Resurrecting the New Keynesian Model: (Un)conventional Policy and the Taylor rule

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September 2019

Abstract

This paper explores the ability of the New Keynesian (NK) model to explain the recent periods of quiet and stable inflation at near-zero nominal interest rates. We show that temporary and permanent shocks to the natural rate (and inflation) are sufficient for the ability of the simple NK model to explain the recent facts. Based on the identified shocks, we show that the model can replicate key macroeconomic variables in accordance with the term structure of interest rates. We find that the term structure helps to identify permanent shocks. Our analysis is restricted to an active role of monetary policy and the traditional regions of (local) determinacy. We also show that capturing highly nonlinear dynamics can be useful to generate a prolonged period of near-zero interest rates as a policy choice.

Keywords: Monetary economics, Calvo pricing, Term structure of interest rates
JEL classification numbers: E32, E12, C61

*Universität Hamburg, Department of Economics, Von-Melle-Park 5, 20146 Hamburg, Germany. This paper supersedes a working paper titled "Solving the New Keynesian Model in Continuous Time" and has benefited from helpful contributions by Jesús Fernández-Villaverde and Juan Rubio-Ramírez. We are also grateful to Anton Braun, Michael Bauer, John Cochrane, Thomas Hintermaier, Alejandro Justiniano, Ricardo Reis, Pedro Silos, Eric Swanson, and Klaus Wälde for comments. The author appreciates financial support from the German Research Foundation (DFG), and the Center for Research in Econometric Analysis of Time Series (CREATES), which is funded by the Danish National Research Foundation.

1. Introduction

> "Theories ultimately rise and fall on their ability to organize and interpret facts." (Cochrane, 2011, p.566)

In the aftermath of the financial crisis, New Keynesian (NK) theory has fallen on hard times. Once being a pillar of macroeconomics, in particular monetary economics, it has been criticized on both the theoretical and the empirical ends. Consider the simplest three-equation NK model with rational expectations and active monetary policy, and the cut in interest rates from 5.25% in 2007 to 0.25% by the end of 2008 (cf. Figure 1). When interpreting this cut as an exogenous but transitory monetary policy shock, the NK model predicts a counterfactual rise of inflation to more than 4 percent. Others would argue that the fall in interest rates is a *response* to some other shock, usually to the natural rate. However, the subsequent episode of an apparently binding zero-lower bound (ZLB), which is referred to as the zero-interest-rate policy (ZIRP) period, even intensified the criticism. If the economy entered a liquidity-trap scenario, the NK model would predict a deep recession with deflation (cf. Werning, 2012; Cochrane, 2017b). But nothing happened. If anything, core inflation (excluding food and energy) declined moderately to values around 1 percent in 2010. So what happened? Is the Taylor principle applicable in a world where interest rates stopped moving more than one-for-one with inflation? Cochrane (2017a) shows that alternative doctrines, including old-Keynesian models and the monetarist view, fail in explaining the ZIRP period, when the Fed drastically decreased interest rates and embarked on immense (unconventional) open market operations.

So the open question is on the ability of the NK model to organize our thoughts and interpret the recent facts. Can we reconcile the dynamics of key macroeconomic variables and the term structure of interest rates with the model predictions? Del Negro, Giannoni, and Schorfheide (2015) challenge the criticism by showing that a medium-sized NK model of Smets and Wouters (2007), based on Christiano, Eichenbaum, and Evans (2005), with time-varying inflation target and financial frictions is able to predict a sharp decline in output without forecasting a large drop in inflation. Based on this finding, do we have to abandon the three-equation NK model? Central to this question is the ability to replicate the yield curve, which from the expectation hypothesis relates to the market perceptions about future interest rates. In this paper, we allow for both temporary and permanent shocks to the interest rate and inflation. Based on this tweak, we show that the simple NK model can be used to interpret the data. We also shed light on potential sources.

Our contribution is to show that the ability to explain the facts crucially depends on the way we interpret and solve the model. We take a fresh look at the standard NK model under active monetary policy and show that it supports both a negative and a positive response of inflation to a 'monetary policy shock', once the definition includes temporary and/or permanent components. We show how the yield curve helps to identify permanent shocks to interest rates and inflation. The variability at the long-end of the yield curve can be triggered by changes in uncertainty, concrete policy action and/or by changing expectations (e.g., reflecting changes in the conduct of monetary policy). This strategy accounts for the enlarged set of policy instruments of the monetary authority. Based on the identified shocks to inflation and the natural rate, we show that the simple NK model is able to replicate the key macroeconomic variables and the term structure of interest rates. In particular, shocks to monetary policy and the natural rate may result into a ZIRP period, and by the same arguments, inflation may rebound while nominal interest rates being kept near-zero values. A nonlinear (and global) solution, which accounts for potentially strong nonlinearities capturing non-normal times, can be useful to generate immobile interest rates near-zero and stable quiet inflation with a single shock.

The quest on the ability of the simple NK model to explain the recent episodes has a deeper motivation. We investigate whether financial frictions are required to reconcile the recent facts with the theory. Although formulating and solving medium-scale NK models is important (cf. Christiano, Motto, and Rostagno, 2014; Del Negro, Giannoni, and Schorfheide, 2015), there is need for a parsimonious specification to provide a simple device to develop intuition, conceptualize, and facilitate the way we think about problems in economics. This paper merely shows that temporary and permanent shocks in the interest rate (and inflation) enable the simple NK model to replicate the data. We also provide an analytical investigation of the effects of uncertainty in the nonlinear NK model and show how it affects the natural rate. Our results confirm that uncertainty shocks are isomorphic to discount factor shocks (Barsky, Justiniano, and Melosi, 2014), so they provide an attractive structural interpretation of the permanent shocks.

Our arguments are motivated by the strong empirical evidence of shifting end-points in the yield curve, which may just reflect the private sector's perception of the inflation target rate (cf. Kozicki and Tinsley, 2001; Gürkaynak, Sack, and Swanson, 2005).[1] There is also empirical evidence in the macroeconomic literature on time-varying inflation target rates (cf. Ireland, 2007; Fève, Matheron, and Sahuc, 2010). Empirical results for the US and Japan is also confirmative of the counteracting effects resulting from transitory and permanent shocks to the interest rate (cf. Uribe, 2017).

In contrast to the ZLB literature, we focus on equilibria with active monetary policy, in which the enlarged set of policy instruments includes long-end target rates (the liquidity trap scenario is studied in Werning, 2012; Wieland, 2015; Cochrane, 2017b). This paper fills the gap in the literature by providing an investigation of the simple NK model in times when the traditional arguments seem to fail. This is highly relevant since the mode of criticism relates to the case that the ZIRP period reflected a binding constraint. The

[1]Linking the policy target rates to the long-end of the yield curve is not new and received increasing attention (see Gürkaynak and Wright, 2012, and the references therein). Time-variation in the inflation target is needed to capture the evolution of inflation expectations (cf. Del Negro and Eusepi, 2011).

bottom line is that changes in longer rates (possibly through unconventional policies) help to explain the recent episodes within the simple NK framework, while nonlinearities play an important role to generate the ZIRP period as a policy choice.

The rest of the paper is organized as follows. First, in Section 2 we present the simple NK model and explore the ability to explain the recent facts. In Section 3 we present the full nonlinear analysis by introducing shocks, and show how near-zero interest rates can be reconciled within the framework and may result as a policy choice. Section 4 concludes. Further results and illustrations are available in an accompanying web appendix.

2. Simplified Framework

In this section we present the continuous-time specification of the standard NK model. This simple framework is used to answer our questions regarding the ability of the model to explain the facts. In the next section we show how the equilibrium dynamics follow from the standard micro-founded rational-expectation solution and shed light on the effects of uncertainty, and potential sources of permanent shocks to interest rates and inflation.

The simplest version of the NK model reads:

$$\mathrm{d}x_t = (i_t - r_t - \pi_t)\mathrm{d}t \tag{1}$$

$$\mathrm{d}\pi_t = (\rho(\pi_t - \pi_t^*) - \kappa x_t)\mathrm{d}t \tag{2}$$

We denote x_t as the output gap (percentage deviations), i_t is the nominal interest rate, r_t is the natural rate, which coincides with the rate of time preference ρ, once transitory shocks have abated, $r_t^* = \rho$, and π_t is inflation, where κ controls the degree of price stickiness with $\kappa \to \infty$ as the frictionless (flexible price) limit. This system summarizes the linearized equilibrium dynamics around zero-inflation target $\pi_t^* = 0$ (or full indexation). Note that the appearance of the inflation target π_t^* in (2) ensures that the long-run equilibrium coincides with the nonlinear solution (cf. Section A.1).

The equation (1) follows directly from the consumption Euler equation representing the optimal investment/saving (IS) decision, often referred to as the IS curve, whereas (2) is the NK forward-looking Phillips curve. Solving forward it expresses inflation in terms of future output gaps,

$$\pi_t - \pi_t^* = \kappa \int_t^\infty e^{-\rho(v-t)} x_v \, \mathrm{d}v.$$

Hence, the *current* rate of inflation and *expected* rate of inflation are the same variable in continuous time. In this model it is useful to think of the path of expected future inflation and other variables (e.g., marginal cost) determining events at time t.

We close the model by specifying a rule which determines the (equilibrium) interest rates. In this perfect-foresight model both inflation dynamics and the output gap are fully determined by the Taylor rule. In what follows we analyze two alternative setups, which we refer to as the traditional feedback model:

$$i_t = \phi(\pi_t - \pi_t^*) + i_t^*, \quad \phi > 0, \tag{3a}$$

and the partial adjustment model (similar to Sims, 2004; Cochrane, 2017b):

$$\mathrm{d}i_t = \theta(\phi(\pi_t - \pi_t^*) - (i_t - i_t^*))\mathrm{d}t, \quad \theta > 0, \tag{3b}$$

which reflects both a response to inflation and a desire to smooth interest rates. The rules (3a) and (3b) show the attitude of the monetary authority towards either the long-run nominal interest rate or the target rate of inflation (one target is isomorphic to the other). In this paper, we consider the inflation target as a policy parameter, but abstract from specifying a specific process. We interpret unexpected changes in target rates as to capture changes in the conduct of monetary policy. Empirically, variations in the long rates are crucial for understanding the dynamics of yields (cf. Bauer and Rudebusch, 2017). One potential interpretation of those changes is that economic agents infer target rates from observed interest rate and inflation dynamics: A large interest rate cut may also trigger a decrease in the long-run interest rate (or inflation) target.

The rule (3b) specifies an explicit time lag between the inflation rate π_t and the policy rate i_t. The delay will be small if the parameter θ is large:

$$i_t - i_t^* = \phi\theta \int_{-\infty}^{t} e^{-\theta(t-k)}(\pi_k - \pi_t^*)\,\mathrm{d}k,$$

which makes i_t a *state* variable, given by past inflation rates. While the rule (3a) may seem simpler, it has some undesirable properties in continuous time. Among others, the clear distinction between inflation (expected future inflation) that the interest rate controls and inflation to which the Fed responds vanishes in continuous time.

Before we can meaningfully study shocks to the interest rate it is important to answer the question about local determinacy and thus the possibility of sunspot equilibria. We define an active monetary policy if $\phi > 1$ and refer to monetary policy as passive if $\phi < 1$. In what follows we focus on an active monetary policy ensures the existence of a unique locally bounded solution (cf. Appendix A.1.1 and the web appendix).[2]

We extend the three-equation NK model by allowing for temporary and permanent shocks to the natural rate (and inflation). Both, the inflation target π_t^* and the long-run (Wicksellian) natural rate r_t^*, are considered as exogenous parametric values. Below we offer some economic interpretations which may reflect such exogenous shocks.

[2]Note that the indeterminacy regions typically depend on the modelling frequency (Hintermaier, 2005). Hence, the findings for the discrete-time model with a presumed timing convention cannot simply be translated to different decision horizons, in particular to the continuous-time limit.

2.1. Which policy instruments?

The recent episodes shed light on the set of central bank instruments. They demonstrated that the nominal interest rate, once considered as the most important (conventional) instrument, cannot be used as a sufficient description of monetary policy. A large body of literature and anecdotal evidence show that unconventional policies, in particular forward guidance and quantitative easing (QE), are important monetary policy instruments too. Unless one adds financial frictions (e.g., Gertler and Karadi, 2011), or assumes imperfect substitutability between different maturities (cf. Chen, Cúrdia, and Ferrero, 2012), the NK model predicts that arbitrary QE operations are irrelevant. This is important because inflation seems to be unaffected by the large-scale asset purchase (LSAP) programmes. Hence, QE as such is *not* considered a separate policy instrument.[3] In contrast, forward guidance, which also includes the communication of the inflation target, has strong effects in the standard NK model (Del Negro, Giannoni, and Patterson, 2015; Campbell, Fisher, Justiniano, and Melosi, 2016). While the traditional instrument targets the short-term interest rate, the unconventional policy measures are commonly targeting interest rates at higher maturities (or the longer-end of the yield curve).

Beside changing the short-term interest rate, the monetary authority may focus on other longer maturities.[4] Such policies would need to control the long-ends of either the nominal and/or the real yield curve. As the inflation target is under the discretion of the monetary authority, there might be changes in its perception by economic agents due to communication or other measures. So we may consider the inflation target π_t^* as a policy instrument.[5] In our analysis, a 'target shock' simply reflects (unexpected) changes to π_t^*, which is interpreted to representing a different 'regime', and thus may induce transitional dynamics. A second source of variability are shocks to the natural rate r_t^*. Below we offer some structural interpretation of such shocks beyond the central banks control.

There is also an important difference with respect to forward guidance for the two Taylor rules specified in (3a) and (3b). Pure 'communication' about future policy induces a reaction of the interest rate in the feedback model due to the effect on inflation, while in the partial adjustment model interest rates are immobile on impact (pre-determined), e.g., with respect to changes in long-run targets. So an immediate challenge for empirical research is to identify permanent shocks, and also to which extent an observed monetary policy shocks contain information about (perceived) changes in long-run targets.

[3] As a caveat, LSAPs could affect term premia, a channel which is absent in the simple NK model and will be discussed later. Moreover, the LSAPs could also affect agents expectations of the future course of monetary policy (cf. Wright, 2012), which may be captured by 'shocks' to the long-run target rates.

[4] Swanson and Williams (2014) find that interest rates with a year or more to maturity were surprisingly unconstrained and responsive to news throughout 2008 to 2010.

[5] Note that the simplifying assumption of constant target rates will not be relevant for our arguments. Alternative approaches such as the regime-switching framework (see Sims and Zha, 2006), or time-varying inflation targets (e.g., Ireland, 2007) would be more realistic, at the cost of more technical details.

2.2. Do higher interest rates raise or lower inflation?

Following the discussion on the policy instruments we now address the question of whether higher interest rates raise or lower inflation. In fact, the NK model for $\phi > 1$ makes sharp predictions regarding the systematic link between interest rates and inflation, but at the same time can explain both the short-run negative response *and* the long-run positive Fisher effect. As shown below, the minimal set of ingredients, in a forward-looking general equilibrium framework with active monetary policy, $\phi > 1$, to produce a negative short-run impact of interest rates on inflation is the partial adjustment model.

