Existence and Uniqueness of Perturbation Solutions in DSGE Models [*]

Hong Lan [†] Alexander Meyer-Gohde[§]

This Version: September 5, 2012

Abstract

We prove the existence of unique solutions for all undetermined coefficients of nonlinear perturbations of arbitrary order in a wide class of discrete time DSGE models under standard regularity and saddle stability assumptions for linear approximations. Our result follows from the straightforward application of matrix analysis to our perturbation derived with Kronecker tensor calculus. Additionally, we relax the assumptions needed for the local existence theorem of perturbation solutions and prove that the local solution is independent of terms first order in the perturbation parameter.

JEL classification: C61, C63, E17

Keywords: Perturbation; DSGE; nonlinear; Sylvester equations; matrix calculus; Bézout theorem

---

[*]We are grateful to Michael Burda, Monique Ebell, Jesús Fernández-Villaverde, Ken Judd, Helmut Lütkepohl, Magali Marx, and Peter Zadrozny, as well as to participants of CEF 2012 and EEA-ESEM 2012 and of research seminars at the HU Berlin and to two anonymous referees for useful comments, suggestions, and discussions. This research was supported by the DFG through the SFB 649 “Economic Risk”. Any and all errors are entirely our own.

[†]Humboldt-Universität zu Berlin, Institut für Wirtschaftstheorie II, Spandauer Straße 1, 10178 Berlin, Germany; Tel.: +49-30-2093 1466; E-Mail:lanhong@cms.hu-berlin.de

[§]Humboldt-Universität zu Berlin, Institut für Wirtschaftstheorie II, Spandauer Straße 1, 10178 Berlin, Germany; Tel.: +49-30-2093 5720; E-Mail: alexander.meyer-gohde@wiwi.hu-berlin.de

1 Introduction

Macroeconomists are increasingly using nonlinear methods to analyze dynamic stochastic general equilibrium (DSGE) models. One such method, perturbation,[1] successively differentiates the equilibrium conditions to recover the coefficients of a higher order Taylor expansion of the policy function. As emphasized by Gaspar and Judd (1997), Judd (1998, ch. 13), and Jin and Judd (2002), solvability/nonsingularity conditions must be fulfilled to ensure the existence of unique solutions for these undetermined coefficients of higher order terms. Current perturbation analyses proceed under the seemingly tenuous assumption that these solvability conditions hold generically, as no general set of conditions has been proven. We corroborate this approach by proving that the standard assumptions imposed on linear approximations to guarantee a unique stable solution are already sufficient to guarantee the existence and uniqueness of solutions for all the unknown coefficients of DSGE perturbations of an arbitrarily high order.

Our main result builds on the Sylvester equation representation common to many perturbation studies[2] by representing all of the linear equations in the undetermined coefficients at all orders of approximation in a Sylvester form. We confirm the result of Jin and Judd (2002) that the solvability conditions (i.e., invertibility of these linear maps or coefficient matrices) change as the order of approximation changes: at each order, the lone trailing matrix in the Sylvester equation is a Kronecker power of the linear transition matrix of the state space. Thus, the change in the solvability conditions is systematic and and the unit-root stability of this lone order dependant matrix is directly dependant on the eigenvalues of the matrix quadratic problem at first order.[3] The generalized Bézout theorem can be applied to deflate the quadratic equation with the unique stable first order solution to relate the set of remaining unstable eigenvalues to a generalized eigenvalue problem, which forms the remaining homogenous coefficients in the series of Sylvester equations. Due to the separation induced by saddle stability, the spectra of these pencils in the generalized Sylvester equation necessarily form a disjoint set, satisfying the necessary and sufficient conditions for the existence and uniqueness of solutions of Chu (1987) to the entire sequence of Sylvester equations. Likewise appealing to an eigenvalue sep-

[1]Perturbation in macro DSGE modeling initiated by Gaspar and Judd (1997) and Judd and Guu (1997) has been successfully applied to a variety of applications with a few recent examples including the effects of time varying volatility in interest rates for small open economies in Fernández-Villaverde, Guerrón-Quintana, Rubio-Ramírez, and Uribe (2011), to multi country real business cycle models in Kollmann, Kim, and Kim (2011), to the yield curve with recursive preferences and long run risks in Rudebusch and Swanson (2012).

[2]Beyond second order perturbations, Juillard and Kamenik (2004) and Kamenik (2005) provide a Sylvester representation for many of the unknown coefficients in their higher order perturbation.

[3]This matrix quadratic—see, e.g., Uhlig (1999)—is the sole exception to the Sylvester representation.

aration, Kim, Kim, Schaumburg, and Sims (2008) demonstrate the solvability of a portion of a second-order perturbation—our results show that an encompassing Sylvester representation can be used to extend their result to all coefficients at all orders of approximation. Thus, we prove that the solvability conditions do hold generically, as saddle stability at the first order ensures the invertibility of all subsequent linear maps regardless of the order of approximation.

Throughout, we take the existence and smoothness of the policy function as given and solve directly for unknown coefficients of its Taylor expansion. Assuming analyticity,[4] our result underlines that successive differentiation of the equilibrium conditions recovers the policy function inside its domain of convergence. Our factorization eliminates the solvability assumption in Jin and Judd's (2002) local existence theorem for solutions to nonlinear DSGE models.[5] Schmitt-Grohé and Uribe (2004) and others have argued that the first derivative of the policy function with respect to the perturbation parameter ought to be zero. However, they assume the invertibility of the mappings they show to be homogenous; we prove this invertibility.

The paper is organized as follows. Section 2 contains a nonlinear multivariate DSGE model and the preliminaries for the approximation to its policy function. We derive a perturbation of arbitrary order and present our main result—solvability of all coefficients given a unique stable first order solution—in section 3. Section 4 presents the proof, with the factored matrix quadratic at first order pivotal for the solvability of the sequence of equations for higher order coefficients. We turn to the proof of the local existence of the policy function and of its first-order independence from the perturbation parameter in section 5. Finally, section 6 concludes.

2 DSGE Problem Statement and Policy Function

We begin with our class of models, a system of (nonlinear) second order expectational difference equations, and a Taylor approximation of the policy function we take as a solution.

2.1 Model Class

We analyze a family of discrete-time rational expectations models given by

(1) $$0 = E_t[f(y_{t+1}, y_t, y_{t-1}, \varepsilon_t)]$$

the vector function $f : \mathbb{R}^{n_y} \times \mathbb{R}^{n_y} \times \mathbb{R}^{n_y} \times \mathbb{R}^{n_e} \to \mathbb{R}^{n_y}$ is assumed C^M with respect to all its arguments, where M is the order of approximation to be introduced subsequently; $y_t \in \mathbb{R}^{n_y}$

[4] See, e.g., Jin and Judd (2002) and Anderson, Levin, and Swanson (2006).

[5] See Woodford (1986) for an alternate approach in the space of infinite sequences of innovations.

endogenous and exogenous variables; and $\varepsilon_t \in \mathbb{R}^{n_e}$ exogenous shocks.[6] We assume that ε_t is i.i.d. with $E\left[\varepsilon_t\right] = 0$ and $E\left[\varepsilon_t^{\otimes[m]}\right]$ finite $\forall m \leq M$.[7]

2.2 Perturbation Solution

As is usual in perturbation methods, we introduce an auxiliary parameter $\sigma \in [0,\ 1]$ to scale the uncertainty in the model.[8] The stochastic model under study corresponds to $\sigma = 1$ and $\sigma = 0$ represents the deterministic version of the model. Indexing solutions likewise with σ

$$y_t = y(\sigma, z_t), \quad y : \mathbb{R}^+ \times \mathbb{R}^{n_z} \to \mathbb{R}^{n_y} \tag{2}$$

with the state vector z_t given by

$$z_t = \begin{bmatrix} y_{t-1} \\ \varepsilon_t \end{bmatrix} \in \mathbb{R}^{n_z \times 1}, \text{ where } n_z = n_y + n_e \tag{3}$$

Assuming time invariance of the policy function and scaling uncertainty give

$$y_{t+1} = \tilde{y}(\sigma, z_{t+1}),\ z_{t+1} = \begin{bmatrix} y_t \\ \sigma\varepsilon_{t+1} \end{bmatrix} \in \mathbb{R}^{n_z \times 1},\ \tilde{y} : \mathbb{R}^+ \times \mathbb{R}^{n_z} \to \mathbb{R}^{n_y} \tag{4}$$

The notation, y and $\tilde{y}$, is adopted to track the source (through y_t or y_{t+1}) of derivatives of the policy function. This is necessary as (i) the z_{t+1} argument of $\tilde{y}$ is itself a function of y through its dependance on y_t, and (ii) σ scales ε_{t+1} in the z_{t+1} argument of $\tilde{y}$, but not ε_t in the z_t argument of y. This follows from the conditional expectations in (1): Conditional on t, ε_t has been realized and is known with certainty—hence, it is not scaled with σ; ε_{t+1}, however, has not yet been realized and is the source of uncertainty—hence, it is scaled with σ.[9]

Inserting the policy functions for y_t and y_{t+1}—equations (2) and (4)—into (1) yields

$$0 = E_t\left[f\left(\tilde{y}\left(\sigma, \begin{bmatrix} y(\sigma, z_t) \\ \sigma\varepsilon_{t+1} \end{bmatrix}\right), y(\sigma, z_t), z_t\right)\right] = F(\sigma, z_t) \tag{5}$$

a function with arguments σ and z_t.[10] We will construct a Taylor series approximation of the solution (2) around a deterministic steady state defined as

Definition 2.1. *Deterministic Steady State*

[6] This model class encompasses competitive equilibria and dynamic programming problems, as well as models with finitely many heterogenous agents, see Judd and Mertens (2012). Nonlinearity or serial correlation in exogenous processes can be captured in the function f and the processes themselves are included in the vector y_t.

