH preferences
κ_V	Vacancy posting cost	0.257	Chosen to match $\sigma(\theta)/\sigma(p) = 7.56$
κ_N	Disutility scaling factor	0.888	Deduced

Table 2: Unconditional standard deviation comparison under Baseline Calibration

variable	constant vol.	stoch. vol.	diff. in %
k	1.1976	1.2202	1.89
y	0.0825	0.0840	1.82
c	0.0456	0.0465	1.97
i	0.0404	0.0411	1.73
n	0.0044	0.0045	2.27
v	0.0037	0.0037	0.00

Table 3: Unconditional Standard deviation comparison under high risk aversion, higher Frisch elasticity and lower job separation rate

variable	constant vol.	stoch. vol.	diff. in %
k	3.7962	3.8698	1.94
y	0.1430	0.1459	2.03
c	0.0603	0.0616	2.16
i	0.0828	0.0844	1.93
n	0.0075	0.0077	2.67
v	0.0061	0.0062	1.64

Table 4: Unconditional standard deviation comparison of the extended model

variable	constant vol.	stoch. vol.	diff. in %
k	1.9689	2.0060	1.88
y	0.0800	0.0815	1.88
c	0.0473	0.0482	1.90
i	0.0432	0.0440	1.85
n	0.0032	0.0033	3.12
v	0.0027	0.0027	0.00

Table 5: Unconditional standard deviation decomposition of the extended model

variable	stoch. vol.	$\omega_z = 0$		$\omega_i = 0$		$\omega_g = 0$		$\omega_b = 0$	
		Value	Diff.in%	Value	Diff.in%	Value	Diff.in%	Value	Diff.in%
k	2.0060	1.9824	1.19	1.9962	0.49	2.0041	0.09	2.0045	0.07
y	0.0815	0.0803	1.49	0.0814	0.12	0.0815	0.00	0.0815	0.00
c	0.0482	0.0475	1.47	0.0480	0.42	0.0481	0.21	0.0481	0.21
i	0.0440	0.0434	1.38	0.0439	0.23	0.0440	0.00	0.0440	0.00
n	0.0033	0.0033	0.00	0.0033	0.00	0.0033	0.00	0.0033	0.00
v	0.0027	0.0027	0.00	0.0027	0.00	0.0027	0.00	0.0027	0.00

