Government Debt Maturity Structure, Fiscal Policy, and Default*

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Abstract

I develop a tractable model to study the optimal debt maturity structure and fiscal policy in an environment with incomplete markets, lack of commitment, and opportunity to default by the government. The default on public debt is endogenous and the real interest rate reflects the default risk and the marginal rate of substitution between present and future consumption. The maturity is used to resolve the time-consistency problem: The present government can incentivize future governments to stick to an ex ante optimal sequence of fiscal policies and interest rates. I show that if both risk-free interest rates and risk premiums can be manipulated, the optimal maturity structure tends to have a decaying profile: The government issues debt at all maturity dates, but the distribution of payments over time is skewed toward the short-term end. Debt maturity data across countries are consistent with model predictions.

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1 Introduction

Debt maturity structure is an important element of optimal fiscal policy, especially in light of recent sovereign debt crises. The consensus is that debt maturity is used to minimize the costs of lack of commitment. In their seminal paper, Lucas and Stokey (1983) derive the classic result that in an environment with endogenous risk-free interest rates and no default the government should issue consol bonds, i.e., the optimal maturity is spread out or flat. In contrast, Aguiar et al. (2016) study an open economy with default but exogenous risk-free interest rates. The authors demonstrate that the time-consistency problem can be resolved if the government issues only short-term debt and abstains from any active issuance or repurchase of long-term liabilities.

In this paper, I combine both sources of time inconsistency - manipulation of risk-free interest rates and debt dilution due to option to default - within a unified framework. I develop a tractable model to study the optimal fiscal policy and optimal debt maturity structure in an environment with incomplete markets, lack of commitment to fiscal policies, and endogenous default on public debt, and show that, if a government can alter both risk-free rates and risk premiums, the optimal maturity structure exhibits a decaying profile, i.e., total payments due at a later maturity date are lower. This prediction is in line with empirical data as observed term structures of most countries are neither flat nor short but skewed toward the short end.

The model features a benevolent government and a continuum of atomistic households with strictly concave utility functions over private consumption. Households are the only lenders to the government. The government cannot commit to either future fiscal policies or to repay its debt, and sets fiscal policies, restructures its debt portfolio, and decides whether or not to default sequentially. The markets are incomplete, and the set of financial instruments is limited to bonds with various maturities. Interest rates reflect both the probability of default and marginal rate of substitution between present and future consumption.

Default is modeled as a stochastic outside option that can be exercised at the beginning of every period. Whenever the value of the outside option exceeds the value of repaying debt, the default option is triggered. Default is costly to sustain a positive amount of debt in equilibrium. The value of the outside option is the only shock in the model. In addition, the value of default is continuously distributed to allow smoothness in default probability.

I analyze the Markov perfect competitive equilibrium in which all decisions are made sequentially and are functions of payoff-relevant state variables: the outstanding debt at various maturities and the value of the outside option. I characterize the optimal allocation by considering the modified commitment problem as the benchmark: A contract that allows the government to commit to predetermined fiscal policies but not to abstain from default. In other words, the planner simultaneously makes the fiscal decisions for all future periods and can promise to pursue the plan: however, it cannot promise to repay debt if the outside option is preferred. The optimal allocation of the modified commitment problem defines the fiscal plan: the sequence of budget surpluses needed to repay debt contingent on no prior default.

In this model, the Markov perfect competitive equilibrium is efficient in the sense that the sequential policy maker follows the ex ante optimal fiscal plan and sticks to ex ante optimal risk-free interest rates and default probabilities. Even though the government cannot commit to future policies, it can set the term structure of its liabilities so that it has no incentive to deviate from the plan in the future. Why might the government be willing to distort the ex ante optimal allocation in the future? At every date, the value of outstanding debt must be financed by future budget surpluses. Therefore, a deviation from the plan can be ex post beneficial if the market value of outstanding debt is decreased. For example, if debt is mostly short term, then reallocation of budget surpluses leading to a decrease in the value of short-term debt at the expense of an increase in the value of long-term debt might be optimal ex post, and vice versa. However, the government can structure its debt maturity such that any such distortion strictly reduces the budget set of the government by increasing the value of outstanding debt and, hence, deviations are not optimal.

The main result is that in the presence of default risk, the government issues more short-term debt than long-term debt. Moreover, the optimal maturity structure has a decaying profile: The government issues debt at all maturity dates, but the distribution of payments over time is skewed toward the short-term end. The average maturity depends on the relative sensitivity of risk-free interest rates and risk premiums. The term structure is shorter if risk-free interest rates are less responsive to changes in government policies. On the other hand, if a deviation from a fiscal plan has a negligible effect on default risk, then the optimal maturity structure is approximately flat.

To gain intuition, suppose that the government can distort only risk-free rates and the default risk is absent. Then the optimal maturity is flat, meaning that the total payments due at different maturity dates are constant. Any deviation from the fiscal plan that increases the budget surplus in one period and decreases it in another period does not lead to a decrease in the value of outstanding debt; this is because changes in risk-free rates are proportional and offset each other. However, if debt is skewed toward the short- or long-term end, decreasing the price of a larger stock of debt at the expense of increasing the price of a lower stock of debt allows the government to reduce the value of total debt.

Now consider an environment in which risk-free rates are exogenous, but the default risk is positive and increasing in total debt issued. The next period, the government can affect default probabilities in future periods by increasing or decreasing the budget surplus. Thus, it can affect the value of debt that matures in subsequent periods. However, the government cannot manipulate the price of a one-period debt issued in the preceding period, because it cannot alter the default probability in the current period. The reason is that all government fiscal policies are conducted conditional on no prior default in that period. Therefore, the optimal debt policy prescribes issuance of one-period bonds only.

Finally, suppose the government can manipulate risk-free interest rates and default probabilities. In such an environment, the price of debt with longer maturity is more sensitive to potential future distortions compared to the price of debt with shorter maturity. Consider a deviation from ex ante optimal fiscal plan that implies reallocation of budget surpluses between two subsequent periods, keeping the market value of budget surpluses constant. This perturbation causes proportional changes in risk-free interest rates. The probability of default in the later period changes as a higher or lower budget surplus in that period corresponds to a higher or lower value of pursuing the fiscal plan and, hence, to a lower or higher default risk. However, this deviation does not affect the default probability in the earlier period, as the deviation described does not change government welfare in the earlier period. Therefore, change in the price of debt with longer maturity reflects distortions in both the risk-free interest rate and risk premium, while the price of debt with shorter maturity varies due only to changes in the risk-free interest rate. A deviation from the ex ante fiscal plan has an offsetting effect on that value of shorter- and longer-term debt if the stock of debt with shorter maturity is larger. Extending this result to a finite-period model leads to the conclusion that the optimal term structure must be decreasing in maturity date.

The benefit of using the modified commitment problem is that it allows me to characterize the optimal maturity structure in a multi-period model with various maturities available to the government. First, I solve for the optimal path of fiscal policies. Then I recursively solve for the maturity structure in every period that renders the ex ante plan incentive-compatible for future governments.

In a quantitative exercise, I consider a six-period model. In the initial period the government is hit with a high taste parameter for government spending and optimally chooses the budget deficit. This budget deficit must be financed by bonds with different maturity dates. I show that the optimal maturity structure is not only tilted toward the short end, but also has a decaying profile: Total payments owed by the government decrease as the maturity date goes up. Moreover, this downward-sloping structure persists in subsequent periods, and future governments issue positive debt with all remaining maturities.

My analysis implies that the data on the maturity structure of developed economies is broadly consistent with normative analysis of the optimal debt policy under lack of commitment and opportunity to default. According to this model, lengthening government debt maturity would cause an increase in long-term risk-free rates and default probabilities, as such term restructuring would incentivize future governments to over-borrow compared to the ex ante optimal policy.

Related Literature

As already mentioned, the paper bridges the gap between two literatures that study lack of commitment due to risk-free rates manipulation and risk premiums manipulation in isolation. I build on the work of Aguiar, Amador, Hopenhayn and Werning (2016) by introducing endogenous risk-free interest rates as in Lucas and Stokey (1983).

This paper also relates to the literature that investigates time consistency of fiscal *and* monetary policy. Alvarez et al. (2004) show that Ramsey policy can be made time consistent under the Friedman rule, i.e., zero nominal interest rate is optimal. Persson et al. (2006) argue that time consistency can be achieved by structuring government nominal and indexed debt in an environment where positive nominal interest rates are optimal. In my paper, the focus is on the option of outright government default which is missing from the discussed studies, however, nominal debt and, hence, government's ability to inflate away debt is absent in my paper. I find that the fiscal policy is time-consistent in a weaker sense, as discussed in Aguiar et al. (2016): A government follows an optimal sequence of fiscal policy decisions conditional on no prior default.

Maturity structure can be also used to hedge a government against fiscal shocks. Angeletos (2002) shows that in an environment with perfect commitment but incomplete markets, state-contingent debt can be replicated by maturity structure of non-contingent debt providing complete insurance to the government. According to quantitative exercises discussed in Buera and Nicolini (2004), such an insurance requires very large debt positions relative to GDP. However, Debortoli et al. (2017) show that such large debt positions are not sustainable in an environment with lack of commitment as a government has an incentive to distort risk-free interest rates to alter the value of outstanding debt. Moreover, the authors find that the optimal maturity structure is approximately flat because minimizing the costs associated with the lack of commitment is quantitatively much more important than minimizing the costs associated with the lack of insurance. The latter conclusion rationalizes the focus of the paper on the commitment problem and abstraction from hedging motive by setting deterministic fiscal shocks.

Maturity has been studied in international quantitative sovereign debt models. Aguiar and Gopinath (2006) were among the first to present a quantitative model with endogenous default decision in an environment with incomplete markets, as in the seminal paper by Eaton and Gersovitz (1981). Hatchondo and Martinez (2009) and Chatterjee and Eyigungor (2012) find that exogenously lengthening debt maturity by introducing a consol bond with a decaying coupon rate improves quantitative fit of such models. Arellano and Ramanarayanan (2012) extend their framework by allowing a sovereign to choose between consol bonds with different decaying rates, and the authors show that average maturity shortens in an event of a crisis. Short-term debt in these models minimizes an incentive to dilute the value of longer-term debt, while long-term debt serves as a hedging against income shocks. In these models, maturity structure of debt has a decaying profile by construction, while in my model I show that such debt structure is optimal. However, in contrast to the aforementioned studies, the role of long-term debt in this paper is to minimize risk-free interest rate distortions, while hedging motive is absent due to deterministic fiscal shocks and constant endowment.

Open economy and corporate finance literature often emphasizes the disciplining role of short-term debt. Jeanne (2009) demonstrates that short-term debt can incentivize a government to pursue a creditor-friendly policy as debt is rolled-over conditional on policy implementation. In Calomiris and Kahn (1991) and Diamond and Rajan (1991) short-term debt provides a creditor an option to liquidate project. In this model, lenders are atomistic and cannot directly affect government's decisions, instead, a time-inconsistent government uses debt maturity to discipline itself in the future.

The remainder of the paper is organized as follows. Section 2 describes the finite-period model. Section 3 considers the modified commitment problem and shows that the Markov perfect competitive equilibrium is equivalent to the allocation under commitment. In Section 4, I show that maturity is tilted toward the short end in a three-period example. In Section 5, I demonstrate that the maturity has a decaying profile in a multi-period model. Section 6 presents numerical results of three-period and six-period models, and Section 7 concludes.

2 A Finite-period Model without Fiscal Shocks

In this section, I describe a model with the two sources of time inconsistency: manipulation of the risk-free rate and debt dilution due to the probability of default. The economy is closed and consists of a government and a unit mass of atomistic households. The time is discrete and finite $t = 0,\ 1,\ ...,T$.

Preferences and Endowment. A representative household values private consumption and government spending:

$$\mathbb{E}\sum_{t=0}^{T}\beta^t\left(u(c_t)+\theta_t\omega(g_t)\right) \tag{1}$$

where u and ω are continuously differentiable, strictly increasing, concave functions and $\beta \in (0, 1]$ is the discount factor. $\theta_t > 0$ represents taste parameter for public spending. Larger θ implies higher marginal utility of government expenditures and, hence, households would prefer more resources to be spent on public goods. In the basic model all θ_t are deterministic, known at date 0 and $\theta_t \in [\theta_{min}, \theta_{max}]\ \forall t$. The government is benevolent and shares the same preferences.

There is no capital in the economy. Each period a representative household is endowed with $1-\tau$ units of consumption and the government is endowed withτ units of consumption. Every period the resource constraint has to be satisfied

$$c_t + g_t = 1\,\forall t \tag{2}$$

I make the following assumptions about u and ω:

Assumption 1. $\theta_{min}\omega'(\tau) \geq u'(1-\tau)$.

Assumption 1 ensures that the government has an incentive to raise funds from households. Moreover, it implies that government utility is strictly decreasing in the budget surplus if the latter is positive.

Bond Markets and Default. The government can borrow from households. I assume that state-contingent bonds are not available, and the set of financial instruments is limited to bonds with different maturities. Define by b_t^{t+k} the government debt held by a household that is issued at date t and promises to pay one unit of consumption at $t+k$ and let q_t^{t+k} be the price of the bond. Without loss of generality the government rebalances its portfolio each period, i.e., it buys back all the outstanding debt and issues new debt at all maturities.

The government can default on its debt and the value of default is stochastic. Partial default is not allowed. Following Aguiar et al. (2016), I assume that in every period the government has an outside option V_t^{def} that can be achieved by default. V_t^{def} is defined as $V_t^{def} = \frac{1-\beta^{T-t+1}}{1-\beta}\cdot v_t^d$, where v_t^d is drawn from continuous distribution F that has bounded support $[v_{min}; v_{max}]$. I make the following assumptions about the outside option.

Assumption 2. *Outside option:*
(i) $v_{max} < u(1-\tau) + \theta_{min}\omega(\tau)$*;*
(ii) $\exists g_{min} > 0$*:* $v_{min} > u(1-g_{min}) + \theta_{min}\omega(g_{min})$*;*
(iii) F *is strictly increasing on* (v_{min}, v_{max}) *and* $f(v_{max}) = 0$*;*
(iv) v_t^d *is independent across time and independent of debt portfolio.*

Restriction (i) ensures that the government will never choose an outside option if debt positions are zero. In addition, it guarantees that some positive level of debt can be sustained in equilibrium. Restriction (ii) implies that the government always defaults if the debt position is high enough and government spending is sufficiently low. Assumption (iii) allows to avoid kinks in the pricing functions which ensures that the equilibrium can be characterized by first-order necessary conditions. The assumption of independence in (iv) is made to abstract from hedging motives.

One way to justify such a structure of outside options is to assume that default excludes governments from bond markets and incurs endowment cost χ in every period. The value of default is defined as $V_t^{def} = \frac{1-\beta^{T-t+1}}{1-\beta} \cdot (u(1-\tau) + \omega(\tau - \chi))$. Then it is possible to find a stochastic endowment cost $\chi \sim F_\chi$, $\chi \in [\chi_{min}; \chi_{max}]$ that defines the outside value objects (v_{min}, v_{max} and F): $v_{min} = u(1-\tau) + \omega(\tau - \chi_{max})$, $v_{max} = u(1-\tau) + \omega(\tau - \chi_{min})$ and $F(v) = 1 - F_\chi(\chi(v))$ where $\chi(v)$ satisfies $v = u(1-\tau) + \omega(\tau - \chi(v))$.[1]

Timing and Government Problem. At the beginning of every period, the government whether or not to default on its debt. If it defaults, it receives the outside option value V_t^{def}. Otherwise, the government sets government expenditures, buys back existing debt and issues new debt. The default decision precedes any fiscal decisions, and the government is not allowed to default until the beginning of the next period once new debt has been issued. This timing rules out the possibility of self-fulfilling debt crises, as discussed by Cole and Kehoe (2000).

I focus on a Markov perfect competitive equilibrium in which the government makes decisions sequentially as functions of payoff-relevant variables: the outstanding bond holdings and period t. Denote by $\boldsymbol{b}_t = (b_t^{t+1}, ..., b_t^T)$ the vector of bond holdings issued at period t where T stands for the index of the final period. The size of the vector decreases by one every period and reduces to zero in the very last period. Let $\boldsymbol{q}_t = (q_t^{t+1}, ..., q_t^T)$ be the vector of corresponding bond prices.

[1] Alternatively, one can assume that default implies utility loss ξ so that $V_t^{def} = \frac{1-\beta^{T-t+1}}{1-\beta} \cdot (u(1-\tau) + \omega(\tau) - \xi)$. Then analogically to χ shock, it is straightforward to define $\xi \in [\xi_{min}, \xi_{max}]$, $\xi \sim F_\xi$ that leads to the outside value objects.