For the partial adjustment model, the inflation rate is a negative function of the interest rate (cf. Figure D.5).[6] The figure plots inflation for different interest rates, which shows the short-run negative relationship. The intuition is that the interest rate depends positively on the level of inflation, but negatively on its time derivative,

$$i_t = \phi(\pi_t - \pi_t^*) + i_t^* - \theta^{-1}\,\mathrm{d}i_t/\,\mathrm{d}t, \quad \theta > 0. \tag{4}$$

For a given value $\mathrm{d}i_t/\,\mathrm{d}t \neq 0$, the larger the central bank's desire to smooth interest rates over time (the lower θ), the larger the second effect: Suppose that after a contractionary monetary policy shock $i_t > i_t^*$, so the (after-shock) time-derivative of the interest rate is negative $\mathrm{d}i_t/\,\mathrm{d}t < 0$, which reflects the slope of the impulse response function. Higher interest rates are related to lower inflation rates, because the inflation rate is determined by both the (long-run) Fisher relation *and* the mean reversion back to the target level. In our solution, inflation falls by 0.5 percentage points on impact for an 1 percentage point increase in interest rates. To summarize, the short-run response of inflation rates on impact is *negative*, while the *positive* relationship (higher inflation targets imply higher interest rates) is still given by the long-run Fisher relation $i_t^* = \rho + \pi_t^*$. Higher interest rates unambiguously imply higher yields to maturity of long-term bonds.

So what happens if central banks raise interest rates? If the increase is considered by agents not only as temporary, but after all reflects a permanent change in the target rate, inflation stability in the Fisher equation will result in higher long-run inflation. But can higher permanent interest rates reduce inflation in the short run? Indeed this is possible if the 'target shock' is accompanied by concrete policy action, i.e., a raise in the short-term interest rate.[7] In the partial adjustment model, this induces the traditional negative effect on inflation, which may even dominate the long-run Fisher effect temporarily. However, inflation *cannot* temporarily decrease in the simple feedback model. Unless we consider a persistent shock to the feedback rule any deviation from the equilibrium instantaneously

[6] In the simplified framework, we solve a standard boundary-value problem (perfect-foresight solution) and plot the initial values for different starting values (cf. Section 3 for a detailed description).

[7] Another possibility is to add long-term debt and use the fiscal theory (FTPL) to pin down inflation (following McCallum, 2001; Del Negro and Sims, 2015). Cochrane (2017a) shows that the FTPL produces a temporary reduction in the inflation due to the decline in the nominal market value of the debt.

jumps back. Any temporary shock would evaporate, and the interest rate accommodates its equilibrium level (infinitely fast). Only for the case where $\theta < \infty$, a temporary change induces some persistence and thus own equilibrium dynamics.

Let us consider a concrete example. Suppose that variables in the simple NK model are at steady state and the long-run interest rate i_t^* (or the inflation target π_t^*) is lower by 50 basis points (bp), and also the short-term interest rate i_t is *decreased* by 250 bp. The concrete policy action is 250 bp (observed), but only a fraction 1/5 of the interest rate cut is permanent (discretionary) leaving the remainder 4/5 being only temporary and not reflecting changes in policy targets. In the long run we expect lower inflation due to the Fisher relation $i_t^* = \rho + \pi_t^*$, but temporarily the traditional negative trade-off dominates the Fisher effect (cf. Figure D.17). Our simulation exercise shows that on impact the inflation rate *increases* to 2.5% and then both inflation and interest rates accommodate their new equilibrium levels after about 10 quarters. This perspective on 'monetary policy shocks' consisting of temporary and permanent shocks offers an alternative explanation for the so-called 'prize puzzle' (going back to Sims, 1992; Eichenbaum, 1992).[8] So at the risk of oversimplifying: Higher short-term interest rates (Fed Funds) decrease inflation, whereas higher long-run interest rates (inflation target) increase inflation.

2.3. Term structure of interest rates

A large body of empirical research shows that in particular unconventional monetary policy is targeting interest rates at the long-end of the yield curve, and is trying to influence market expectations about the future macro aggregates including inflation, interest rates and output gaps. Financial data can shed light on expectations, which is highly relevant when the standard instrument of monetary policy is not available.

Let us consider a nominal (zero-coupon) bond with unity payoff at maturity N:

$$P_t^{(N)} = \mathbb{E}_t \left(e^{-\rho N} \lambda_{t+N} / \lambda_t e^{-\int_t^{t+N} \pi_s ds} \right), \tag{5}$$

where λ_t is the marginal value of wealth, or the present value shadow price, consistent with equilibrium dynamics of macro aggregates.[9] The equilibrium bond price can be obtained from the fundamental pricing equation for the price $P_t^{(N)}$:

$$\mathbb{E}_t \left((\,\mathrm{d}P_t^{(N)}) / P_t^{(N)} \right) - \left(1/P_t^{(N)} (\partial P_t^{(N)} / \partial N) + i_t \right) \mathrm{d}t = 0. \tag{6}$$

Observe that in equilibrium, the bond price $P_t^{(N)}$ is a function of the state variables, so

[8] Similarly, a cost-channel in addition to the demand channel is likely to generate a positive response on impact, but has little empirical support (see Castelnuovo, 2012, and the references therein). Castelnuovo and Surico (2010) show that accounting for *expected* inflation may also explain the 'puzzle'.

[9] In this simplified framework, we abstract from a term premium to focus on the expectation channel of the NK model. Below we discuss the term premium and the full model with shocks (cf. Section 3).

in the partial adjustment model $P_t^{(N)} = P^{(N)}(i_t)$, and after some algebra we obtain the partial differential equation (henceforth *PDE approach*) for the bond:

$$\theta(\phi(\pi_t - \pi_t^*) - (i_t - i_t^*))(\partial P_t^{(N)}/\partial i_t) - (\partial P_t^{(N)}/\partial N) - i_t P_t^{(N)} = 0. \tag{7}$$

The solution to the pricing equation implies the complete term structure of interest rate for any given interest rate and maturity (we study the term premium in Section 3):

$$y_t^{(N)} \equiv y^{(N)}(i_t) = -\log P^{(N)}(i_t)/N. \tag{8}$$

The traditional *expectation approach* is to simulate the bond price (5) for a given interest rate and obtain the N-period ahead distribution in order to approximate the first moment (cf. Cochrane, 2005). While the approach is easy to implement, approximating moments for more state variables and longer maturities becomes computationally infeasible.

There are efficient algorithms to numerically solve the fundamental pricing equation (7). Our strategy is to use collocation, so we approximate the function $P_t^{(N)} \approx \Phi(N, i_t)v$, in which v is an n-vector of coefficients and Φ denotes the known $n \times n$ basis matrix, and can compute the unknown coefficients from a *linear* interpolation equation:

$$\theta(\phi(\pi_t - \pi_t^*) - (i_t - i_t^*))\Phi_2'(N, i_t)v - \Phi_1'(N, i_t)v = i_t\Phi(N, i_t)v,$$

or

$$(\theta(\phi(\pi_t - \pi_t^*) - (i_t - i_t^*))\Phi_2' - \Phi_1' - i_t\Phi)\, v = 0_n,$$

where $n = n_1 \cdot n_2$ with the boundary condition $\Phi(0, i_t)v = 1_n$. So we concatenate the two matrices and solve the linear equation for the unknown coefficients:

$$\begin{bmatrix} \Phi_1' - \theta(\phi(\pi_t - \pi_t^*) - (i_t - i_t^*))\Phi_2' + i_t\Phi, & \Phi(0, A_t) \end{bmatrix} v \quad = \quad \begin{bmatrix} 0_n \\ 1_n \end{bmatrix}.$$

For the *feedback model*, with no relevant state variables, we obtain the trivial solution that without shocks, the yield curve is flat, $i_t \equiv i_t^* = \rho + \pi_t^*$. The analytical solution of the feedback model, however, is useful for studying the long-end of both the nominal and real yield curves in the partial adjustment model and thus for interpreting the data:

$$\lim_{N\to\infty} y_t^{(N)} = i_t^* = \rho + \pi_t^*, \quad \lim_{N\to\infty} r_t^{(N)} = i_t^* - \pi_t^* = \rho \equiv r_t^*,$$

which show the expectation component of the long-end yields. The inflation target does not affect the long-run risk-free rate, but the nominal rate, $i_t^* = \rho + \pi_t^*$. For comparison with the data we consistently define the model-implied 10-years to maturity yields of (nominal and inflation-protected) zero-coupon bonds $y_t^{(10)}$ and $r_t^{(10)}$, respectively. Figure D.5 shows the model-implied 10-years to maturity yield of a zero-coupon bond $y_t^{(10)} = y^{(10)}(i_t)$ and the 10-years to maturity inflation-protected yield $r_t^{(10)} = r^{(10)}(i_t)$.

2.4. Can we explain the recent episodes?

In this section, we study the ability of the simple NK model to explain the recent episodes during the new century, including the financial crisis episodes. This sheds light on some anecdotal evidence, which suggests that the traditional arguments are flawed.

Our data is from the Federal Reserve Bank of St. Louis Economic Dataset (FRED), i.e., the US Effective Federal Funds Rate (Fed Funds), the 10-Year Treasury Constant Maturity Rate (10Y Govt), the Consumer Price Index (Core CPI), seasonally adjusted, the 10-Year Treasury Inflation Protected Securities Rate (10Y TIPS), and the Output gap (HP Filter) from 1990 through 2017. In particular, we focus on four episodes brought forward by prominent economists, and discussed in the literature.

First, to the unaided eye, the data suggests a reversal of the interest rates-inflation tradeoff in the period 2001-2007, supporting the alternative hypothesis that inflation and interest rates are positively related. If anything, inflation decreased in response to the interest rate cuts. Second, in the subsequent period from 2007 the Fed Funds rate has remained near-zero until Dec 2015, to which we refer to as the zero-interest-rate policy (ZIRP) period, but inflation kept stable and quiet (cf. Cochrane, 2017a). Third, despite interest rates near-zero through 2015, inflation rebounded already in 2011, with about the same pattern as before. While the short rate seems immobile over the recent episode, the long-end of the yield curve has considerable variation and declines over time. From the expectation hypothesis, we may also read this as changes of market perceptions about future interest rates and/or monetary policy. Fourth, there is anecdotal evidence of term structure anomalies between 2004 and 2005: The Fed Funds rate increased by 150 bp, but the 10Y Govt decreased by about 70 bp (cf. Backus and Wright, 2007). So can we explain the recent episodes and term structure anomalies within the NK model?

Our approach is to study whether the NK model (with traditional parameterization) is able to explain the recent dynamics of macroeconomic aggregates *and* whether they are consistent with the term structure (expectations of future interest rates). This simple accounting exercise sheds light on the size of shocks, conditional on the fixed parameters. We allow for temporary autoregressive shocks d_t (preference shocks):

$$\mathrm{d}d_t \quad = \quad -\rho_d(d_t - 1)\mathrm{d}t, \tag{9}$$

such that $r_t = r_t^* + \rho_d(d_t - 1)$ defines the 'natural rate' of interest (e.g., Werning, 2012). Formally, given a set of (nonlinear) functions and observable variables, our goal is to find a vector d_t that makes the difference between model-implied values and observable variables zero. Our numerical routine attempts to solve the system of equations by minimizing the sum of squares of the function components. If the equations can be solved (the residuals are negligible), we say that the NK model is able to explain the data.

More precisely, given the dynamics in (9), the model implies inflation $\pi_t = \pi(i_t, d_t)$, output gap $x_t = x(i_t, d_t)$, together with the 10-year yields to maturity of nominal bonds, $y_t^{(10)} = y^{(10)}(i_t, d_t)$ and inflation-protected bonds $r_t^{(10)} = r^{(10)}(i_t, d_t)$, as functions of the interest rate and the preference shock. So we can identify the shocks d_t (or natural rate r_t), given the set of observed data and parameterization.[10] For example, the observed data may include both the interest rate i_t^{obs} and inflation $\pi_t^{obs} = \pi(i_t^{obs}, d_t)$, where the observation frequency is set by the availability of macroeconomic data, either monthly or quarterly, depending of whether the output gap $x_t^{obs} = x(i_t^{obs}, d_t)$ is added.[11]

First, we identify the shocks d_t required to replicate the interest rate (Fed Funds), the inflation rate (Core CPI), and the output gap (Output gap) for the fixed parameters $\rho = 0.03$, $\kappa = 0.8842$, $\phi = 4$, $\theta = 0.5$, $\pi_{ss} = 0.02$, and $\rho_d = 0.4214$. This particular set of parameters is within plausible estimates discussed in the literature. In any case, only one admissible set from the parameters space is needed, which is able to replicate the data. If this set was empty, any estimation strategy would be doomed. Most importantly, this parameterization implies values $r_t^* = 0.03$ and $\pi_t^* = 0.02$. Based on the identified natural rate r_t (see Figure 2), we find that the model is able to explain (at least) interest rate and inflation dynamics, but dramatically fails to generate sufficient variability in 10-year treasury rates (see Figures 3 and 4). Hence, the expectations are not consistent with the data and/or variations in term premia (which is studied in Section 3) are relevant.[12]

Suppose that in addition to temporary shocks we allow for permanent shocks to the interest rate and inflation (by allowing time-variability in r_t^* and π_t^*) in order to account for the variability of the long-end of the nominal and the real yield curve. So we compute the model-implied inflation rate $\pi_t = \pi(i_t, d_t; \rho, \pi_{ss})$, output gap $x_t = x(i_t, d_t; \rho, \pi_{ss})$, together with the 10-year yields to maturity of nominal bonds, $y_t^{(10)} = y^{(10)}(i_t, d_t; \rho, \pi_{ss})$ and inflation-protected bonds $r_t^{(10)} = r^{(10)}(i_t, d_t; \rho, \pi_{ss})$, as functions of the interest rate, the preference shock, the long-run (Wicksellian) natural rate and the inflation target. We identify the natural rate r_t, together with the shocks for r_t^* and π_t^* to replicate the observed data by also including 10-Year Treasury rates (10Y Govt and 10Y TIPS). Based on the identified shocks r_t, π_t^* and r_t^* (see Figure 5), we find that the model is able to replicate interest rate, inflation dynamics, the slope of the yield curves, even when including output in the set of observable variables (see Figures 6 and 7). Note that the first identification

[10] Our identification differs from the schemes used in the literature (cf. Ramey, 2016; Uribe, 2017).

[11] See Kamber, Morley, and Wong (2018) for the various ways to extract the series from observed output data. For our illustrations, we use the HP filtered data (see Figure D.1).

[12] Christiano, Motto, and Rostagno (2014) include the slope of the term structure of interest rates (i.e., the difference of the long-term bond and the federal funds rate), by including an exogenous measurement error shock on the long-term bonds, which they interpret as a term premium shock.

of r_t without permanent shocks (for $\pi_t^* = 0.02$ an $r_t^* = 0.03$) is an restricted version of the approach followed here. From the identified series of shocks, the permanent shocks to the interest rate seem more important than the permanent shocks to the inflation.