[7] $\varepsilon_t^{\otimes[m]}$ is the m'th fold Kronecker product of ε_t with itself: $\underbrace{\varepsilon_t \otimes \varepsilon_t \cdots \otimes \varepsilon_t}_{m\ times}$.

[8] Our formulation follows Adjemian, Bastani, Juillard, Mihoubi, Perendia, Ratto, and Villemot's (2011) Dynare, Anderson, Levin, and Swanson's (2006) PerturbationAIM, Juillard (2011), and Lombardo (2010). Jin and Judd's (2002) or Schmitt-Grohé and Uribe's (2004) model classes can be rearranged to fit (1).

[9] See also Anderson, Levin, and Swanson (2006) and Juillard (2011) for similar discussions.

[10] Note that ε_{t+1} is not an argument of F as it is the variable of integration inside the expectations. I.e.,

$$F(\sigma, z_t) = \int_\Omega f\left(\tilde{y}\left(\sigma, \begin{bmatrix} y(\sigma, z_t) \\ \sigma\varepsilon_{t+1} \end{bmatrix}\right), y(\sigma, z_t), z_t\right)\phi\left(\varepsilon_{t+1}\right) d\varepsilon_{t+1}$$

where Ω is the support and ϕ the p.d.f. of ε_{t+1}. Thus, when $\sigma = 0$, ε_{t+1} is no longer an argument of f and the integral (and hence the expectations operator) is superfluous, yielding the deterministic version of the model.

Let $\bar{y} \in \mathbb{R}^{n_y}$ be a vector such that

$$0 = F(0, \bar{z}), \text{ where } \bar{z} = \begin{bmatrix} \bar{y} \\ 0 \end{bmatrix} \tag{6}$$

solving (5) in the absence of both uncertainty $(\sigma = 0)$ and shocks ($\varepsilon_t = 0$).[11]

The policy function evaluated at the deterministic steady state is thus $\bar{y} = y(0, \bar{z})$ and $\bar{y}$ likewise solves $0 = f(\bar{y}, \bar{y}, \bar{y}, 0)$. We will admit models that possess unit root solutions in the first order approximation and do not require the deterministic steady state to be unique.[12]

2.3 Taylor Series Approximation

Since y is a vector valued function, its partial derivatives form a hypercube. We use the method of Lan and Meyer-Gohde (2012b) that differentiates conformably with the Kronecker product, allowing us to maintain standard linear algebraic structures to derive our results.

Definition 2.2. *Matrix Derivatives*
Let $A(B) : \mathbb{R}^{s\times 1} \to \mathbb{R}^{p\times q}$ be a matrix-valued function that maps an $s \times 1$ vector B into an $p \times q$ matrix $A(B)$, the derivative structure of $A(B)$ with respect to B is defined as

$$A_B \equiv \mathscr{D}_{B^T}\{A\} \equiv \begin{bmatrix} \frac{\partial}{\partial b_1} & \ldots & \frac{\partial}{\partial b_s} \end{bmatrix} \otimes A \tag{7}$$

where b_i denotes i'th row of vector B, ^{T} indicates transposition; n'th derivatives are

$$A_{B^n} \equiv \mathscr{D}_{(B^T)^n}\{A\} \equiv \left(\begin{bmatrix} \frac{\partial}{\partial b_1} & \ldots & \frac{\partial}{\partial b_s} \end{bmatrix}^{\otimes[n]} \right) \otimes A \tag{8}$$

We leave the details of the associated calculus that generalizes familiar chain and product rules as well as Taylor approximations to multidimensional settings to the Appendix.

With definition 2.2 and assuming (2) is C^M with respect to all its arguments, we can write a Taylor series approximation of $y_t = y(\sigma, z_t)$ at a deterministic steady state as

$$y_t = \sum_{j=0}^{M} \frac{1}{j!} \left[\sum_{i=0}^{M-j} \frac{1}{i!} y_{z^j \sigma^i} \sigma^i \right] (z_t - \bar{z})^{\otimes[j]} \tag{9}$$

where $y_{z^j \sigma^i} \in \mathbb{R}^{n_y \times n_z^j}$ is the partial derivative of the vector function y with respect to the state vector z_t j times and the perturbation parameter σ i times evaluated at the deterministic steady state. Here $\left[\sum_{i=0}^{M-j} \frac{1}{i!} y_{z^j \sigma^i} \sigma^i\right]$ collects all the coefficients associated with the j'th fold Kronecker product of the state vector, $(z_t - \bar{z})$. Higher orders of σ correct the Taylor series coefficients for uncertainty by successively opening the coefficients to higher moments in the distribution of future shocks.[13] We need to generate and solve equations that determine these $y_{z^j \sigma^i}$.

[11] Accordingly, the stochastic or "risky" steady state would solve $0 = F(1, \bar{z})$.

[12] The degenerate nonuniqueness as studied, for example, in Coeurdacier, Rey, and Winant (2011) and Juillard (2011), however, cannot be studied with the standard perturbation approach of this paper.

[13] A similar interpretation can be found in Judd and Mertens (2012) for univariate expansions and in Lan and Meyer-Gohde (2012b) for expansions in infinite sequences of innovations.

3 Higher Order Perturbation: Existence and Uniqueness

3.1 Equations Characterizing the Coefficients

Following general practice, we pin down the coefficient matrices $y_{z^j\sigma^i}$ in (9) though repeated application of an implicit function theorem by successively differentiating (5) and solving the resulting systems of equations. It is the existence and uniqueness of solutions to these equations (and hence for the coefficients in the Taylor series) that is the focus of our analysis.

A standard result in the literature, noted by Judd (1998, ch. 13), Jin and Judd (2002), Schmitt-Grohé and Uribe (2004) and others, is that the higher order terms of the Taylor expansion are solutions to linear problems taking the coefficients from lower orders as given. A Sylvester form for these linear equations has been identified in previous studies,[14] to our knowledge, however, ours is the first representation that (i) expresses all equations of an arbitrary order perturbation as Sylvester equations and (ii) provides a closed form representation of the order dependency of the homogenous part of the equations (Kronecker products in $z_y y_z$).

3.1.1 Deterministic First Order Term y_z and Matrix Quadratic

To recover y_z, we first differentiate f in (5) with respect to z_t

$$\mathscr{D}_{z_t^T}\{f\} = f_{\tilde{y}}\tilde{y}_z z_y y_z + f_y y_z + f_z \tag{10}$$

Evaluating this at the deterministic steady state and setting its expectation to zero yields

$$E_t\left[\mathscr{D}_{z_t^T}\{f\}\right]\Big|_{\substack{z_t=\bar{z}\\ \sigma=0}} = f_{\tilde{y}} y_z z_y y_z + f_y y_z + f_z = 0 \tag{11}$$

Taking $y_z z_y$ as given, y_z then solves

$$(f_{\tilde{y}} y_z z_y + f_y) y_z + f_z = 0 \tag{12}$$

Postmultiplying the foregoing with z_y yields

$$f_{\tilde{y}}(y_z z_y)^2 + f_y y_z z_y + f_z z_y = 0 \tag{13}$$

This is the familiar matrix quadratic equation (in $y_z z_y$) from linear analyses.[15]

3.1.2 Arbitrary Order Terms $y_{z^j\sigma^i}$

For all other coefficients, we successively differentiate (5) with respect to the state vector z_t and the perturbation parameter σ, evaluate the resulting expressions at the deterministic steady state

[14] The Sylvester form in second order context of, e.g., Kim, Kim, Schaumburg, and Sims (2008) or Gomme and Klein (2011) aside, Juillard and Kamenik (2004) and Kamenik (2005) show explicitly that many of the unknown coefficients of a perturbation of arbitrary order can be cast as Sylvester equations.

[15] See, e.g., Uhlig (1999).

and set their expectations equal to zero. This generates a set of generalized Sylvester equations,

Lemma 3.1. *For all* $j, i \in \mathbb{N}^0$ *such that* $j+i \geqslant 1$ *except the case* $j=1$ *and* $i=0$*, the undetermined coefficients* $y_{z^j\sigma^i}$ *solve the following generalized Sylvester equation*

$$f_{\tilde{y}} y_{z^j\sigma^i} (z_y y_z)^{\otimes[j]} + (f_y + f_{\tilde{y}} y_z z_y) y_{z^j\sigma^i} + A(j,i) = 0 \tag{14}$$

where $A(j,i)$ *is a function of coefficients from lower orders and given moments* $E\left[\varepsilon_t^{\otimes[k]}\right]$, $k \leq i$*.*

Proof. See the Appendix. □

This representation provides an explicit formulation of the homogenous structure of the equations that the unknown coefficients of each order of approximation must fulfill,[16] which will facilitate the analysis of solvability using linear algebra. At each order, the leading matrix coefficients, $f_{\tilde{y}}$ and $f_y + f_{\tilde{y}} y_z z_y$, remain unchanged and are formed by the coefficients of unstable factorization of the matrix quadratic as will be detailed in proposition 4.6. The trailing matrix coefficient, $(z_y y_z)^{\otimes[j]}$, is a Kronecker power of the linear transition matrix of the state space and changes with the order of approximation. This is the source for the dependence of the solvability conditions on the order of approximation identified by Jin and Judd (2002).[17] However, this dependence is systematic and has a convenient closed form.