To simplify notation, it is useful to define the contingent budget surplus as the difference between endowment of and spending by the government if it does not default:

$$s_t = \tau - g_t$$

Consumption is then defined as $c_t = 1 - \tau + s_t$ and government spending is $g_t = \tau - s_t$. Therefore, setting contingent budget surpluses is equivalent to choosing government expenditures. Conditional on no default the budget constraint in every period satisfies:

$$s_t + \boldsymbol{q}_t(s_t,\, \boldsymbol{b}_t) \cdot \boldsymbol{b}_t \geq (1,\, \boldsymbol{q}_t(s_t,\, \boldsymbol{b}_t)) \cdot \boldsymbol{b}_{t-1} \tag{3}$$

Let $V_t(\boldsymbol{b}_{t-1})$ be the value of the government if it does not prefer the outside option V_t^{def}. Then at any $t < T$ it can be defined recursively as

$$V_t(\boldsymbol{b}_{t-1}) = \max_{s_t,\, \boldsymbol{b}_t} \left\{ u(1 - \tau + s_t) + \theta_t \omega(\tau - s_t) + \beta \cdot \mathbb{E} \max \left\{ V_{t+1}(\boldsymbol{b}_t);\, V_{t+1}^{def} \right\} \right\} \tag{4}$$

subject to $s_t \in (-(1-\tau),\, \tau)$ and the budget constraint (3)

At $t = T$, if government does not default, it just pays the outstanding debt b_{T-1}^T and $V_T(b_{T-1}^T) = u(1 - \tau + b_{T-1}^T) + \theta_T \omega(\tau - b_{T-1}^T)$. Denote by $\rho_t(\boldsymbol{b}_{t-1}) = \{s_t^\star(\boldsymbol{b}_{t-1}),\, \boldsymbol{b}_t^\star(\boldsymbol{b}_{t-1})\}$ the optimal government fiscal and debt policies, conditional on no default at t.

Household Optimization and Bond Prices. In any competitive equilibrium, household optimality conditions must be satisfied. A representative household takes into account the future government policies that are reflected in risk-free interest rates and risk premiums. The price of a bond that matures in $k \geq 1$ periods can be defined recursively as

$$q_t^{t+k}(s_t,\, \boldsymbol{b}_t) = \beta \frac{u'(1 - \tau + s_{t+1}^\star(\boldsymbol{b}_t))}{u'(1 - \tau + s_t)} \cdot (1 - \pi_{t+1}(\boldsymbol{b}_t)) \cdot q_{t+1}^{t+k}(\rho_{t+1}(\boldsymbol{b}_t)) \tag{5}$$

where $\pi_{t+1}(\boldsymbol{b}_t)$ defines the probability of default in the next period:

$$\pi_{t+1}(\boldsymbol{b}_t) = \text{Prob}(V_{t+1}^{def} > V_{t+1}(\boldsymbol{b}_t)) = 1 - F(V_{t+1}(\boldsymbol{b}_t) \cdot \frac{1-\beta}{1-\beta^{T-t}}) \tag{6}$$

Definition of Markov Perfect Competitive Equilibrium. The Markov Perfect Competitive Equilibrium of the economy consists of the value function $V_t(\boldsymbol{b}_{t-1})$, the fiscal policy function $\rho_t(\boldsymbol{b}_{t-1})$ and the pricing function $\boldsymbol{q}_t(s_t,\, \boldsymbol{b}_t)$ such that:

(i) the value function $V_t(\boldsymbol{b}_{t-1})$ solves the Bellman equation (4) given the fiscal policy function $\rho_t(\boldsymbol{b}_{t-1})$ and the pricing function $\boldsymbol{q}_t(s_t, \boldsymbol{b}_t)$;

(ii) the fiscal policy function $\rho_t(\boldsymbol{b}_{t-1})$ maximizes the right-hand side of (4) subject to the budget constraint (3), taking into account the pricing function $\boldsymbol{q}_t(s_t, \boldsymbol{b}_t)$;

(iii) the pricing function $\boldsymbol{q}_t(s_t, \boldsymbol{b}_t)$ satisfies the first-order condition of household utility maximization (5) given the fiscal policy function $\rho_t(\boldsymbol{b}_{t-1})$.

3 Time Consistency of a Markov Perfect Competitive Equilibrium

In this section, I show that the time consistency of fiscal policy carries over in environments in which interest rates reflect both the default probability, as in Aguiar et al. (2016), and the endogenous marginal rate of substitution between present and future consumption, as in Lucas and Stokey (1983).

Lucas and Stokey (1983) argue that if debt commitments are binding, then in an environment with no capital the discretionary fiscal policy is time consistent. However, they show that time consistency does not carry over in a monetary economy, in which future governments can inflate away the value of outstanding debt. Analogous conclusion applies to the model studied in this paper: If default risk is positive, it is reflected in bond prices and, hence, the first-best allocation is generally not attainable for a Markov perfect competitive equilibrium. Nevertheless, the optimal fiscal policy can still be characterized by considering the modified commitment problem in which fiscal commitments are binding but debt commitments are not. Then the fiscal policy is "time consistent," in a sense that the sequential government sticks to the ex ante optimal fiscal plan and follows it as long as the government does not default.

3.1 The Modified Commitment Problem

To characterize the optimal maturity debt structure of a government that cannot commit (a Markov government), it is useful to consider the following planning problem. The government (a planner) can commit to fiscal policies conditional on sequential default decisions not being preferred. In other words, at date 0 the planner simultaneously makes fiscal decisions for all periods, and it can promise to follow the plan; however, it defaults whenever the value of the outside option is higher than the value of pursuing the fiscal plan. I call this the modified commitment problem due to the planner's inability to commit to pay its debt.

Suppose that the planner does not opt to default at the beginning of $t = 0$. Then the planner sets fiscal policies for current and all future periods. Define by a fiscal plan

$\boldsymbol{s}_t = (s_t, s_{t+1}, ..., s_T)$ a sequence of contingent budget surpluses from period t to T. Let $W_t(\boldsymbol{s}_t)$ be the value of fiscal plan $\boldsymbol{s}_t$, that is defined recursively as

$$W_t(\boldsymbol{s}_t) = u(1 - \tau + s_t) + \theta_t \omega(\tau - s_t) + \beta \cdot \mathbb{E} \max \left\{ W_{t+1}(\boldsymbol{s}_{t+1}),\ V_{t+1}^{def} \right\} \tag{7}$$

Any fiscal plan $\boldsymbol{s}_0$ uniquely determines bond prices. Iterating equation (5) forward we can derive the bond prices at the initial period:

$$q_0^t(\boldsymbol{s}_0) = \beta^t \frac{u'(1 - \tau + s_t)}{u'(1 - \tau + s_0)} \cdot Pr_0^t(\boldsymbol{s}_0) \tag{8}$$

where $Pr_0^t(\boldsymbol{s}_0)$ defines the probability of repaying debt issued at date 0, which matures at t, i.e., the probability that the planner does not default at dates 1, 2, ..., t with $Pr_0^0(\boldsymbol{s}_0) = 1$:

$$Pr_0^t(\boldsymbol{s}_0) = \prod_{k=1}^{t} \text{Prob}(W_k(\boldsymbol{s}_k) \geq V_k^{def}) = \prod_{k=1}^{t} F(W_k(\boldsymbol{s}_k) \cdot \frac{1 - \beta}{1 - \beta^{T-k+1}}) \tag{9}$$

Any fiscal plan $\boldsymbol{s}_0$ must satisfy the dynamic budget constraint:

$$s_0 + \sum_{k=1}^{T} \beta^k \frac{u'(1 - \tau + s_k)}{u'(1 - \tau + s_0)} \cdot Pr_0^k(\boldsymbol{s}_0) \cdot s_k \geq b_{-1}^0 + \sum_{k=1}^{T} \beta^k \frac{u'(1 - \tau + s_k)}{u'(1 - \tau + s_0)} \cdot Pr_0^k(\boldsymbol{s}_0) \cdot b_{-1}^k$$

or equivalently

$$\sum_{k=0}^{T} \beta^k u'(1 - \tau + s_k) \cdot Pr_0^k(\boldsymbol{s}_0) \cdot s_k \geq \sum_{k=0}^{T} \beta^k u'(1 - \tau + s_k) \cdot Pr_0^k(\boldsymbol{s}_0) \cdot b_{-1}^k \tag{10}$$

The left-hand side represents the present value of contingent budget surpluses (adjusted by $u'(1 - \tau + s_0)$), while the right-hand side is the market value of outstanding debt, i.e., any outstanding debt must be financed by future budget surpluses.

Let $\hat{V}_0(\boldsymbol{b}_{-1})$ be the value of the planner at the initial period if the planner prefers not to default:

$$\hat{V}_0(\boldsymbol{b}_{-1}) = \max_{\{s_0\}} W_0(\boldsymbol{s}_0) \tag{11}$$

subject to $s_t \in (-(1 - \tau),\ \tau)$, $\forall t$ and the dynamic budget constraint (10)

Therefore, the allocation associated with the modified commitment problem consists of the fiscal plan $\hat{\boldsymbol{s}}_0(\boldsymbol{b}_{-1})$ and the value function of the planner $\hat{V}_0(\boldsymbol{b}_{-1})$ such that:

(i) the fiscal plan $\hat{\boldsymbol{s}}_0(\boldsymbol{b}_{-1})$ maximizes (11) subject to the dynamic budget constraint (10);

(ii) the value function $\hat{V}_0(\boldsymbol{b}_{-1})$ satisfies (11).

Importantly, the maturity structure is completely irrelevant for the planner. As long as the government can commit to the sequence of budget surpluses, default probabilities and risk-free interest rates remain constant. Therefore, bond prices do not change and there are infinitely many ways to implement the allocation, with multiple maturities available every period. This does not apply to the Markov government, as the inherited maturity structure affects government decisions in future periods.

Characterization of the Modified Commitment Problem

Throughout the paper, I will assume that the solution to the modified commitment problem (i) is interior, (ii) can be described by first-order necessary conditions, and (iii) the budget constraint holds with equality. Assumption 2 (iii) guarantees that the value and pricing functions are continuously differentiable.

Define by $B_0(\boldsymbol{s}_0,\, \boldsymbol{b}_{-1})$ the right-hand side of (10), i.e. the market value of outstanding debt $\boldsymbol{b}_{-1}$ at date 0 if the planner pursues fiscal plan $\boldsymbol{s}_0$. Similarly, let $S_0(\boldsymbol{s}_0)$ define the market value of budget surpluses (the left-hand side of (10)):

$$B_0(\boldsymbol{s}_0,\, \boldsymbol{b}_{-1}) = \sum_{k=0}^{T} \beta^k u'(1-\tau+s_k) \cdot Pr_0^k(\boldsymbol{s}_0) \cdot b_{-1}^k,$$

$$S_0(\boldsymbol{s}_0) = \sum_{k=0}^{T} \beta^k u'(1-\tau+s_k) \cdot Pr_0^k(\boldsymbol{s}_0) \cdot s_k$$

Let $MRS^0_{t,\,t+k}(\boldsymbol{s}_0)$ be the marginal rate of substitution between contingent budget surpluses at period t and $t+k$ from the perspective of the planner at period 0, $0 \le t < t+k$. The marginal rate of substitution shows by how much the budget surplus at period t can be decreased if the planner increases the budget surplus at $t+k$ by one unit, keeping value in (7) constant. Note that the marginal rate of substitution depends only on fiscal plan $\boldsymbol{s}_0$ and does not depend on the initial debt composition $\boldsymbol{b}_{-1}$:

$$MRS^0_{t,\,t+k}(\boldsymbol{s}_0) = -\frac{\triangle s_t}{\triangle s_{t+k}}|_{keeping\, W_0(\boldsymbol{s}_0^T)\, constant} =$$

$$= \frac{\frac{\partial}{\partial s_{t+k}} W_0(\boldsymbol{s}_0)}{\frac{\partial}{\partial s_t} W_0(\boldsymbol{s}_0)} \tag{12}$$

Define by $MRT^0_{t,\,t+k}(\boldsymbol{s}_0,\, \boldsymbol{b}_{-1})$ the marginal rate of transformation between contingent budget surpluses at period t and $t+k$, i.e. the rate at which the planner at period $0 \le t < t+k$

can transfer budget surpluses from $t+k$ to t, keeping its budget constraint satisfied with equality:

$$MRT^0_{t,\,t+k}(\boldsymbol{s}_0,\,\boldsymbol{b}_{-1}) = -\frac{\triangle s_t}{\triangle s_{t+k}}|_{budget\,constraint\,(10)\,at\,t=0\,is\,satisfied\,with\,equality} =$$

$$= \frac{\frac{\partial}{\partial s_{t+k}}\left(S_0(\boldsymbol{s}_0) - B_0(\boldsymbol{s}_0,\,\boldsymbol{b}_{-1})\right)}{\frac{\partial}{\partial s_t}\left(S_0(\boldsymbol{s}_0) - B_0(\boldsymbol{s}_0,\,\boldsymbol{b}_{-1})\right)} \tag{13}$$

In equilibrium, marginal rates of substitution must be equal to the corresponding marginal rates of transformation, i.e. $MRS^0_{t,\,t+1}(\boldsymbol{s}_0) = MRT^0_{t,\,t+1}(\boldsymbol{s}_0,\,\boldsymbol{b}_{-1})\ \forall t = 0,\,1,\,...,\,T-1$. Therefore, the optimal fiscal plan $\hat{\boldsymbol{s}}_0(\boldsymbol{b}_{-1})$ solves

$$\begin{cases} MRS^0_{t,\,t+1}(\boldsymbol{s}_0) = MRT^0_{t,\,t+1}(\boldsymbol{s}_0,\,\boldsymbol{b}_{-1}) \quad \text{for } t = 0,\,...,\,T-1 \\ S_0(\boldsymbol{s}_0) = B_0(\boldsymbol{s}_0,\,\boldsymbol{b}_{-1}) \end{cases} \tag{14}$$

Intuitively, any marginal deviation from the optimal fiscal plan keeping the budget constraint (10) satisfied with equality should not lead to an increase in the planner's value.

3.2 Equivalence of the Markov and Modified Commitment Problems

In this subsection, I demonstrate that the Markov government can replicate the planner's allocation. First of all, note that the value of the modified commitment problem is an upper bound for the value of the Markov problem.

Lemma 1. $\hat{V}_0(\boldsymbol{b}_{-1}) \geq V_0(\boldsymbol{b}_{-1})\ \forall T \geq 1$.

The proof of Lemma 1 is simple. Any Markov allocation has to satisfy budget constraint (3) at every period $t = 0,\,1,\,...,\,T$. Combining all sequential budget constraints (3) yields dynamic budget constraint (10). Therefore, any Markov allocation is feasible to the planner and can be replicated by the modified commitment contract. The allocation associated with the modified commitment problem is, thus, at least as good as the Markov perfect competitive equilibrium allocation. Lemma 1 also implies that if the Markov government can replicate the planner's optimal allocation, then this allocation must be optimal for the Markov government.

At date 0 the Markov government can choose budget deficit as under the modified commitment contract. However, it cannot commit to the sequence of ex ante optimal budget surpluses, as future decisions will be made by the future governments and their actions depend on the maturity of outstanding debt.

Let $\boldsymbol{s}_0^\star(\boldsymbol{b}_{-1}) = (s_0^\star,\, s_1^\star, ...,\, s_T^\star)$ be the sequence of contingent budget surpluses associated with the Markov perfect competitive equilibrium and let $\hat{\boldsymbol{s}}_0(\boldsymbol{b}_{-1}) = (\hat{s}_0,\, \hat{s}_1, ...,\, \hat{s}_T)$ be the optimal fiscal plan associated with the modified commitment problem. Proposition 1 states that the Markov government achieves exactly the same allocation as the planner.

Proposition 1. *Suppose the solution to (11) is interior and given by (14). Then* $\boldsymbol{s}_0^\star(\boldsymbol{b}_{-1}) = \hat{\boldsymbol{s}}_0(\boldsymbol{b}_{-1})$ *and* $V_0(\boldsymbol{b}_{-1}) = \hat{V}_0(\boldsymbol{b}_{-1})\ \forall T \geq 1$.

The proof of Proposition 1 is inductive. Notice that Proposition 1 holds trivially for $T = 1$. If default is not optimal at the beginning of the last period, a government only repays the outstanding debt. Moreover, Proposition 1 is then satisfied for $T = 2$, as no fiscal decision is made in the last period. Therefore, the Markov government and the planner's problem are identical for $T = 2$. To complete the proof, it is sufficient to show that if Proposition 1 holds for a T period model then it is also true for a $T+1$ period model.