In an alternative specification we allow for variations in r_t^* only for robustness of our results, keeping the inflation target constant ($\pi_t^* = 0.02$). So we identify the natural rate r_t, together with the shocks for r_t^* to replicate the observed data, by including the nominal 10-Year Treasury rates (10Y Govt). Still, the simple NK model is able to replicate interest rates, inflation dynamics, and both long-end yields, even though the inflation-protected 10-Year Treasury rates (10Y TIPS) are not used for identification (cf. Figure D.3). For most of observations output dynamics can be replicated (cf. Figure D.4). The bottom line is that both temporary and permanent shocks are required (also sufficient) for the ability of the simple NK model with $\phi > 1$ to explain the facts.

Let us now turn to the most prominent features of the data. We may use the NK model to interpret the episodes: (i) with an apparent sign reversal (2001-2007), (ii) including a zero-interest-rate policy (2007-2015), (iii) with an inflation rebound *and* near-zero interest rates (2011), and (iv) including an apparent term structure anomaly (2004-2005).

2.4.1. Sign reversal

While the academic discourse about the effects of the nominal interest rate on the inflation rate has some tradition in macroeconomics, motivated by the 'price puzzle', it received public attention in 2008, when the interest rates in the US (followed by the ECB in 2014) hit essentially zero. Consider the period 2001-2007, right before the financial crisis.

In Jan 2001 the Fed Funds rate was at 6 percent (5.98%), the 10Y Govt at 5 percent (5.16%). In Sep 2007 the Fed Funds rate was slightly below 5 percent (4.94%), the 10Y Govt at 4.5 percent (4.52%). In the meantime, the Fed Funds rate has been sharply decreased and raised to and from 1 percent. Over the same period, the Core CPI inflation followed a similar $\vee$ pattern and decreased slightly from 2.5 percent (2.57%) to values around 2 percent (2.10%). When the Fed Funds rate dipped at 1 percent (0.98%) in Dec 2003, inflation also had its lowest value of 1 percentage point (1.09%) with 10Y Govt at 4 percent (4.27%). Can we reconcile this pattern with the NK model?

If we interpreted the $\vee$ pattern as two consecutive temporary monetary policy shocks, the NK model predicts that inflation should have followed a counterfactual $\wedge$ pattern. A transitory (negative) monetary policy shock of 500 bp would imply inflation to increase by about 250 bp in 2003.[13] This summarizes the puzzling 'sign reversal' and strikingly fails to explain the decline in long yields. However, the same transitory shock together with a (negative) permanent shock of 150 bp would account for the observed pattern for inflation and would predict a decline in yields to longer maturities. Similarly, a (positive) temporary

[13]For details and transitional dynamics, we refer to the figures in the accompanying web appendix.

monetary policy shock of 400 bp together with restoring the announced inflation target rate in 2007 has the opposite effect and may have generated the observed $\vee$ pattern of Fed Funds, 10Y yields and Core CPI inflation in the period.

Using the identified shocks suggests a different story. Suppose in Jan 2001 the target rate shock was slightly below 3 percent (2.83%), and the natural rate at 2.5 percent (2.55%). Until Dec 2003 the natural rate dropped by about 4.5 percent (−3.16%), mainly driven by a large temporary preference shock (−15.33%). In Sep 2007, the preference shock disappeared and the target rate shock was at 2 percent (2.24%), and the natural rate back at 2.5 percent (2.37%). This exactly replicates the observed $\vee$ pattern of the Fed Funds, the 10Y Govt, and the Core CPI inflation (and Output gap).

To summarize, the observed pattern indeed can be reconciled with the simple NK model, when allowing for a $\vee$ pattern to either the inflation target or the natural rate. From the identified shocks, the latter explanation seems more plausible.

2.4.2. ZIRP period

We next consider the zero-interest-rate policy (ZIRP) period 2007-2015, right after the start of the financial crisis in Sep 2007 until the 'liftoff' in Dec 2015, with the end-point marking the start of the Fed's 'normalization' of monetary policy (Williamson, 2016).

In Sep 2007, the Fed Funds rate was at 5 percent (4.94%), the 10Y Govt at 4.5 percent (4.52%), while in Jan 2009 the Fed Funds rate was at 0.25 percent (0.15%), the long rate 10Y at 2.5 percent (2.52%) and stayed there. Over the same period, the Core CPI inflation decreased from 2 percent (2.10%) to values way below the announced target rate around 0.5 percent (0.60%) in Oct 2010, and then bounced back in Aug 2011 to values around the announced target of 2 percent (1.97%). At a first glance, things look pretty much like the sharp decrease during the 2001-2003 period. This time, however, the (short-run) nominal interest rate was quite close to the ZLB and did not return to higher values for a while. Can we generate a ZIRP period within the NK model?

An interest rate cut by about 475 bp, for an inflation target of 2 percent, we should have expected inflation rates of more than 4 percent. If anything, Core CPI inflation declined from slightly above 2 percent to values around 0.5 percent in 2010, and then rebounded to 2 percent 2011. If we borrow the inflation target rate shock explanation, the sharp decrease may reflect a change in the target rate by 200 bp (10Y Govt declined by about 200 bp). Inflation would jump by 0.5% and then after about 2.5 years decline to zero. This sounds reasonable. But it does *not* explain the near-zero values of the interest rate. One subtle issue is that such a hypothetical series of shocks seems inconsistent with the underlying process (9) to generate the identified shocks. Although we can replicate the observed variables on impact (including the nominal yield curve), and inflation rates eventually approaching zero, the simple NK model predicts a counterfactual strong tendency of the interest rate to revert back to its steady-state value.

To explain the ZIRP episode with a *single* shock we would need to modify the shock dynamics. Our simulation results confirm this conjecture: Adding a negative preference shock of roughly 10 percent to both monetary policy shocks helps to fix the yield curve and inflation, but does not generate a ZIRP period. Even with higher persistence the assumed shock process (9) would *not* imply that interest rates do remain close to zero.

From the identified shocks, indeed the inflation target rate dropped from about 2.5 percent by 150 bp to 0.5 percent (0.75%) in Jan 2009, and the natural rate slightly dropped to 1.5 percent (1.53%). Then, the inflation target increased back to values about 2 percent in Oct 2010, accompanied by a large negative preference shock (-14.09%) which dragged the natural rate by about 580 bp to values around -4.5 percent and stayed negative until the liftoff in Dec 2015. In fact, this exactly replicates the observed dynamics of the Fed Funds, the 10Y Govt, and the Core CPI inflation (and Output gap).

2.4.3. Inflation rebound (near-zero interest rates)

Elaborating on the previous results, it gets even more challenging to reconcile the facts with the simple NK model, when we consider that after the long decline since 2007, the start of the ZIRP period, inflation suddenly rebounds to levels around 2 percent in 2011, while interest rates remain immobile and near-zero at least throughout 2015.

Let us turn to the facts. In Oct 2010, the Fed Funds rate was at 0.25 percent (0.19%), the 10Y Govt at 2.5 percent (2.54%), and Core CPI inflation dipped at 0.5 percent (0.60%), while in Jun 2011 the Fed Funds rate was close to zero (0.09%), whereas the 10Y Govt was at 3 percent (3.00%) and Core CPI inflation increased to 1.5 percent (1.58%), with tendency to revert back to values about the 'official' inflation target at 2 percent.

If we considered a target shock of 200 bp, so we re-established the announced target rate around 2011, while interest rates at near-zero values, the NK model could explain the inflation rebound without any effects on nominal interest rates *on impact* with the partial adjustment model only (the feedback model cannot generate this for $\phi > 1$). If anything, the Fed Funds rate in fact was lowered by 10 bp, which seems too small to account for the rebound of inflation rates. Consider now a simultaneous negative preference shock of about 15 percent. Without this shock to the natural rate, inflation would jump to values around 4.5 percent for the presumed inflation target shock. A similar (counterfactual) picture arose at the longer-end of the yield curve, if we would try to match the inflation figures with a shock to the natural rate only. So we conclude that the simple NK model fails to replicate the observed pattern in the data with a single shock. One plausible scenario is that the economy experiences a series of negative shocks to the natural rate, which keeps inflation at reasonable levels after the inflation target shock.

From the identified shocks, in Aug 2011 the inflation target shock was indeed back to

2 percent (2.15%), but accompanied by a large negative permanent shock to the natural rate by 80 bp to values 1 percent (0.85%), which let inflation approach values around the target rate, though nominal interest rate remain immobile and near-zero values. Again, this exactly replicates the key macroeconomic aggregates and the yield curves.

2.4.4. Term structure anomalies

The discussion on preference shocks vs. target shocks has shown that it is important to consider both, the short and the longer-end of the yield curve in order to interpret the data through theoretical arguments. We now present some anecdotal evidence that shocks may indeed arise simultaneously. If a monetary policy shock is accompanied by a preference shock, some of the 'anomalies' observed in the data arise in the standard NK model.

Let us consider the period between 2004 and 2005, when a rotation in the yield curves gave rise to what Alan Greenspan's called a 'conundrum' (cf. Backus and Wright, 2007). In Jun 2004, the Fed Funds rate was at 1 percent (1.03%), the 10Y Govt at 4.5 percent (4.73%), while in Feb 2005, the Fed Funds rate was at 2.5 percent (2.50%), the 10Y Govt at 4 percent (4.17%). Over the same period, Core CPI inflation increased from slightly below its announced target rate of 2 percent (1.87%) to about 2.5 percent (2.31%).

The 'conundrum' is that the federal funds rate increased by 150 bp, but the ten-year yield decreased by about 50 bp. Can we reconcile the rotation of the yields, to which we refer as term structure 'anomalies', with the standard NK model? Given the previous discussion, we may conjecture that a positive monetary policy shock was accompanied by a negative shock to the natural rate, keeping the inflation target about the same level, such that the standard negative relationship between interest rates and inflation is as expected (with a tendency to revert back to the target rate). We simulate a positive monetary policy shock of 150 bp which is accompanied by a negative preference shock of about 10 percent. Both shocks generate the rotation in the yields as observed in the data. While a rotation in the nominal yield curve could also be obtained by a contemporaneous negative target shock, two reasons speak against this hypothesis for the period 2004-2005: First, if anything, we would expect that a *rise* in nominal interest rates may trigger a *rise* in the target rate. Second, the predicted real yield curve would not show a rotation as observed in the data. Hence, a rotation in both yield curves suggests that the monetary policy shock was accompanied by a shock to the natural rate.

From the identified shocks, the positive monetary policy shock was accompanied in fact by an increase of the natural rate by about 250 bp, which mainly is attributed to a reversion of the temporary component from values around −15 percent (−14.36%) to values around −5 percent (−5.51%). This replicates the observed rotation in the yields.

2.5. Discussion and open questions

So the bottom line is a partial remedy of the NK model to interpret the data. We show how target shocks (loosely interpreted as unconventional policy), and shocks to the natural rate in addition to the traditional policy instrument improves the ability of the model to explain the facts. Hence, the simple NK model helps us to organize our thoughts, so abandon the model might be too shortsighted: Allowing the shocks to have transitory and permanent components, we may explain the $\vee$ pattern of the Fed Funds rate *and* Core CPI inflation (and the Output gap) in the data. These predictions are also in line with the predictions for the yield curve. It also helps to explain the ZIRP period and the inflation rebound in 2011, while interest rates being immobile and near-zero.

One open question remains because the NK model fails to replicate the ZIRP period with a single shock, it seems that the shock dynamics are not consistent with the underlying shock process. According to the identified shocks we would need a large shock to the natural rate *and* that this shock keeps the natural rate negative for a while, before eventually reverting back to its steady state. The simple NK model (with $\phi > 1$) is unable to account for potential nonlinearities, so the paper does not stop here.

Perhaps we need to distinguish between normal times and non-normal times, where the dynamics are different from those at the intended equilibrium point? This is what we learn from Brunnermeier and Sannikov (2014): In normal times, the equilibrium system is near the steady state, where the system is resilient to most shocks near the steady state. Unusually large shocks, however, may induce completely different dynamics of macro aggregates. Once in a crisis state (non-normal times), also smaller shocks are subject to amplification. A nonlinear framework may be an alternative interpretation in which a single preference shock accounts for the ZIRP period. In what follows, we set up a parsimonious model, where the dynamics of large negative shocks are different from those around the steady state, at which the model is observational equivalent to (9).

In what follows we formulate and solve the nonlinear version of the NK model. We show that an alternative shock process, which is observational equivalent in normal times (with small shocks), has quiet different dynamics in non-normal times (large shocks). We also allow for stochastic shocks and show how uncertainty shocks will affect the natural rate and the long-end of the yield curve even if the inflation target rate is constant.

3. Nonlinear New Keynesian Model with Shocks

We describe now the environment for our investigation. It is the continuous-time version of the standard NK model (cf. Woodford, 2003). We summarize the equilibrium dynamics, show how to compute impulse responses, compute the effects of uncertainty, and how to solve the model in the policy function space. Throughout the paper, we keep the nonlinear structure of the model, which turns out to be quite relevant for non-normal times when considering large shocks and/or large deviations from the point of approximation.

3.1. The model

The basic structure of the model is as follows. A representative household consumes, saves, and supplies labor. The final output is assembled by a final good producer, who uses as inputs a continuum of intermediate goods manufactured by monopolistic competitors. The intermediate good producers rent labor to manufacture their good, and face the constraint that they can only adjust the price following Calvo's pricing rule (Calvo, 1983). Finally, there is a monetary authority that fixes the short-term nominal interest rate through open market operations with public debt, and a fiscal authority that taxes and consumes. We introduce four stochastic shocks, one to preferences (which can be loosely interpreted as a shock to aggregate demand, temporarily affecting the real interest rate), one to technology (interpreted as a shock to aggregate supply), one to monetary policy, and one to fiscal policy. For simplicity, we do not explicitly model a shock to the inflation target, which is considered a policy instrument under the discretion of the central bank.

3.1.1. Households

There is a representative household in the economy that maximizes the following lifetime utility function, which is separable in consumption, c_t and hours worked, l_t:

$$\mathbb{E}_0 \int_0^\infty e^{-\rho t} d_t \left\{ \log c_t - \psi \frac{l_t^{1+\vartheta}}{1+\vartheta} \right\} \mathrm{d}t, \quad \psi > 0, \tag{10}$$

where ρ is the subjective rate of time preference, ϑ is the inverse of Frisch labor supply elasticity, and d_t is a preference shock, with $\log d_t$ following an Ornstein-Uhlenbeck (OU) process (the continuous-time analog of a first-order autoregression):

$$\mathrm{d} \log d_t = -\rho_d \log d_t \mathrm{d}t + \sigma_d \mathrm{d}B_{d,t}. \tag{11}$$

The process $B_{d,t}$ is a standard Brownian motion, such that by Itô's lemma:

$$\mathrm{d}d_t = -\left(\rho_d \log d_t - \tfrac{1}{2}\sigma_d^2\right) d_t \mathrm{d}t + \sigma_d d_t \mathrm{d}B_{d,t}.$$

Below, for this shock and the other exogenous stochastic processes, we will use both the formulation in level and in logs depending on the context and ease of notation.