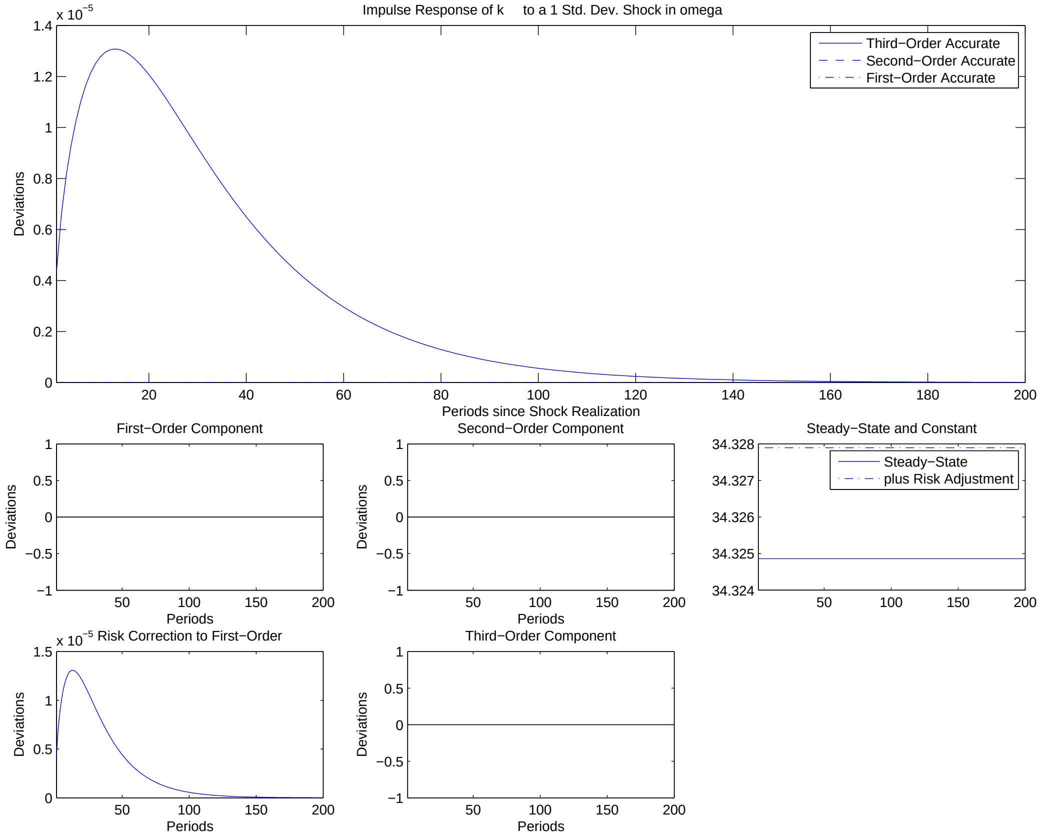

Figure 1: Capital IRF: Volatility Shock, Baseline Model of Section 2

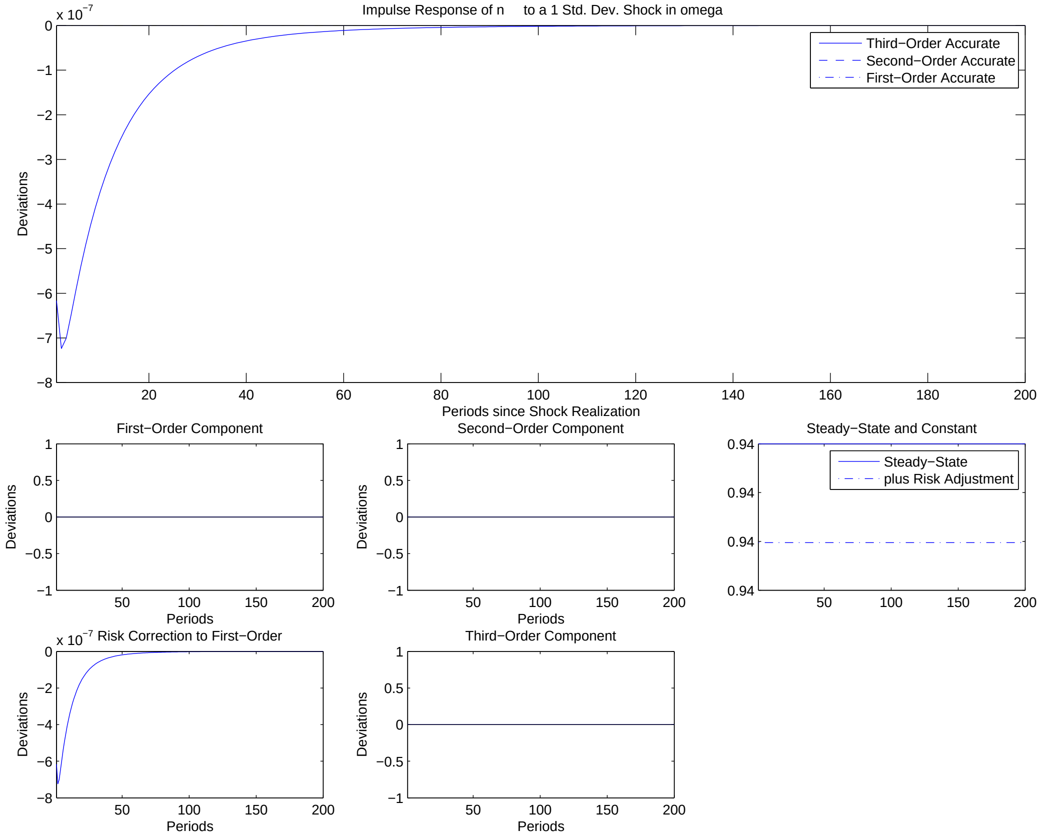

Figure 2: Employment IRF: Volatility