Consider a T period economy with time indexed as $t = 1,\, 2,\, ...,\, T$. Suppose that Proposition 1 holds for T. Define by $\boldsymbol{s}_1^\star(\boldsymbol{b}_0) = (s_1^\star(\boldsymbol{b}_0),\, ...,\, s_T^\star(\boldsymbol{b}_0))$ the allocation under Markov Perfect Competitive Equilibrium and by $\hat{\boldsymbol{s}}_1(1,\, \boldsymbol{b}_0) = (\hat{s}_1(1,\, \boldsymbol{b}_0),\, ...,\, \hat{s}_T(1,\, \boldsymbol{b}_0))$ the optimal fiscal plan made by a planner at period 1 given outstanding debt $\boldsymbol{b}_0$. The Markov allocation is, therefore, equivalent to the fiscal plan made at $t = 1$, i.e. $\boldsymbol{s}_1^\star(\boldsymbol{b}_0) = \hat{\boldsymbol{s}}_1(1,\, \boldsymbol{b}_0)$, moreover, $V_1(\boldsymbol{b}_0) = \hat{V}_1(\boldsymbol{b}_0) \equiv W_1(\hat{\boldsymbol{s}}_1(1,\, \boldsymbol{b}_0))$.

Now consider a $T+1$ period model with $t = 0,\, 1,\, ...,\, T$. The Markov problem at period 0, conditional on default not being optimal at $t = 0$, can be rewritten as follows:

$$V_0(\boldsymbol{b}_{-1}) = \max_{\{s_0,\, \boldsymbol{b}_0\}} \Big\{ u(1-\tau+s_0) + \theta_0\omega(\tau - s_0) + \beta \cdot \mathbb{E} \max \Big\{ W_1(\hat{\boldsymbol{s}}_1(1,\, \boldsymbol{b}_0));\, V_1^{def} \Big\} \Big\} \tag{15}$$

subject to $s_0 \in (-(1-\tau),\, \tau)$ and

$$u'(1-\tau+s_0)\cdot s_0 = u'(1-\tau+s_0)\cdot b_{-1}^0 + \sum_{k=1}^{T} \beta^k u'(1-\tau+\hat{s}_k(1,\, \boldsymbol{b}_0))\cdot Pr_0^k(\hat{\boldsymbol{s}}_1(1,\, \boldsymbol{b}_0))\cdot \Big(b_{-1}^k - b_0^k\Big) \tag{16}$$

The government optimally chooses the budget deficit at date 0 that equals to the change in the market value of debt. Bond prices depend on the fiscal plan set by the next period planner and the fiscal plan in turn is affected by the maturity choice of the current period government. Lemma 2 states that the Markov government can use maturity to discipline the future policy maker.

Lemma 2. *Suppose the solution to (11) is interior and given by (14). Then there exists a (generally unique) maturity structure* $\boldsymbol{b}_0^\star$ *such that if* $s_0 = \hat{s}_0(\boldsymbol{b}_{-1})$ *budget constraint (13) is satisfied and* $\hat{\boldsymbol{s}}_1(1, \boldsymbol{b}_0^\star) = \hat{\boldsymbol{s}}_1(\boldsymbol{b}_{-1})$.

Therefore, any optimal fiscal plan $\hat{\boldsymbol{s}}_0(\boldsymbol{b}_{-1})$ can be replicated by the Markov government. The government can structure its debt portfolio such that the next period planner chooses exactly the same allocation as the planner at period 0. Every period, the number of maturities is sufficient so that all first-order necessary conditions and the budget constraint of the next-period planner are satisfied by the ex ante optimal fiscal plan. This completes the proof of Proposition 1.

3.3 The Optimal Maturity Structure

To better understand the intuition behind Lemma 2 and the role of maturity, first, note that for a given budget surplus at period 0 $\hat{s}_0$, the planners at period 0 and period 1 maximize exactly the same objective function $W_1(\boldsymbol{s}_1)$. Therefore, the marginal rates of substitution given by (12) are identical for the planners. Lemma A.2 in Appendix A formally proves this statement.

However, the marginal rates of transformation as defined by (13) are generally different. The reason is that by adjusting the fiscal plan, the government at period 1 not only alters the market value of contingent budget surpluses but also affects the market value of outstanding debt, which is generally different from the outstanding debt in preceding period. To see this, rewrite the budget constraint at $t = 0$ (10) as

$$\sum_{k=1}^{T} \beta^{k-1} \cdot u'(1-\tau+s_k) \cdot Pr_1^k(\boldsymbol{s}_1) \cdot (s_k - b_{-1}^k) = -\frac{\beta}{Pr_0^1(\boldsymbol{s}_1)} \cdot u'(1-\tau+s_0) \cdot (s_0 - b_{-1}^0) \quad (17)$$

where the left hand-side represents the difference between market values[2] of budget surpluses and outstanding debt from period 1 to T, and and the right-hand side is analogical difference for period 0. As defined in (9), $Pr_1^k(\boldsymbol{s}_1)$ is the probability of repaying debt issued at date 1 which matures at period k, $Pr_0^1(\boldsymbol{s}_0)$ defines the probability of no default at $t = 1$

[2]adjusted by $\frac{\beta}{Pr_0^1(\boldsymbol{s}_1)} \cdot u'(1-\tau+s_0)$

given fiscal plan $\boldsymbol{s}_0$.

Analogously, the budget constraint (10) at $t = 1$ can be rewritten as

$$\sum_{k=1}^{T} \beta^{k-1} \cdot u'(1-\tau+s_k) \cdot Pr_1^k(\boldsymbol{s}_1) \cdot (s_k - b_{-1}^k) = \sum_{k=1}^{T} \beta^{k-1} \cdot u'(1-\tau+s_k) \cdot Pr_1^k(\boldsymbol{s}_1) \cdot (b_0^k - b_{-1}^k) \quad (18)$$

Note that the left-hand sides of (17) and (18) are identical. Combining the right-hand sides of the equations yields the budget constraint of Markov government at $t = 0$ (16), i.e., the value of the budget deficit at $t = 0$ equals to the difference in the market values of outstanding debt at $t = 0$ and $t = 1$.

Consider a marginal deviation from the optimal plan $\hat{\boldsymbol{s}}_1 = (\hat{s}_1, ..., \hat{s}_T)$ keeping $\hat{s}_0$ and $W_1(\hat{\boldsymbol{s}}_1)$ constant. The planner's optimality conditions imply that budget constraint (17) remains satisfied with equality. Importantly, the right-hand side of equation (17) is unaffected, because the no default probability $Pr_0^1(\hat{\boldsymbol{s}}_1) = F(W_1(\hat{\boldsymbol{s}}_1) \cdot \frac{1-\beta}{1-\beta^{T-1}})$ depends exclusively on $W_1(\hat{\boldsymbol{s}}_1)$, which remains constant. Consequently, the left-hand side of (17) must also be unaffected.

Let us now discuss how the same deviation can affect the budget constraint at $t = 1$ (18). Note that the left-hand side of (18) stays constant because it is identical to the left-hand side of (17). However, the right-hand side of (18)–the value of net debt issued at period 0–can be manipulated. For example, suppose that the debt is mostly short term. Then a reallocation in budget surpluses that decreases the price of short-term debt at the expense of an increase in the price of long-term debt could decrease the market value of total debt.

The optimal conditions of modified commitment problem at period 1 require that any marginal deviations from $\hat{\boldsymbol{s}}_1$ that keep $W_1(\hat{\boldsymbol{s}}_1)$ constant do not allow the government to relax budget constraint (18). Consequently, the optimal maturity structure $\boldsymbol{b}_0^\star$ has to be such that any changes in $\hat{\boldsymbol{s}}_1$ that keep $W_1(\hat{\boldsymbol{s}}_1)$ constant do not cause a reduction in the value of debt issued at period 1.

Proposition 2. The Optimal Maturity Structure. *Suppose the solution to (11) is interior and given by (14). Then the optimal maturity structure* $\boldsymbol{b}_0^\star(\boldsymbol{b}_{-1}) = (b_0^1(\boldsymbol{b}_{-1}), ..., b_0^T(\boldsymbol{b}_{-1}))$ *satisfies:*

$$\frac{\frac{\partial \beta \cdot u'(1-\tau+\hat{s}_{t+1}) \cdot Pr_t^{t+1}(\hat{s}_{t+1})}{\partial \hat{s}_{t+1}}}{\frac{\partial u'(1-\tau+\hat{s}_t)}{\partial \hat{s}_t}} \cdot \frac{b_0^{t+1}(\boldsymbol{b}_{-1}) - b_{-1}^{t+1}}{b_0^t(\boldsymbol{b}_{-1}) - b_{-1}^t} = MRS_{t,t+1}^0(\hat{\boldsymbol{s}}_t(\boldsymbol{b}_{-1})) \quad (19)$$

for $t = 0, 1, ..., T-1$, where $\hat{\boldsymbol{s}}_t(\boldsymbol{b}_{-1})$ is the optimal fiscal plan and $MRS_{t,t+1}^0(\hat{\boldsymbol{s}}_t(\boldsymbol{b}_{-1}))$ is given by (12).

The complete proof of Proposition 2 is provided in Appendix A. Intuitively, debt maturity is structured such that any marginal alterations in bond prices cannot reduce the value of debt issued in the initial period. For example, consider an increase in the budget surplus at $t+1$, $\triangle s_{t+1}$, and a decrease in the budget surplus at t, $-\triangle s_t$, such that government value $W_1(\boldsymbol{s}_1)$ stays constant. Note that such perturbation does not affect the value of debt maturing before t, because marginal utilities $u'(1-\tau+s_k)$ for $k<t$ are unchanged. In addition, since $W_k(\boldsymbol{s}_k)$ for $k<t$ stays constant the default probabilities in periods $k=0$, 1, ..., $t-1$ are unaffected. Similarly, this deviation does not alter the prices of bonds that mature after $t+1$.

Thus, a variation in budget surpluses s_t and s_{t+1} keeping $W_1(\boldsymbol{s}_1)$ constant affects only the market value of debt maturing at t and $t+1$. It is worth highlighting that the default probability at t is constant because $W_t(\boldsymbol{s}_t)$ is unchanged. However, as $W_{t+1}(\boldsymbol{s}_{t+1})$ depends on s_{t+1} but not on s_t, the probability of default at date $t+1$ changes. Therefore, the described deviation from a fiscal plan manipulates the risk-free rate of debts maturing at t and $t+1$ and the risk premium of debt maturing at $t+1$ but not at t. The latter implies that if default probability is positive the price of debt maturing at a later period is more elastic compared to an obligation with shorter maturity. In other words, a marginal deviation from a path of fiscal policies has a higher impact on longer-term debt.

Along with the elasticities of bond prices, the optimal maturity structure depends on the ex ante optimal fiscal plan that is designed by taking into account the initial period inherited debt and future public spending taste parameters. In Sections 4 and 5, I consider an example with zero initial debt and stationary future taste parameters that allows me to characterize the shape of optimal maturity analytically.

4 A Three-period Analytical Model: Issue More Short-Term Debt

In the previous section I demonstrated that maturity is used to discipline future governments. The question that arises is how maturity should be structured if both risk-free interest rates and risk premiums can be altered. In this section I consider a tractable three-period model and show that if default risk is increasing in debt, then the optimal maturity is skewed toward the short-term end.

4.1 The Model

Consider a three-period model, as in Section 2. In contrast to the benchmark model, I make one simplifying assumption: the only uncertainty in this model concerns the outside value at period 2; default never occurs in periods 0 or 1. More formally, V_0^{def} and V_1^{def} are deterministic, known at $t = 0$ and $V_0^{def} = V_1^{def} \to -\infty$.

Notice that this assumption does not affect the implications for the optimal maturity structure because any perturbations in budget surpluses at $t = 1$ and $t = 2$ do not change the default probability at $t = 1$. The Markov government at period 1 makes its fiscal decisions conditional on no prior default. Once the government has decided not to take the outside option at $t = 1$, it cannot manipulate the risk of default in the intermediate period.

Let $V_2^{def} = v^d$ where v^d is as defined in Section 2: it is drawn from continuous distribution F, $v^d \in [v_{min}, v_{max}]$ and Assumption 2 is satisfied. To abstract away from effect of exogenously given inherited debt, I set the initial debt to be equal to zero. Taste parameter for government spending $\theta_1 = \theta_2 = 1$, but $\theta_0 > 1$ to ensure that the government has an incentive to issue some positive debt, which must be repaid at the future dates.

At the beginning of period 2, the government decides whether to default or not. Notice that the budget constraint at $t = 2$ always holds with equality for any $b_1^2 \geq 0$ due to Assumption 1. Hence, conditional on no default the government sets the budget surplus equal to the outstanding debt $s_2 = b_1^2$. The government defaults whenever $V_2^{def} > u(1 - \tau + b_1^2) + \omega(\tau - b_1^2)$.

Given that $V_2^{def} \in [v_{min}, v_{max}]$ and taking into account Assumption 2 (i) and (ii), the debt space can be divided into three regions. Let $\underline{b}$ denote the largest value of debt for which the government never defaults and $\bar{b}$ be the smallest amount of debt for which the government defaults with probability 1, i.e. $\underline{b}$ and $\bar{b}$ satisfy:

$$u(1 - \tau + \underline{b}) + \omega(\tau - \underline{b}) = v_{max}$$

$$u(1 - \tau + \bar{b}) + \omega(\tau - \bar{b}) = v_{min}$$

Then for any $b_1^2 \leq \underline{b}$ default never occurs because $u(1 - \tau + b_1^2) + \omega(\tau - b_1^2) \geq v_{max}$. Analogously, for any $b_1^2 > \bar{b}$ the government always defaults because the value of outside option is higher. For $b_1^2 \in (\underline{b}, \bar{b})$ the default decision depends on the realization of the outside option value shock, and the ex-ante default probability is $\pi(b_1^2) = 1 - F(u(1 - \tau + b_1^2) + \omega(\tau - b_1^2))$. Following the literature, I refer to these regions as to safe, default and crisis regions respectively.

The intermediate-period government inherits legacy debt[3] $(b_0^1,\, b_0^2)$ and chooses what fraction of outstanding debt is repaid in the current period (the budget surplus at period 1 s_1) and leaves the rest to be paid in the last period (contingent budget surplus at period 2 $s_2 = b_1^2$), taking into account the default decision of the next-period government. The budget constraint implies that the present value of budget surpluses has to be equal to the present value of outstanding debt. The government's problem at $t = 1$ is

$$V_1(b_0^1,\, b_0^2) = \max_{s_1,\, s_2} u(1-\tau+s_1)+\omega(\tau-s_1)+\int \max\{u(1-\tau+s_2)+\omega(\tau-s_2);\; v^d\}dF(v^d) \quad (20)$$

subject to $s_0 \in (-(1-\tau),\, \tau)$ and

$$u'(1-\tau+s_1)\cdot\left(s_1 - b_0^1\right) + u'(1-\tau+s_2)(1-\pi(s_2))\cdot(s_2 - b_0^2) \geq 0 \quad (21)$$

Let $s_1^\star(b_0^1,\, b_0^2)$ and $s_2^\star(b_0^1,\, b_0^2)$ be the optimal policy functions of problem (20). The period 0 government takes into account the decisions made by the future governments, and chooses the budget deficit at date 0 $-s_0$, short-term debt b_0^1 and long-term debt b_0^2 to solve:

$$V_0 = \max_{s_0,\, b_0^1,\, b_0^2} u(1-\tau+s_0) + \theta_0\omega(\tau - s_0) + V_1(b_0^1,\, b_0^2) \quad (22)$$

subject to $s_0 \in (-(1-\tau),\, \tau)$ and

$$u'(1-\tau+s_0)\cdot s_0 + u'(1-\tau+s_1^\star(b_0^1,\, b_0^2))\cdot b_0^1 + u'(1-\tau+s_2^\star(b_0^1,\, b_0^2))(1-\pi(s_2^\star(b_0^1,\, b_0^2)))\cdot b_0^2 \geq 0 \quad (23)$$

According to Proposition 1, the Markov government can structure debt maturity such that the next government is incentivized to follow the ex ante optimal sequence of contingent budget surpluses. Thus, in order to characterize the optimal maturity structure $\boldsymbol{b}_0^\star = (b_0^{1\star},\, b_0^{2\star})$ I first solve for the optimal fiscal plan of the modified commitment contract as explained in Section 3. Then I find the maturity that makes the optimal fiscal plan incentive-compatible for the next-period government.