Let a_t denote real financial wealth, the household's real wealth evolves according to:

$$\mathrm{d}a_t = \left((i_t - \pi_t)a_t - c_t + w_t l_t + T_t + \digamma_t\right) \mathrm{d}t, \tag{12}$$

in which i_t is the nominal interest rate on government bonds, π_t the rate of inflation of the price level p_t (or price of the consumption good), w_t is the real wage, T_t is a lump-sum transfer, and $\digamma_t$ are the profits of the firms in the economy.

3.1.2. The final good producer

There is one final good produced using intermediate goods with the following production function:

$$y_t = \left(\int_0^1 y_{it}^{\frac{\varepsilon-1}{\varepsilon}} \, \mathrm{d}i \right)^{\frac{\varepsilon}{\varepsilon-1}}, \tag{13}$$

where ε is the elasticity of substitution.

Final good producers are perfectly competitive and maximize profits subject to the production function (13), taking as given all intermediate goods prices p_{it} and the final good price p_t. Hence, the input demand functions associated with this problem are:

$$y_{it} = \left(\frac{p_{it}}{p_t} \right)^{-\varepsilon} y_t \qquad \forall i,$$

and

$$p_t = \left(\int_0^1 p_{it}^{1-\varepsilon} \mathrm{d}i \right)^{\frac{1}{1-\varepsilon}} \tag{14}$$

is the (aggregate) price level.

3.1.3. Intermediate good producers

Each intermediate firm produces differentiated goods out of labor using:

$$y_{it} = A_t l_{it},$$

where l_{it} is the amount of the labor input rented by the firm and where A_t follows:

$$\mathrm{d} \log A_t = -\rho_A \log A_t \mathrm{d}t + \sigma_A \mathrm{d}B_{A,t}. \tag{15}$$

Therefore, the marginal cost of the intermediate good producer is the same across firms:

$$mc_t = w_t / A_t. \tag{16}$$

The monopolistic firms engage in infrequent price setting à la Calvo. We assume that intermediate good producers reoptimize their prices p_{it} only when a price-change signal is received. The probability (density) of receiving such a signal h periods from today is assumed to be independent of the last time the firm got the signal, and to be given by:

$$\delta e^{-\delta h}, \quad \delta > 0.$$

Thus $e^{-\delta(\tau-t)}$ denotes the probability of not having received a signal during $\tau - t$,

$$1 - \int_t^\tau \delta e^{-\delta(h-t)}\,\mathrm{d}h = 1 - \left(-e^{-\delta(\tau-t)} + 1\right) = e^{-\delta(\tau-t)}. \tag{17}$$

A fraction of firms will receive the price-change signal per unit of time. All other firms cannot reoptimize their price, but (partially) index their price to π_t^*:

$$\mathrm{d}\tilde{p}_{it} = \chi\pi_t^*\tilde{p}_{it}\,\mathrm{d}t$$

Indexation is controlled by the parameter $\chi \in [0,1]$. This implies that if a firm cannot change its price at t for a period length of $\tau - t$, its price at τ is $p_{i\tau} = p_{it}e^{\int_t^\tau \chi\pi_s^*\,\mathrm{d}s}$. Note that the higher the parameter δ, the lower price rigidities, in the frictionless case $\delta \to \infty$. Hence, for $\delta < \infty$ prices are set to maximize the expected discounted profits:

$$\max_{p_{it}} \mathbb{E}_t \int_t^\infty \frac{\lambda_\tau}{\lambda_t} e^{-(\rho+\delta)(\tau-t)} \left(\frac{p_{i\tau}}{p_\tau} y_{i\tau} - mc_\tau y_{i\tau}\right) \mathrm{d}\tau \quad \text{s.t. } y_{i\tau} = \left(\frac{p_{i\tau}}{p_\tau}\right)^{-\varepsilon} y_\tau,$$

or

$$\max_{p_{it}} \mathbb{E}_t \int_t^\infty \frac{\lambda_\tau}{\lambda_t} e^{-(\rho+\delta)(\tau-t)} \left(\left(\frac{p_{it}}{p_\tau}\right)^{1-\varepsilon} e^{\int_t^\tau (1-\varepsilon)\chi\pi_s^*\,\mathrm{d}s} y_\tau - mc_\tau \left(\frac{p_{it}}{p_\tau}\right)^{-\varepsilon} e^{-\int_t^\tau \varepsilon\chi\pi_s^*\,\mathrm{d}s} y_\tau\right) \mathrm{d}\tau$$

After dropping constants, we may write the first-order condition as:

$$\begin{aligned}
&\mathbb{E}_t \int_t^\infty \lambda_\tau e^{-(\rho+\delta)(\tau-t)} (1-\varepsilon) \left(\frac{p_t}{p_\tau}\right)^{1-\varepsilon} p_{it} e^{\int_t^\tau (1-\varepsilon)\chi\pi_s^*\,\mathrm{d}s} y_\tau \mathrm{d}\tau \\
&+\mathbb{E}_t \int_t^\infty \lambda_\tau e^{-(\rho+\delta)(\tau-t)} mc_\tau \varepsilon \left(\frac{p_t}{p_\tau}\right)^{-\varepsilon} e^{-\int_t^\tau \varepsilon\chi\pi_s^*\,\mathrm{d}s} p_t y_\tau \mathrm{d}\tau = 0
\end{aligned}$$

We may write the first-order condition as:

$$p_{it}x_{1,t} = \frac{\varepsilon}{\varepsilon-1} p_t x_{2,t} \quad \Rightarrow \quad \Pi_t^* = \frac{\varepsilon}{\varepsilon-1}\frac{x_{2,t}}{x_{1,t}}$$

in which $\Pi_t^* \equiv p_{it}/p_t$ is the ratio between the optimal new price (common across all firms that can reset their prices) and the price of the final good and where we define the auxiliary variables (interpreted as expected discounted marginal revenue and marginal costs):

$$x_{1,t} \equiv \mathbb{E}_t \int_t^\infty \lambda_\tau e^{-(\rho+\delta)(\tau-t)} \left(\frac{p_t}{p_\tau}\right)^{1-\varepsilon} e^{\int_t^\tau (1-\varepsilon)\chi\pi_s^*\,\mathrm{d}s} y_\tau \mathrm{d}\tau, \tag{18}$$

$$x_{2,t} \equiv \mathbb{E}_t \int_t^\infty \lambda_\tau e^{-(\rho+\delta)(\tau-t)} mc_\tau \left(\frac{p_t}{p_\tau}\right)^{-\varepsilon} e^{-\int_t^\tau \varepsilon\chi\pi_s^*\,\mathrm{d}s} y_\tau \mathrm{d}\tau. \tag{19}$$

Both variables are forward looking (or jump variables) and determined in equilibrium.

Differentiating $x_{1,t}$ with respect to time gives:

$$\mathrm{d}x_{1,t} \;=\; ((\rho+\delta+(1-\varepsilon)(\pi_t-\chi\pi_t^*))x_{1,t}-\lambda_t y_t)\,\mathrm{d}t \tag{20}$$

in which the rate of inflation $\pi_t = \mathrm{d}p_t/p_t$. Accordingly:

$$\mathrm{d}x_{2,t} \;=\; ((\rho+\delta-\varepsilon(\pi_t-\chi\pi_t^*))x_{2,t}-\lambda_t mc_t y_t)\,\mathrm{d}t \tag{21}$$

Note that we assume that the dynamics of the inflation index does *not* contribute to the dynamics of the auxiliary variables (taken as parametric to the firm).

Assuming that the price-change is stochastically independent across firms gives:

$$p_t^{1-\varepsilon} = \int_{-\infty}^{t} \delta e^{-\delta(t-\tau)} \left(p_{i\tau} e^{\int_\tau^t \chi\pi_s^*\,\mathrm{d}s}\right)^{1-\varepsilon} \mathrm{d}\tau,$$

making the price level p_t a predetermined variable at time t, its level being given by past price quotations (Calvo's insight). Differentiating with respect to time gives:

$$\mathrm{d}p_t^{1-\varepsilon} \;=\; \left(\delta p_{it}^{1-\varepsilon} - (\delta-(1-\varepsilon)\chi\pi_t^*)p_t^{1-\varepsilon}\right)\mathrm{d}t$$

and

$$\frac{1}{\mathrm{d}t}\mathrm{d}p_t^{1-\varepsilon} = (1-\varepsilon)\,p_t^{-\varepsilon}\frac{\mathrm{d}p_t}{\mathrm{d}t}.$$

Then

$$(1-\varepsilon)\;\mathrm{d}p_t = \left(\delta p_{it}^{1-\varepsilon}p_t^{\varepsilon-1} - (\delta-(1-\varepsilon)\chi\pi_t^*)\right)p_t$$

which implies

$$\pi_t - \chi\pi_t^* = \frac{\delta}{1-\varepsilon}\left((\Pi_t^*)^{1-\varepsilon}-1\right) \tag{22}$$

Differentiating (22) with respect to time, we obtain the inflation dynamics as:

$$\mathrm{d}(\pi_t-\chi\pi_t^*) = -(\delta+(1-\varepsilon)(\pi_t-\chi\pi_t^*))\,(\pi_t-\chi\pi_t^*+(mc_t/x_{2,t}-1/x_{1,t})\lambda_t y_t)\;\mathrm{d}t, \tag{23}$$

which is interpreted as the NK Phillips-curve.

3.1.4. The government problem

We assume that the government sets the nominal interest rate i_t through open market operations according to two alternative setups, i.e., the feedback model:

$$i_t - i_t^* = \phi_\pi(\pi_t-\pi_t^*) + \phi_y(y_t/y_{ss}-1), \quad \phi_\pi > 0,\; \phi_y \geq 0, \tag{24a}$$

or the partial adjustment model:

$$\mathrm{d}i_t = \theta(\phi_\pi(\pi_t - \pi_t^*) + \phi_y(y_t/y_{ss} - 1) - (i_t - i_t^*))\mathrm{d}t + \sigma_i \mathrm{d}B_{i,t}, \quad \theta > 0, \tag{24b}$$

which includes a response to inflation and output, and a desire to smooth interest rates. Similar to equation (3b), the rule in (24b) specifies a time lag between the inflation rate and the interest rate, and allows for an output response and monetary policy shocks.[14]

The coupon payments of the government treasury bills $T_t^b = -i_t a_t$ are financed through lump-sum taxes. Suppose transfers finance a given stream of government consumption expressed in terms of its constant share of output, $s_g s_{g,t}$, with a mean s_g and a stochastic component $s_{g,t}$ that follows another Ornstein-Uhlenbeck process[15]:

$$\mathrm{d}\log s_{g,t} = -\rho_g \log s_{g,t}\mathrm{d}t + \sigma_g \mathrm{d}B_{g,t}, \tag{25}$$

such that

$$g_t - T_t^b = s_g s_{g,t} y_t - T_t^b \equiv -T_t.$$

3.1.5. Aggregation

First, we derive an expression for aggregate demand:

$$y_t = c_t + g_t. \tag{26}$$

In other words, there is no possibility to transfer the output good intertemporally. With this value, the demand for each intermediate good producer is

$$y_{it} = (c_t + g_t)\left(\frac{p_{it}}{p_t}\right)^{-\varepsilon} \qquad \forall i. \tag{27}$$

Using the production function we may write:

$$A_t l_{it} = (c_t + g_t)\left(\frac{p_{it}}{p_t}\right)^{-\varepsilon}.$$

We integrate both sides:

$$A_t \int_0^1 l_{it}\mathrm{d}i = (c_t + g_t)\int_0^1 \left(\frac{p_{it}}{p_t}\right)^{-\varepsilon}\mathrm{d}i$$

[14] Given our previous discussion, we will mainly focus on the partial adjustment model. Nevertheless, for the ease of comparison with the literature, we highlight some of the results for the feedback model.

[15] While we could have $s_g s_{g,t} > 1$, our calibration of s_g and σ_g is such that this event will happen with a negligibly small probability. Alternatively we could specify a stochastic process with support (0,1).

to get an expression:

$$c_t + g_t = y_t = \frac{A_t}{v_t} l_t,$$

in which we define:

$$v_t = \int_0^1 \left(\frac{p_{it}}{p_t} \right)^{-\varepsilon} \mathrm{d}i$$

as the aggregate loss of efficiency induced by price dispersion of the intermediate goods. Similar to the price level, v_t is a predetermined variable (Calvo's insight):

$$v_t = \int_{-\infty}^{t} \delta e^{-\delta(t-\tau)} \left(\frac{p_{i\tau}}{p_t} \right)^{-\varepsilon} e^{-\varepsilon \int_\tau^t \chi \pi_s^* \, \mathrm{d}s} \mathrm{d}\tau. \tag{28}$$

Differentiating this expression with respect to time gives:

$$\mathrm{d}v_t \;\; = \;\; \left(\delta \left(\Pi_t^* \right)^{-\varepsilon} + \left(\varepsilon(\pi_t - \chi \pi_t^*) - \delta \right) v_t \right) \, \mathrm{d}t. \tag{29}$$

Finally, as shown in the appendix, in equilibrium aggregate profits can be written as a function of other variates:

$$\digamma_t = (1 - mc_t v_t) y_t. \tag{30}$$

3.2. The flexible-price case

An important benchmark solution to the NK model is the case where prices become more flexible. In the frictionless limit, $\delta \to \infty$, the firms set prices to maximize profits:

$$\max_{p_{it}} \left(\frac{p_{it}}{p_t} y_{it} - mc_t y_{it} \right) \quad \text{s.t. } y_{it} = \left(\frac{p_{it}}{p_t} \right)^{-\varepsilon} y_t,$$

After dropping constants, we may write the first-order condition as:

$$\frac{p_{it}}{p_t} = \frac{\varepsilon}{\varepsilon - 1} mc_t$$

Hence, in the flexible-price case, all firms set prices $p_{it} = p_t$. Given the analytical value of marginal costs, we compute optimal consumption and hours from (37):

$$c_t = (1 - s_g s_{g,t})^{\frac{\vartheta}{1+\vartheta}} A_t ((\varepsilon - 1)/(\varepsilon \psi))^{\frac{1}{1+\vartheta}} \tag{31}$$

such that $l_t = ((\varepsilon - 1)/((1 - s_g s_{g,t}) \varepsilon \psi))^{\frac{1}{1+\vartheta}}$.

3.3. Equilibria with price stickiness

For the case of $\delta < \infty$, we obtain the first-order conditions by defining the state space $U_z \subseteq \mathbb{R}^n$ and the control region $U_x \subseteq \mathbb{R}^m$, the *reward function* $f : U_z \times U_x \to \mathbb{R}$, the *drift function* $g : U_z \times U_x \to \mathbb{R}^n$, and the *diffusion function* $\sigma : U_z \to \mathbb{R}^{n \times n}$. Given our description of the problem, we define the household's value function:

$$V(\mathbb{Z}_0; \mathbb{Y}_0) \equiv \max_{\{\mathbb{X}_t\}_{t=0}^{\infty}} \mathbb{E}_0 \int_0^{\infty} e^{-\rho t} f(\mathbb{Z}_t, \mathbb{X}_t)\, \mathrm{d}t,$$

in which $\mathbb{Z}_t \in U_z$ denotes the n-vector of states, $\mathbb{X}_t \in U_x$ denotes the m-vector of controls, and $\mathbb{Y}_t = \mathbb{Y}(\mathbb{Z}_t)$ is a vector of variates to be determined in equilibrium as a function of the state variables, but taken as parametric by the representative household,

$$s.t. \quad \mathrm{d}\mathbb{Z}_t = g(\mathbb{Z}_t, \mathbb{X}_t; \mathbb{Y}_t)\, \mathrm{d}t + \sigma(\mathbb{Z}_t)\, \mathrm{d}B_t,$$

where B_t is a given vector of independent standard Brownian motions. The instantaneous covariance matrix of $\mathbb{Z}_t$ is $\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top$, which may be less than full rank n.