Shock, Baseline Model of Section 2

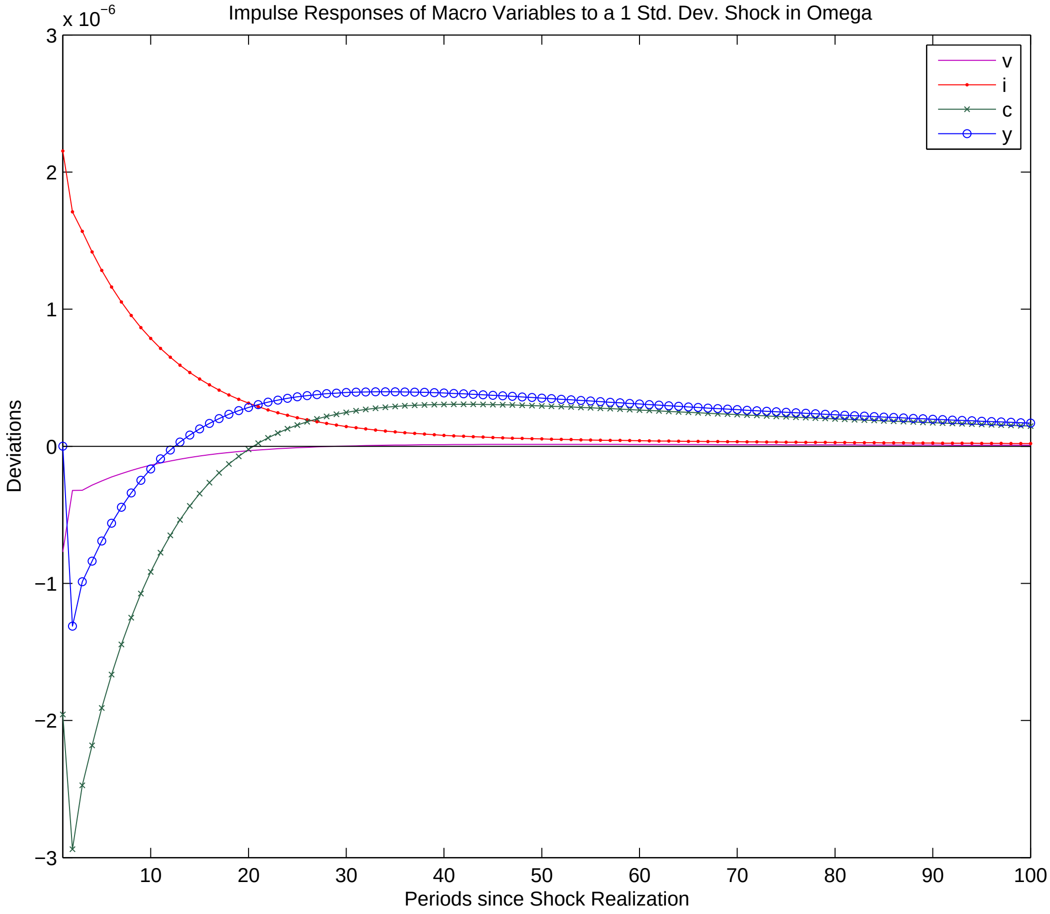

Figure 3: Macro IRFs: Volatility Shock, Baseline Model of Section 2

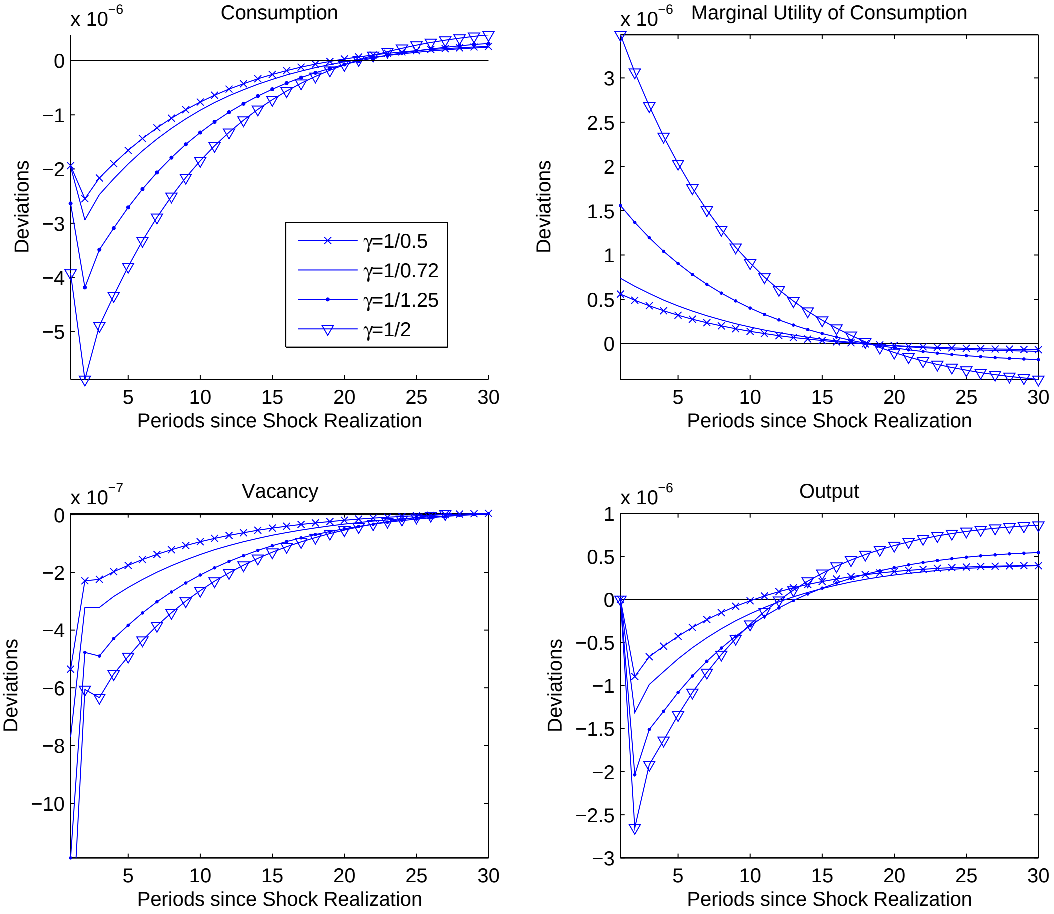