[3] I assume that the debt position $(b_0^1,\, b_0^2)$ is feasible in a sense that there exist s_1 and s_2, $s_{1,2} \in (-(1-\tau),\, \tau)$ that satisfy the budget constraint (21). Notice that if the government at period 0 issues large positions of debt such that it can not be repaid by the future government and default occurs then the bond prices are zeros. The initial period government is then strictly better-off by issuing no debt at all.

4.2 Optimal Fiscal Plan

Consider the modified commitment problem, as discussed in Section 3. The planner can commit to a fiscal plan; however, it defaults whenever the outside option exceeds the value of pursuing the fiscal plan. At period 0 the planner chooses the sequence of contingent budget surpluses $\boldsymbol{s}_0 = (s_0,\, s_1,\, s_2)$ that satisfies the dynamic budget constraint:

$$u'(1-\tau+s_0)\cdot s_0 + u'(1-\tau+s_1)\cdot s_1 + u'(1-\tau+s_2)Pr_1^2(s_2)\cdot s_2 \geq 0 \tag{24}$$

where $Pr_1^2(s_2) \equiv (1-\pi(s_2)) = F(u(1-\tau+s_2)+\omega(\tau-s_2))$ is the probability of no default in period 2.

The intuition for the above constraint is that the budget deficit at period 0 is financed by the budget surplus at period 1 and the contingent budget surplus at period 2. Notice that the maturity structure $(b_0^1,\, b_0^2)$ does not enter the budget constraint, and, therefore, is completely irrelevant for the planner. As long as the government can commit to the sequence of budget surpluses, bond prices do not change and the budget constraint is satisfied. This does not apply to the Markov government, as the inherited maturity structure affects the government's decision at period 1 and, hence, the prices at period 0.

Let $W_0(\boldsymbol{s}_0)$ be the expected value of pursuing the fiscal plan $\boldsymbol{s}_0 = (s_0,\, s_1,\, s_2)$:

$$\begin{aligned} W_0(\boldsymbol{s}_0) = u(1-\tau+s_0) + \theta_0\omega(\tau-s_0) + u(1-\tau+s_1) + \omega(\tau-s_1) \\ + \textstyle\int \max\{u(1-\tau+s_2)+\omega(\tau-s_2);\, v^d\}dF(v^d) \end{aligned} \tag{25}$$

Then the planner's problem in the initial period is

$$\hat{V}_0 = \max_{\boldsymbol{s}_0} W_0(\boldsymbol{s}_0) \tag{26}$$

$$\text{subject to } s_0 \in (-(1-\tau),\, \tau) \text{ and (24)}$$

Let $\hat{\boldsymbol{s}}_0 = (\hat{s}_0,\, \hat{s}_1,\, \hat{s}_2)$ be the optimal allocation under the modified commitment problem. As discussed in Section 3.2, any optimal plan satisfies the following optimality condition: the marginal rate of substitution between budget surpluses at period 1 and period 2 $(MRS_{1,2}^0(\boldsymbol{s}_0))$ equals to the marginal rate of transformation $(MRT_{1,2}^0(\boldsymbol{s}_0))$:

$$MRS_{1,2}^0(\boldsymbol{s}_0) = -\frac{\triangle s_1}{\triangle s_2}|_{keeping\, W_0(s_0)\, constant} =$$

$$= \frac{\omega'(\tau-s_2)-u'(1-\tau+s_2)}{\omega'(\tau-s_1)-u'(1-\tau+s_1)}Pr_1^2(s_2) \tag{27}$$

$$MRT^0_{1,\,2}(\boldsymbol{s}_0) = -\frac{\triangle s_1}{\triangle s_2}|_{budget\, constraint\,(25)\, is\, satisfied\, with\, equality} =$$

$$= \frac{u'(1-\tau+s_2) + \left[u''(1-\tau+s_2) - u'(1-\tau+s_2)\cdot \frac{\frac{\partial Pr_1^2(s_2)}{\partial s_2}}{Pr_1^2(s_2)}\right]\cdot s_2}{u'(1-\tau+s_1) + u''(1-\tau+s_1)\cdot s_1} Pr_1^2(s_2) \tag{28}$$

Suppose that a representative household has a constant relative risk aversion function:

Assumption 3. $u(c) = \frac{c^{1-\gamma_C}}{1-\gamma_C}$, $\gamma_C \geq 0$.

Proposition 3 summarizes the optimal allocation in the intermediate and last periods.

Proposition 3. The Optimal Fiscal Plan
(i) if $\theta_0 \in (1,\, \bar{\theta}]$, where $\bar{\theta}$ is such that $\hat{s}_2(\bar{\theta}) = \underline{b}$, the planner sets $\hat{s}_1 = \hat{s}_2$;
(ii) if $\theta_0 > \bar{\theta}$, the optimal fiscal plan implies $\hat{s}_1 > \hat{s}_2$.

The first result of Proposition 3 relates to the safe region. Suppose θ_0 is relatively small so that the government never defaults at the beginning of period 2. Then periods 1 and 2 are identical and the optimal policy for the planner is to smooth consumption and government spending over time. As θ_0 goes up and the government sets a higher budget deficit in period 0, the planner eventually enters the crisis region in which a marginal increase in $\hat{s}_2$ continuously raises the default risk. Then setting equal budget surpluses is no longer optimal. If $\hat{s}_1 = \hat{s}_2$ the marginal rate of substitution (27) equals the probability of repaying debt $Pr_1^2(\hat{s}_2)$. However, the marginal rate of transformation (28) is less than $Pr_1^2(\hat{s}_2)$, since a marginal decrease in contingent budget surplus s_2 decreases the default probability. Intuitively, the planner prefers to repay a larger fraction of debt in the intermediate period to decrease default probability in the last period.

4.3 The Optimal Maturity Structure

Recall that dynamic budget constraint (24) is

$$u'(1-\tau+s_1)\cdot s_1 + u'(1-\tau+s_2)Pr_1^2(s_2)\cdot s_2 \geq -u'(1-\tau+s_0)\cdot s_0 \tag{29}$$

and the budget constraint of period 1 (21) can be rewritten as

$$u'(1-\tau+s_1)\cdot s_1+u'(1-\tau+s_2)Pr_1^2(s_2)\cdot s_2 \geq u'(1-\tau+s_1)\cdot b_0^1+u'(1-\tau+s_2)Pr_1^2(s_2)\cdot b_0^2 \quad (30)$$

The left-hand sides of (29) and (30) are identical and represent the market value of budget surpluses. The right-hand side of (30) is fixed: The planner optimally chooses budget surpluses to finance some optimally chosen budget deficit in the initial period. However, the right-hand side of (30), the market value of outstanding debt, can be manipulated by the intermediate government.

Following the discussion in Section 3.3, the optimal debt structure $(b_0^{1\star},\, b_0^{2\star})$ has to be such that any marginal deviations from the ex ante optimal fiscal plan along the indifference curve cannot decrease the market value of outstanding debt (the left-hand side of (30)). More formally, $(b_0^{1\star},\, b_0^{2\star})$ has to satisfy

$$\frac{\partial}{\partial s_1}u'(1-\tau+\hat{s}_1)\cdot b_0^{1\star}\cdot MRS_{1,\,2}^0(\hat{\boldsymbol{s}}_0)+\frac{\partial}{\partial s_2}u'(1-\tau+\hat{s}_2)Pr_1^2(\hat{s}_2)\cdot b_0^{2\star}=0 \text{ or}$$

$$\frac{\frac{\partial}{\partial s_2}u'(1-\tau+s_2)Pr_1^2(\hat{s}_2)}{\frac{\partial}{\partial s_1}u'(1-\tau+\hat{s}_1)}\cdot\frac{b_0^{2\star}}{b_0^{1\star}}=MRS_{1,\,2}^0(\hat{\boldsymbol{s}}_0) \quad (31)$$

The focus of my analysis is on the crisis region, where the government can manipulate both risk-free interest rates and the risk premium, i.e. $\theta_0 > \bar{\theta}$. Proposition 4 states that the optimal maturity is tilted toward the short-term end, i.e., the government issues more short-term debt than long-term debt.

Proposition 4. The Optimal Maturity Structure in a Three-period Model. *If the economy is in the crisis region ($\theta_0 > \bar{\theta}$), the government issues positive short-term and long-term debt, but the term structure is skewed toward the short end:*

$b_0^{1\star} > 0,\, b_0^{2\star} > 0$ *and* $\frac{b_0^{2\star}(\theta_0)}{b_0^{1\star}(\theta_0)} \in (0,\,1)$. *In addition,* $b_0^{1\star} > \hat{s}_1 > \hat{s}_2 > b_0^{2\star}$.

To better understand this result, it is useful to consider two limiting cases. The first corresponds to the safe region ($\pi_t = \pi_t' = 0$): The government can manipulate risk-free interest rates, but it cannot affect the default risk. The second limiting case is $\gamma_C \to 0$: Lenders are risk-neutral, risk-free interest rates equal β, and cannot be manipulated by the government. However, the default risk is positive, and the government can dilute the value of outstanding debt.

The first limiting case mirrors the findings of Stokey and Lucas (1983). According to Proposition 3, if the probability of default is zero then the government prefers to smooth budget surpluses over time, $s_1 = s_2$. Given the fiscal plan and that the default risk is absent, the price of a one-period bond and a two-period bond are equal to each other. Then the only maturity structure that does not allow the future government to decrease the value of outstanding debt is flat: $b_0^1 = b_0^2$. Under such a term structure, any marginal change in risk-free rates has an equal effect on the value of outstanding short-term debt and long-term debt, i.e. $u''(1-\tau+\hat{s}_1)\cdot b_0^1 = u''(1-\tau+\hat{s}_2)\cdot b_0^2$, and therefore the government has no incentive to deviate from ex-ante optimal allocation. Also, notice that the outstanding debt equals to the corresponding budget surplus, $\hat{s}_1 - b_0^1 = \hat{s}_2 - b_0^2 = 0$, and the government does not have to actively manage its debt in the intermediate period.

The second limiting case reflects the result of Aguiar et al. (2016). Assume that risk-free interest rates are exogenous, and only risk premiums can be manipulated. Therefore, the future government can affect the value of outstanding long-term debt but cannot alter the value of short-term debt. As already mentioned, the short-term default premium cannot be manipulated by the intermediate period government. Therefore, the only maturity that prevents the future government from being able to decrease the market of value of outstanding debt is issuing one-period debt only. Whenever $b_0^2 > 0$, the government at period 1 has an incentive to dilute the value of long-term debt by deviating from the ex ante optimal allocation and issuing more b_1^2. If, on the other hand, $b_0^2 < 0$, the government would want to issue less debt to increase the market value of its long-term savings. More formally, $MRT^0_{1,2}(\boldsymbol{s}_0) = Pr_1^2(s_2) + \frac{\partial Pr_1^2(s_2)}{\partial s_2}\cdot s_2$ but $MRT^1_{1,2}(\boldsymbol{s}_1) = Pr_1^2(s_2) + \frac{\partial Pr_1^2(s_2)}{\partial s_2}\cdot(s_2 - b_0^2)$. The optimal maturity structure is then $b_0^{2\star} = 0$ implying positive net debt issuance in the intermediate period.

Now suppose that the government can affect both risk-free interest rates and default risk. Recall that the government in the intermediate period can manipulate only the long-term risk premium and cannot manipulate the short-term risk premium. Consequently, the price of long-term bonds is more elastic than the price of short-term debt. The flat maturity is not optimal in this case. If the structure were nearly flat, then marginally increasing the contingent budget surplus at period 2 and decreasing the budget surplus at period 1 would lead to a larger decrease in the market value of long-term debt compared to an increase in the value of short-term debt. Issuing only short-term debt is also not optimal, as the government can distort the short-term risk-free rate. The higher elasticity of the long-term debt price implies that the stock of short-term debt must be larger than the long-term debt position.

Moreover, the net issuance of debt in the intermediate period is positive, i.e. $b_1^2 = s_2 > b_0^2$. Suppose by contradiction that $b_0^1 = s_1 > s_2 = b_0^2$. Recall from Proposition 2 that he government prefers to repay a larger fraction of debt at period 1 to decrease default probability at period 2, i.e. $s_1 > s_2$. In equilibrium, the marginal rate of substitution is below the default probability: $\frac{\omega'(\tau-s_2)-u'(1-\tau+s_2)}{\omega'(\tau-s_1)-u'(1-\tau+s_1)}Pr_1^2(s_2) < Pr_1^2(s_2)$. However, the ex post marginal rate of transformation is larger than default probability: $\frac{u'(1-\tau+s_2)}{u'(1-\tau+s_1)}Pr_1^2(s_2) > Pr_1^2(s_2)$. Intuitively, if net debt issued is zero at the intermediate period then the Markov government does not internalize the adverse effect of increasing default risk on the value of already issued debt. Therefore, optimal $(b_0^{1\star}, b_0^{2\star})$ has to satisfy $b_0^1 > s_1 > s_2 > b_0^2$ concluding that the net debt issued is positive in the intermediate period.

5 The Decaying Profile of Maturity

In this section I extend the three-period model to a general $T+1$ period model where default can occur in any period and $\beta \leq 1$. I continue to assume that the initial debt is zero and $\theta_0 > 1$, $\theta_1 = \theta_2 = ... = \theta_T = 1$ to ensure that the government has an incentive to issue some positive debt that must be repaid at the future dates. I show that the optimal maturity structure is decaying: Debt position is lower if maturity date is later.

Obviously, if $\theta_0 = 1$ then the government does not need to borrow any debt from households and $s_0^\star = s_1^\star = ... = s_T^\star = 0$. Importantly, the default probability is zero because due to Assumption 2 (i), $u(1-\tau)+\omega(\tau) > v_{max}$. If $\theta_0 > 1$, then the government prefers to run a budget deficit in the initial period that has to be financed by budget surpluses in future periods. As long as default probability is zero, the government prefers to smooth consumption over time and sets equal budget surplus in periods 1 to T. Let $\hat{s}(\theta_0)$ be the budget surplus in consequent periods as a function of θ_0.

Similarly to Section 4, there is $\bar{\theta}$ that satisfies $u(1-\tau+\hat{s}(\bar{\theta}))+\omega(\tau-\hat{s}(\bar{\theta})) = v_{max}$ so if $\theta_0 > \bar{\theta}$, the default probability becomes positive. Let $\hat{\boldsymbol{s}} = (\hat{s}_0, \hat{s}_1, ..., \hat{s}_T)$ be the optimal fiscal plan as defined in Section 3. Proposition 5 states that if $\theta_0 > \bar{\theta}$ then the government prefers to repay a larger fraction of debt in earlier periods. Thus, the result in Proposition 3 for the three-period model extends to the model.

Proposition 5. The Optimal Fiscal Plan. *Suppose that* $\theta_0 > \bar{\theta}$. *Then* $\hat{s}_1 > \hat{s}_2 > ... > \hat{s}_T$.

This result reflects the trade-off between smoothing consumption over time and decreasing default probability. Evenly distributed budget surpluses are not optimal, because a marginal increase in $\hat{s}_t$ and marginal decrease in $\hat{s}_{t+1}$ would decrease the default probability in period $t+1$ while having no effect on default risk in period t. This increases the expected value of the outside option conditional on default, while having a negligible effect on government's value conditional on no default. Therefore, the planner has an incentive to frontload net payments to creditors.

Now we are ready to characterize the optimal term structure. Let $\boldsymbol{b}_0^\star = (b_0^1, ..., b_0^T)$ be the optimal debt issuances by the Markov government at date 0. Proposition 6 establishes that if the economy is in the crisis region, with debt probability being positive, then the optimal maturity has a decaying profile: Total payments are lower if the maturity date is later.

Proposition 6. The Optimal Maturity Structure. *Suppose that* $\theta_0 > \bar{\theta}$. *Then* $b_0^1 > b_0^2 > ... > b_0^T > 0$.