3.2 Existence and Uniqueness of the Coefficients

Here, we present our main result that the existence of a unique stable solution at first order guarantees the existence and uniqueness of the unknown coefficients of a Taylor expansion of arbitrary order.

We guarantee a unique stable solution at first order with Blanchard and Kahn's (1980) order and rank conditions.[18] The order condition assumes a full set of latent roots with half on or inside and half outside the unit circle

Assumption 3.2. *Order*

There exists $2n_y$ *latent roots of* $f_{\tilde{y}}\lambda^2 + f_y\lambda + f_z z_y$*—that is,* $\lambda \in \mathbb{R} : \det(f_{\tilde{y}}X^2 + f_y X + f_z z_y) = 0$*— of which* n_y *lie inside or on the unit circle and* n_y *outside.*

[16] The derivations for the second order expansion and the three resulting Sylvester equations of (14) in y_{z^2}, $y_{z\sigma}$, and y_{σ^2} can be found in the Appendix and are needed to initialize the induction that proves lemma 3.1.

[17] Specifically, Jin and Judd (2002) first develop a deterministic perturbation in z_t only and then perturb stochastically with respect to σ. They point out that the change in the solvability conditions occurs only in a change in the order of approximation in the deterministic perturbation. This is reflected in (14) as the only change in the homogenous components occurs with j, the order of the perturbation with respect to the state vector z_t.

[18] In the working paper version, Lan and Meyer-Gohde (2012a), we derive the results from the assumptions of Klein (2000) on the companion linearized pencil of the matrix quadratic and its generalized Schur decomposition. We hasten the exposition by imposing the existence of a unique stable solution and regularity directly.

We then assume that a solution can be constructed containing these stable eigenvalues

Assumption 3.3. *Rank*
There exists an $X \in \mathbb{R}^{n_y \times n_y}$ *such that* $f_{\tilde{y}} X^2 + f_y X + f_z z_y = 0$ *and* $|eig(X)| \leq 1$.

We now state our main result,

Theorem 3.4. *Let the assumptions 3.2 and 3.3 be fulfilled and set* $y_z z_y$ *equal to this stable solvent, then there exist unique solutions,* $y_{z^j \sigma^i}$ *for all* $j, i \in \mathbb{N}^0$ *such that* $j + i \geqslant 1$*, for (12) and the generalized Sylvester equations (14) in lemma 3.1*

Proof. From lemmata 4.10 and 4.11, the conditions of proposition 4.8 are fulfilled for for all $j, i \in \mathbb{N}^0$ such that $j + i \geqslant 1$. See the following section. □

While the solvability of coefficients outside the matrix quadratic in linear models is guaranteed by any separation (and not just unit root) of eigenvalues, we must be more careful in nonlinear models. Kim, Kim, Schaumburg, and Sims (2008) require that the square of the largest eigenvalue in the linear transition matrix be smaller than the smallest unstable eigenvalue for their second order solution. Moving to an M'th order of approximation, the smallest unstable root in assumption 3.2 would analogously need to be larger than the M'th power of the largest eigenvalue in $y_z z_y$, the largest stable root in assumption 3.2. Requiring stability with respect to the unit circle at the first order, of course, eliminates this problem and ensures solvability for perturbations of arbitrary order.[19]

4 Solvents, Sylvesters, and Proof of Theorem 3.4

After laying out some preliminaries, we factor the matrix quadratic into two regular pencils with disjoint spectra by deflating the matrix quadratic (13) according to the Generalized Bézout Theorem with the stable solvent of assumption 3.3. We then apply this factorization to the sequence of generalized Sylvester equations (14) in lemma 3.1 and prove that the existence of unique solutions is guaranteed by assumptions 3.2 and 3.3.

4.1 A Factorization of the Matrix Quadratic

To derive our factorization, we begin by formalizing the matrix quadratic equation (13). Our analysis will proceed initially in the complex plane, but the results carry over when we restrict

[19] These separations are merely sufficient. The necessary disjointness of lemma 4.11 would still be satisfied up to M'th order if there is no M'th order or less product of eigenvalues of $y_z z_y$ equal to an eigenvalue of $P_U(z)$.

solutions to be real valued due to the eigenvalue separation in assumption 3.2, see also Klein (2000).

Definition 4.1. *Matrix Quadratic Problem*

For $f_{\tilde{y}}$, f_y, *and* $f_z z_y \in \mathbb{R}^{n_y \times n_y}$, *a matrix quadratic* $M(X) : \mathbb{C}^{n_y \times n_y} \to \mathbb{C}^{n_y \times n_y}$ *is defined as*

(15) $$M(X) \equiv f_{\tilde{y}} X^2 + f_y X + f_z z_y$$

A solution to the matrix quadratic (15) is called a solvent and is defined as

Definition 4.2. *Solvent of Matrix Quadratic*

$X \in \mathbb{C}^{n_y \times n_y}$ *is a solvent of the matrix quadratic (15) if and only if* $M(X) = 0$

The eigenvalues of solvents of (15) are latent roots of the associated lambda matrix,[20]

Definition 4.3. *Lambda Matrix*

The lambda matrix $M(\lambda) : \mathbb{C} \to \mathbb{C}^{n \times n}$ *(of degree two) associated with (15) is given by*

(16) $$M(\lambda) \equiv f_{\tilde{y}} \lambda^2 + f_y \lambda + f_z z_y$$

Its latent roots are values of λ *such that* $\det M(\lambda) = 0$.

We are now prepared to link lambda matrices and solvents through the generalized Bézout theorem, repeated in the Appendix, which states that a lambda matrix divided on the right by a binomial in a matrix has as a remainder the matrix polynomial associated with the lambda matrix evaluated at the matrix of the binomial. As noted by Gantmacher (1959, vol. I, ch. 4) and repeated in Lancaster (1966) and Higham and Kim (2000), if this matrix in the binomial is a solvent of the matrix polynomial, the division is without remainder, yielding a factorization of the matrix polynomial. For our matrix quadratic, the lambda matrix can then be factored as

Corollary 4.4. *Let* $y_z z_y$ *be the stable solvent of assumption 3.3, then (16) has the following factorization*

(17) $$M(\lambda) = \underbrace{(\lambda f_{\tilde{y}} + f_{\tilde{y}} y_z z_y + f_y)}_{\equiv P_U(\lambda)} \underbrace{(I_{n_y} \lambda - y_z z_y)}_{\equiv P_S(\lambda)}$$

Proof. Apply theorem A.4 in the Appendix to (15), set $A = y_z z_y$, and note that $M(y_z z_y) = 0$ as $y_z z_y$ is a solvent of $M(X)$. □

Note that the latent roots of $M(\lambda)$ are given by λ's such that

(18) $$\det(\lambda f_{\tilde{y}} + f_{\tilde{y}} y_z z_y + f_y) \det(I_{n_y} \lambda - y_z z_y) = 0$$

The latter determinant gives the eigenvalues associated with the solvent $y_z z_y$ and the former determinant gives a generalized eigenvalue problem in the coefficients of $M(X)$ and the solvent

[20] See, e.g., J. E. Dennis, Traub, and Weber (1976, p. 835) or Gantmacher (1959, vol. I, p. 228).

$y_z z_y$.[21] This former determinant is simply a multidimensional lambda matrix analog to Viéte's formula, which relates the two solutions, x_1 and x_2, of the scalar quadratic $ax^2+bx+c=0$ through $a(x_1+x_2)+b=0$. We can now use assumption 3.2 on the number of eigenvalues to establish the regularity of $P_U(\lambda)$ and $P_S(\lambda)$

Lemma 4.5. *The pencils $P_U(\lambda)$ and $P_S(\lambda)$ are both regular.*

Proof. As $\det M(\lambda)$ is vanishing for only $2n_y$ values (respecting multiplicities) in $\mathbb{C}$, $\det P_U(\lambda)$ and $\det P_S(\lambda)$ are likewise vanishing for only n_y values (respecting multiplicities) in $\mathbb{C}$. Thus, there exists $\lambda \in \mathbb{C}$ such that $\det P_U(\lambda) \neq 0$ and likewise such that $\det P_U(\lambda) \neq 0$. □

Additionally, assumption 3.2 restricts the eigenvalues of $P_U(\lambda)$.

Proposition 4.6. *Let $y_z z_y$ be the stable solvent of assumption 3.3, the eigenvalues of $P_U(\lambda)$ are contained entirely outside the closed unit circle.*

Proof. From assumption 3.2, there are exactly n_y latent roots of $M(\lambda)$ inside or on the unit circle and exactly n_y outside the unit circle. The n_y eigenvalues of the pencil $P_S(\lambda)$ are all inside or on the unit circle by assumption 3.3. Hence, the n_y eigenvalues of $P_U(\lambda)$ are the n_y remaining latent roots of $M(\lambda)$, which must be outside the unit circle. □

From proposition 4.6, there exists a unique solution to (12)

Corollary 4.7. *Let $y_z z_y$ be the stable solvent of assumption 3.3, there exists a unique y_z that solves (12), given by*

$$y_z = -\left(f_{\tilde{y}} y_z z_y + f_y\right)^{-1} f_z \tag{19}$$

Proof. At issue is whether $f_{\tilde{y}} y_z z_y + f_y$ is nonsingular. As the eigenvalues of $P_U(\lambda)$ are all outside the unit circle following proposition 4.6, $\det\left(\lambda f_{\tilde{y}} + f_{\tilde{y}} y_z z_y + f_y\right) \neq 0$ for $|\lambda| \leq 1$. This applies, of course to $\lambda = 0$, from which we can see that $\det\left(f_{\tilde{y}} y_z z_y + f_y\right) \neq 0$ □

The regularity of these pencils and the disjointness of their spectra will be central to the solvability of the undetermined coefficients of perturbations of arbitrary order.