Figure 4: Role of Frisch Elasticity

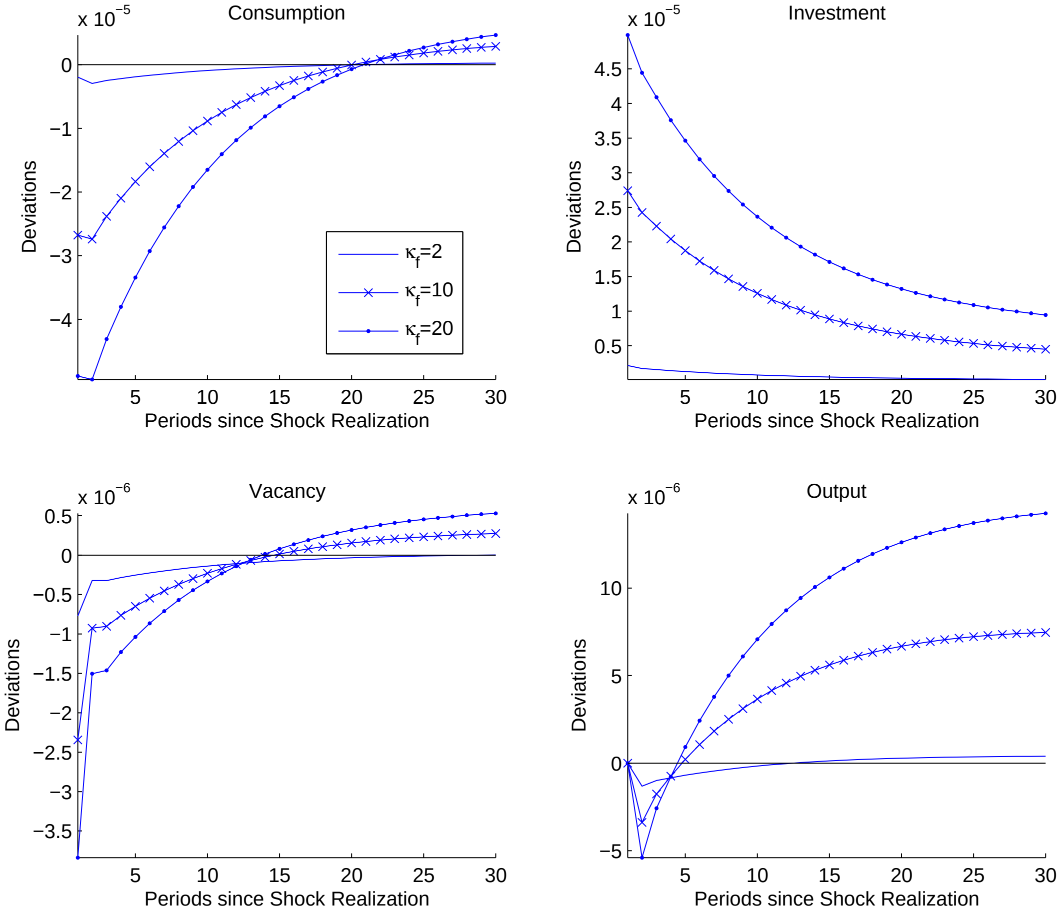

Figure 5: Role of Risk Aversion

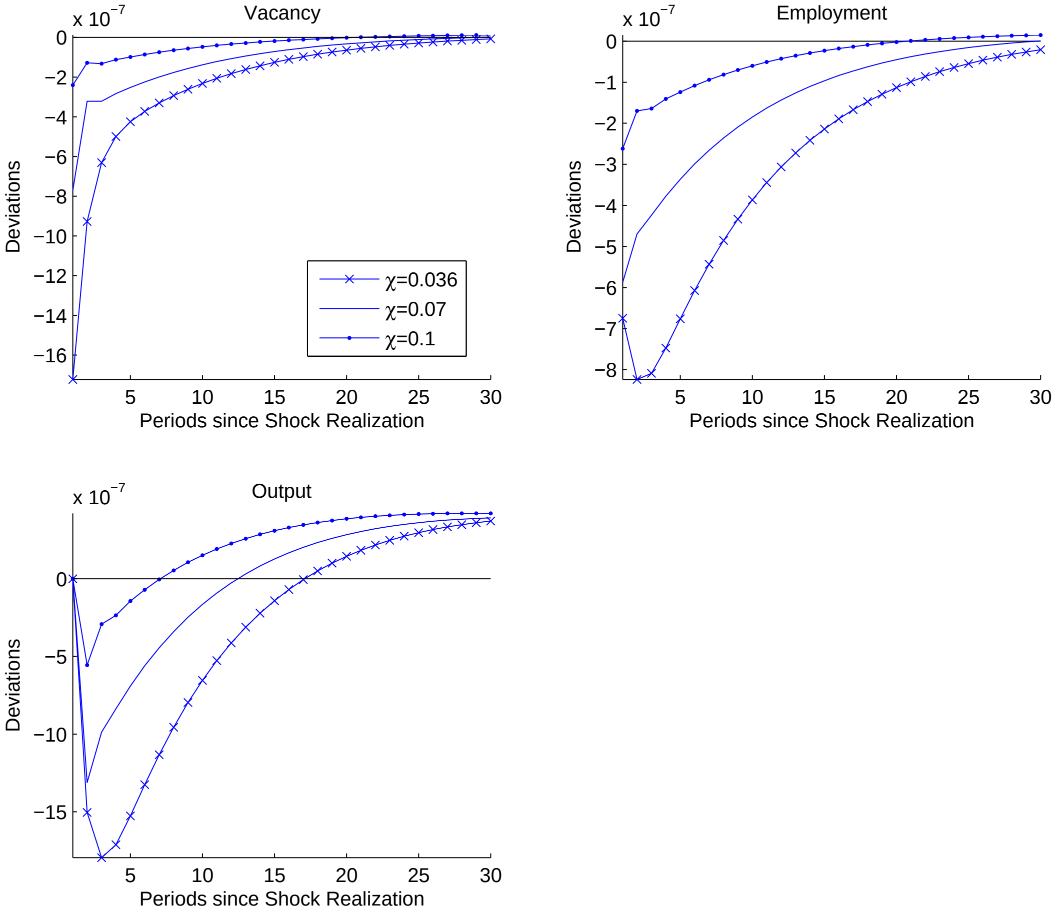

Figure 6: Role of Job Separation Rate