In particular, the vector of state variables is $\mathbb{Z}_t = (a_t, i_t, v_t, d_t, A_t, s_{g,t})^\top$ and equilibrium variables $\mathbb{Y}_t = (y_t, mc_t, w_t, \pi_t, x_{1,t}, x_{2,t}, \Pi_t^*, \lambda_t, T_t, Ⅎ_t)^\top$ to be determined endogenously, and $\mathbb{X}_t = (c_t, l_t)^\top$ is the vector of controls. In our case, the *reward function* reads:

$$f(\mathbb{Z}_t, \mathbb{X}_t) = d_t \log c_t - d_t \psi \frac{l_t^{1+\vartheta}}{1+\vartheta}.$$

From the discussion above, we define the *drift function* (with partial adjustment):

$$g(\mathbb{Z}_t, \mathbb{X}_t; \mathbb{Y}_t) = \begin{bmatrix} (i_t - \pi_t)a_t - c_t + w_t l_t + T_t + Ⅎ_t \\ \theta\phi_\pi(\pi_t - \pi_t^*) + \theta\phi_y(y_t/y_{ss} - 1) - \theta(i_t - i_t^*) \\ \delta\,(\Pi_t^*)^{-\varepsilon} + (\varepsilon(\pi_t - \chi\pi_t^*) - \delta)\, v_t \\ -(\rho_d \log d_t - \frac{1}{2}\sigma_d^2)d_t \\ -(\rho_A \log A_t - \frac{1}{2}\sigma_A^2)A_t \\ -(\rho_g \log s_{g,t} - \frac{1}{2}\sigma_g^2)s_{g,t} \end{bmatrix} \begin{matrix} (12) \\ (24b) \\ (29) \\ (11) \\ (15) \\ (25) \end{matrix},$$

and the *diffusion function* of the state transition equations:

$$\sigma(\mathbb{Z}_t) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma_i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_d d_t & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_A A_t & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_g s_{g,t} \end{bmatrix} \begin{matrix} (12) \\ (24b) \\ (29) \\ (11) \\ (15) \\ (25) \end{matrix}.$$

By choosing the control $\mathbb{X}_t \in \mathbb{R}_+^2$ at time t, the HJB equation reads:

$$\rho V(\mathbb{Z}_t; \mathbb{Y}_t) = \max_{\{\mathbb{X}_t\}_{t=0}^{\infty}} \left\{ f(\mathbb{Z}_t, \mathbb{X}_t) + g(\mathbb{Z}_t, \mathbb{X}_t; \mathbb{Y}_t)^\top V_{\mathbb{Z}} + \tfrac{1}{2}\mathrm{tr}\left(\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top V_{\mathbb{Z}\mathbb{Z}}\right) \right\}, \qquad (32)$$

where $V_{\mathbb{Z}}$ is an n-vector, $V_{\mathbb{Z}\mathbb{Z}}$ is a $n \times n$ matrix, and $\text{tr}(\cdot)$ denotes the trace of a matrix. A neat result about the formulation of our problem in continuous time is that the HJB equation (32) is, in effect, a deterministic functional equation. In the discrete-time version, we need to numerically approximate expectations (or the n-dimensional integral).

The first-order conditions with respect to c_t and l_t for any interior solution are:

$$\begin{aligned} \frac{d_t}{c_t} &= V_a, && (33) \\ d_t \psi l_t^{\vartheta} &= V_a w_t, && (34) \end{aligned}$$

or, eliminating the costate variable (for $\psi \neq 0$):

$$\psi l_t^{\vartheta} c_t = w_t,$$

which is the standard static optimality condition between labor and consumption.

Most notably, the first-order conditions (33) and (34) yield *optimal controls*:

$$\mathbb{X}_t = \mathbb{X}(\mathbb{Z}_t, V_{\mathbb{Z}}(\mathbb{Z}_t; \mathbb{Y}_t); \mathbb{Y}_t) \equiv \begin{bmatrix} c(\mathbb{Z}_t, V_{\mathbb{Z}}(\mathbb{Z}_t; \mathbb{Y}_t); \mathbb{Y}_t) \\ l(\mathbb{Z}_t, V_{\mathbb{Z}}(\mathbb{Z}_t; \mathbb{Y}_t); \mathbb{Y}_t) \end{bmatrix} = \begin{bmatrix} (V_a(\mathbb{Z}_t; \mathbb{Y}_t))^{-1} d_t \\ (V_a(\mathbb{Z}_t; \mathbb{Y}_t) w_t / (d_t \psi))^{1/\vartheta} \end{bmatrix}.$$

Thus, the first-order conditions (33) and (34) make the optimal controls functions of the states, $c_t = c(\mathbb{Z}_t; \mathbb{Y}_t)$, $l_t = l(\mathbb{Z}_t; \mathbb{Y}_t)$. Hence, the concentrated HJB equation reads:

$$\begin{aligned} \rho V(\mathbb{Z}_t; \mathbb{Y}_t) &= f(\mathbb{Z}_t, \mathbb{X}(\mathbb{Z}_t, V_{\mathbb{Z}}(\mathbb{Z}_t; \mathbb{Y}_t)) + g(\mathbb{Z}_t, \mathbb{X}(\mathbb{Z}_t, V_{\mathbb{Z}}(\mathbb{Z}_t; \mathbb{Y}_t)); \mathbb{Y})^{\top} V_{\mathbb{Z}} \\ &\quad + \tfrac{1}{2} \text{tr} \left(\sigma(\mathbb{Z}_t) \sigma(\mathbb{Z}_t)^{\top} V_{\mathbb{Z}\mathbb{Z}} \right). && (35) \end{aligned}$$

Note that $V_a(\mathbb{Z}_t; \mathbb{Y}_t) = \lambda_t$ in (33) and (34) is readily interpreted as the marginal value of wealth or the current value of a unit of consumption in period t, and thus determines the asset pricing kernel in this economy. In what follows, we provide the asset pricing kernel or the stochastic discount factor (SDF) consistent with equilibrium dynamics of macro aggregates, which can be used to price any asset in the economy.

Defining the recursive-competitive equilibrium (see Appendix C.3), it is instructive to revisit the pure mechanics of the NK model. We start from market clearing:

$$c_t = y_t - g_t = (1 - s_g s_{g,t}) y_t = (1 - s_g s_{g,t}) A_t l_t / v_t, \qquad (36)$$

such that the combined first-order condition reads:

$$w_t = \psi l_t^{\vartheta} c_t \;\Leftrightarrow\; v_t w_t = \psi l_t^{1+\vartheta} (1 - s_g s_{g,t}) A_t \;\Leftrightarrow\; l_t^{1+\vartheta} = \frac{v_t w_t}{(1 - s_g s_{g,t}) A_t \psi},$$

and from (34):

$$c_t = ((1 - s_g s_{g,t})/v_t)^{\frac{\vartheta}{1+\vartheta}} A_t (mc_t/\psi)^{\frac{1}{1+\vartheta}}, \tag{37}$$

or

$$mc_t = \psi l_t^{1+\vartheta}(1 - s_g s_{g,t})/v_t.$$

For a given level of marginal cost (or wages), the solution is known analytically. In contrast to the flexible-price benchmark, the firms now take into account current marginal cost *and* expected future marginal cost. Hence, the equilibrium value for marginal costs in the sticky-price solution is an unknown function of all states, $mc_t = mc(\mathbb{Z}_t)$.

As we show in the appendix, the marginal value of wealth evolves according to:

$$\begin{aligned} \mathrm{d}\lambda_t &= (\rho - i_t + \pi_t)\lambda_t \mathrm{d}t \\ &\quad +\sigma_d d_t \lambda_d \mathrm{d}B_{d,t} + \sigma_A A_t \lambda_A \mathrm{d}B_{A,t} + \sigma_g s_{g,t}\lambda_g \mathrm{d}B_{g,t} + \sigma_i \lambda_i \mathrm{d}B_{i,t}, \end{aligned} \tag{38}$$

which determines the equilibrium SDF (see Hansen and Scheinkman, 2009):

$$m_s/m_t = e^{-\rho(s-t)}\frac{V_a(\mathbb{Z}_s;\mathbb{Y}_s)}{V_a(\mathbb{Z}_t;\mathbb{Y}_t)}, \quad \text{and} \quad m_t \equiv e^{-\rho t}\lambda_t, \tag{39}$$

or, equivalently, the present value shadow price. Under the risk-neutral measure $\mathbb{Q}$ we may increase the drift of each price process by its covariance with the discount factor, and write a risk-neutral discount factor (see Cochrane, 2005, p.52):

$$\mathrm{d}\lambda_t^{\mathbb{Q}} = (\rho - i_t + \pi_t)\lambda_t^{\mathbb{Q}}\mathrm{d}t. \tag{40}$$

After some algebra (see Appendix C.2), we arrive at the Euler equation, which shows the equilibrium dynamics of consumption:

$$\begin{aligned} \mathrm{d}c_t &= -(\rho - i_t + \pi_t - \sigma_A^2\tilde{c}_A^2 - \sigma_g^2\tilde{c}_g^2 - \sigma_i^2\tilde{c}_i^2 + \rho_d \log d_t + (\tilde{c}_d(1-\tilde{c}_d) - \tfrac{1}{2})\sigma_d^2)c_t\mathrm{d}t \\ &\quad +\sigma_d\tilde{c}_d c_t \mathrm{d}B_{d,t} + \sigma_A\tilde{c}_A c_t\mathrm{d}B_{A,t} + \sigma_g\tilde{c}_g c_t \mathrm{d}B_{g,t} + \sigma_i\tilde{c}_i c_t \mathrm{d}B_{i,t}, \end{aligned} \tag{41}$$

where $\tilde{c}_i \equiv c_i/c_t$, $\tilde{c}_d \equiv c_d d_t/c_t$, $\tilde{c}_g \equiv c_g s_{g,t}/c_t$, and $\tilde{c}_A \equiv c_A A_t/c_t$, reflecting the slope of the consumption function with respect to the state variables that are driven by shocks.

3.3.1. Equilibrium dynamics

So we arrive at a system of 5 endogenous processes, i.e., for the auxiliary variables $x_{1,t}$, $x_{2,t}$, price dispersion v_t, the Taylor rule i_t, and the consumption Euler equation c_t, and 3 exogenous shock processes for $s_{g,t}, d_t, A_t$, which summarize equilibrium dynamics:

$$
\begin{aligned}
\mathrm{d}c_t &= -(\rho - i_t + \pi_t - \sigma_A^2 \tilde{c}_A^2 - \sigma_g^2 \tilde{c}_g^2 - \sigma_i^2 \tilde{c}_i^2 + \rho_d \log d_t + (\tilde{c}_d(1-\tilde{c}_d) - \tfrac{1}{2})\sigma_d^2) c_t \mathrm{d}t \\
&\quad + \sigma_d \tilde{c}_d c_t \mathrm{d}B_{d,t} + \sigma_A \tilde{c}_A c_t \mathrm{d}B_{A,t} + \sigma_g \tilde{c}_g c_t \mathrm{d}B_{g,t} + \sigma_i \tilde{c}_i c_t \mathrm{d}B_{i,t} \\
\mathrm{d}x_{1,t} &= ((\rho + \delta + (1-\varepsilon)(\pi_t - \chi \pi_t^*)) x_{1,t} - d_t/(1 - s_g s_{g,t}))\, \mathrm{d}t \\
\mathrm{d}x_{2,t} &= ((\rho + \delta - \varepsilon(\pi_t - \chi \pi_t^*)) x_{2,t} - mc_t d_t/(1 - s_g s_{g,t}))\, \mathrm{d}t \\
\mathrm{d}i_t &= \theta(\phi_\pi(\pi_t - \pi_t^*) + \phi_y(y_t/y_{ss} - 1) - (i_t - i_t^*))\mathrm{d}t + \sigma_i \mathrm{d}B_{i,t} \\
\mathrm{d}v_t &= (\delta(1 + (1-\varepsilon)(\pi_t - \chi\pi_t^*)/\delta)^{-\frac{\varepsilon}{1-\varepsilon}} + (\varepsilon(\pi_t - \chi \pi_t^*) - \delta)\, v_t)\, \mathrm{d}t
\end{aligned}
$$

in which $(1 + (1-\varepsilon)(\pi_t - \chi\pi_t^*)/\delta)^{\frac{1}{1-\varepsilon}} = \varepsilon/(\varepsilon - 1)(x_{2,t}/x_{1,t})$ determines inflation and

$$
\begin{aligned}
d_t/c_t &= ((1 - s_g s_{g,t})/v_t)^{-\frac{\vartheta}{1+\vartheta}} (mc_t/\psi)^{-\frac{1}{1+\vartheta}} d_t/A_t \\
\Leftrightarrow\ mc_t &= \psi((d_t/c_t)(A_t/d_t))^{-(1+\vartheta)} (v_t/(1 - s_g s_{g,t}))^{\vartheta},
\end{aligned} \tag{42}
$$

pins down marginal costs. Given a solution to the system of dynamic equations augmented by the stochastic processes (11), (15), and (25), the general equilibrium policy functions (as a function of relevant state variables) can be computed.

3.3.2. Numerical solution of the (conditional) deterministic system

In what follows we solve the NK model using the (conditional) deterministic system, which demands that we need to account appropriately for risk. This is obtained if the (nonlinear) solution to the HJB equation implies the same policy function of the boundary value problem. The solution of the deterministic model is contained as a special case.