4.2 Existence and Uniqueness in Sylvester Equations

The necessary and sufficient conditions proposed by Theorem 1 of Chu (1987) for the existence and uniqueness of solutions to generalized Sylvester equations requires the two matrix pencils formed by the leading and trailing matrix coefficients to be regular and have disjoint spectra. We prove here that they are fulfilled for all our equations in lemma 3.1 as a direct consequence

[21] The Appendix contains a definition of a pencil, $P(\lambda)$, and its spectrum or set of generalized eigenvalues, $\rho(P)$.

of the existence of the unique stable solution at first order. We adapt his theorem, adopting his notation temporarily, to our purposes in the following

Proposition 4.8. *There exists a unique solution, $X \in \mathbb{R}^{m\times n}$, for the Sylvester equation*

$$AXB+CXD+E=0$$

where $A,C \in \mathbb{R}^{m\times m}$ and $D,B \in \mathbb{R}^{n\times n}$, if and only if

1. *$P_{AC}(z) \equiv Az+C$ and $P_{DB}(z) \equiv Dz-B$ are regular matrix pencils, and*
2. *$\rho(P_{AC}) \cap \rho(P_{DB}) = \emptyset$, e.g. their spectra are disjoint*

Proof. See Chu (1987). Notice the rearrangement and redefinition of terms. ☐

Before we examine the general case, we will highlight the intuition behind proposition 4.8 using a scalar version of (14), when $f_{\tilde{y}}$, $f_y, y_z z_y$, and $z_y y_z \in \mathbb{R}$ and $A(j,i)$ is a scalar function of known terms.[22] In this case, (14) can be arranged as

$$\left[f_{\tilde{y}}\,(z_y y_z)^j + (f_y + f_{\tilde{y}} y_z z_y)\right] y_{z^j\sigma^i} + A(j,i) = 0 \tag{20}$$

From, e.g., Strang (2009), the foregoing has a unique solution if and only if the leading coefficient is not zero, i.e., $\left[f_{\tilde{y}}\,(z_y y_z)^j + (f_y + f_{\tilde{y}} y_z z_y)\right] \neq 0$. As otherwise there is either no solution (when $A(j,i) \neq 0$) or there exists infinitely many solutions (when $A(j,i) = 0$). The conditions in proposition 4.8 classify the two ways this coefficient can be equal to zero.

The regularity condition in the scalar case precludes both coefficients in either of the pencils being equal to zero: either $f_{\tilde{y}} = f_y + f_{\tilde{y}} y_z z_y = 0$ or $1 = (z_y y_z)^j = 0$. Obviously, both coefficients in the trailing pencil cannot be zero and this general regularity holds in the matrix case as well. The second condition, disjoint spectra, rules out the remaining hurdle that the sum of all the coefficients is zero, which can be rearranged as $\frac{f_y + f_{\tilde{y}} y_z z_y}{f_{\tilde{y}}} \neq (z_y y_z)^j$. Recognize that the two terms correspond to the eigenvalues of the scalar regular pencils $P_U(\lambda)$ and $P_{IS}(\lambda)$, hence their set of eigenvalues (or spectra) must not contain any identical elements (be disjoint).

Returning to the general case, we first define the leading and trailing matrix pencils and then establish their regularity and the disjointness of their spectra.

Definition 4.9. *For all $j \in \mathbb{N}^0$, the leading and trailing matrix pencils, respectively, of the generalized Sylvester equation (14) in lemma 3.1 are*

1. *$\lambda f_{\tilde{y}} + f_{\tilde{y}} y_z z_y + f_y = P_U(\lambda)$, see proposition 4.6*
2. *$P_{IS}(\lambda) \equiv \lambda I_{n_z^j} - (z_y y_z)^{\otimes[j]}$*

The regularity of both the pencils is straightforward and is summarized in the following

[22]This special case, of course, is not useful practically. Either all shocks or the presence of y_{t-1} has to be shut down, but the mechanisms behind the matrix case are usefully illustrated in this case.

Lemma 4.10. *For all $j \in \mathbb{N}^0$, $P_U(\lambda)$ and $P_{IS}(\lambda)$ and are regular*

Proof. For $P_U(\lambda)$, see lemma 4.5. For $P_{IS}(\lambda)$, this follows from its leading matrix being the identity matrix, see Gantmacher (1959, vol. II, pp. 25–27). □

The spectral disjointness follows nearly directly from the factorization of the matrix quadratic in corollary 4.4, with the spectrum of the leading pencil $P_U(\lambda)$ being outside and that of the trailing pencil $P_{IS}(\lambda)$ being inside the closed unit circle. From corollary 4.4, the pencil $P_S(\lambda) = I_{n_y}\lambda - y_z z_y$ is stable, but noting that z_y and z_ε are two constant matrices with all their entries being either unity or zero

$$z_y \equiv \mathscr{D}_{y_{t-1}^T}\{z_t\} = \mathscr{D}_{y_t^T}\{z_{t+1}\} = \begin{bmatrix} I_{n_y} \\ 0_{n_e \times n_y} \end{bmatrix},\ z_\varepsilon \equiv \mathscr{D}_{\varepsilon_t^T}\{z_t\} = \mathscr{D}_{\sigma\varepsilon_{t+1}^T}\{z_{t+1}\} = \begin{bmatrix} 0_{n_y \times n_e} \\ I_{n_e} \end{bmatrix} \tag{21}$$

the matrix $z_y y_z$ in $P_{IS}(\lambda)$ is

$$z_y y_z = \begin{bmatrix} y_z z_y & y_z z_\varepsilon \\ 0_{n_e \times n_y} & 0_{n_e \times n_e} \end{bmatrix} \tag{22}$$

and it follows directly[23] that the the eigenvalues of $P_{IS}(\lambda)$ are all stable with respect to the closed unit circle, and thus those of an arbitrary Kronecker power too. We summarize the disjointness in the following

Lemma 4.11. *For all $j \in \mathbb{N}^0$, the spectra of $P_U(\lambda)$ and $P_{IS}(\lambda)$ form a disjoint set.*

Proof. See Appendix. □

From lemmata 4.10 and 4.11, proposition 4.8 applies and the existence and uniqueness of solutions to the generalized Sylvester equations (14) in lemma 3.1 is immediate, completing—along with corollary 4.7—the proof of our main result in theorem 3.4.

5 Applications

Jin and Judd (2002) provide a local existence theorem for the solution to stochastic models. We eliminate their solvability assumption, as their assumption of a unique locally asymptotically stable solution enables us to apply our factorization to confirm that their solvability assumption is necessarily fulfilled, analogously to our theorem 3.4.

Theorem 5.1. *Simplified Local Existence Theorem of Jin and Judd (2002)*
If (i) the function f in (1) exists and is analytic for all ε_t in some neighborhood of $\bar{z}$ defined in (6), (ii) there exists a unique deterministic solution $y(0,z_t)$ locally analytic in z_t and locally asymptotically stable, (iii) $E[\varepsilon_t] = 0$, and (iv) ε_t has bounded support, then there is an $r > 0$ such that for all (z_t,σ) in a ball with radius r centered at $(0,\bar{z})$ there exists a unique solution

[23] See, e.g., Golub and Loan (1996, p. 311).

$y(\sigma,z_t)$ to (5). Furthermore, all derivatives of $y(\sigma,z_t)$ exist in a neighborhood of $(0,\bar{z})$ and can be solved by implicit differentiation.

Proof. See the Appendix. □

All told, what is needed for the local existence of a solution to a stochastic problem is sufficient differentiability of the equilibrium conditions, the existence of a solution to the deterministic variant of the model and restrictions on the moments and support of the stochastic processes that ensure the model remains well defined.[24]

Previous studies have conjectured the independence of the policy function from first order effects of the perturbation parameter ($y_{z^j\sigma^i} = 0$ for $i = 1$), as the equations that these coefficients solve are homogenous. The conjecture lies in the solvability of these systems: Schmitt-Grohé and Uribe (2004) to second, Andreasen (2012) to third, and Jin and Judd (2002) to arbitrary order prove that the unknown coefficients involving the perturbation parameter solve homogeneous equations. Of course, the zero solution solves these equations, but the claim that the solution is uniquely zero requires solvability in addition to homogeneity—see, e.g., Strang (2009). Theorem 3.4 adds the missing link, showing not only that zero is a solution (as follows from homogeneity), but that it is the only solution for a linearly saddle stable model. With the first moment of exogenous shocks and all $y_{z^k\sigma}$ for $k < j$ zero, the generalized Sylvester equations in $y_{z^j\sigma}$ are homogenous

$$f_{\tilde{y}} y_{z^j\sigma} (z_y y_z)^{\otimes[j]} + (f_y + f_{\tilde{y}} y_z z_y) y_{z^j\sigma} = 0 \tag{23}$$

As the zero matrix is always a solution to (23) and the solution must be unique following theorem 3.4, $y_{z^j\sigma} = 0$ is the unique solution for all j. We formalize this in the following

Proposition 5.2. *Let the conditions for theorem 3.4 hold, then $y_{z^j\sigma} = 0$ for all $j \in \mathbb{N}^0$.*

Proof. See the Appendix. □

The intuition behind this is simple: the unknown coefficient $y_{z^j\sigma}$ is the comparative static matrix measuring the impact of the first moment of future exogenous shocks on the policy function y (and its derivatives with respect to the state vector z_t). As the first moment of future exogenous shocks is assumed to be zero, it has no impact at all. Thus, our main result confirms the conjecture of Jin and Judd (2002) and Schmitt-Grohé and Uribe (2004) by providing the necessary solvability so as to add uniqueness to their existence of the zero solution.