The government structures its debt portfolio so that the future policy maker cannot benefit from altering the market value of debt. Recall from (19) that the optimal maturity given no initial debt satisfies

$$\frac{\frac{\partial \beta \cdot u'(1-\tau+\hat{s}_{t+1}) \cdot Pr_t^{t+1}(\hat{\boldsymbol{s}}_{t+1})}{\partial \hat{s}_{t+1}}}{\frac{\partial u'(1-\tau+\hat{s}_t)}{\partial \hat{s}_t}} \cdot \frac{b_0^{t+1}}{b_0^t} = MRS_{t,t+1}^0(\hat{\boldsymbol{s}}_t) \tag{32}$$

The proposition 5 implies that $MRS_{t,t+1}^0(\hat{\boldsymbol{s}}_t) < \beta \cdot Pr_t^{t+1}(\hat{\boldsymbol{s}}_{t+1})$ because $\hat{s}_t > \hat{s}_{t+1}$. Therefore, (32) can be rewritten as

$$\frac{u''(1-\tau+\hat{s}_{t+1}) + u'(1-\tau+\hat{s}_{t+1}) \cdot \frac{\frac{\partial Pr_t^{t+1}(\hat{\boldsymbol{s}}_{t+1})}{\partial \hat{s}_{t+1}}}{Pr_t^{t+1}(\hat{\boldsymbol{s}}_{t+1})}}{u''(1-\tau+\hat{s}_t)} \cdot \frac{b_0^{t+1}}{b_0^t} < 1 \tag{33}$$

The numerator on the left-hand side of (33) consists of two parts. The first part[4] corresponds to the change in the risk-free interest rate at date $t+1$. The second part[5] represents the change in the risk premium at $t+1$. The denominator is the change in the risk-free interest rate at date t. As already discussed, a marginal perturbation is s_t and s_{t+1} that keeps the value $W_1(\hat{s}_1)$ constant has no effect on default probability at t.

The distribution of debt over maturity is therefore skewed toward the short end. The reason is that a marginal perturbation in the fiscal plan has a larger effect on the price of bonds with longer maturity. Any such deviation affects the long-term default risk, while the effect on the short-term risk premium is negligible, as the government's value of pursuing the fiscal plan is not changed. Therefore, to preserve the value of debt from being manipulated

[4] $u''(1-\tau+\hat{s}_{t+1})$

[5] $u'(1-\tau+\hat{s}_{t+1}) \cdot \frac{\frac{\partial Pr_t^{t+1}(\hat{\boldsymbol{s}}_{t+1}))}{\partial \hat{s}_{t+1}}}{Pr_t^{t+1}(\hat{\boldsymbol{s}}_{t+1})}$

by future governments, the optimal term structure must have a decaying profile: Payments due are decreasing in maturity date.

Importantly, the frontloading result of Proposition 5 is not crucial for the decaying maturity structure. Suppose that $\hat{s}_1 = \hat{s}_2 = ... = \hat{s}_T = \hat{s}$. Then after simplifying (32) the optimal maturity structure satisfies

$$(1 + \frac{u'(1-\tau+\hat{s})}{u''(1-\tau+\hat{s})} \cdot \frac{\frac{\partial Pr_t^{t+1}(\hat{\boldsymbol{s}}_{t+1})}{\partial \hat{s}_{t+1}}}{Pr_t^{t+1}(\hat{\boldsymbol{s}}_{t+1})}) \cdot \frac{b_0^{t+1}}{b_0^t} = 1.$$

The maturity scheme is still skewed to the short end because $\frac{\partial Pr_t^{t+1}(\hat{\boldsymbol{s}}_{t+1})}{\partial \hat{s}_{t+1}} < 0$ reflecting the fact that the longer-term debt price is more sensitive to deviations from the fiscal plan.

The optimal maturity structure thus depends on the sensitivity of risk-free interest rates and risk premiums. If risk-free interest rates are almost constant, while default risk changes considerably, then the optimal maturity is very short. At the limit, the government issues only short-term debt because $u''(1-\tau+\hat{s}_t) = 0$. On the other hand, if a marginal deviation from the fiscal plan has significant impact on risk-free interest rates but negligible effect on the default risk, then the optimal term structure is approximately flat.

6 Quantitative Exercise

In this section I numerically solve for the optimal maturity and discuss how it is affected by the debt-to-GDP ratio, risk aversion, and default risk. In addition, the time consistency of fiscal policy allows me to analyze the evolution of debt maturity in a multi-period model with multiple maturities available for the government.

6.1 Functional Forms

Throughout this section I assume that conditional on no prior default, the government's per-period payoff is

$$\frac{c^{1-\gamma_C}-1}{1-\gamma_C} + \kappa \frac{g^{1-\gamma_G}-1}{1-\gamma_G}$$

where $\gamma_C = \gamma_G = 1$, $\tau = 0.25$ and κ satisfies $(1-\tau)^{-\gamma_C} = \kappa \tau^{-\gamma_G}$. I assume that the outside option V_t^{def} is equivalent to being excluded from the bond market and incurring output loss χ in every period. I model output loss as the proportional decrease in consumption and government spending, i.e. $V_t^{def} = \frac{1-\beta^{(T-t+1)}}{1-\beta} \cdot u^d(\chi)$, where $u^d(\chi)$ is defined as

$$u^d(\chi) = \frac{((1-\tau)\cdot(1-\chi))^{1-\gamma_C} - 1}{1-\gamma_C} + \kappa\frac{(\tau\cdot(1-\chi))^{1-\gamma_G} - 1}{1-\gamma_G}$$

Estimates of output drops following default vary vastly in the empirical literature. For example, Aguiar and Gopinath (2006) use 2% output loss in their model, while Hebert and Schreger (2016) find that Argentina's cost of default corresponds to a 9.4% permanent decrease in output. For this exercise I set $\chi_{min} = 0.005$ and $\chi_{max} = 0.095$, i.e., the permanent loss of output varies from 0.5% to 9.5%, and thereby $v_{min} = u^d(\chi_{max})$ and $v_{max} = u^d(\chi_{min})$. The per-period payoff of being in default v^d has a probability density distribution f:

$$f(v) = \alpha_F \cdot (v - v_{min})^2 \cdot (v - v_{max})^2 \text{ for } v \in [v_{min}, v_{max}]$$

where α_F is such that $\int_{v_{min}}^{v_{max}} \alpha_F \cdot (v - v_{min})^2 \cdot (v - v_{max})^2 = 1$. The distribution satisfies $f(v_{min}) = f(v_{max}) = 0$; it also implies a smooth increase in default probability as the economy enters the crisis region.

6.2 Three-period Model

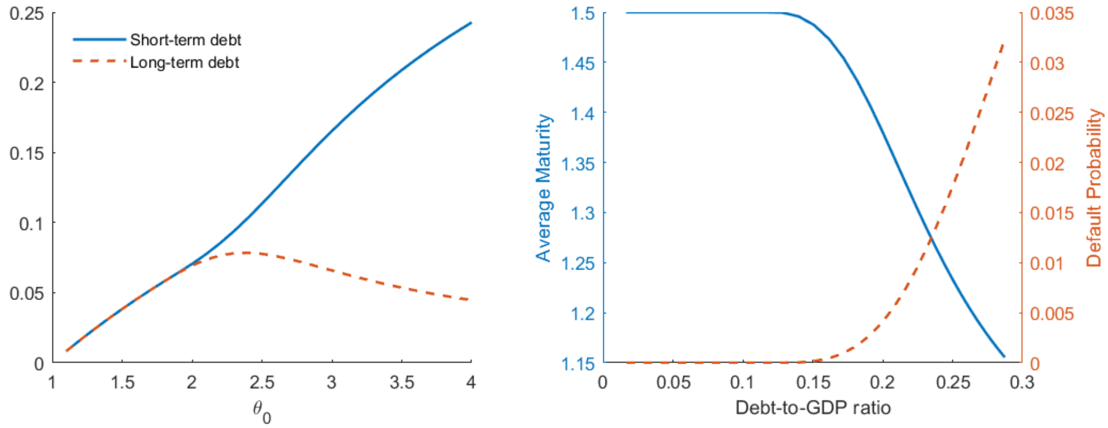
6.2.1 The Benchmark Model

I first consider the three-period problem as discussed in Section 4. The default can occur only at date 2. The discount factor β is 1.

Figure 1 displays the optimal maturity structure in the three-period model. The left panel shows the short-term and long-term debt positions as taste parameter θ_0 increases. The right panel shows the average maturity and default probability for different levels of debt-to-GDP ratio. If the debt maturity is flat, then the average maturity is 1.5 years; if debt consists of short-term bonds only, then the average maturity is 1 year.

For a relatively low θ_0 the economy is in the safe region, the default probability is zero. The optimal maturity structure is flat, i.e., the government issues an equal amount of one-period and two-period debt. However, as θ_0 increases and debt-to-GDP ratio goes up the government eventually faces an increasing default risk. For $\theta_0 \approx 2$ and debt-to-GDP≈ 0.13 the economy enters the crisis region. The optimal maturity then shortens significantly, and therefore covariates negatively with the default probability. At a 23% debt-to-GDP ratio the average maturity is approximately 1.25 years; this implies that about 75% of debt is in one-period bonds. Importantly, such a skewed maturity structure corresponds to a default risk of only (approximately) 1%.

Figure 1: The optimal maturity structure in the three-period model



6.2.2 Relative Risk Aversion and the Hazard Rate

The two limiting cases considered in Section 4.2 imply that the optimal maturity structure is flat if marginal reallocation of budget surpluses does not change the default risk, and the optimal debt policy implies the issuance of only one-period bonds if lenders are risk-neutral. Combining strict concavity of household utility functions and continuously increasing default probability leads to an intermediate maturity structure, i.e., the average maturity is in between 1 year and 1.5 years. In the next two numerical exercises I demonstrate the effect of change in risk aversion and the default hazard rate on the optimal term structure.

Figure 2: Average Maturity under Different Relative Risk Aversion

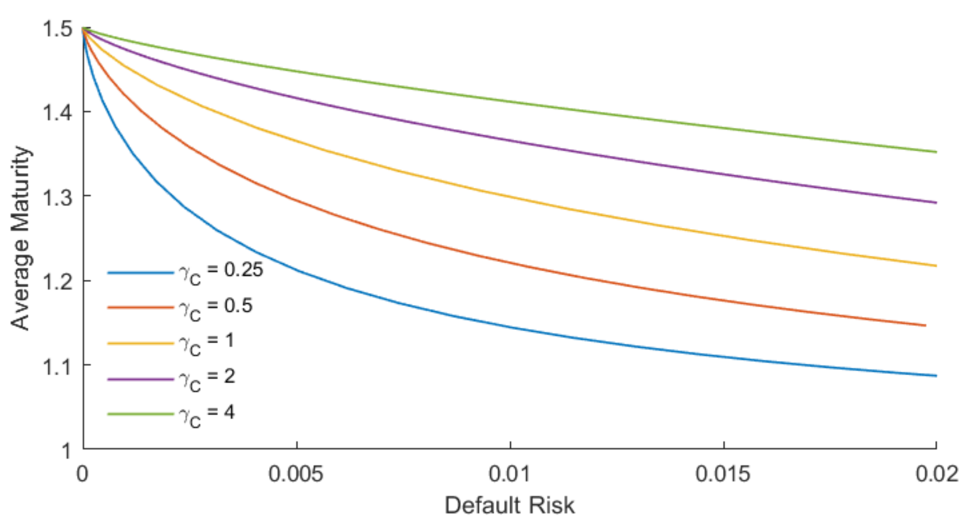
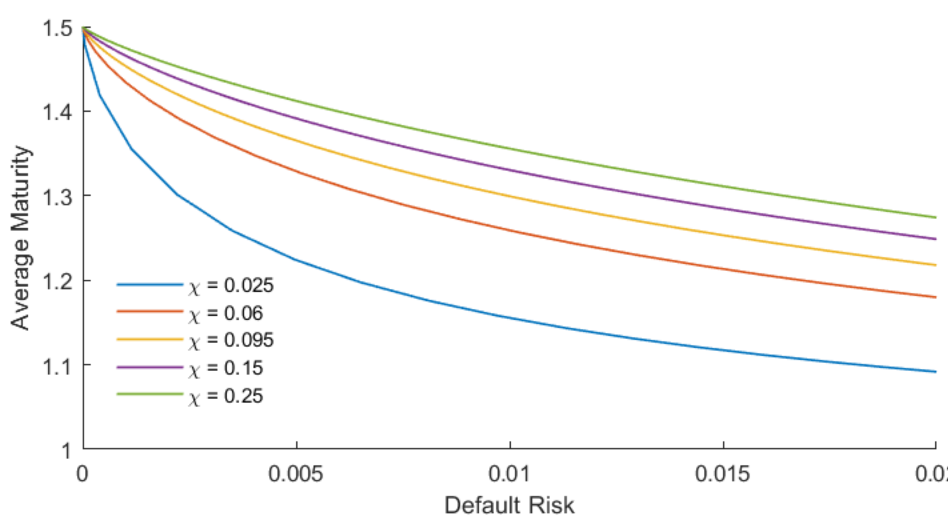


Figure 2 shows how average maturity depends on relative risk aversion γ_C. Since changing γ_C also shifts the crisis region–i.e. the debt-to-GDP ratio at which the default probability becomes positive–I make the comparison for different levels of default risk. The middle line ($\gamma_C = 1$) corresponds to the benchmark model. If default risk is zero, then the optimal term structure is flat. As default risk increases, the average maturity gradually decreases. This feature is common for various values of γ_C; however, keeping the probability of default fixed the average maturity is different. A larger γ_C is associated with a flatter term structure. For example, if default risk is 1% and $\gamma_C = 1$, then the average maturity is approximately 1.3 years. However, if $\gamma_C = 4$, the average maturity is about 1.45 years, and if $\gamma_C = 0.25$, the average maturity drops below 1.2 years.

Intuitively, as risk aversion increases, a deviation from the ex ante optimal fiscal plan has a larger impact on risk-free interest rates. Therefore, a flatter maturity minimizes government's ability to decrease the market value of outstanding debt by manipulating risk-free rates. As γ_C goes to zero, the risk-free interest rates become almost constant, and therefore the government can manipulate only the risk premium. The optimal maturity then shortens so that the government could not alter the market value of outstanding debt by manipulating the default risk.

Figure 3: Average Maturity under Different Default Hazard Rates



In the second exercise, I manipulate the default hazard rate $\frac{\pi'(s_2)}{1-\pi(s_2)}$ by considering different values of maximum output loss χ_{max}. As χ_{max} decreases and gets close to χ_{min}, a marginal increase in the contingent budget surplus leads to a larger rise in the default risk, which implies a higher default hazard rate and vice versa. Also, notice that $\chi_{max} \to 1$, so that $v_{min} \to -\infty$ would correspond to a no-default case, as for any finite $v > -\infty$, $f(v) = 0$.

Figure 3 shows the average maturity structure at given default risk for different values of χ_{max}. The middle line corresponds to the benchmark model parameters. As χ_{max} increases and, hence, the default hazard rate decreases, the average maturity increases. The reason is that a marginal increase in the budget surplus at period 2 leads to a relatively lower increase in default probability and, therefore, the price of long-term debt becomes relatively less elastic.

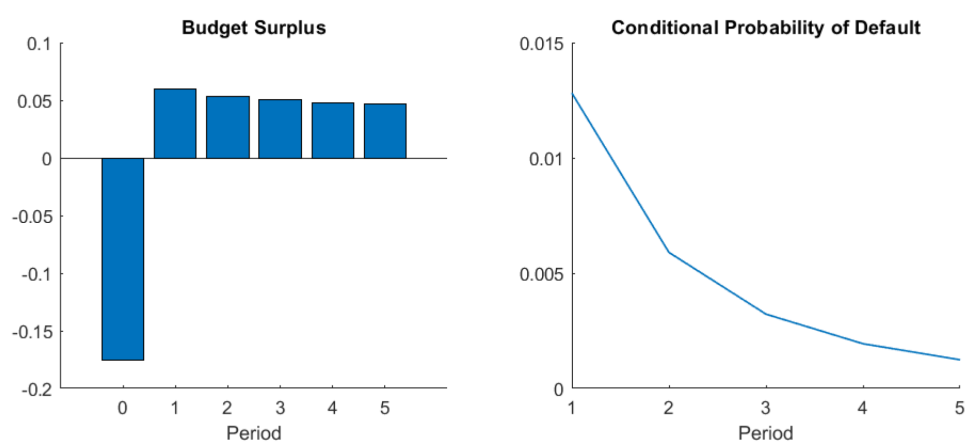
6.3 A Six-period Model

6.3.1 The Benchmark Model

In this subsection I study the optimal fiscal and debt policy in a six period model. Consider a general model as discussed in Section 2. In contrast to the previous section, the government can default in every period. I assume that at the initial date the government has a temporary higher taste parameter for public spending, $\theta_0 = 4$, followed by $\theta_t = 1\ \forall t = 1, ..., 5$. The initial debt is zero and $\beta = 0.99$. For this exercise, I set $\chi_{min} = 0.005$ and $\chi_{max} = 0.03$.