We start from the HJB equation (35) or the detailed version (C.9) (cf. Appendix C.1), and find that for $V_{aa}(\mathbb{Z}_t; \mathbb{Y}_t) \neq 0$

$$
\begin{aligned}
c(\mathbb{Z}_t;\mathbb{Y}_t) &= (i_t - \pi_t)a_t + w_t l(\mathbb{Z}_t;\mathbb{Y}_t) + T_t + \Gamma_t - (\rho - (i_t - \pi_t))\frac{V_a(\mathbb{Z}_t;\mathbb{Y}_t)}{V_{aa}(\mathbb{Z}_t;\mathbb{Y}_t)} \\
&\quad + (\theta\phi_\pi(\pi_t - \pi_t^*) + \theta\phi_y(y_t/y_{ss} - 1) - \theta(i_t - i_t^*))\frac{V_{ia}(\mathbb{Z}_t;\mathbb{Y}_t)}{V_{aa}(\mathbb{Z}_t;\mathbb{Y}_t)} + \tfrac{1}{2}\sigma_i^2 \frac{V_{iia}(\mathbb{Z}_t;\mathbb{Y}_t)}{V_{aa}(\mathbb{Z}_t;\mathbb{Y}_t)} \\
&\quad + \left(\delta(1 + (1-\varepsilon)(\pi_t - \chi\pi_t^*)/\delta)^{-\frac{\varepsilon}{1-\varepsilon}} + (\varepsilon(\pi_t - \chi\pi_t^*) - \delta)v_t\right)\frac{V_{va}(\mathbb{Z}_t;\mathbb{Y}_t)}{V_{aa}(\mathbb{Z}_t;\mathbb{Y}_t)} \\
&\quad - (\rho_d \log d_t - \tfrac{1}{2}\sigma_d^2) d_t \frac{V_{da}(\mathbb{Z}_t;\mathbb{Y}_t)}{V_{aa}(\mathbb{Z}_t;\mathbb{Y}_t)} + \tfrac{1}{2}\sigma_d^2 d_t^2 \frac{V_{dda}(\mathbb{Z}_t;\mathbb{Y}_t)}{V_{aa}(\mathbb{Z}_t;\mathbb{Y}_t)} \\
&\quad - (\rho_A \log A_t - \tfrac{1}{2}\sigma_A^2) A_t \frac{V_{Aa}(\mathbb{Z}_t;\mathbb{Y}_t)}{V_{aa}(\mathbb{Z}_t;\mathbb{Y}_t)} + \tfrac{1}{2}\sigma_A^2 A_t^2 \frac{V_{AAa}(\mathbb{Z}_t;\mathbb{Y}_t)}{V_{aa}(\mathbb{Z}_t;\mathbb{Y}_t)} \\
&\quad - (\rho_g \log s_{g,t} - \tfrac{1}{2}\sigma_g^2) s_{g,t} \frac{V_{ga}(\mathbb{Z}_t;\mathbb{Y}_t)}{V_{aa}(\mathbb{Z}_t;\mathbb{Y}_t)} + \tfrac{1}{2}\sigma_g^2 s_{g,t}^2 \frac{V_{gga}(\mathbb{Z}_t;\mathbb{Y}_t)}{V_{aa}(\mathbb{Z}_t;\mathbb{Y}_t)},
\end{aligned} \tag{43}
$$

which we will use to define the Euler equation errors below.

In what follows, we compute the solution to the HJB equation from a deterministic system of differential equations (a boundary value problem), which works in continuous time since the HJB equation itself becomes a deterministic equation (cf. Chang, 2004).[16] Nevertheless, we have to account appropriately for risk. So the idea is to transform the system of SDEs into a system of PDEs, which also solves the HJB equation. Assume the existence of a consumption function $c_t = c(\mathbb{Z}_t)$, and use Itô's formula to arrive at:

$$\begin{aligned} \mathrm{d}c_t &= c_a\,\mathrm{d}a_t + c_i\,\mathrm{d}i_t + \tfrac{1}{2}c_{ii}\sigma_i^2\,\mathrm{d}t + c_v\,\mathrm{d}v_t \\ &\quad + c_d\,\mathrm{d}d_t + \tfrac{1}{2}c_{dd}(\sigma_d d_t)^2\mathrm{d}t + c_A\,\mathrm{d}A_t + \tfrac{1}{2}c_{AA}(\sigma_A A_t)^2\mathrm{d}t + c_g\,\mathrm{d}s_{g,t} + \tfrac{1}{2}c_{gg}(\sigma_g s_{g,t})^2\mathrm{d}t. \end{aligned}$$

This leads us to the following proposition.

Proposition 1. *By subtracting the Itô second-order terms from the Euler equation (41),*

$$\begin{gathered} dc_t - \tfrac{1}{2}c_{ii}\sigma_i^2\,dt - \tfrac{1}{2}c_{dd}(\sigma_d d_t)^2dt - \tfrac{1}{2}c_{AA}(\sigma_A A_t)^2dt - \tfrac{1}{2}c_{gg}(\sigma_g s_{g,t})^2dt = \\ c_a\,da_t + c_i\,di_t + c_v\,dv_t + c_A\,dA_t + c_d\,dd_t + c_g\,ds_{g,t}, \end{gathered}$$

and inserting dc_t from (41) we may eliminate time (and stochastic shocks) and together with $c_t = d_t V_a^{-1}$ yields (43) from the HJB equation.

Proof. Appendix C.4 ■

A system of PDEs which implies the same policy function is constructed using (41) and Proposition 1 by subtracting Itô terms from the Euler equation (accounting for risk) and setting $\mathrm{d}B_{d,t} = \mathrm{d}B_{A,t} = \mathrm{d}B_{g,t} = \mathrm{d}B_{i,t} = 0$ (in the absence of shocks),

$$\begin{aligned} \mathrm{d}c_t &= -(\rho - (i_t - \pi_t))c_t\mathrm{d}t + \tilde{c}_d^2\sigma_d^2 c_t\mathrm{d}t + \tilde{c}_A^2\sigma_A^2 c_t\mathrm{d}t + \tilde{c}_g^2\sigma_g^2 c_t\mathrm{d}t + \tilde{c}_i^2\sigma_i^2 c_t\mathrm{d}t \\ &\quad - \tfrac{1}{2}\tilde{c}_{dd}\sigma_d^2 c_t\mathrm{d}t - \tfrac{1}{2}\tilde{c}_{AA}\sigma_A^2 c_t\mathrm{d}t - \tfrac{1}{2}\tilde{c}_{gg}\sigma_g^2 c_t\mathrm{d}t - \tfrac{1}{2}\tilde{c}_{ii}\sigma_i^2 c_t\,\mathrm{d}t \\ &\quad - c_t\rho_d \log d_t\mathrm{d}t + \tfrac{1}{2}\sigma_d^2 c_t\mathrm{d}t - \tilde{c}_d\sigma_d^2 c_t\mathrm{d}t \end{aligned}$$

where we define $\tilde{c}_{ii} \equiv c_{ii}/c_t$, $\tilde{c}_{dd} \equiv c_{dd}d_t^2/c_t$, $\tilde{c}_{gg} \equiv c_{gg}s_{g,t}^2/c_t$, and $\tilde{c}_{AA} \equiv c_{AA}A_t^2/c_t$ reflecting curvature of the consumption function with respect to the state variables that are driven by shocks, such that $\mathrm{d}c_t = c_a\,\mathrm{d}a_t + c_i\,\mathrm{d}i_t + c_v\,\mathrm{d}v_t + c_A\,\mathrm{d}A_t + c_d\,\mathrm{d}d_t + c_g\,\mathrm{d}s_{g,t}$ solves (43).

[16] In contrast, the discrete-time HJB equation requires the analyst needs to evaluate the state space not only at the current information set, but also at future expected values, so the continuous-time approach does not require to numerically compute expectations (a burdensome step in discrete-time models).

So we refer to the following system of PDEs as the *conditional* deterministic system:

$$\begin{aligned}
\mathrm{d}c_t &= -(\rho-(i_t-\pi_t))c_t\mathrm{d}t + \tilde{c}_d^2\sigma_d^2 c_t\mathrm{d}t + \tilde{c}_A^2\sigma_A^2 c_t\mathrm{d}t + \tilde{c}_g^2\sigma_g^2 c_t\mathrm{d}t + \tilde{c}_i^2\sigma_i^2 c_t\mathrm{d}t \\
&\quad -\tfrac{1}{2}\tilde{c}_{dd}\sigma_d^2 c_t\mathrm{d}t - \tfrac{1}{2}\tilde{c}_{AA}\sigma_A^2 c_t\mathrm{d}t - \tfrac{1}{2}\tilde{c}_{gg}\sigma_g^2 c_t\mathrm{d}t - \tfrac{1}{2}\tilde{c}_{ii}\sigma_i^2 c_t\,\mathrm{d}t \\
&\quad -(\rho_d\log d_t - \tfrac{1}{2}\sigma_d^2)c_t\mathrm{d}t - \tilde{c}_d\sigma_d^2 c_t\mathrm{d}t \qquad (44) \\
\mathrm{d}i_t &= \theta(\phi_\pi(\pi_t-\pi_t^*) + \phi_y(y_t/y_{ss}-1) - (i_t-i_t^*))\,\mathrm{d}t \\
\mathrm{d}v_t &= (\delta\,(1+(1-\varepsilon)(\pi_t-\chi\pi_t^*)/\delta)^{-\frac{\varepsilon}{1-\varepsilon}} + (\varepsilon(\pi_t-\chi\pi_t^*)-\delta)v_t)\,\mathrm{d}t \\
\mathrm{d}d_t &= -\left(\rho_d\log d_t - \tfrac{1}{2}\sigma_d^2\right)d_t\mathrm{d}t \\
\mathrm{d}A_t &= -\left(\rho_A\log A_t - \tfrac{1}{2}\sigma_A^2\right)A_t\mathrm{d}t \\
\mathrm{d}s_{g,t} &= -\left(\rho_g\log s_{g,t} - \tfrac{1}{2}\sigma_g^2\right)s_{g,t}\mathrm{d}t \\
\mathrm{d}x_{1,t} &= ((\rho+\delta-(\varepsilon-1)(\pi_t-\chi\pi_t^*))x_{1,t} - d_t/(1-s_g s_{g,t}))\,\mathrm{d}t \\
\mathrm{d}x_{2,t} &= ((\rho+\delta-\varepsilon(\pi_t-\chi\pi_t^*))\,x_{2,t} - mc_t d_t/(1-s_g s_{g,t}))\,\mathrm{d}t
\end{aligned}$$

So the Euler equation (44) of the conditional deterministic system is used to obtain the conditional deterministic (or stochastic) steady state.[17] Recall that the inflation rate π_t is endogenously determined from (22), and the jump variables $x_{1,t}$ and $x_{2,t}$. We restrict our attention to the solution which leads the economy towards the (stochastic) steady state, in which $\pi_t \to \pi_t^*$. By solving for the time paths, the solution satisfies both the initial and the transversality condition (TVC) and characterizes the stable manifold. We iterate computing controls and updating the derivatives until convergence (cf. Table 1).[18]

Intuitively, the conditional deterministic system summarizes the dynamics under the presence of uncertainty, that agents internalize into their consumption-saving decision, conditional on no further shocks (conditional on the current information set). So agents would not change their (optimal) decision as long as they remain on the stable manifold summarized by the dynamic system, which is idle at the (stochastic) steady state value. Hence, the impulse response functions based on the conditional deterministic system summarize the paths of the stable manifold as implied by the HJB equation.

It is important to note that as long as $\|(\sigma_d,\sigma_A,\sigma_g,\sigma_i)\| \neq 0$, the term $\mathrm{d}c_t$ of the conditional deterministic system (44) does *not* coincide with the term $\mathrm{d}c_t$ of the Euler equation (41), which is an abuse of notation only needed for the numerical solution. Once we derived the policy functions, the original Euler equation is used to simulate the model and/or to make statistical inference, by allowing for the arrival of stochastic shocks.

We solve the system of PDEs by the Waveform Relaxation algorithm. In this way, we can separate the solution in the time dimension from the solution in the policy space, which

[17] Though there will be a steady-state distribution, we follow the convention in the literature and define the fix point of this system as the 'stochastic steady state', and thus use both terms interchangeably.

[18] It is important to note that a recursion as set out in Table 1 is only required if we are interested in the solution of the stochastic model where $\|(\sigma_d,\sigma_A,\sigma_g,\sigma_i)\| \neq 0$.

turns out to be computationally more robust and less expensive.[19] Following the idea in Posch and Trimborn (2013) we obtain the unknown derivatives starting from the solution of the deterministic system, then iteratively define $\tilde{c}_i(\mathbb{Z}_t;\mathbb{Y}_t) \equiv c_i/c_t$, $\tilde{c}_{ii}(\mathbb{Z}_t;\mathbb{Y}_t) \equiv c_{ii}/c_t$, $\tilde{c}_d(\mathbb{Z}_t;\mathbb{Y}_t) \equiv c_d d_t/c_t$, $\tilde{c}_{dd}(\mathbb{Z}_t;\mathbb{Y}_t) \equiv c_{dd} d_t^2/c_t$, $\tilde{c}_g(\mathbb{Z}_t;\mathbb{Y}_t) \equiv c_g s_{g,t}/c_t$, $\tilde{c}_{gg}(\mathbb{Z}_t;\mathbb{Y}_t) \equiv c_{gg} s_{g,t}^2/c_t$, $\tilde{c}_A(\mathbb{Z}_t;\mathbb{Y}_t) \equiv c_A A_t/c_t$, and $\tilde{c}_{AA}(\mathbb{Z}_t;\mathbb{Y}_t) \equiv c_{AA} A_t^2/c_t$, and solve the system of ODEs. The initial value for the control and/or jump variables is used to approximate the solution in the policy function space (using tensor products of univariate grids as initial values), then update the solution, and iterate until convergence.

In the boundary value problem (BVP) we seek a function $x : [0, T] \mapsto \mathbb{R}^k$ that satisfies the (conditional) deterministic system consisting of the Euler equation (44) determining c_t, and the law of motion for $x_{1,t}, x_{2,t}, i_t, v_t, d_t, A_t$, and $s_{g,t}$ (which gives $k = 8$), together with the given initial conditions for the states $(i_0, v_0, d_0, A_0, s_{g,0})$ and the TVC assuming that variables approach their (stochastic) steady state values. One complication is that the time horizon is infinite, so we use the following transformation of time:

$$t = \frac{\tau}{\nu(1-\tau)} \quad \text{for} \quad \tau \in [0, 1),$$

where ν is a positive (nuisance) parameter, such that for $t \to \infty$ we have that $\tau \to 1$. Alternatively, we may set T sufficiently large but finite number.[20]

3.3.3. Numerical solution in the policy function space

In what follows, we show how we may alternatively solve the HJB equation (35) directly by collocation based on the Matlab `CompEcon` toolbox (Miranda and Fackler 2002).

Since the functional form of the solution is unknown, an alternative strategy for solving the HJB equation is to approximate $V(\mathbb{Z}_t;\mathbb{Y}_t) \approx \phi(\mathbb{Z}_t;\mathbb{Y}_t)v$, in which v is an n-vector of coefficients and ϕ is the $n \times n$ basis matrix. The computational burden can be reduced when replacing the tensor product by sparse grids (Winschel and Krätzig, 2010). Starting from the HJB equation (35), we may approximation of the value function and/or control variables for given set of collocation nodes and basis functions $\phi(\mathbb{Z}_t;\mathbb{Y}_t)$:

$$\rho\phi(\mathbb{Z}_t;\mathbb{Y}_t)v = f(\mathbb{Z}_t,\mathbb{X}_t) + g(\mathbb{Z}_t,\mathbb{X}_t)^\top \phi_{\mathbb{Z}}(\mathbb{Z}_t;\mathbb{Y}_t)v + \tfrac{1}{2}\text{tr}\left(\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top \phi_{\mathbb{Z}\mathbb{Z}}(\mathbb{Z}_t;\mathbb{Y}_t)v\right),$$

or

$$v = \left(\rho\phi(\mathbb{Z}_t;\mathbb{Y}_t) - g(\mathbb{Z}_t,\mathbb{X}_t)^\top \phi_{\mathbb{Z}} - \tfrac{1}{2}\text{tr}\left(\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top \phi_{\mathbb{Z}\mathbb{Z}}\right)\right)^{-1} f(\mathbb{Z}_t,\mathbb{X}_t),$$

which yields the coefficients based on a Newton method. This approach, however, requires

[19] It is possible to parallelize the computation by allocating the grid of state variables to workers.