[24]See Woodford (1986) for a related result in the space of infinite sequences of innovations.

6 Conclusion

We have proven the existence and uniqueness of solutions for the undetermined coefficients in perturbations of an arbitrarily high order. Thus, solvability of the higher order terms in a nonlinear perturbation as questioned by Gaspar and Judd (1997), Judd (1998, ch. 13), and Jin and Judd (2002) is guaranteed if the model possesses a unique stable solution at first order. That is, successive differentiation of the equilibrium condition of a linearly saddle stable model leads to a unique set of coefficients for a Taylor expansion of the policy function.

With the recent proliferation of interest in nonlinear methods and general familiarity of economists with the first order perturbation—i.e., (log-)linearization, our results should provide confidence to researchers refining their approximations to incorporate nonlinearity that their perturbations of arbitrary order will necessarily be associated with a unique solution if the linear approximation has a unique stable solution. For users of numerical perturbation algorithms, we have answered two questions. First, given a nonlinear perturbation solution from a numerical algorithm, is this solution the only solution? Second, should a numerical algorithm fail to deliver a solution: does a solution not exist at all or did the numerical algorithm simply fail to find a solution? Given a unique stable solution at first order, our results provide a definitive assurance that solutions for all unknown coefficients in the perturbation exist and are unique.

References

ADJEMIAN, S., H. BASTANI, M. JUILLARD, F. MIHOUBI, G. PERENDIA, M. RATTO, AND S. VILLEMOT (2011): "Dynare: Reference Manual, Version 4," Dynare Working Papers 1, CEPREMAP.

ANDERSON, G. S., A. LEVIN, AND E. SWANSON (2006): "Higher-Order Pertubation Solutions to Dynamic Discrete-Time Rational Expectations Models," Federal Reserve Bank of San Francisco Working Paper Series 2006-01.

ANDREASEN, M. M. (2012): "On the Effects of Rare Disasters and Uncertainty Shocks for Risk Premia in Non-Linear DSGE Models," *Review of Economic Dynamics*, 15(3), 295316.

BLANCHARD, O. J., AND C. M. KAHN (1980): "The Solution of Linear Difference Models under Rational Expectations," *Econometrica*, 48(5), 1305–1311.

CHU, K.-W. E. (1987): "The Solution of the Matrix Equations $AXB-CXD=E$ and $(YA-DZ,YC-BZ)=(E,F)$," *Linear Algebra and its Applications*, 93, 93–105.

COEURDACIER, N., H. REY, AND P. WINANT (2011): "The Risky Steady State," *American Economic Review*, 101(3), 398–401.

FERNÁNDEZ-VILLAVERDE, J., P. A. GUERRÓN-QUINTANA, J. RUBIO-RAMÍREZ, AND M. URIBE (2011): "Risk Matters: The Real Effects of Volatility Shocks," *American Economic Review*, 101(6), 2530–61.

GANTMACHER, F. R. (1959): *The Theory of Matrices*, vol. I&II. Chelsea Publishing Company, New York, NY.

GASPAR, J., AND K. L. JUDD (1997): "Solving Large-Scale Rational-Expectations Models," *Macroeconomic Dynamics*, 1(01), 45–75.

GOLUB, G. H., AND C. F. V. LOAN (1996): *Matrix Computations*. The Johns Hopkins University Press, Baltimore, MD, and London, UK, 3 edn.

GOMME, P., AND P. KLEIN (2011): "Second-Order Approximation of Dynamic Models without the Use of Tensors," *Journal of Economic Dynamics and Control*, 35(4), 604–615.

HIGHAM, N. J., AND H.-M. KIM (2000): "Numerical Analysis of a Quadratic Matrix Equation," *IMA Journal of Numerical Analysis*, 20, 499–519.

J. E. DENNIS, J., J. F. TRAUB, AND R. P. WEBER (1976): "The Algebraic Theory of Matrix Polynomials," *SIAM Journal on Numerical Analysis*, 13(6), 831–845.

JIN, H.-H., AND K. L. JUDD (2002): "Pertubation Methods for General Dynamic Stochastic Models," Mimeo April.

JUDD, K. L. (1998): *Numerical Methods in Economics*. MIT Press, Cambridge, MA.

JUDD, K. L., AND S.-M. GUU (1997): "Asymptotic Methods for Aggregate Growth Models," *Journal of Economic Dynamics and Control*, 21(6), 1025–1042.

JUDD, K. L., AND T. M. MERTENS (2012): "Equilibrium Existence and approximation of Incomplete Market Models with Substantial Heterogeneity," Mimeo March.

JUILLARD, M. (2011): “Local Approximation of DSGE Models around the Risky Steady State,” Mimeo October.

JUILLARD, M., AND O. KAMENIK (2004): “Solving Stochastic Dynamic Equilibrium Models: a k-Order Perturbation Approach,” Mimeo Autumn.

KAMENIK, O. (2005): “Solving SDGE Models: A New Algorithm for the Sylvester Equation,” *Computational Economics*, 25(1), 167–187.

KIM, J., S. KIM, E. SCHAUMBURG, AND C. A. SIMS (2008): “Calculating and Using Second-Order Accurate Solutions of Discrete Time Dynamic Equilibrium Models,” *Journal of Economic Dynamics and Control*, 32(11), 3397–3414.

KLEIN, P. (2000): “Using the Generalized Schur Form to Solve a Multivariate Linear Rational Expectations Model,” *Journal of Economic Dynamics and Control*, 24(10), 1405–1423.

KOLLMANN, R., J. KIM, AND S. H. KIM (2011): “Solving the Multi-Country Real Business Cycle Model Using a Perturbation Method,” *Journal of Economic Dynamics and Control*, 35(2), 203–206.

LAN, H., AND A. MEYER-GOHDE (2012a): “Existence and Uniqueness of Perturbation Solutions to DSGE Models,” SFB 649 Discussion Papers 2012-015.

——— (2012b): “Solving DSGE Models with a Nonlinear Moving Average,” Mimeo July.

LANCASTER, P. (1966): *Lambda-Matrices and Vibrating Systems*. Pergamon Press, Oxford, UK.

LOMBARDO, G. (2010): “On Approximating DSGE Models by Series Expansions,” Working Paper Series 1264, European Central Bank.

MAGNUS, J. R., AND H. NEUDECKER (1979): “The Commutation Matrix: Some Properties and Applications,” *The Annals of Statistics*, 7(9), 383–394.

RUDEBUSCH, G. D., AND E. T. SWANSON (2012): “The Bond Premium in a DSGE Model with Long-Run Real and Nominal Risks,” *AEJ: Macroeconomics*, 4(1), 105–43–61.

SCHMITT-GROHÉ, S., AND M. URIBE (2004): “Solving Dynamic General Equilibrium Models Using a Second-Order Approximation to the Policy Function,” *Journal of Economic Dynamics and Control*, 28(4), 755–775.

STRANG, G. (2009): *Introduction to Linear Algebra*. Wellesley-Cambridge Press and SIAM, Wellesley, MA, 4 edn.

UHLIG, H. (1999): “A Toolkit for Analysing Nonlinear Dynamic Stochastic Models Easily,” in *Computational Methods for the Study of Dynamic Economies*, ed. by R. Marimon, and A. Scott, chap. 3, pp. 30–61. Oxford University Press.

VETTER, W. J. (1973): “Matrix Calculus Operations and Taylor Expansions,” *SIAM Review*, 15(2), 352–369.

WOODFORD, M. (1986): “Stationary Sunspot Equilibria: The Case of Small Fluctuations around a Deterministic Steady State,” Mimeo September.

A Appendices

A.1 A Multidimensional Calculus and Taylor Approximation

Theorem A.1. *A Multidimensional Calculus*

Given the vector $B \in \mathbb{R}^{s\times 1}$ and the matrix-valued functions $F : B \to \mathbb{R}^{p\times q}$, $G : B \to \mathbb{R}^{q\times u}$, $H : B \to \mathbb{R}^{u\times v}$ and given the vector-valued function $C : B \to \mathbb{R}^{u\times 1}$, $J : C \to \mathbb{R}^{p\times 1}$ and the matrix-valued function $A : C \to \mathbb{R}^{p\times q}$, the following rules of calculus hold

1. *Matrix Product Rule: $\mathscr{D}_{B^T}\{FG\} = F_B(I_s \otimes G) + FG_B$, where I_s is an $s\times s$ identity matrix*
2. *Matrix Chain Rule: $\mathscr{D}_{B^T}\{A(C)\} = A_C\left(C_B \otimes I_q\right)$, where I_q is an $q\times q$ identity matrix*
3. *Matrix Kronecker Product Rule: $\mathscr{D}_{B^T}\left\{\underset{p\times q}{F} \otimes \underset{u\times v}{H}\right\} = F_B \otimes H + (F \otimes H_B)\left(K_{q,s} \otimes \underset{v\times v}{I}\right)$, where $K_{q,s}$ is a $qs\times qs$ commutation matrix (see Magnus and Neudecker (1979)).*