The focus of the analysis is on optimal maturity structure if multiple maturities are available and how term structure changes over time. I start by discussing the optimal fiscal plan and optimal endogenous default risk. I then solve for the maturity structure at every period that makes the ex ante optimal fiscal plan incentive-compatible for the Markov government.

Figure 4: Optimal Fiscal Plan and Default Probabilities Conditional on No Prior Default



The right panel of Figure 4 presents the optimal fiscal plan. The left panel displays the default probability conditional on no default in preceding periods. At period 0 the government runs a budget deficit of 17.5%. In subsequent periods the government chooses positive contingent budget surpluses to finance the deficit. Budget surpluses are slightly skewed toward the left-end as established in Proposition 5. The budget surplus at period 1, 6%, gradually decreases in the following periods and is equal to 4.6% in the last period. Such skewness reflects decaying default risk. The conditional default risk in period 1 is 1.27%, and only 0.1% in the last period.

As discussed in Section 5, the government prefers such paths for budget surpluses for the following reason. By reallocating budget surpluses from the last period to the first period, the government can decrease the default probability in the last period as it increases the government's utility in that period. However, such perturbation has a negligible effect on default risk in the first period. The reason is that an increase in utility in the last period is compensated by a decrease in utility in the first period. Thus, such reallocation does not change the value of pursuing the fiscal plan in period 1. By distributing budget surpluses more evenly the government would not be able to decrease default risk in period 1; however, it would increase default risk in the last period. Therefore, the government faces a trade-off between smoothing consumption over time and decreasing default probabilities in later periods.

Figure 5: Optimal Maturity Structure

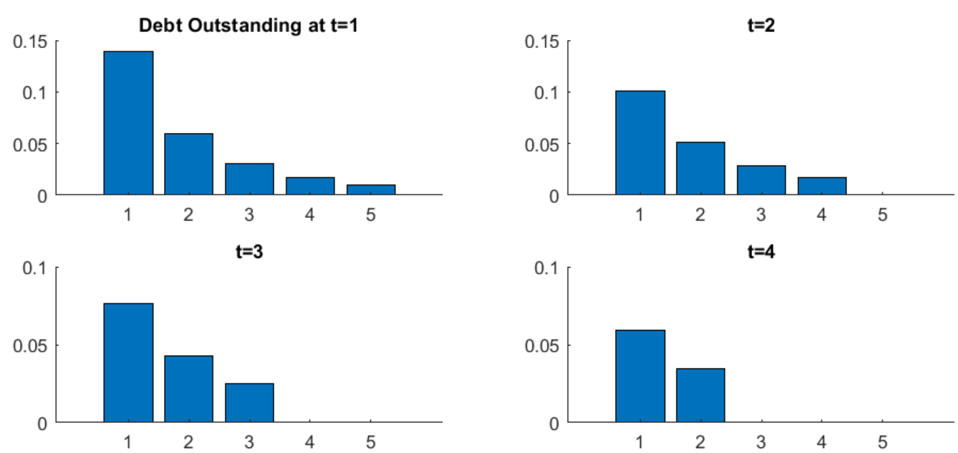
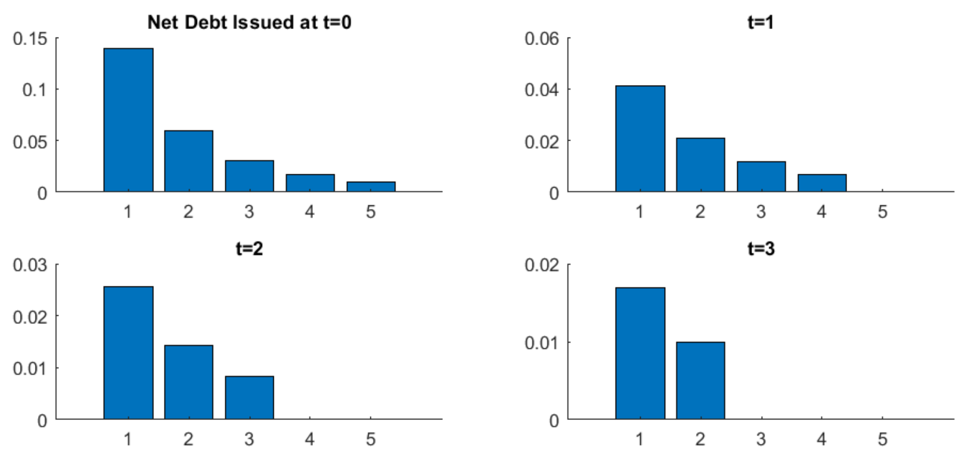


Figure 5 displays the maturity structure at different dates. The top left panel shows the term structure of outstanding debt at the beginning of period 1. The X axis corresponds to maturity. As predicted by the analytical model, maturity is tilted toward the short-term end and has a decaying profile: Payments due at later dates are lower than payments at earlier dates. For example, at period 0 the government issues 0.14 units of one-period bond, 0.06 units of two-period bond and only 0.01 of five-period bond. In addition, as seen from the other panels, term structure remains decaying in the subsequent periods as well. However, it is worth noting that maturity structure flattens over time. This tendency reflects the decreasing conditional probability of default as highlighted in the right panel of Figure 4.

Figure 6 demonstrates the net debt issued at dates 0, 1, 2 and 3. The top left panel is identical to the outstanding debt at $t = 1$ due to zero initial debt. Importantly, at every date the government issues positive net debt at all maturities. Moreover, net debt issued is also skewed towards the short end.

Figure 6: Net Debt Issued



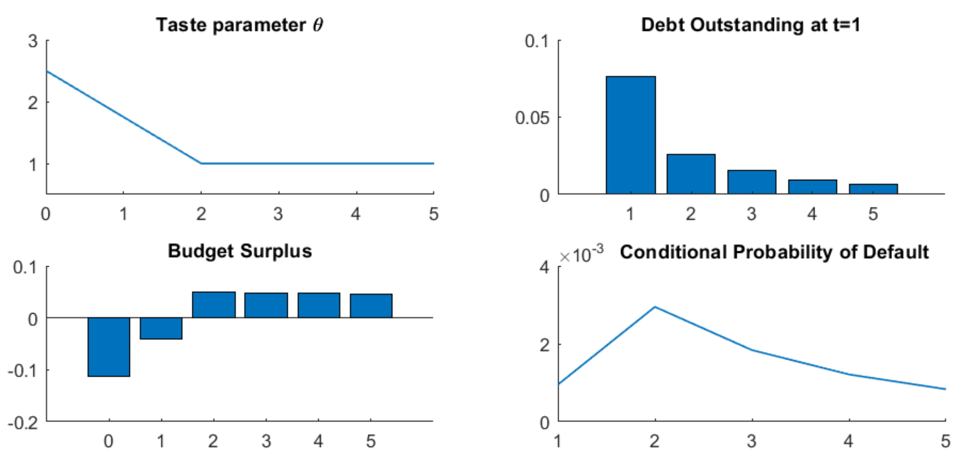
6.3.2 Persistent Fiscal Shock

In the previous exercise the positive fiscal shock at the initial date is assumed to be temporary. In reality, many shocks are persistent over time. In this exercise, I analyze how optimal maturity changes if the initial shock to taste parameter is persistent. I set $\theta_0 = 3$ and $\theta_1 = 2$. At date 2 and at all future dates the parameter is one: $\theta_2 = ... = \theta_5 = 1$. Initial debt is zero.

Figure 7 displays taste parameter θ, the optimal maturity structure, the fiscal plan, and the conditional default risk at every date. The bottom left panel demonstrates that the government prefers to run a budget deficit in first two periods, which must be repaid in the future. Due to this reason, the conditional risk of default is higher in period 2 compared to the default probability in period 1, as presented on the right bottom panel. Nonetheless, the right top panel shows the debt issued at $t = 0$ still has a decaying structure.

The exercise demonstrates that a persistent fiscal shock does not alter the implications for the optimal maturity structure. In the previous exercise, the optimal fiscal plan consists of budget surpluses that slightly decrease over time causing the conditional probability of default to decrease over time. In this exercise, the next period budget surplus is negative and, thus, the conditional default risk is not monotonically decreasing. Still, the stock of one-period debt is much larger than the stock of two-period debt. Such maturity is optimal because the government at date 1 cannot manipulate the default risk at date 1.

Figure 7. Persistent Fiscal Shock

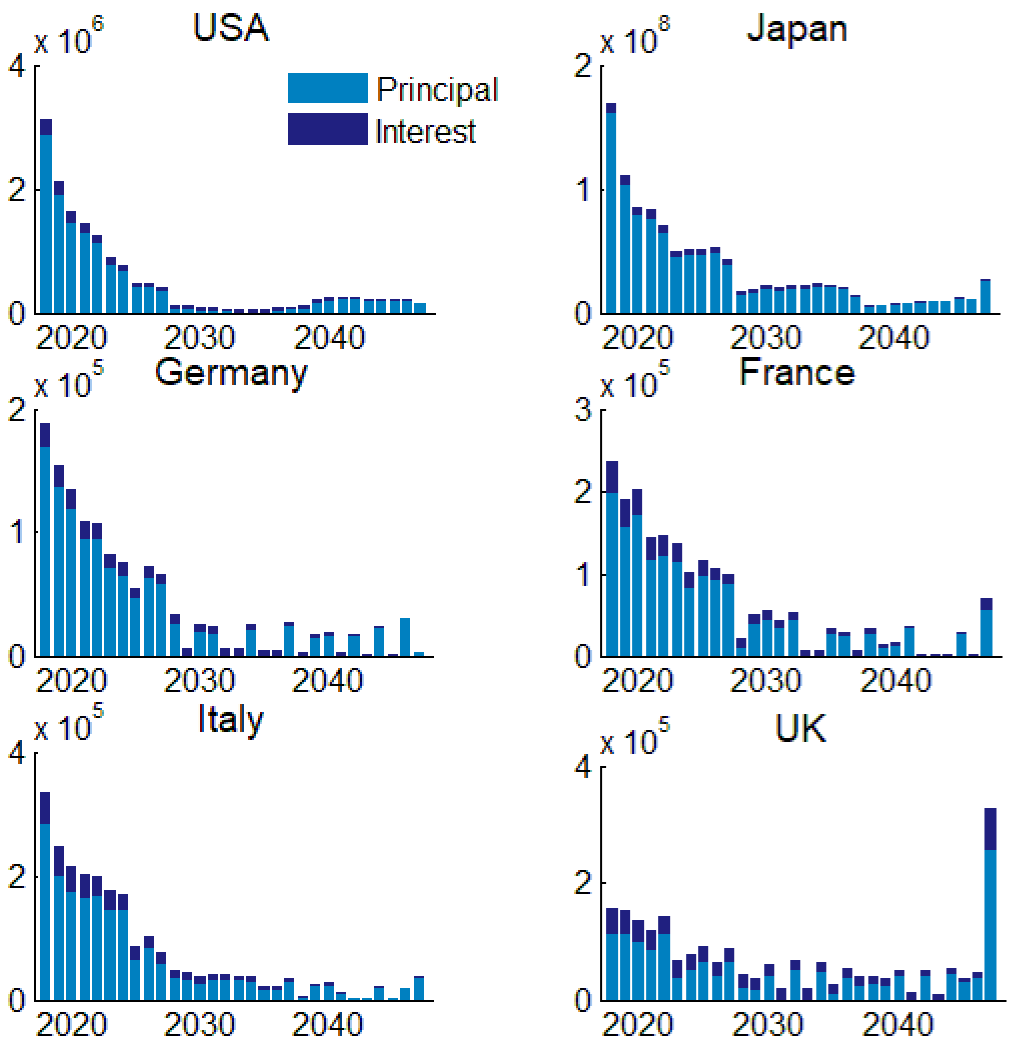


6.4 The Maturity Structure of Developed Countries

Figure 8 displays the maturity structure of marketable bonds for the following countries: the US, Japan, Germany, France, Italy and the UK. The US debt is in millions USD, the debts of the German, Italian and French governments are in millions EUR, the UK debt is in millions GBP and the Japanese debt is in millions JPY. The data was collected on 17th of October, 2017 and each bar shows the principal and interest payments owed by a government that has to be paid by the government in a given year as of October 17, 2017. I, thus, skip the payments due in the end of October, November and December 2017 and start with 2018. In each panel the first bar represents the total payments owed by a government due in 2018, the second bar represents the total payments due in 2019 and so on.

The exception is the very last bar in each panel that includes payments due in 2047 and all future years. The last bar is somewhat higher for Italy, Japan and France, but most importantly it represents a considerable part of debt for the United Kingdom. The main reason is that the British Government actively issued consol bonds during the Industrial Revolution (see Mokyr, 2011). Due to this aggregation of debt I ignore the last bar in the discussion of maturity data.

Figure 8: Maturity Structure of the USA, Japan, Germany, France, Italy and the UK (Source: Bloomberg)



The maturity statistics is broadly consistent with the predictions of the model. First of all, debt is skewed toward the short-term end. Even though the countries issue bonds maturing in 30 years and later, the average maturity for the US, Japan, Germany, France and Italy is 5.79, 7.74, 6.8, 7.83 and 6.8 years respectively. It is worth noting that the maturity structure of the UK government debt is much flatter, and the average maturity is 14.97 years. For each country the stock of debt maturing in one year is the largest[6]. In 2018 the US government has to pay (or roll-over) more than 20% of the total debt. The debt to be paid by the US government in the next five years constitutes 62% of the total debt. Similarly, total amount of debt maturing in the next 5 years constitutes approximately 50% of total debt for Japan, Germany, France and Italy[7].

Moreover, maturity structure has a decaying profile as predicted by the model. The US maturity structure exhibits decaying profile for 15 years: payments due in 2018-2033 are strictly decreasing. Then the term structure does display some increasing trend, however, note that the total debt due in 2034 and all later years is lower than the debt due in 2018. Debt term structures of Japan, Germany, France and Italy have similar patterns. Even much flatter UK debt has a tendency to decline over maturity date: The total debt maturing in 1-5 years amounts to approximately 32%, while the total debt to be paid in 6-10 years constitutes less than 18%.

One of the reasons maturity structure is not perfectly decaying is that the number of issuance is limited and most of them are short-term. For example, the number of French debt active issuances is only 95[8]. There is no principal payments due at 2033, 2034, 2037 and some subsequent years. The number of issuances can be limited due to some fixed costs or other frictions.

7 Summary

This paper shows that in an environment with endogenous risk-free interest rates and endogenous default premiums the optimal maturity structure has a decaying profile. An important assumption is that marginal changes in fiscal policies lead to a marginal change in the risk-free interest rate and the default risk.

In this model, fiscal policy is time consistent in a sense that the government pursues the ex ante optimal fiscal plan. However, the consistency depends on deterministic nature of fiscal

[6]recall that I ignore the last bar which aggregates total payments due in 2047 and later years.
[7]it is 48%, 53%, 46% and 52% for Japan, Germany, France and Italy respectively.
[8]Source: Bloomberg

shocks assumed in the model. If fiscal shocks are stochastic then the allocation associated with the modified commitment problem is generally not feasible for the Markov planner at least if markets are incomplete. To study the optimal maturity structure in such an environment, ideally, we want to consider a multiple period model with multiple maturities available to the government. However, such analysis is infeasible in infinite horizon model with an infinite choice of maturities. Moreover, the problem is complicated even if there are more than two instruments with different maturities.

There are several interesting avenues for future research. First, this paper assumes that the government cannot default within the period once new debt has been issued. The optimal debt policy under lack of commitment and positive default risk implies issuance of a large stock of short-term debt. This in turn increases the likelihood of self-fulfilling debt crisis if the latter is possible. Thus, allowing for self-fulfilling debt crises could lead to an interesting trade-off between short-term and long-term debt in such an environment. Second, the government is assumed to be able to default on its debt, but partial default is not allowed in this model. Therefore, it would be interesting to investigate the optimal fiscal and debt policies if the government issues nominal debt that can be inflated away rather than real debt.

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Appendix A. Proofs.

Proof of Lemma 1.