[20] Trimborn, Koch, and Steger (2008) introduced the relaxation algorithm to applications in economics. In contrast to their approach, we use projection methods to solve the boundary value problem, which turns out to be relatively efficient and (even for a few approximation points) highly accurate.

a good initial guess, but is extremely useful to verify whether the implied solution obtained from the conditional deterministic system indeed solves the HJB equation.

3.3.4. Impulse responses

To compute the impulse response functions (IRFs), we initialize the state variables, given the solution $V(\mathbb{Z}_t;\mathbb{Y}(\mathbb{Z}_t)) \approx \phi(\mathbb{Z}_t;\mathbb{Y}(\mathbb{Z}_t)v$, or the consumption function (43), and solve the resulting system of ODEs following Posch and Trimborn (2013). Because we use a global (and nonlinear) solution technique, in principle, we may initialize the system at any state vector. Hence, we do not need to restrict our analysis to situations, where the economy is assumed to be in the close neighborhood of the steady state (or normal times). This is particularly important since we want to study the equilibrium dynamics in a situation where the nominal interest rate is close to zero and/or the economy is hit by large shocks (non-normal times). In fact, the computed IRF is the equilibrium time path of economic variables, which reflect a single transitional path to the (stochastic) steady state.

3.4. Effects of uncertainty and the natural rate of interest

Our results confirm that the effects of uncertainty in the standard NK model are small, and primarily a level shift (cf. Appendix D.2, Figures D.8 to D.12). Both fixed points of deterministic steady state and stochastic steady state are very close.

After having obtained the consumption function (either of the two approaches above) and its derivatives in the nonlinear model, we can proceed with Itô's formula and obtain the Euler equation as:

$$\begin{aligned} \mathrm{d}c_t \;=\;& \theta(\phi_\pi(\pi_t-\pi_t^*)+\phi_y(y_t/y_{ss}-1)-(i_t-i_t^*))\tilde{c}_i c_t \mathrm{d}t+\sigma_i\tilde{c}_i c_t \mathrm{d}B_{i,t} \\ &+(\delta(1+(1-\varepsilon)(\pi_t-\chi\pi_t^*)/\delta)^{-\frac{\varepsilon}{1-\varepsilon}}+(\varepsilon(\pi_t-\chi\pi_t^*)-\delta)\,v_t)c_v\,\mathrm{d}t \\ &-\left(\rho_d\log d_t-\tfrac{1}{2}\sigma_d^2\right)\tilde{c}_d c_t\mathrm{d}t+\sigma_d\tilde{c}_d c_t \mathrm{d}B_{d,t} \\ &-\left(\rho_A\log A_t-\tfrac{1}{2}\sigma_A^2\right)\tilde{c}_A c_t\mathrm{d}t+\sigma_A\tilde{c}_A c_t \mathrm{d}B_{A,t} \\ &-\left(\rho_g\log s_{g,t}-\tfrac{1}{2}\sigma_g^2\right)\tilde{c}_g c_t\mathrm{d}t+\sigma_g\tilde{c}_g c_t \mathrm{d}B_{g,t} \\ &+\tfrac{1}{2}\tilde{c}_{ii}\sigma_i^2 c_t\,\mathrm{d}t+\tfrac{1}{2}\tilde{c}_{dd}\sigma_d^2 c_t\mathrm{d}t+\tfrac{1}{2}\tilde{c}_{AA}\sigma_A^2 c_t\mathrm{d}t+\tfrac{1}{2}\tilde{c}_{gg}\sigma_g^2 c_t\mathrm{d}t \end{aligned} \tag{45}$$

where we used that in general equilibrium $\mathrm{d}a_t=0$. Using equation (45) together with the Euler equation (41), we are ready to pin down the equilibrium inflation rate as:

$$\begin{aligned}\pi_t &= i_t-\rho+\sigma_A^2\tilde{c}_A^2+\sigma_g^2\tilde{c}_g^2+\sigma_i^2\tilde{c}_i^2+\tilde{c}_d^2\sigma_d^2-\tilde{c}_d\sigma_d^2-(\rho_d\log d_t-\tfrac{1}{2}\sigma_d^2)\\ &\quad -\theta(\phi_\pi(\pi_t-\pi_t^*)+\phi_y(y_t/y_{ss}-1)-(i_t-i_t^*))\tilde{c}_i\\ &\quad -(\delta(1+(1-\varepsilon)(\pi_t-\chi\pi_t^*)/\delta)^{-\frac{\varepsilon}{1-\varepsilon}}+(\varepsilon(\pi_t-\chi\pi_t^*)-\delta)\,v_t)c_v/c_t\\ &\quad +\left(\rho_d\log d_t-\tfrac{1}{2}\sigma_d^2\right)\tilde{c}_d+\left(\rho_A\log A_t-\tfrac{1}{2}\sigma_A^2\right)\tilde{c}_A+\left(\rho_g\log s_{g,t}-\tfrac{1}{2}\sigma_g^2\right)\tilde{c}_g\\ &\quad -\tfrac{1}{2}\tilde{c}_{ii}\sigma_i^2-\tfrac{1}{2}\tilde{c}_{dd}\sigma_d^2-\tfrac{1}{2}\tilde{c}_{AA}\sigma_A^2-\tfrac{1}{2}\tilde{c}_{gg}\sigma_g^2, \end{aligned}\tag{46}$$

to study the effects of uncertainty on the consumption-saving decision and interest rates, compute impulse response functions, and the implied term structure of interest rates.

In general equilibrium, the risk-free rate is defined as (cf. Posch, 2011):

$$\rho-\frac{1}{\mathrm{d}t}\mathbb{E}\left[\frac{\mathrm{d}u'(c_t)}{u'(c_t)}\right]=i_t-\pi_t\equiv r_t^f.$$

Observe that the *implicit* risk premium in the economy is zero, the (instantaneous) return to the government bond is riskless. Even though the zero-coupon bond is default-free, in the general case it is still risky in the sense that its price can covary with the households marginal utility of consumption (cf. Rudebusch and Swanson, 2008, p.115).

In the short-run, only the nominal interest rate is under the control of the monetary authority. Our numerical results show how the equilibrium *real* interest rate r_t^f is affected by the different state variables (cf. Appendix D.2), which together with (46) reads:

$$\begin{aligned}r_t^f &= \rho-\left(\sigma_A^2\tilde{c}_A^2+\sigma_g^2\tilde{c}_g^2+\sigma_i^2\tilde{c}_i^2+\tilde{c}_d^2\sigma_d^2-\tilde{c}_d\sigma_d^2-(\rho_d\log d_t-\tfrac{1}{2}\sigma_d^2)\right)\\ &\quad +\theta(\phi_\pi(\pi_t-\pi_t^*)+\phi_y(y_t/y_{ss}-1)-(i_t-i_t^*))\tilde{c}_i\\ &\quad +(\delta(1+(1-\varepsilon)(\pi_t-\chi\pi_t^*)/\delta)^{-\frac{\varepsilon}{1-\varepsilon}}+(\varepsilon(\pi_t-\chi\pi_t^*)-\delta)\,v_t)c_v/c_t\\ &\quad -\left(\rho_d\log d_t-\tfrac{1}{2}\sigma_d^2\right)\tilde{c}_d-\left(\rho_A\log A_t-\tfrac{1}{2}\sigma_A^2\right)\tilde{c}_A-\left(\rho_g\log s_{g,t}-\tfrac{1}{2}\sigma_g^2\right)\tilde{c}_g\\ &\quad +\tfrac{1}{2}\tilde{c}_{ii}\sigma_i^2+\tfrac{1}{2}\tilde{c}_{dd}\sigma_d^2+\tfrac{1}{2}\tilde{c}_{AA}\sigma_A^2+\tfrac{1}{2}\tilde{c}_{gg}\sigma_g^2. \end{aligned}\tag{47}$$

Following Barsky, Justiniano, and Melosi (2014), we define the natural rate as the real interest rate prevailing in an economy with flexible prices, or the second-best equilibrium (cf. Blanchard and Galí, 2007), which is:

$$\begin{aligned}r_t &= \rho-(\sigma_A^2+\tilde{c}_g^2\sigma_g^2)+\tfrac{1}{2}\tilde{c}_{gg}\sigma_g^2\\ &\quad +\rho_d\log d_t-\tfrac{1}{2}\sigma_d^2-\left(\rho_A\log A_t-\tfrac{1}{2}\sigma_A^2\right)-\left(\rho_g\log s_{g,t}-\tfrac{1}{2}\sigma_g^2\right)\tilde{c}_g\end{aligned}\tag{48}$$

where from (31) we obtain $\tilde{c}_g=-\frac{\vartheta}{1+\vartheta}\frac{s_g}{1-s_g s_{g,t}}s_{g,t}$, and $\tilde{c}_{gg}=-\frac{\vartheta}{(1+\vartheta)^2}\frac{s_g^2}{(1-s_g s_{g,t})^2}s_{g,t}^2$.Defining the (Wicksellian) natural rate as the (stochastic) steady-state interest rate once transitory shocks have abated, $r_t^*=\rho-(\sigma_A^2+\tilde{c}_g^2\sigma_g^2)+\frac{1}{2}\tilde{c}_{gg}\sigma_g^2$, we shed light on potential sources of shocks to the natural rate (temporary or permanent). An increase in uncertainty of either technology or fiscal policy shocks depresses the natural rate. These results are in accordance and complementary to Barsky, Justiniano, and Melosi (2014), who find that increases in patience, i.e., declines in ρ (often referred to as discount factor shocks), lower the natural rate r_t^* and are isomorphic to higher uncertainty about future productivity. Indeed, the uncertainty shocks provide an attractive structural interpretation.

Our analysis clearly shows that the spread between the equilibrium risk-free rate and the interest rate under certainty are affected by staggered price setting. For example, while $\tilde{c}_A = 1$ and $\tilde{c}_{AA} = 0$ in the frictionless limit, the staggered price equilibrium has $\tilde{c}_A \approx 0.57$ and $\tilde{c}_{AA} = 0.28$ (at stochastic steady-state). So while uncertainty about future technology depresses the *natural rate* relative to the interest rate under certainty by about 4 bp, uncertainty does increase the (long-run) real interest rate in the NK model by about 3 bp (for the parameterization see Table 2). Here, the negative effect on the equilibrium risk-free rate in the flexible-price scenario compares to the effects on r in the endowment economy (cf. Posch, 2011, Corollary 2.1).

The definition of the natural rate is consistent with the economy operating at its full potential (natural output), i.e., the level of output that would have prevailed in an economy without price rigidities:

$$y_t^n = (1 - s_g s_{g,t})^{-\frac{1}{1+\vartheta}} A_t((\varepsilon - 1)/(\varepsilon\psi))^{\frac{1}{1+\vartheta}}. \tag{49}$$

Based on the definition, the output gap is readily available from $x_t \equiv y_t/y_t^n - 1$.

3.5. Term structure of interest rates

Following Rudebusch and Swanson (2012), the term premium on long-term nominal bonds compensates investors for inflation and consumption risk over the lifetime of the bond. The term premium can be defined by comparing the equilibrium price under the physical and the risk-neutral probability measure. Consider a (zero-coupon) bond with unity payoff at maturity N. Using the expectation approach, the equilibrium price reads:

$$P_t^{(N)} = \mathbb{E}_t\left(m_{t+N}/m_t e^{-\int_t^{t+N}\pi_s ds}\right), \tag{50}$$

which from (38) and (39) can be solved, by simulating for a given maturity N:

$$\begin{aligned} P_t^{(N)} &= \mathbb{E}_t\Big(e^{-\int_t^{t+N}\left(r_s^f+\pi_s+\frac{1}{2}\sigma_d^2(1-\tilde{c}_d)^2+\frac{1}{2}\sigma_A^2\tilde{c}_A^2+\frac{1}{2}\sigma_g^2\tilde{c}_g^2+\frac{1}{2}\tilde{c}_i^2\sigma_i^2\right)\mathrm{d}s} \\ &\quad \times e^{\int_t^{t+N}\sigma_d(1-\tilde{c}_d)\mathrm{d}B_{d,s}-\sigma_A\tilde{c}_A\mathrm{d}B_{A,s}-\sigma_g\tilde{c}_g/\lambda_s\mathrm{d}B_{g,s}-\tilde{c}_i\sigma_i\mathrm{d}B_{i,s}}\Big) \end{aligned}$$

forward a few thousand times, and take the average. Similarly, we obtain the hypothetical bond price under the risk-neutral probability measure , i.e., using a risk-neutral discount factor (or equivalently the risk-free rate) rather than the household's SDF:

$$\begin{aligned}\tilde{P}_t^{(N)} &= \mathbb{E}_t^{\mathbb{Q}}\left(m_{t+N}/m_t e^{-\int_t^{t+N}\pi_s ds}\right) \\ &= \mathbb{E}_t^{\mathbb{Q}}\left(e^{-\int_t^{t+N} i_s\,\mathrm{d}s}\right) = \mathbb{E}_t^{\mathbb{Q}}\left(e^{-\int_t^{t+N}(r_s^f+\pi_s)\,\mathrm{d}s}\right),\end{aligned} \tag{51}$$

where $\mathbb{E}_t^{\mathbb{Q}}$ denotes the expectation under the risk-neutral probability measure and thus the risk-neutral evaluation of the bond price (Rudebusch and Swanson, 2008, 2012). Then, the yield is the (fictional) interest rate that justifies the quoted price, such that the log price $p_t^{(N)} \equiv \log P_t^{(N)}$ satisfies $y_t^{(N)} = -(1/N)p_t^{(N)}$, and the yield curve is a plot of the yields as a function of their maturity. Hence, the expectation approach is easily adapted for the perfect-foresight solution (the term premium would be zero). The term premium measures the 'riskiness' of long-term bonds, and reflects the compensation that investors require for bearing the risk that r_t^f does not evolve as expected.

The expectation approach is quite common in macroeconomic models of the term structure (Gürkaynak, Sack, and Swanson, 2005; Rudebusch and Swanson, 2008, 2012; Andreasen, Fernández-Villaverde, and Rubio-Ramírez, 2018). One drawback is that the computation of the whole term structure is quite challenging. It requires the simulation in the time dimension to study the resulting N-periods ahead distribution of $P_t^{(N)}$, for a given *particular* state. Naturally, it puts a limit to study the long-end of the yield curve, which therefore cannot easily be explored.