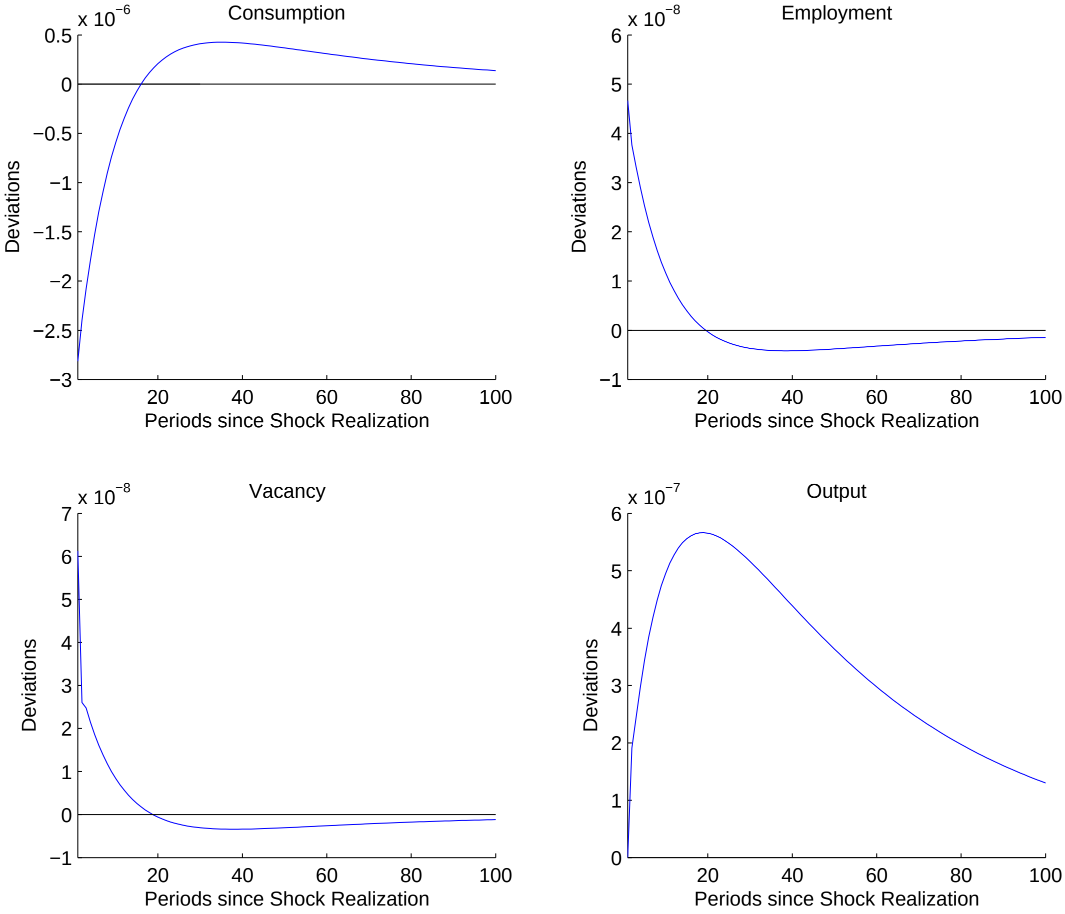

Figure 7: IRF of Macros with KPR Preferences

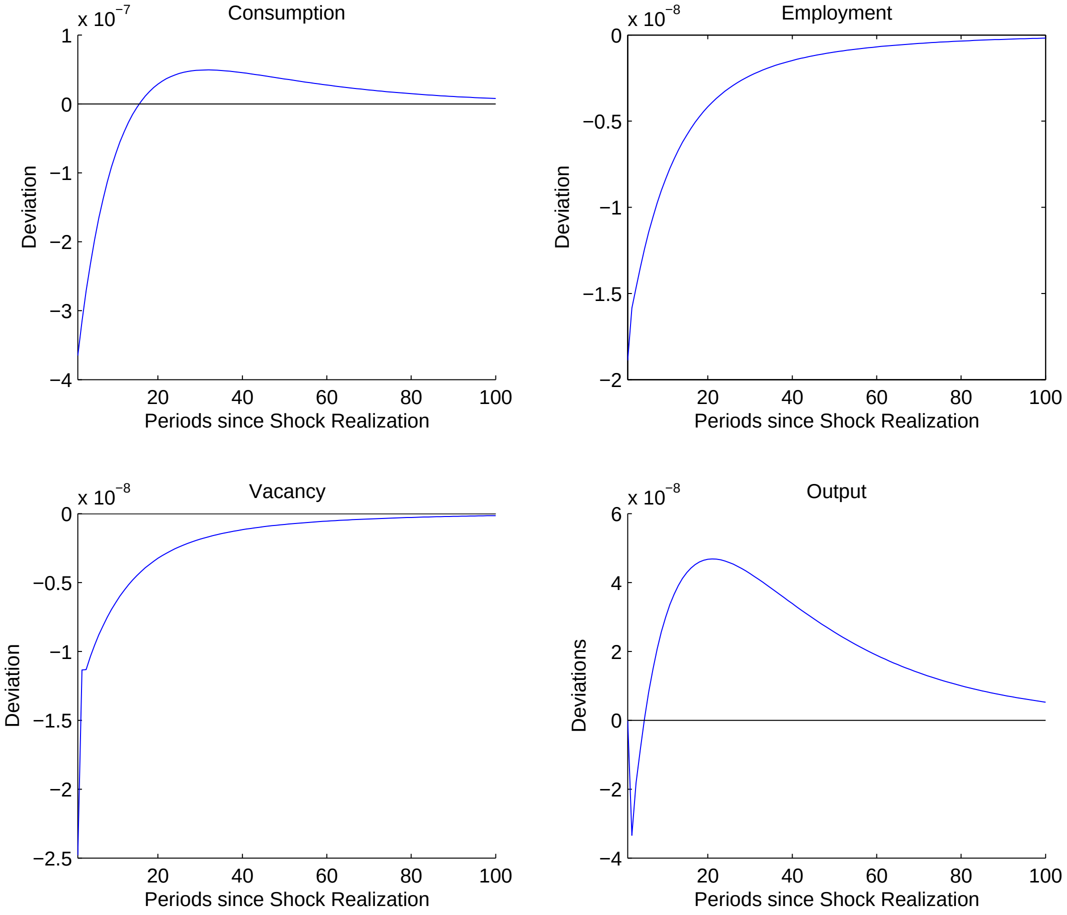

Figure 8: IRF of Macros with KPR Preferences