Let $\boldsymbol{s}_0^\star(\boldsymbol{b}_{-1}) = (s_0^\star,\, s_1^\star, ...,\, s_T^\star)$ be the sequence of contingent budget surpluses associated with the Markov perfect competitive equilibrium and $\boldsymbol{b}_t^\star(\boldsymbol{b}_{t-1})$ be the optimal debt issuances at $t = 0,\, ...,\, T$. $\boldsymbol{s}_0^\star(\boldsymbol{b}_{-1})$ must satisfy budget constraint (3) at every period $t = 0,\, ...,\, T$. Combining budget constraints (3) at every period yields the dynamic budget constraint (10). It implies that $\boldsymbol{s}_0^\star(\boldsymbol{b}_{-1})$ is always feasible for the planner. Therefore, the value of a planner is at least as good as the value of a Markov government: $\hat{V}(\boldsymbol{b}_{-1}) \geq V(\boldsymbol{b}_{-1})$. ■

Lemma A1. *For* $1 \leq k \leq T - t$

$$\frac{\partial W_t(\boldsymbol{s}_t)}{\partial s_{t+k}} = -\beta^k \cdot Pr_t^{t+k}(\boldsymbol{s}_{t+1}) \cdot \left(\theta_{t+k}\omega'(\tau - s_{t+k}) - u'(1 - \tau + s_{t+k})\right).$$

Proof.

Recall that $W_t(\boldsymbol{s}_t)$ is defined recursively as

$$W_t(\boldsymbol{s}_t) = u(1 - \tau + s_t) + \theta_t\omega(\tau - s_t) + \beta \cdot \mathbb{E}\max\left\{W_{t+1}(\boldsymbol{s}_{t+1}),\, V_{t+1}^{def}\right\}.$$

.

Note that

$$\frac{\partial W_t(\boldsymbol{s}_t)}{\partial s_t} = -\left(\theta_t\omega'(\tau - s_t) - u'(1 - \tau + s_t)\right)$$

because s_t does not affect budget surpluses or default probabilities in future periods. I first show that

$$\frac{\partial W_t(\boldsymbol{s}_t)}{\partial s_{t+k}} = \beta \cdot Pr_t^{t+1}(\boldsymbol{s}_{t+1}) \cdot \frac{\partial W_{t+1}(\boldsymbol{s}_{t+1})}{\partial s_{t+k}}$$

for $1 \leq k \leq T - t$ where

$$Pr_t^{t+1}(\boldsymbol{s}_{t+1}) = F(W_{t+1}(\boldsymbol{s}_{t+1}) \cdot \frac{1 - \beta}{1 - \beta^{T-t}}).$$

Decompose the last component of $W_t(\boldsymbol{s}_t)$ as

$$\mathbb{E}\max\left\{W_{t+1}(\boldsymbol{s}_{t+1}),\, V_{t+1}^{def}\right\} = (1 - Pr_t^{t+1}(\boldsymbol{s}_{t+1})) \cdot \mathbb{E}[V_{t+1}^{def} | V_{t+1}^{def} > W_{t+1}(\boldsymbol{s}_{t+1})] + Pr_t^{t+1}(\boldsymbol{s}_{t+1}) \cdot W_{t+1}(\boldsymbol{s}_{t+1})$$

where

$$\mathbb{E}[V_{t+1}^{def}|V_{t+1}^{def} > W_{t+1}(\boldsymbol{s}_{t+1})] = \frac{1}{1 - Pr_t^{t+1}(\boldsymbol{s}_{t+1})} \cdot \frac{1 - \beta^{T-t}}{1 - \beta} \cdot \int_{v'(\boldsymbol{s}_{t+1})}^{v_{max}} vdF(v)$$

is the conditional expected value of outside value option, and $v'(\boldsymbol{s}_{t+1}) = W_{t+1}(\boldsymbol{s}_{t+1}) \cdot \frac{1-\beta}{1-\beta^{T-t}}$ is the minimum value of per-period payoff of the outside option, at which the economy enters the crisis region. Then

$$\frac{\partial W_t(\boldsymbol{s}_t)}{\partial s_{t+k}} = \beta \cdot \frac{\partial (1 - Pr_t^{t+1}(\boldsymbol{s}_{t+1})) \cdot \mathbb{E}[V_{t+1}^{def}|V_{t+1}^{def} > W_{t+1}(\boldsymbol{s}_{t+1})]}{\partial s_{t+k}} + \beta \cdot \frac{\partial Pr_t^{t+1}(\boldsymbol{s}_{t+1}) \cdot W_{t+1}(\boldsymbol{s}_{t+1})}{\partial s_{t+k}}$$

$$\frac{\partial (1 - Pr_t^{t+1}(\boldsymbol{s}_{t+1})) \cdot \mathbb{E}[V_{t+1}^{def}|V_{t+1}^{def} > W_{t+1}(\boldsymbol{s}_{t+1})]}{\partial s_{t+1}} = -v'(\boldsymbol{s}_{t+1}) \cdot f(v'(\boldsymbol{s}_{t+1})) \cdot \frac{\partial v'(\boldsymbol{s}_{t+1})}{\partial s_{t+k}} =$$

$$= -\frac{1-\beta}{1-\beta^{T-t}} \cdot W_{t+1}(\boldsymbol{s}_{t+1}) \cdot f\left(W_{t+1}(\boldsymbol{s}_{t+1}) \cdot \frac{1-\beta}{1-\beta^{T-t}}\right) \cdot \frac{\partial W_{t+1}(\boldsymbol{s}_{t+1})}{\partial s_{t+k}},$$

$$\frac{\partial Pr_t^{t+1}(\boldsymbol{s}_{t+1}) \cdot W_{t+1}(\boldsymbol{s}_{t+1})}{\partial s_{t+k}} = Pr_t^{t+1}(\boldsymbol{s}_{t+1}) \cdot \frac{\partial W_{t+1}(\boldsymbol{s}_{t+1})}{\partial s_{t+k}} +$$

$$+ \frac{1-\beta}{1-\beta^{T-t}} \cdot W_{t+1}(\boldsymbol{s}_{t+1}) \cdot f\left(W_{t+1}(\boldsymbol{s}_{t+1}) \cdot \frac{1-\beta}{1-\beta^{T-t}}\right) \cdot \frac{\partial W_{t+1}(\boldsymbol{s}_{t+1})}{\partial s_{t+k}}.$$

Thus yielding

$$\frac{\partial W_t(\boldsymbol{s}_t)}{\partial s_{t+k}} = \beta \cdot Pr_t^{t+1}(\boldsymbol{s}_{t+1}) \cdot \frac{\partial W_{t+1}(\boldsymbol{s}_{t+1})}{\partial s_{t+k}}.$$

Iterating forward leads to

$$\frac{\partial W_t(\boldsymbol{s}_t)}{\partial s_{t+k}} = \beta^k \cdot \Pi_{j=1}^k Pr_{t+j-1}^{t+j}(\boldsymbol{s}_{t+j}) \cdot \frac{\partial W_{t+k}(\boldsymbol{s}_{t+k})}{\partial s_{t+k}} =$$

$$= -\beta^k \cdot Pr_t^{t+k}(\boldsymbol{s}_t) \cdot (\theta_{t+k}\omega'(\tau - s_{t+k}) - u'(1 - \tau + s_{t+k})).$$

■

Lemma A.2. *Let* $\boldsymbol{s}_1 = (s_1, ..., s_T)$*. Then* $MRS^0_{1,1+t}(\boldsymbol{s}_1) = MRS^1_{1,1+t}(\boldsymbol{s}_1)$*.*

Proof.

Recall from (12) and (13) that

$$MRS^0_{1,\,1+t}(\boldsymbol{s}_1) = \frac{\frac{\partial W_0((s_0,\boldsymbol{s}_1))}{\partial s_{1+t}}}{\frac{\partial W_0((s_0,\boldsymbol{s}_1))}{\partial s_1}} \text{ and } MRS^1_{1,\,1+t}(\boldsymbol{s}_1) = \frac{\frac{\partial W_1(\boldsymbol{s}_1)}{\partial s_{1+t}}}{\frac{\partial W_1(\boldsymbol{s}_1)}{\partial s_1}}$$

The proof follows from Lemma A.1 which implies $\frac{\partial W_0(s_0,\boldsymbol{s}_1)}{\partial s_{1+t}} = \beta Pr^1_0(\boldsymbol{s}_1)\cdot\frac{\partial W_1(\boldsymbol{s}_1)}{\partial s_{1+t}}$ for $0 \le t \le T-1$, hence,

$$\frac{\frac{\partial W_0((s_0,\boldsymbol{s}_1))}{\partial s_{1+k}}}{\frac{\partial W_0((s_0,\boldsymbol{s}_1))}{\partial s_1}} = \frac{\beta Pr^1_0(\boldsymbol{s}_1)\cdot\frac{\partial W_1(\boldsymbol{s}_1)}{\partial s_{1+t}}}{\beta Pr^1_0(\boldsymbol{s}_1)\cdot\frac{\partial W_1(\boldsymbol{s}_1)}{\partial s_1}} =$$

$$= \frac{\frac{\partial W_1(\boldsymbol{s}_1)}{\partial s_{1+t}}}{\frac{\partial W_1(\boldsymbol{s}_1)}{\partial s_1}}.$$

■

Proof of Lemma 2.

Let $\hat{\boldsymbol{s}}_0(\boldsymbol{b}_{-1})$ be the optimal fiscal plan designed by a planner at period 0 given the outstanding debt $\boldsymbol{b}_{-1}$. Let $\hat{\boldsymbol{s}}_1(1,\,\boldsymbol{b}_0)$ be the optimal fiscal plan chosen by a planner at period 1 given the outstanding debt $\boldsymbol{b}_0$.

$\hat{\boldsymbol{s}}_0(\boldsymbol{b}_{-1})$ has to satisfy the first-order necessary conditions:

$$\begin{cases} MRS^0_{0,\,1}(\boldsymbol{s}_0) = MRT^0_{0,\,1}(\boldsymbol{s}_0,\,\boldsymbol{b}_{-1}) \\ MRS^0_{t,\,t+1}(\boldsymbol{s}_0) = MRT^0_{t,\,t+1}(\boldsymbol{s}_0,\,\boldsymbol{b}_{-1}) \quad \text{for } t = 1,\,...,\,T-1 \\ S_0(\boldsymbol{s}_0) = B_0(\boldsymbol{s}_0,\,\boldsymbol{b}_{-1}) \end{cases}$$

where $MRS^0_{t,\,t+1}(\boldsymbol{s}_0)$ and $MRT^0_{t,\,t+1}(\boldsymbol{s}_0)$ are as defined by (12) and (13) .

In turn, $\hat{\boldsymbol{s}}_1(1,\,\boldsymbol{b}_0)$ satisfies the first-order necessary conditions at period 1:

$$\begin{cases} MRS^1_{t,\,t+1}(\boldsymbol{s}_1) = MRT^1_{t,\,t+1}(\boldsymbol{s}_1,\,\boldsymbol{b}_0) \quad \text{for } t = 1,\,...,\,T-1 \\ S_1(\boldsymbol{s}_1) = B_1(\boldsymbol{s}_1,\,\boldsymbol{b}_0) \end{cases}$$

where $MRS^1_{t,\,t+1}(\boldsymbol{s}_0)$ and $MRT^1_{t,\,t+1}(\boldsymbol{s}_1)$ are defined analogously to (12) and (13).

Recall from Lemma A.2 that if $\boldsymbol{s}_1 \in \boldsymbol{s}_0$then $MRS^0_{t,\,t+1}(\boldsymbol{s}_0) = MRS^1_{t,\,t+1}(\boldsymbol{s}_1)$, $t = 1,\,...,\,T-1$.

Therefore, if $\boldsymbol{b}_0$ is such that the following system of equations is satisfied where $\hat{\boldsymbol{s}}_1(\boldsymbol{b}_{-1}) \in \hat{\boldsymbol{s}}_0(\boldsymbol{b}_{-1})$, i.e., $\hat{\boldsymbol{s}}_0(\boldsymbol{b}_{-1}) = (\hat{s}_0(\boldsymbol{b}_{-1}),\,\hat{\boldsymbol{s}}_1(\boldsymbol{b}_{-1}))$:

$$\begin{cases} MRT^0_{t,\,t+1}(\hat{\boldsymbol{s}}_1(\boldsymbol{b}_{-1}),\,\boldsymbol{b}_0) = MRT^1_{t,\,t+1}(\hat{\boldsymbol{s}}_1(\boldsymbol{b}_{-1}),\,\boldsymbol{b}_{-1}) \quad \text{for } t = 1,\,...,\,T-1 \\ S_1(\hat{\boldsymbol{s}}_1(\boldsymbol{b}_{-1})) = B_1(\hat{\boldsymbol{s}}_1(\boldsymbol{b}_{-1}),\,\boldsymbol{b}_0) \end{cases} \tag{34}$$

then the planner chooses $\hat{\boldsymbol{s}}_1(\boldsymbol{b}_{-1})$ at $t = 1$ because the plan solves the planner's optimality conditions. Note that this system is linear. To see it, Note that (I set $\hat{\boldsymbol{s}}_t(\boldsymbol{b}_{-1}) = \hat{\boldsymbol{s}}_t$ to simplify notation)

$$MRT^9_{t,\,t+1}(\hat{\boldsymbol{s}}_t(\boldsymbol{b}_{-1}),\,\boldsymbol{b}_{-1}) = \frac{\frac{\partial \sum_{k=0}^{T} \beta^k u'(1-\tau+\hat{s}_k)\cdot Pr_0^k(\hat{s}_0)\cdot\left(\hat{s}_k - b_{-1}^k\right)}{\partial \hat{s}_{t+1}}}{\frac{\partial \sum_{k=0}^{T} \beta^k u'(1-\tau+\hat{s}_k)\cdot Pr_0^k(\hat{s}_0)\cdot\left(\hat{s}_k - b_{-1}^k\right)}{\partial \hat{s}_t}} = \beta\cdot Pr_t^{t+1}(\hat{\boldsymbol{s}}_{t+1})\times$$

$$\times\,\frac{u'(1-\tau+\hat{s}_{t+1}) + \left[u''(1-\tau+\hat{s}_{t+1}) + u'(1-\tau+\hat{s}_{t+1})\frac{\frac{\partial Pr_t^{t+1}(\hat{s}_{t+1})}{\partial \hat{s}_{t+1}}}{Pr_t^{t+1}(\hat{s}_{t+1})}\right]\cdot(\hat{s}_{t+1} - b_{-1}^{t+1})}{u'(1-\tau+\hat{s}_t) + u''(1-\tau+\hat{s}_t)\cdot(\hat{s}_t - b_{-1}^t)} \tag{35}$$

The reason is that changes in budget surpluses at dates t and $t+1$ keeping the value $W_1(\hat{\boldsymbol{s}}_1(\boldsymbol{b}_{-1}))$ constant does not causes changes in default probabilities in all periods except $t+1$.

Analogously, the MRT for the planner at period 1 is

$$MRT^1_{t,\,t+1}(\hat{\boldsymbol{s}}_t(\boldsymbol{b}_{-1}),\,\boldsymbol{b}_0) = \frac{\frac{\partial \sum_{k=1}^{T} \beta^{k-1} u'(1-\tau+\hat{s}_k)\cdot Pr_1^k(\hat{s}_1)\cdot\left(\hat{s}_k - b_0^k\right)}{\partial \hat{s}_{t+1}}}{\frac{\partial \sum_{k=1}^{T} \beta^{k-1} u'(1-\tau+\hat{s}_k)\cdot Pr_1^k(\hat{s}_1)\cdot\left(\hat{s}_k - b_0^k\right)}{\partial \hat{s}_t}} = \beta\cdot Pr_t^{t+1}(\hat{\boldsymbol{s}}_{t+1})\times$$

$$\times\,\frac{u'(1-\tau+\hat{s}_{t+1}) + \left[u''(1-\tau+\hat{s}_{t+1}) + u'(1-\tau+\hat{s}_{t+1})\frac{\frac{\partial Pr_t^{t+1}(\hat{s}_{t+1})}{\partial \hat{s}_{t+1}}}{Pr_t^{t+1}(\hat{s}_{t+1})}\right]\cdot(\hat{s}_{t+1} - b_0^{t+1})}{u'(1-\tau+\hat{s}_t) + u''(1-\tau+\hat{s}_t)\cdot(\hat{s}_t - b_0^t)} \tag{36}$$

Equating (35) and (36), and given that $MRS^0_{t,\,t+1}(\hat{\boldsymbol{s}}_t(\boldsymbol{b}_{-1})) = MRT^9_{t,\,t+1}(\hat{\boldsymbol{s}}_t(\boldsymbol{b}_{-1}),\,\boldsymbol{b}_{-1})$ yields (19):

$$\frac{\frac{\partial \beta\cdot u'(1-\tau+\hat{s}_{t+1})\cdot Pr_t^{t+1}(\hat{s}_{t+1})}{\partial \hat{s}_{t+1}}}{\frac{\partial u'(1-\tau+\hat{s}_t)}{\partial \hat{s}_t}}\cdot\frac{b_0^{t+1}(\boldsymbol{b}_{-1}) - b_{-1}^{t+1}}{b_0^t(\boldsymbol{b}_{-1}) - b_{-1}^t} = MRS^0_{t,\,t+1}(\hat{\boldsymbol{s}}_t(\boldsymbol{b}_{-1}))$$

Therefore, the system (34) consists of T equations with T unknowns.