Proof. See Lan and Meyer-Gohde (2012b). □

The M-th order Taylor approximation of (2) at the deterministic steady state (6) is[25]

Corollary A.2. *An M-th order Taylor Approximation of (2) is written as*

$$y_t = \sum_{j=0}^{M} \frac{1}{j!}\left[\sum_{i=0}^{M-j}\frac{1}{i!}y_{z^j\sigma^i}\sigma^i\right](z_t - \bar{z})^{\otimes[j]} \tag{A-1}$$

Proof. From Vetter (1973), a multidimensional Taylor expansion is given by

$$\underset{(p\times 1)}{M}\big(\underset{(s\times 1)}{B}\big) = M(\bar{B}) + \sum_{n=1}^{N}\frac{1}{n!}\mathscr{D}^n_{B^{T^n}}M(\bar{B})\left(B-\bar{B}\right)^{\otimes[n]} + R_{N+1}\left(\bar{B},B\right) \tag{A-2}$$

$$\text{where } R_{N+1}\left(\bar{B},B\right) = \frac{1}{N!}\int_{\xi=\bar{B}}^{B}\mathscr{D}^{N+1}_{B^{T^{N+1}}}M(\xi)\left(I_s \otimes (B-\xi)^{\otimes[N]}\right)d\xi \tag{A-3}$$

Differentiating (2) M times,a Taylor approximation at the deterministic steady state $\bar{z}$ is

$$\begin{aligned}
y_t = &\frac{1}{0!}\left(\frac{1}{0!}\bar{y} + \frac{1}{1!}y_\sigma\sigma + \frac{1}{2!}y_{\sigma^2}\sigma^2 + \ldots + \frac{1}{M!}y_{\sigma^M}\sigma^M\right)\\
&+\frac{1}{1!}\left(\frac{1}{0!}y_z + \frac{1}{1!}y_{z\sigma}\sigma + \frac{1}{2!}y_{z\sigma^2}\sigma^2 + \ldots + \frac{1}{(M-1)!}y_{z\sigma^{M-1}}\sigma^{M-1}\right)(z_t-\bar{z})\\
&+\frac{1}{2!}\left(\frac{1}{0!}y_{z^2} + \frac{1}{1!}y_{z^2\sigma}\sigma + \frac{1}{2!}y_{z^2\sigma^2}\sigma^2 + \ldots + \frac{1}{(M-2)!}y_{z^2\sigma^{M-2}}\sigma^{M-2}\right)(z_t-\bar{z})^{\otimes[2]}\\
&\vdots\\
&+\frac{1}{M!}\frac{1}{0!}y_{z^M}(z_t-\bar{z})^{\otimes[M]}
\end{aligned}$$

Writing the foregoing more compactly yields (A-1). □

A.2 Pencils and Spectra

Definition A.3. *Matrix Pencil and Spectrum*

[25] We leave this dependency implicit in the following and adopt the notation of definition 2.2.

Let $P : \mathbb{C} \to \mathbb{C}^{n\times n}$ be a matrix-valued function of a complex variable; a matrix pencil. Its set of generalized eigenvalues or spectrum $\rho(P)$ is defined via $\rho(P) = \{z \in \mathbb{C} : \det P(z) = 0\}$.

A.3 The Generalized Bézout Theorem

Theorem A.4. *The Generalized Bézout Theorem*

The arbitrary lambda matrix

$$M(\lambda) = M_0\lambda^m + M_1\lambda^{m-1} + \cdots + M_m,\ \textit{where } M_0 \neq \underset{(n\times n)}{0}$$

when divided on the right by the binomial $I_n\lambda - A$ yields

$$M(\lambda) = Q(\lambda)\,(I_n\lambda - A) + M(A)$$

where $Q(\lambda) = M_0\lambda^{m-1} + (M_0A + M_1)\,\lambda^{m-2} + \cdots + M_0A^{m-1} + M_1A^{m-2} + \cdots + M_m$

Proof. See Gantmacher (1959, vol. I). □

A.4 Proof of Lemma 3.1

We will first show that for all $j, i \in \mathbb{N}^0$ such that $j + i \geqslant 1$ except the case $j = 1$ and $i = 0$, successive differentiation of the function f with respect to its arguments, z_t and σ, yields

(A-4) $$\mathscr{D}_{z_t^{T\,j}\sigma^i}\{f\} = f_{\tilde{y}}\tilde{y}_{z^j\sigma^i}\,(z_y y_z)^{\otimes[j]} + (f_y + f_{\tilde{y}}\tilde{y}_z z_y)\,y_{z^j\sigma^i} + B(j,i)$$

where the function $B(j,i)$ is (i) linear in ε_{t+1} up to and including i-th Kronecker power and contains (ii) products involving derivatives of y and $\tilde{y}$ with respect to z_t $j+i$ or less times and σ i or less times except for the unknown $y_{z^j\sigma^i}$ under consideration

(A-5) $$B(j,i) = B\left(\tilde{y}_{z^l\sigma^k}, y_{z^l\sigma^k}, \varepsilon_{t+1}^{\otimes[k]}\right)$$

(A-6) where $l = 0,1,2,\ldots, j+i$; $k = 0,1,2,\ldots,i$; $l + k \leq j+i$; but not $l = j$ and $k = i$

The index rule (A-6) ensures that $B(j,i)$ contains only terms given by previous calculations with the unknown, $y_{z^j\sigma^i}$, excluded by $l = j$ and $k = i$ simultaneously having been disallowed.

We will proceed inductively by differentiating (A-4) with respect to z_t and σ respectively and confirming that the two resulting expressions take the form of (A-4). Beginning with z_t

$$\begin{aligned}\mathscr{D}_{z_t^{T\,j+1}\sigma^i}\{f\} = &f_{\tilde{y}}\tilde{y}_{z^{j+1}\sigma^i}\,(z_y y_z)^{\otimes[j+1]} + (f_y + f_{\tilde{y}}\tilde{y}_z z_y)\,y_{z^{j+1}\sigma^i}\\ &+ \mathscr{D}_{z_t^T}\{f_{\tilde{y}}\}\left(I_{n_z}\otimes\left[\tilde{y}_{z^j\sigma^i}(z_y y_z)^{\otimes[j]}\right]\right) + f_{\tilde{y}}\tilde{y}_{z^j\sigma^i}\mathscr{D}_{z_t^T}\left\{(z_y y_z)^{\otimes[j]}\right\}\\ &+ \mathscr{D}_{z_t^T}\{f_y\}\left(I_{n_z}\otimes y_{z^j\sigma^i}\right) + \mathscr{D}_{z_t^T}\{f_{\tilde{y}}\tilde{y}_z z_y\}\left(I_{n_z}\otimes y_{z^j\sigma^i}\right)\\ &+ \mathscr{D}_{z_t^T}\left\{B\left(\tilde{y}_{z^l\sigma^k}, y_{z^l\sigma^k}, \varepsilon_{t+1}^{\otimes[k]}\right)\right\}\end{aligned} \tag{A-7}$$

The second and third lines of the foregoing contain products involving the derivatives of y and $\tilde{y}$ with respect to z_t $j+i$ or less times and σ i or less times, given by previous calculations.

The last line contains products of the derivatives of y and $\tilde{y}$ with respect to z_t $j+i+1$ or fewer and σ i or fewer times, as is revealed by differentiating through $B(j,i)$ in the last line with respect to z_t in which

(A-8) $$\mathscr{D}_{z_t^T}\left\{\tilde{y}_{z^l\sigma^k}\right\} = \tilde{y}_{z^{l+1}\sigma^k}[(z_y y_z)\otimes I_{z^l}],\ \mathscr{D}_{z_t^T}\left\{y_{z^l\sigma^k}\right\} = y_{z^{l+1}\sigma^k}[(z_y y_z)\otimes I_{z^l}]$$

where $l=0,1,2,\ldots,j+i$; $k=0,1,2,\ldots,i$; $l+k\leq j+i$; but not $l=j$ and $k=i$

Importantly, the unknown under consideration upon differentiation, $y_{z^{j+1}\sigma^i}$, is excluded by advancing the exclusion in the index rule: with no $y_{z^j\sigma^i}$ in $B(j,i)$, there is no $y_{z^{j+1}\sigma^i}$ in $B(j+1,i)$. Furthermore, the terms are linear in ε_{t+1} up to and including the i-th Kronecker power as differentiating $\varepsilon_{t+1}^{\otimes[k]}$ in the last line does not advance the index i. Hence (A-7) can be rewritten

(A-9) $$\mathscr{D}_{z_t^{T\,j+1}\sigma^i}\{f\} = f_{\tilde{y}}\tilde{y}_{z^{j+1}\sigma^i}(z_y y_z)^{\otimes[j+1]} + (f_y + f_{\tilde{y}}\tilde{y}_z z_y)y_{z^{j+1}\sigma^i} + B(j+1,i)$$

Hence differentiation with respect to z_t confirms the form of (A-4).