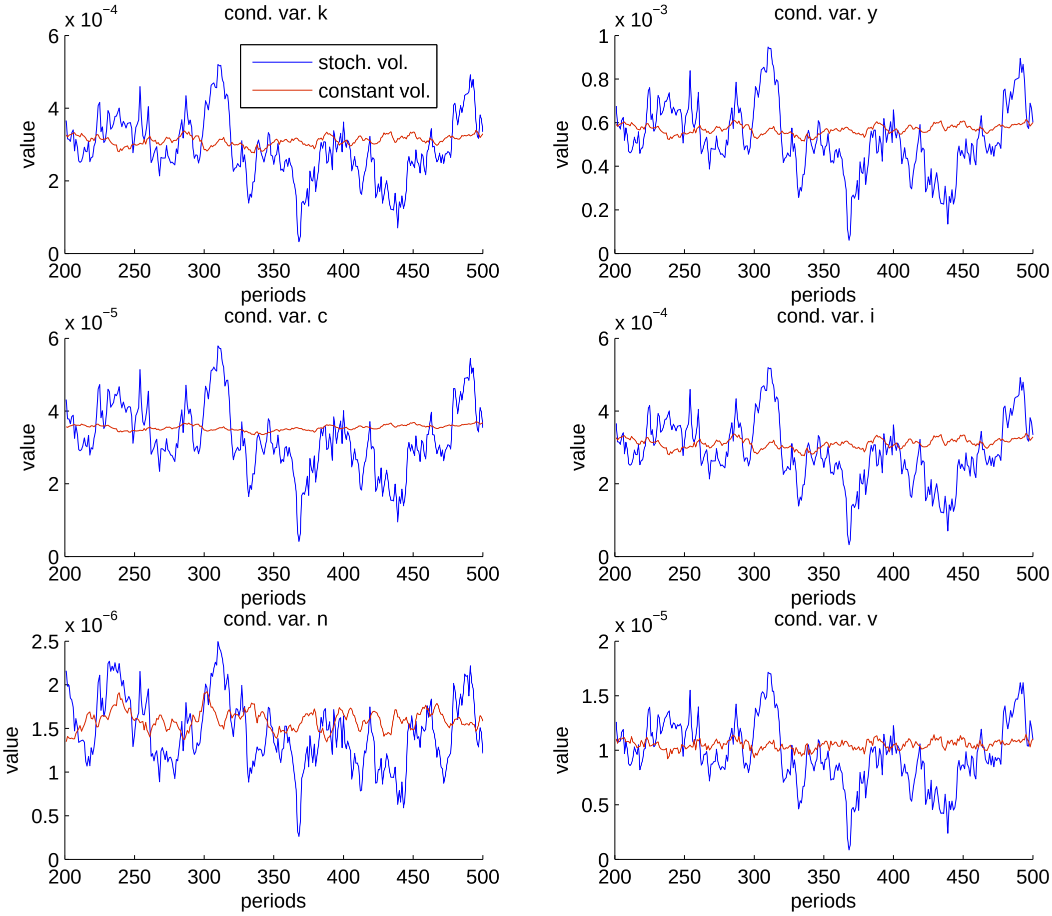

Figure 9: Conditional Variance Comparison, Baseline Model of Section 2

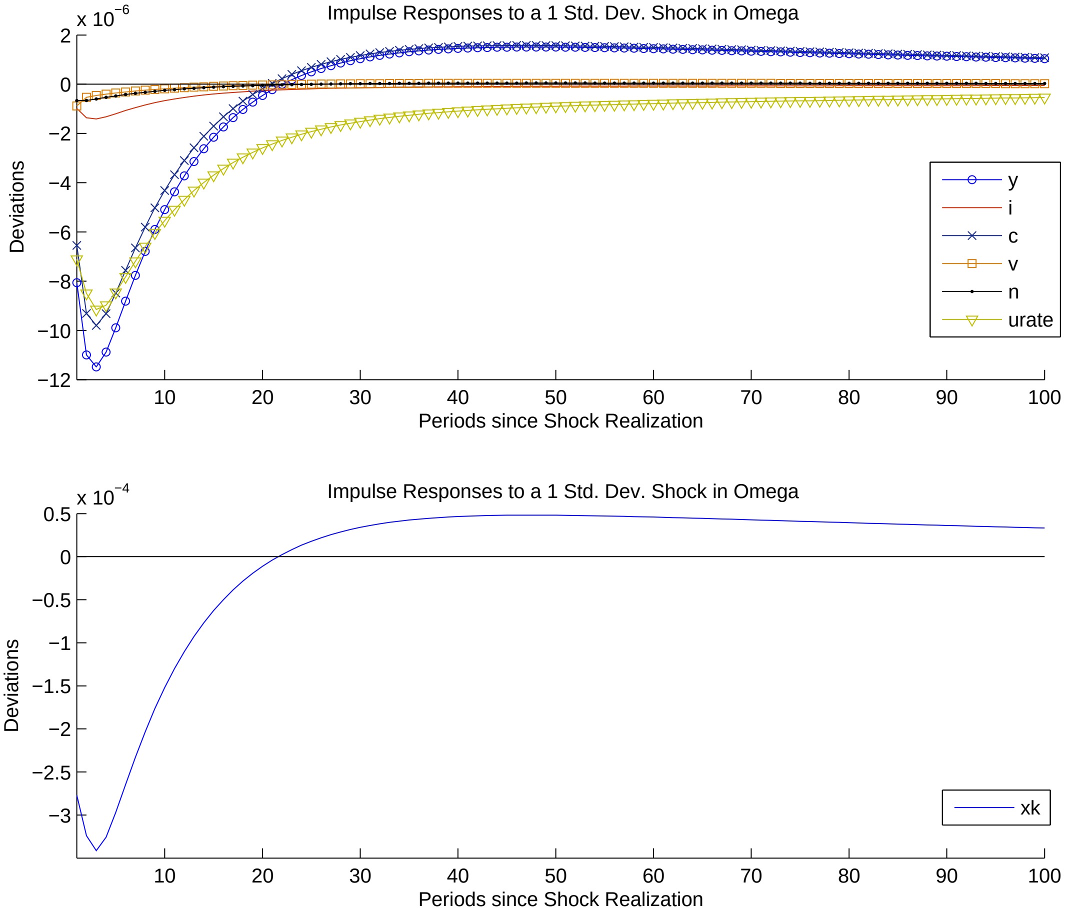

Figure 10: Macro IRFs: Volatility