Let $\boldsymbol{b}_0^\star(\boldsymbol{b}_{-1})$ be the solution. Finally, I show that $(\hat{s}_0(\boldsymbol{b}_{-1}),\, \boldsymbol{b}_0^\star(\boldsymbol{b}_{-1}))$ satisfies the budget constraint (16). Note that $S_0(\boldsymbol{s}_0) = u'(1-\tau+s_0)\cdot s_0 + \beta\cdot(1-\pi_1(\boldsymbol{s}_1))\cdot S_1(\boldsymbol{s}_1)$. Plugging in $S_1(\hat{\boldsymbol{s}}_1(\boldsymbol{b}_{-1})) = B_1(\hat{\boldsymbol{s}}_1(\boldsymbol{b}_{-1}),\, \boldsymbol{b}_0^\star(\boldsymbol{b}_{-1}))$ and $S_0(\hat{\boldsymbol{s}}_0(\boldsymbol{b}_{-1})) = B_0(\hat{\boldsymbol{s}}_0(\boldsymbol{b}_{-1}),\, \boldsymbol{b}_{-1})$ yields (16). ■

Proof of Proposition 1.

The proof is inductive. Proposition 1 holds trivially for $T=1$ and $T=2$.

Suppose the Proposition holds for a T period model with timing $t = 1,\, ...,\, T$, i.e. $V_1(\boldsymbol{b}_0) = \hat{V}_1(\boldsymbol{b}_0)$ and $\boldsymbol{s}_1^\star(\boldsymbol{b}_0) = \hat{\boldsymbol{s}}_1(\boldsymbol{b}_0)$.

Consider a $T+1$ model. According to Lemma 2 there exists feasible $\boldsymbol{b}_0$ such that $\hat{\boldsymbol{s}}_1(\boldsymbol{b}_0) = \hat{\boldsymbol{s}}_1(\boldsymbol{b}_{-1})$, hence, $V_0(\boldsymbol{b}_{-1}) = \hat{V}_0(\boldsymbol{b}_{-1})$ and $\boldsymbol{s}_0^\star(\boldsymbol{b}_{-1}) = \hat{\boldsymbol{s}}_0(\boldsymbol{b}_{-1})$. ■

Proof of Proposition 2.

See proof of Lemma 2. Follows from (35) and (36), and given that $MRS^0_{t,\,t+1}(\hat{\boldsymbol{s}}_t(\boldsymbol{b}_{-1})) = MRT^9_{t,\,t+1}(\hat{\boldsymbol{s}}_t(\boldsymbol{b}_{-1}),\, \boldsymbol{b}_{-1})$ in equilibrium.

■

Lemma A.3. *If $u'(1-\tau+s)\cdot s$ is increasing in s for $s \in (-(1-\tau),\, \tau)$ then $\frac{\partial u'(1-\tau+s)\cdot s}{\partial s}$ is strictly decreasing in s.*

Proof

Recall from Assumption 3 that $u(c) = \frac{c^{1-\gamma_C}}{1-\gamma_C}$, $\gamma_C > 0$. Then

$$\frac{\partial u'(1-\tau+s)\cdot s}{\partial s} = (1-\tau+s)^{-\gamma_C-1}\cdot(1-\tau+s-\gamma_C\cdot s)$$

$\frac{\partial u'(1-\tau+s)\cdot s}{\partial s} \geq 0$ implies $1-\tau+s-\gamma_C\cdot s \geq 0$, therefore,

$$\frac{\partial^2 u'(1-\tau+s)\cdot s}{\partial s^2} = 2\cdot u''(1-\tau+s) + (-1-\gamma_C)u''(1-\tau+s)\cdot\frac{s}{1-\tau+s} =$$

$$= u''(1-\tau+s)\cdot(1-\tau+(1-\tau+s-\gamma_C\cdot s)) < 0.$$

■

Lemma A.4. *Suppose the solution to (11) is interior and given by (14). Besides, suppose that the budget constraint holds with equality, initial debt is zero and $\hat{s}_t > 0$ for $t \geq 1$. Then in equilibrium $u'(1-\tau+\hat{s}_t)\cdot\hat{s}_t$ is increasing in $\hat{s}_t$.*

Proof

The proof is by contradiction. Suppose that $u'(1-\tau+\hat{s}_t)\cdot\hat{s}_t$ is decreasing in $\hat{s}_t$ for some optimal fiscal plan $\hat{\boldsymbol{s}}_0$. Then marginal decrease in $\hat{s}_t$ decreases the market value of budget surpluses:

$$\frac{\partial\sum_{k=0}^{T}\beta^k u'(1-\tau+\hat{s}_k)\cdot Pr_0^k(\hat{\boldsymbol{s}}_0)\cdot\hat{s}_k}{\partial\hat{s}_t}=\sum_{k=1}^{t}\beta^k u'(1-\tau+\hat{s}_k)\cdot\frac{\partial Pr_0^k(\hat{\boldsymbol{s}}_0)}{\partial\hat{s}_t}\cdot\hat{s}_k+$$

$$+\beta^t\cdot Pr_0^t(\hat{\boldsymbol{s}}_0)\cdot(u'(1-\tau+\hat{s}_t)+u''(1-\tau+\hat{s}_t)\cdot\hat{s}_t)\leq 0$$

The first term represents decrease in market value of budget surpluses before t. It is negative because default probabilities (weakly) increase due to decrease in $W_k(\hat{\boldsymbol{s}}_k)$ for $k=1,\ ...,\ t$:

$$\frac{\partial Pr_0^k(\hat{\boldsymbol{s}}_0)}{\partial\hat{s}_t}=\sum_{k=1}^{t}\frac{f(W_k(\boldsymbol{s}_k)\cdot\frac{1-\beta}{1-\beta^{T-k+1}})}{F(W_k(\boldsymbol{s}_k)\cdot\frac{1-\beta}{1-\beta^{T-k+1}})}\cdot Pr_0^k(\hat{\boldsymbol{s}}_0)\cdot\frac{1-\beta}{1-\beta^{T-k+1}}\cdot\frac{\partial W_k(\hat{\boldsymbol{s}}_k)}{\partial\hat{s}_t}\leq 0$$

The second term is the change in market value of budget surplus at period t (keeping the default probability constant). The second term is (weakly) negative if $u'(1-\tau+\hat{s}_t)\cdot\hat{s}_t$ is decreasing in $\hat{s}_t$.

Therefore, a marginal decrease in $\hat{s}_t$ keeping all other surpluses constant is feasible. Such deviation strictly increases government's value $W_0(\hat{\boldsymbol{s}}_0)$. However, it contradicts the optimality of $\hat{\boldsymbol{s}}_0$. Therefore, in equilibrium $u'(1-\tau+\hat{s}_t)\cdot\hat{s}_t$ is increasing in $\hat{s}_t$.$\blacksquare$

Proof of Proposition 3.

(i) Suppose that $\frac{\partial Pr_1^2(s_2)}{\partial s_2}=Pr_1^2(s_2)=0$.Then the optimality condition ((27) and (28)) implies :

$$\frac{\omega'(\tau-s_2)-u'(1-\tau+s_2)}{\omega'(\tau-s_1)-u'(1-\tau+s_1)}=\frac{u'(1-\tau+s_2)+u''(1-\tau+s_2)\cdot s_2}{u'(1-\tau+s_1)+u''(1-\tau+s_1)\cdot s_1}$$

As s_2 increases and s_1 decreases the left-hand side of the above equation strictly goes up, however, according to Lemma A.3. the right-hand side strictly goes down. The only solution is $s_1=s_2$.

(ii) Now suppose that $\frac{\partial Pr_1^2(s_2)}{\partial s_2}>0$ and $Pr_1^2(s_2)>0$. The optimality condition requires

$$\frac{\omega'(\tau-s_2)-u'(1-\tau+s_2)}{\omega'(\tau-s_1)-u'(1-\tau+s_1)}Pr_1^2(s_2)=$$

$$= \frac{u'(1-\tau+s_2) + \left[u''(1-\tau+s_2) + \frac{\frac{\partial Pr_1^2(s_2)}{\partial s_2}}{Pr_1^2(s_2)} u'(1-\tau+s_2)\right] \cdot s_2}{u'(1-\tau+s_1) + u''(1-\tau+s_1) \cdot s_1} Pr_1^2(s_2)$$

If $s_1 = s_2$ then the left-hand side (MRS) equals $Pr_1^2(s_2)$, however, the right-hand side (MRT) is

$$\left[1 + \frac{u'(1-\tau+s_2)}{u'(1-\tau+s_2) + u''(1-\tau+s_2) \cdot s_2} \cdot \frac{\frac{\partial Pr_1^2(s_2)}{\partial s_2}}{Pr_1^2(s_2)} \cdot s_2\right] \cdot Pr_1^2(s_2) < Pr_1^2(s_2)$$

because $u'(1-\tau+s_2) + u''(1-\tau+s_2) \cdot s_2 > 0$ and $\frac{\partial Pr_1^2(s_2)}{\partial s_2} < 0$.

Therefore, in the equilibrium the marginal rate of substitution is lower than $Pr_1^2(s_2) \Rightarrow s_1 > s_2$. ■

Proof of Proposition 4.

The optimal maturity structure has to satisfy (31) or

$$\frac{u''(1-\tau+s_2) + u'(1-\tau+s_2) \cdot \frac{\frac{\partial Pr_1^2(s_2)}{\partial s_2}}{Pr_1^2(s_2)}}{u''(1-\tau+s_1)} \cdot \frac{b_0^2}{b_0^1} = MRT^0_{s_1, s_2}(\hat{\boldsymbol{s}}_0) \cdot \frac{1}{Pr_1^2(s_2)}$$

Recall from Proposition 2 that $s_1 > s_2$. From Assumption 3 it follows that $-u''(1-\tau+s_1) \leq -u''(1-\tau+s_2)$ leading to

$$\frac{u''(1-\tau+s_2) + u'(1-\tau+s_2) \cdot \frac{\frac{\partial Pr_1^2(s_2)}{\partial s_2}}{Pr_1^2(s_2)}}{u''(1-\tau+s_1)} > 1$$

On other hand, the (negative) marginal rate of transformation that equals marginal rate of substitution is below $(1-\pi(s_2))$:

$$MRT^0_{s_1, s_2}(\hat{\boldsymbol{s}}_0) = \frac{\omega'(\tau-s_2) - u'(1-\tau+s_2)}{\omega'(\tau-s_1) - u'(1-\tau+s_1)} Pr_1^2(s_2) < Pr_1^2(s_2)$$

This yields $\frac{b_0^2}{b_0^1} < 1$. Besides, recall that $\omega'(\tau-s_2) - u'(1-\tau+s_2) > 0$ for any $s_2 > 0 \Rightarrow \frac{b_0^2}{b_0^1} > 0$ (The government never chooses s_2 such that $Pr_1^2(s_2) = 0$ because then the government is strictly better off by setting $s_2 = 0$).

Next notice that if $s_1 \geq b_0^1$ and $s_2 \leq b_0^2$ then $MRT^1_{1,2}(s_1, s_2)$ is

$$\frac{u'(1-\tau+s_2)+\left[u''(1-\tau+s_2)+u'(1-\tau+s_2)\cdot\frac{\frac{\partial Pr_1^2(s_2)}{\partial s_2}}{Pr_1^2(s_2)}\right]\cdot(s_2-b_0^2)}{u'(1-\tau+s_1)+u''(1-\tau+s_1)\cdot(s_1-b_0^1)}\cdot Pr_1^2(s_2)\geq$$

$$\geq\frac{u'(1-\tau+b_0^2)}{u'(1-\tau+b_0^1)}Pr_1^2(s_2)>Pr_1^2(s_2)$$

that contradicts the optimality condition because $MRS_{1,\,2}^1(s_1,\,s_2)<Pr_1^2(s_2)$, therefore, $s_1<b_0^1$ and $s_2>b_0^2$.■

Proof of Proposition 5

It is sufficient to prove that if $\theta_0>\bar{\theta}$ then $\hat{s}_t>\hat{s}_{t+1}$.

Suppose by contradiction that $\hat{s}_{t+1}\geq\hat{s}_t$. It then implies that

$$MRS_{t,\,t+1}^0=\beta P_t^{t+1}(\hat{\boldsymbol{s}}_{t+1})\cdot\frac{\omega'(\tau-\hat{s}_{t+1})-u'(1-\tau+\hat{s}_{t+1})}{\omega'(\tau-\hat{s}_t)-u'(1-\tau+\hat{s}_t)}\geq\beta P_t^{t+1}(\hat{\boldsymbol{s}}_{t+1})$$

However, $\hat{s}_{t+1}\geq\hat{s}_t$ also implies that

$$MRT_{t,\,t+1}^0=$$

$$=\frac{u'(1-\tau+\hat{s}_{t+1})+\left[u''(1-\tau+\hat{s}_{t+1})+u'(1-\tau+\hat{s}_{t+1})\cdot\frac{\frac{\partial P_t^{t+1}(\hat{\boldsymbol{s}}_{t+1})}{\partial\hat{s}_{t+1}}}{P_t^{t+1}(\hat{\boldsymbol{s}}_{t+1})}\right]\cdot\hat{s}_{t+1}}{u'(1-\tau+\hat{s}_t)+u''(1-\tau+\hat{s}_t)\cdot\hat{s}_t}\beta P_t^{t+1}(\hat{\boldsymbol{s}}_{t+1})<$$

$$<\beta P_t^{t+1}(\hat{\boldsymbol{s}}_{t+1}).$$

The last inequality follows from Lemma A.3 and A.4 because

$$s_{t+1}\geq s_t\;\Rightarrow\;u'(1-\tau+s_t)+u''(1-\tau+s_t)\cdot s_t\geq u'(1-\tau+s_{t+1})+u''(1-\tau+s_{t+1})\cdot s_{t+1}>$$

$$>u'(1-\tau+s_{t+1})+\left[u''(1-\tau+s_{t+1})+u'(1-\tau+s_{t+1})\cdot\frac{\frac{\partial P_t^{t+1}(\boldsymbol{s}_{t+1})}{\partial s_{t+1}}}{P_t^{t+1}(\boldsymbol{s}_{t+1})}\right]\cdot s_{t+1}.$$

■

Proof of Proposition 6

It is sufficient to show that $b_0^{t+1} > b_0^t$ $\forall t = 1, 2, ..., T$. The proposition 5 implies that $MRS^0_{t,t+1}(\hat{\boldsymbol{s}}_t) = \beta P_t^{t+1}(\hat{\boldsymbol{s}}_{t+1}) \cdot \frac{\omega'(\tau-\hat{s}_{t+1})-u'(1-\tau+\hat{s}_{t+1})}{\omega'(\tau-\hat{s}_t)-u'(1-\tau+\hat{s}_t)} < \beta P_t^{t+1}(\hat{\boldsymbol{s}}_{t+1})$ because $\hat{s}_t > \hat{s}_{t+1}$. Therefore, (32) can be rewritten as

$$\frac{b_0^{t+1}}{b_0^t} < \frac{u''(1-\tau+\hat{s}_t)}{u''(1-\tau+\hat{s}_{t+1}) - u'(1-\tau+\hat{s}_{t+1}) \cdot \frac{\frac{\partial P_t^{t+1}(\hat{\boldsymbol{s}}_{t+1})}{\partial \hat{s}_{t+1}}}{P_t^{t+1}(\hat{\boldsymbol{s}}_{t+1})}}$$

Assumption 3 implies that if $\hat{s}_t > \hat{s}_{t+1}$ then $-u''(1-\tau+\hat{s}_t) < -u''(1-\tau+\hat{s}_{t+1}) < -u''(1-\tau+\hat{s}_{t+1}) + u'(1-\tau+\hat{s}_{t+1}) \cdot \frac{\frac{\partial \pi_{t+1}(\hat{\boldsymbol{s}}_{t+1})}{\partial \hat{s}_{t+1}}}{1-\pi_{t+1}(\hat{\boldsymbol{s}}_{t+1})} \Rightarrow \frac{b_0^{t+1}}{b_0^t} < 1$. Additionally, note that $\frac{b_0^{t+1}}{b_0^t} > 0$. Therefore, $b_0^1 > b_0^2 > ... > b_0^T > 0$. ■