Differentiating (A-4) with respect to σ yields

$$\begin{aligned}\mathscr{D}_{z_t^{T\,j}\sigma^{i+1}}\{f\} = {} & f_{\tilde{y}}\tilde{y}_{z^j\sigma^{i+1}}(z_y y_z)^{\otimes[j]} + (f_y + f_{\tilde{y}}\tilde{y}_z z_y)y_{z^j\sigma^{i+1}}\\ & + \mathscr{D}_\sigma\{f_{\tilde{y}}\}\left[\tilde{y}_{z^j\sigma^i}(z_y y_z)^{\otimes[j]}\right] + f_{\tilde{y}}\tilde{y}_{z^{j+1}\sigma^i}(z_y y_z)^{\otimes[j+1]} + f_{\tilde{y}}\tilde{y}_{z^{j+1}\sigma^i}z_\varepsilon\varepsilon_{t+1}(z_y y_z)^{\otimes[j]}\\ & + f_{\tilde{y}}\tilde{y}_{z^j\sigma^i}\mathscr{D}_\sigma\left\{(z_y y_z)^{\otimes[j]}\right\} + \mathscr{D}_\sigma\{f_y\}y_{z^j\sigma^i} + \mathscr{D}_\sigma\{f_{\tilde{y}}\tilde{y}_z z_y\}y_{z^j\sigma^i}\\ & + \mathscr{D}_\sigma\left\{B\left(\tilde{y}_{z^l\sigma^k}, y_{z^l\sigma^k}, \varepsilon_{t+1}^{\otimes[k]}\right)\right\}\end{aligned} \qquad \text{(A-10)}$$

The second and third lines of the foregoing contain products involving the derivatives of y and $\tilde{y}$ with respect to z_t $j+i+1$ or less times and σ i or less times, all known from previous calculations. Note again that the unknown, here $y_{z^j\sigma^{i+1}}$, only appears in the first line.

The last line contains products involving the derivatives of y and $\tilde{y}$ with respect to z_t $j+i+1$ or fewer and σ $i+1$ or fewer times. To see this, differentiate through $B(j,i)$ in the last line with respect to σ in which

(A-11) $$\mathscr{D}_\sigma\left\{\tilde{y}_{z^l\sigma^k}\right\} = \tilde{y}_{z^{l+1}\sigma^k}(z_y y_\sigma + z_\varepsilon\varepsilon_{t+1}) + \tilde{y}_{z^l\sigma^{k+1}},\ \mathscr{D}_\sigma\left\{y_{z^l\sigma^k}\right\} = y_{z^l\sigma^{k+1}}$$

where $l=0,1,2,\ldots,j+i$; $k=0,1,2,\ldots,i$; $l+k\leq j+i$; but not $l=j$ and $k=i$

Importantly, the unknown $y_{z^j\sigma^{i+1}}$ is again not present here either, as when $k=i$ or equivalently, $k+1=i+1$, $l=j$ is not allowed by the index rule: with no $y_{z^j\sigma^i}$ in $B(j,i)$, there can be no $y_{z^j\sigma^{i+1}}$ in $B(j,i+1)$. Notice that an additional ε_{t+1} is included in (A-11). The possibility that this term multiplies with the existing $\varepsilon_{t+1}^{\otimes[k]}$ necessitates the advancement of the index associated with Kronecker powers of ε_{t+1} for $B(j,i+1)$ to remain linear in the set of $\varepsilon_{t+1}^{\otimes[k+1]}$.

All terms in the last three lines of (A-10) form $B(j,i+1)$ and (A-10) can be rewritten

(A-12) $$\mathscr{D}_{z_t^{T\,j}\sigma^{i+1}}\{f\} = f_{\tilde{y}}y_{z^j\sigma^{i+1}}(z_y y_z)^{\otimes[j]} + (f_y + f_{\tilde{y}}\tilde{y}_z z_y)y_{z^j\sigma^{i+1}} + B(j,i+1)$$

Hence differentiation with respect to σ likewise confirms the form of (A-4).

The second step is to evaluate (A-4), having been verified by induction above, with the given moments of ε_{t+1} and at the deterministic steady state. Setting the resulting expression equal to zero and letting $A(j,i) \equiv E_t\left[B(j,i)\right]\Big|_{\substack{z_t=\bar{z}\\ \sigma=0}}$ yields (14) in the text.

All that remains is to address the cases that were excluding by the indexation rule and to initialize the induction. Excluded were: (i) $(j=0, i=0)$ corresponding to the deterministic steady state value of y which was assumed given in the text; and (ii) $(j=1, i=0)$ for y_z, which was solved separately as (12) in the text. The case $(j=0, i=1)$ for y_σ can be handled individually,[26] so that we can start the induction with the three second order terms ($j+i=2$), y_{z^2}, $y_{z\sigma}$, and y_{σ^2}, which are provided in the next section separately.

A.5 Generalized Sylvester Equations for Second Order Terms

From corollary A.2, the second order Taylor expansion of the policy function (2) takes the form

$$y_t = \bar{y} + y_\sigma \sigma + \frac{1}{2} y_{\sigma^2} \sigma^2 + \left(y_z + y_{z\sigma}\sigma\right)\left(z_t - \bar{z}\right) + \frac{1}{2} y_{z^2} \left(z_t - \bar{z}\right)^{\otimes[2]} \tag{A-13}$$

Given coefficients from the first order, there are three unknowns: y_{z^2}, $y_{z\sigma}$ and y_{σ^2}.

To find y_{z^2}, we differentiate (10) with respect to z_t

$$\begin{aligned} \mathscr{D}_{z_t^T z_t^T}\{f\} =& \mathscr{D}_{z_t^T}\left\{f_{\tilde{y}}\right\}\left(I_{n_z} \otimes \tilde{y}_z z_y y_z\right) + f_{\tilde{y}} \tilde{y}_{z^2} \left(z_y y_z\right)^{\otimes 2} + f_{\tilde{y}} \tilde{y}_z z_y y_{z^2} \\ & + \mathscr{D}_{z_t^T}\left\{f_y\right\}\left(I_{n_z} \otimes y_z\right) + f_y y_{z^2} + \mathscr{D}_{z_t^T}\left\{f_z\right\} \end{aligned} \tag{A-14}$$

$$\begin{aligned} \text{where } \ \mathscr{D}_{z_t^T}\left\{f_{\tilde{y}}\right\} &= f_{\tilde{y}^2}\left[\left(\tilde{y}_z z_y y_z\right) \otimes I_{n_y}\right] + f_{y\tilde{y}}\left(y_z \otimes I_{n_y}\right) + f_{z\tilde{y}} \\ \mathscr{D}_{z_t^T}\left\{f_y\right\} &= f_{\tilde{y}y}\left[\left(\tilde{y}_z z_y y_z\right) \otimes I_{n_y}\right] + f_{y^2}\left(y_z \otimes I_{n_y}\right) + f_{zy} \\ \mathscr{D}_{z_t^T}\left\{f_z\right\} &= f_{\tilde{y}z}\left[\left(\tilde{y}_z z_y y_z\right) \otimes I_{n_z}\right] + f_{yz}\left(y_z \otimes I_{n_z}\right) + f_{zz} \end{aligned}$$

Evaluating at the deterministic steady state, the expectation of the foregoing set to zero yields

$$\begin{aligned} 0 = E_t\left[\mathscr{D}_{z_t^T z_t^T}\{f\}\right]\Big|_{\substack{z_t=\bar{z}\\ \sigma=0}} =& f_{\tilde{y}} y_{z^2} \left(z_y y_z\right)^{\otimes 2} + \left(f_{\tilde{y}} y_z z_y + f_y\right) y_{z^2} \\ & + E_t\left[\mathscr{D}_{z_t^T}\left\{f_{\tilde{y}}\right\}\left(I_{n_z} \otimes \tilde{y}_z z_y y_z\right) + \mathscr{D}_{z_t^T}\left\{f_y\right\}\left(I_{n_z} \otimes y_z\right) + \mathscr{D}_{z_t^T}\left\{f_z\right\}\right]\Big|_{\substack{z_t=\bar{z}\\ \sigma=0}} \end{aligned} \tag{A-15}$$

This is (14) with $j=2$ and $i=0$ in lemma 3.1.

To determine $y_{z\sigma}$, we differentiate (26) with respect to z_t

$$\begin{aligned} \mathscr{D}^2_{z_t^T \sigma}\{f\} =& \mathscr{D}_{z_t^T}\left\{f_{\tilde{y}}\right\}\left(I_{n_z} \otimes \left(f_{\tilde{y}}[\tilde{y}_z(z_y y_\sigma + z_\varepsilon \varepsilon_{t+1}) + \tilde{y}_\sigma]\right)\right) \\ & + f_{\tilde{y}} \mathscr{D}_{z_t^T}\left\{\tilde{y}_z\right\}\left[I_{n_z} \otimes \left(z_y y_\sigma + z_\varepsilon \varepsilon_{t+1}\right)\right] + f_{\tilde{y}} \tilde{y}_z z_y y_{z\sigma} \end{aligned}$$

[26] $\mathscr{D}_\sigma\{f\} = f_{\tilde{y}} \tilde{y}_z z_y y_\sigma + f_{\tilde{y}} \tilde{y}_z z_\varepsilon \varepsilon_{t+1} + f_{\tilde{y}} \tilde{y}_\sigma + f_y y_\sigma$, which, when evaluated at the deterministic steady state and with its expectation set to zero, yields $E_t\left[\mathscr{D}_\sigma\{f\}\right]\Big|_{\substack{z_t=\bar{z}\\ \sigma=0}} = f_{\tilde{y}} y_\sigma + \left(f_y + f_{\tilde{y}} y_z z_y\right) y_\sigma + f_{\tilde{y}} y_z z_\varepsilon E_t\left[\varepsilon_{t+1}\right] = 0$.

$$+ f_{\tilde{y}}\tilde{y}_{z\sigma}z_y y_z + \mathscr{D}_{z_t^T}\left\{f_y\right\}(I_{n_z}\otimes y_\sigma) + f_y y_{z\sigma} \tag{A-16}$$

$$\text{where}\quad \mathscr{D}_{z_t^T}\{\tilde{y}_z\} = \tilde{y}_{z^2}(z_y y_z)^{\otimes 2} + \tilde{y}_z z_y y_{z^2}$$