Shock to Productivity, Extended Model of Section 5.1

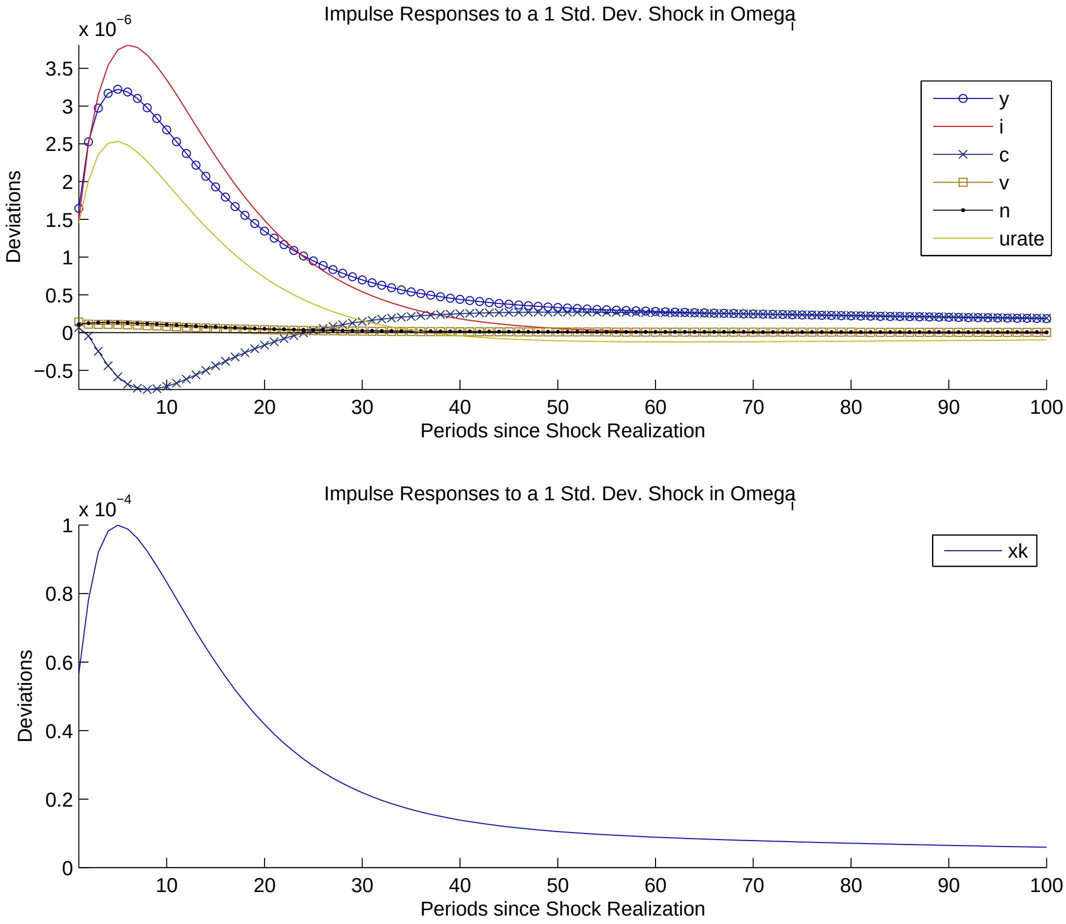

Figure 11: Macro IRFs: Volatility Shock to Investment, Extended Model of Section 5.1

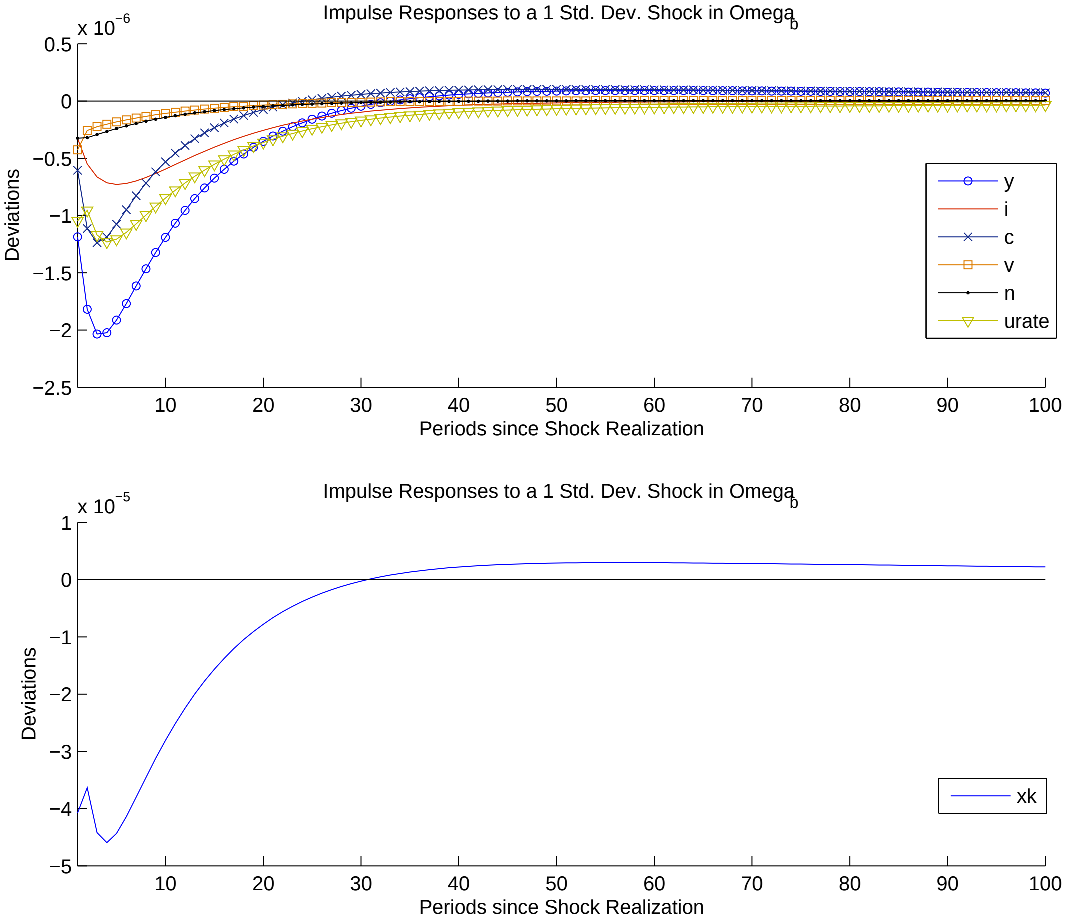

Figure 12: Macro IRFs: Volatility Shock to Preferences, Extended Model of Section 5.1

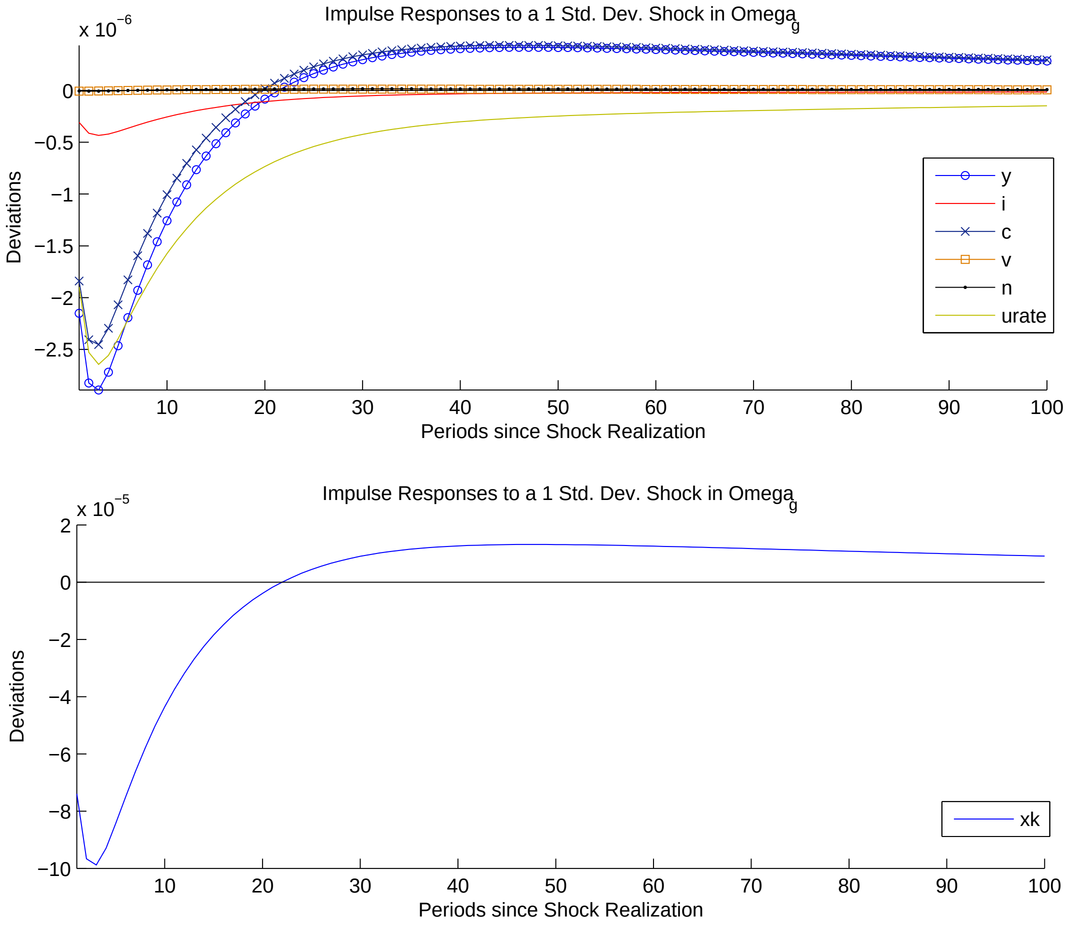

Figure 13: Macro IRFs: Volatility Shock to Government Spending, Extended Model of Section 5.1