Setting the expectation of the foregoing evaluated at the deterministic steady state to zero yields

$$\begin{aligned}
0 = E_t\left[\mathscr{D}_{z_t^T\sigma}\{f\}\right]\Big|_{\substack{z_t=\bar{z}\\ \sigma=0}} =& f_{\tilde{y}}y_{z\sigma}(z_y y_z) + (f_{\tilde{y}}y_z z_y + f_y)y_{z\sigma} \\
&+ E_t\Big[\mathscr{D}_{z_t^T}\left\{f_{\tilde{y}}\right\}\left(I_{n_z}\otimes(f_{\tilde{y}}[\tilde{y}_z(z_y y_\sigma + z_\varepsilon\varepsilon_{t+1}) + \tilde{y}_\sigma])\right) \\
&+ f_{\tilde{y}}\mathscr{D}_{z_t^T}\{\tilde{y}_z\}\left[I_{n_z}\otimes(z_y y_\sigma + z_\varepsilon\varepsilon_{t+1})\right] + \mathscr{D}_{z_t^T}\left\{f_y\right\}(I_{n_z}\otimes y_\sigma)\Big]\Big|_{\substack{z_t=\bar{z}\\ \sigma=0}}
\end{aligned} \tag{A-17}$$

This is (14) with $j = 1$ and $i = 1$ in lemma 3.1.

To determine y_{σ^2}, we differentiate (26) with respect to σ

$$\begin{aligned}
\mathscr{D}^2_{\sigma^2}\{f\} =& \mathscr{D}_\sigma\left\{f_{\tilde{y}}\right\}(\tilde{y}_z z_y y_\sigma + \tilde{y}_z z_\varepsilon\varepsilon_{t+1} + \tilde{y}_\sigma) + f_{\tilde{y}}\mathscr{D}_\sigma\{\tilde{y}_z\}(z_y y_\sigma + z_\varepsilon\varepsilon_{t+1}) \\
&+ f_{\tilde{y}}\tilde{y}_z z_y y_{\sigma^2} + f_{\tilde{y}}\tilde{y}_{\sigma^2} + \mathscr{D}_\sigma\left\{f_y\right\}y_\sigma + f_y y_{\sigma^2}
\end{aligned} \tag{A-18}$$

$$\begin{aligned}
\text{where}\quad \mathscr{D}_\sigma\left\{f_{\tilde{y}}\right\} =& f_{\tilde{y}^2}\left[(\tilde{y}_z(z_y y_\sigma + z_\varepsilon\varepsilon_{t+1}) + \tilde{y}_\sigma)\otimes I_{n_y}\right] + f_{y\tilde{y}}(y_\sigma\otimes I_{n_y}) \\
\mathscr{D}_\sigma\{\tilde{y}_z\} =& \tilde{y}_{z^2}\left[(z_y y_\sigma + z_\varepsilon\varepsilon_{t+1})\otimes I_{n_z}\right] + \tilde{y}_{\sigma z} \\
\mathscr{D}_\sigma\left\{f_y\right\} =& f_{\tilde{y}y}\left[(\tilde{y}_z(z_y y_\sigma + z_\varepsilon\varepsilon_{t+1}) + \tilde{y}_\sigma)\otimes I_{n_y}\right] + f_{y^2}(y_\sigma\otimes I_{n_y})
\end{aligned}$$

Evaluating at the deterministic steady state, the expectation of the foregoing set to zero yields

$$\begin{aligned}
0 = E_t\left[\mathscr{D}^2_{\sigma^2}\{f\}\right]\Big|_{\substack{z_t=\bar{z}\\ \sigma=0}} =& f_{\tilde{y}}y_{\sigma^2} + (f_{\tilde{y}}y_z z_y + f_y)y_{\sigma^2} \\
&+ E_t\Big[\mathscr{D}_\sigma\left\{f_{\tilde{y}}\right\}(\tilde{y}_z z_y y_\sigma + \tilde{y}_z z_\varepsilon\varepsilon_{t+1} + \tilde{y}_\sigma) + f_{\tilde{y}}\mathscr{D}_\sigma\{\tilde{y}_z\}(z_y y_\sigma + z_\varepsilon\varepsilon_{t+1}) \\
&+ \mathscr{D}_\sigma\left\{f_y\right\}y_\sigma\Big]\Big|_{\substack{z_t=\bar{z}\\ \sigma=0}}
\end{aligned} \tag{A-19}$$

This is (14) with $j = 0$ and $i = 2$ in lemma 3.1.[27]

A.6 Proof of Lemma 4.11

From (22), it follows that the eigenvalues of $z_y y_z$ are those of $y_z z_y$ plus a zero eigenvalue with algebraic multiplicity n_e and are, following assumption 3.3 all inside the closed unit circle. As the eigenvalues of the Kronecker product of two matrices are equal to the products of the eigenvalues of the two matrices, all the eigenvalues of $(z_y y_z)^{\otimes[j]}$ for all $j \in \mathbb{N}^0$, and hence the trailing pencil of definition 4.9, are also inside the closed unit circle. The eigenvalues of the leading pencil of definition 4.9 are all outside the closed unit circle from proposition 4.6. The spectra of the two pencils in question are thusly disjoint, being separated by the unit circle.

[27]The second moment of future shocks in (A-19) emerges from the terms under the expectation operator.

A.7 Proof of Theorem 5.1

This is Jin and Judd's (2002) Theorem 6 adapted to our exposition with their assumption (iii) concerning solvability eliminated. Under our problem statement (1), the derivative of Jin and Judd's (2002) operator $\mathcal{N}(y,\sigma)$ has a leading coefficient matrix given by $f_y + f_{\tilde{y}} y_z z_y$ at the steady state. From proposition 4.6, this matrix is necessarily invertible.

A.8 Proof of Proposition 5.2

From the proof of lemma 3.1, we can write the equations governing $y_{z^j\sigma}$, for $j \geq 0$, as

(A-20) $$f_{\tilde{y}} y_{z^j\sigma} (z_y y_z)^{\otimes[j]} + (f_y + f_{\tilde{y}} y_z z_y) y_{z^j\sigma} + A(j,1) = 0$$

where $A(j,1) = E_t[B(j,1)]$. We will proceed inductively over the terms in $B(j,1)$ where the homogeneity of the equations will follow from the solvability proven in theorem 3.4.

To begin, assume that for some $j \geq 0$, $B(j,1)$ is a set of terms involving a product of at least one of $y_{z^k\sigma}$, $k < j$, or ε_{t+1}, but at most one of the latter. As differentiating

(A-21) $$f_{\tilde{y}} y_{z^j\sigma} (z_y y_z)^{\otimes[j]} + (f_y + f_{\tilde{y}} y_z z_y) y_{z^j\sigma} + B(j,1) = 0$$

with respect to z_t only advances the index j, see section A.4, it follows that

(A-22) $$\mathscr{D}_{z_t^T\sigma}\{B(j,1)\} = B(j+1,1)$$

with $B(j+1,1)$ being a set of terms involving a product of at least one of $y_{z^k\sigma}$, $k < j+1$, or ε_{t+1}, but at most one of the latter. To start the induction, note from footnote 26 that

(A-23) $$B(0,1) = f_{\tilde{y}} \tilde{y}_z z_\varepsilon \varepsilon_{t+1}$$

thus, confirming the composition of $B(j,1)$ as a set of terms involving a product of at least one of $y_{z^k\sigma}$, $k < j$, or ε_{t+1}, but at most one of the latter.[28]

Taking expectations

(A-24) $$A(j,1) = E_t[B(j,1)]$$

and as the first moment of ε_t was assumed zero, all terms except those involving only products of $y_{z^k\sigma}$, $k < j$ are eliminated. Thus, if all $y_{z^k\sigma}$, $k < j$ are zero, then $A(j,1)$ is zero and the equation in $y_{z^j\sigma}$ is homogenous. From theorem 3.4 it then follows that $y_{z^j\sigma}$ must also be zero, as a unique solution exists and zero is always a solution of a homogenous equation. Hence by induction, starting from the homogenous equation for y_σ, all $y_{z^j\sigma} = 0$, for $j \geq 0$.

[28] As $k < j$ would admit only negative values of k in $y_{z^k\sigma}$ for $B(0,1)$, it is useful to examine $B(1,1)$ as well to confirm the induction. Examining (A-16) for the second order case, which gives

$$B(1,1) = \mathscr{D}_{z_t^T}\{f_{\tilde{y}}\}\left(I_{n_z} \otimes (f_{\tilde{y}}[\tilde{y}_z(z_y y_\sigma + z_\varepsilon \varepsilon_{t+1}) + \tilde{y}_\sigma])\right) + f_{\tilde{y}} \mathscr{D}_{z_t^T}\{\tilde{y}_z\}[I_{n_z} \otimes (z_y y_\sigma + z_\varepsilon \varepsilon_{t+1})] + \mathscr{D}_{z_t^T}\{f_y\}(I_{n_z} \otimes y_\sigma)$$

where $\mathscr{D}_{z_t^T}\{\tilde{y}_z\} = \tilde{y}_{z^2}(z_y y_z)^{\otimes 2} + \tilde{y}_z z_y y_{z^2}$

notice that all terms involve a product of at least one of y_σ, or ε_{t+1}, but at most one of the latter.