An alternative offers the PDE approach (Cochrane, 2005, chap. 19.4), in which the basic pricing equation for the price $P_t^{(N)}$ reads:

$$\mathbb{E}_t\left((\,\mathrm{d}P_t^{(N)})/P_t^{(N)}\right) - \left(1/P_t^{(N)}(\partial P_t^{(N)}/\partial N) + i_t\right)\,\mathrm{d}t = -\mathbb{E}_t\left((\,\mathrm{d}P_t^{(N)}/P_t^{(N)})\,(\,\mathrm{d}\lambda_t/\lambda_t)\right),$$

or for the price under the risk-neutral probability measure $\mathbb{Q}$

$$\mathbb{E}_t^{\mathbb{Q}}\left((\,\mathrm{d}\tilde{P}_t^{(N)})/\tilde{P}_t^{(N)}\right) - \left(1/\tilde{P}_t^{(N)}(\partial\tilde{P}_t^{(N)}/\partial N) + i_t\right)\,\mathrm{d}t = 0.$$

Observe that in equilibrium, the prices $P_t^{(N)}$ and $\tilde{P}_t^{(N)}$ are functions of the state variables, so we may write $P_t^{(N)} = P^{(N)}(\mathbb{Z}_t)$, and obtain the PDE:

$$\begin{aligned}
&\theta(\phi_\pi(\pi_t - \pi_t^*) + \phi_y(y_t/y_{ss} - 1) - (i_t - i_t^*))(\partial P_t^{(N)}/\partial i_t) + \tfrac{1}{2}\sigma_i^2(\partial^2 P_t^{(N)}/(\partial i_t)^2) \\
&+(\delta(1-(\varepsilon-1)(\pi_t - \chi\pi_t^*)/\delta)^{-\frac{\varepsilon}{1-\varepsilon}} + (\varepsilon(\pi_t - \chi\pi_t^*) - \delta)v_t)(\partial P_t^{(N)}/\partial v_t) \\
&- \left(\rho_d \log d_t - \tfrac{1}{2}\sigma_d^2\right) d_t(\partial P_t^{(N)}/\partial d_t) + \tfrac{1}{2}\sigma_d^2 d_t^2(\partial^2 P_t^{(N)}/(\partial d_t)^2) \\
&- \left(\rho_A \log A_t - \tfrac{1}{2}\sigma_A^2\right) A_t(\partial P_t^{(N)}/\partial A_t) + \tfrac{1}{2}\sigma_A^2 A_t^2(\partial^2 P_t^{(N)}/(\partial A_t)^2) \\
&- \left(\rho_g \log s_{g,t} - \tfrac{1}{2}\sigma_g^2\right) s_{g,t}(\partial P_t^{(N)}/\partial s_{g,t}) + \tfrac{1}{2}\sigma_g^2 s_{g,t}^2(\partial^2 P_t^{(N)}/(\partial s_{g,t})^2) \\
&-(\partial P_t^{(N)}/\partial N) - i_t P_t^{(N)} = -\sigma_d^2 d_t(1-\tilde{c}_d)(\partial P_t^{(N)}/\partial d_t) \\
&+\sigma_i^2 \tilde{c}_i(\partial P_t^{(N)}/\partial i_t) + \sigma_A^2 A_t \tilde{c}_A(\partial P_t^{(N)}/\partial A_t) + \sigma_g^2 s_{g,t}\tilde{c}_g(\partial P_t^{(N)}/\partial s_{g,t}). \qquad (52)
\end{aligned}$$

For the hypothetical price $\tilde{P}_t^{(N)}$, the covariance terms on the RHS are zero.

The solution to the pricing equation implies the complete term structure of interest rate for *any* given state vector and maturity,

$$y_t^{(N)} \equiv y^{(N)}(\mathbb{Z}_t) = -\log P^{(N)}(\mathbb{Z}_t)/N \qquad (53)$$

and the term premium is

$$TP_t^{(N)} \;\; \equiv \;\; y^{(N)}(\mathbb{Z}_t) - \tilde{y}^{(N)}(\mathbb{Z}_t) \qquad (54)$$

Our basic strategy for solving the PDE to approximate the function $P_t^{(N)} \approx \Phi(N, \mathbb{Z}_t)v$ in which v is the vector of coefficients and Φ denotes the basis matrix can be easily extended for the stochastic model, approximating the price function (alternatively we may use finite differences as in Achdou, Han, Lasry, Lions, and Moll, 2017).

For illustration, Figure 10 the term structure of interest rates for the nominal bond and the inflation-protected bond following a negative preference shock of about 10 percent. The difference between the physical and the risk-neutral probability measure defines the term premium, which is about 6 basis points (10 basis points with feedback rule).

3.6. ZIRP period revisited

Let us now consider the monetary policy shocks together with a preference shock. The new insights we get are really due to the nonlinearities. We simulate the monetary policy shock of 475 bp (and target shock of 200 bp) together with a 'preference shock' of about 10 percent, which is assumed to follow the logistic process (cf. Appendix A.2):

$$\mathrm{d}d_t = \rho_d(d_t - \bar{d})\,(1-d_t)\,/(1-\bar{d})\,\mathrm{d}t, \quad d_t > \bar{d}, \qquad (55)$$

with $\bar{d} = 0.9130$ and $\rho_d = 0.975$. It implies that the initial value $d_0 = 0.9220$ is 1 percent above the lower bound.[21] In other words, this shock is 'large' it will have completely different dynamics than small shocks. This particular parameterization has been chosen to show that the implied interest rate process (of the full nonlinear approach) now is a prolonged period of an *apparently* binding ZLB (cf. Figures 8 and 9). Our thought experiment implies a ZIRP period of 5 quarters, but is consistent with $\phi > 1$.

Note that the linearized model around non-zero inflation targets shows quite similar dynamics as the simple NK model, but strikingly fails to capture the nonlinear effects of the logistic process. This result is quite intuitive because the dynamics of the linear model (9) are the same as (55) only for small shocks.

Why researches did not go beyond the OU process (9) so far? A potential explanation is that in 'normal times' with smaller shocks, the local dynamics of the linear approximation could have been quite successful. In normal times, when the lower bound was out of reach, the (unobserved) shocks might have been well described by the simple OU process. In fact, the local dynamics of the logistic process are observationally equivalent to the dynamics of the OU process (cf. Appendix A.2). But the implications for the model dynamics can be different in non-normal times. Moreover, we show that such shocks must be large in order to drag the interest rate close to (potentially below) zero values. The traditional local approximation schemes would be inappropriate for large shocks. We confirm Brunnermeier and Sannikov (2014) that nonlinearities can be important in times of crises.

So distinguishing between normal and non-normal times, in which the dynamics are different from those at the steady state, is one alternative interpretation in which a *single* shock to the natural rate generates the observed pattern in the interest rates. We conclude that (55) is a parsimonious specification where the dynamics of large negative shocks (non-normal times) are different from the dynamics of a small shock (normal times).

3.7. Discussion of the new insights

The full (nonlinear) approach and the local dynamics around positive trend inflation give rise to at least three insights. First, uncertainty about shocks has an effect on the natural rate and is one potential structural explanation for permanent shocks. The effects of risk are negligible in the standard NK model (cf. Parra-Alvarez, Polattimur, and Posch, 2018). Second, the PDE approach is a promising alternative to compute the term structure of interest rates consistent with equilibrium dynamics of macro aggregates. We find that the nominal and real yield curves provide useful information about future expectations and for the identification of shocks, but term premia are also negligible.[22] Third, apart from the

[21] Note that with the assumed logistic process for the preference shock the Euler equation (41) changes for the nonlinear model (cf. Section C.2).

[22] Introducing recursive preferences produces a large and variable term premium without compromising the model's ability to fit key macroeconomic variables (cf. Rudebusch and Swanson, 2012).

alternative specification for the preference shock dynamics, the parsimonious NK model specification does not inherit important nonlinearities (compare Figures D.5 and D.6). Although a preference shock of the same order of magnitude as during the ZIRP period is assumed before, as long as the initial value is sufficiently above its 'natural' lower bound $\bar{d} = 0$, the nonlinear dynamics would be irrelevant. Nevertheless, a nonlinear approach consistently may generate a ZIRP period with a single shock as a policy choice.

4. Conclusion

In this paper we show the ability of the NK model to explain the recent episodes, restricting ourselves to the regions of (local) determinacy. We find that temporary and permanent shocks to the interest rate (and inflation) are required to replicate the dynamics of key macroeconomic aggregates consistently with the term structure of interest rates. We show that the NK model with active monetary policy supports both views, either higher interest rates result into higher long-run inflation (neo-Fisherian view), or higher interest can temporarily reduce inflation (traditional view). One potential interpretation of the nature of shocks is that monetary policy actions (changes in short-term rates) may trigger variations in long-term target rates, a view that is motivated by empirical data. Allowing for temporary (and permanent) shocks to the natural rate allows us to understand several puzzles in the literature, including apparent term structure anomalies. We also show that in a nonlinear approach a single shock can generate a ZIRP period with stable and quiet inflation, fully consistent with the model predictions. This paper is the first to provide a full analytical investigation of the effects of uncertainty and sheds light on how it affects the natural rate. Hence, our results confirm that uncertainty shocks are isomorphic to discount factor shocks (Barsky, Justiniano, and Melosi, 2014), so they provide an attractive structural interpretation of permanent shocks to the natural rate.

We believe that this paper is a starting point for several lines of research. First, our benchmark specification is useful for the comparison with a medium-scale model, allowing for other nominal and/or real frictions, habit formation, variable capacity utilization and adjustment cost as in models used by central banks for policy analysis, and/or by including a financial sector (e.g., Brunnermeier and Sannikov, 2014). Second, we should estimate the structural parameters using empirical data. Though we think that our identification scheme of shocks provides a good benchmark, eventually the data should also be used to pin down the structural parameters. One key advantage of the continuous-time approach is that the model solution is consistent with different frequencies of macro and financial data (cf. Christensen, Posch, and van der Wel, 2016). Here, two promising alternatives are either to apply the continuous-time econometric toolbox from the financial literature to our macroeconomic (or macro-finance) models, or to combine the discrete-time estimation approaches with an Euler discretization scheme of the equilibrium dynamics. Third, we may study the monetary policy transmission in a heterogeneous-agent economy, e.g., with idiosyncratic income shocks (cf. Kaplan, Moll, and Violante, 2018).

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Contents

A. Appendix

A.1. New Keynesian analysis

This section sheds some light on the implications of the NK model for both the IS-curve and the NK Phillips-curve. For illustration, we abstract from the effects of uncertainty by setting variance terms to zero to compare the solution of the nonlinear model with the solution of the linear approximation used in the literature (cf. Figures D.5 and D.6).

We start with the NK forward-looking Phillips-curve, which from (23), the first-order condition $\lambda_t = d_t/c_t$, and the market-clearing condition $c_t = (1 - s_g s_{g,t})y_t$ reads:

$$\mathrm{d}(\pi_t - \chi\pi_t^*) = -(\delta - (\varepsilon - 1)(\pi_t - \chi\pi_t^*))\,(\pi_t - \chi\pi_t^* + (mc_t/x_{2,t} - 1/x_{1,t})\,d_t/(1 - s_g s_{g,t}))\,\mathrm{d}t,$$

which together with $mc_t = \psi l_t^{1+\vartheta}(1 - s_g s_{g,t})/v_t$ and $l_t = y_t v_t/A_t$, among other variables, shows the response of inflation to the output gap.[23] Hence, the linearized Phillips-curve around deterministic steady-state values reads (see Section C.5 for definitions)

$$\begin{aligned}
\mathrm{d}(\pi_t - \chi\pi_t^*) &= -a_2(\pi_t - \chi\pi_t^* - (1-\chi)\pi_{ss})\mathrm{d}t \\
&\quad -a_2\,(\rho + \delta - \varepsilon(1-\chi)\pi_{ss})\,(mc_t/mc_{ss} - 1)\mathrm{d}t \\
&\quad -a_2\,(\rho + \delta + (1-\varepsilon)(1-\chi)\pi_{ss})\,(x_{1,t}/x_{1,ss} - 1)\mathrm{d}t \\
&\quad +a_2\,(\rho + \delta - \varepsilon(1-\chi)\pi_{ss})\,(x_{2,t}/x_{2,ss} - 1)\mathrm{d}t \\
&\quad +a_2(1-\chi)\pi_{ss}(d_t/d_{ss} - 1)\mathrm{d}t \\
&\quad +a_2(1-\chi)\pi_{ss}\,(s_g s_{g,ss}/(1 - s_g s_{g,ss}))\,(s_{g,t}/s_{g,ss} - 1)\mathrm{d}t
\end{aligned}$$

in which $a_2 \equiv \delta + (1-\varepsilon)(1-\chi)\pi_{ss}$ and

$$\pi_t - \chi\pi_t^* - (1-\chi)\pi_{ss} = a_2(x_{2,t}/x_{2,ss} - x_{1,t}/x_{1,ss})$$

It shows in the NK Phillips-curve how the change in inflation depends on marginal costs. We may insert the linearized equation for marginal cost,

$$\begin{aligned}
mc_t/mc_{ss} - 1 &= (1+\vartheta)(y_t/y_{ss} - 1) - (1+\vartheta)(A_t/A_{ss} - 1) \\
&\quad +\vartheta(v_t/v_{ss} - 1) - s_g s_{g,ss}/(1 - s_g s_{g,ss})(s_{g,t}/s_{g,ss} - 1) \\
&= (1+\vartheta)(c_t/c_{ss} - 1) - (1+\vartheta)(A_t/A_{ss} - 1) \\
&\quad +\vartheta(v_t/v_{ss} - 1) + \vartheta(s_g s_{g,ss}/(1 - s_g s_{g,ss}))(s_{g,t}/s_{g,ss} - 1)
\end{aligned}$$

where

$$y_t/y_{ss} = c_t/c_{ss} + (s_g s_{g,ss}/(1 - s_g s_{g,ss}))(s_{g,t}/s_{g,ss} - 1)$$

[23] In order to analyze local dynamics, the traditional approach is to (log-)linearize the variables. We define $\hat{x}_t \equiv (x_t - x_{ss})/x_{ss}$, where x_{ss} is the steady-state value for the variable x_t, such that $x_t = (1+\hat{x}_t)x_{ss}$

to obtain the NK Phillips-curve with respect to output and/or consumption. Moreover,

$$\begin{aligned} \mathrm{d}(v_t/v_{ss}-1) &= \frac{\varepsilon(1-\chi)\pi_{ss}}{\delta+(1-\varepsilon)(1-\chi)\pi_{ss}}(\pi_t-\chi\pi_t^*-(1-\chi)\pi_{ss})\mathrm{d}t \\ &\quad +(\varepsilon(1-\chi)\pi_{ss}-\delta)(v_t/v_{ss}-1)\mathrm{d}t \end{aligned}$$

From (41), the linearized Euler equation reads:

$$\begin{aligned} \mathrm{d}(c_t/c_{ss}-1) &= (i_t-i_t^*-(\pi_t-\pi_t^*)-\rho_d(d_t/d_{ss}-1))\mathrm{d}t \\ &= (i_t-\rho-\pi_t-\rho_d(d_t/d_{ss}-1))\mathrm{d}t \end{aligned}$$

which is readily interpreted as the (micro-founded) NK IS-curve. For comparison with the literature, in the case without technology and government expenditure shocks, this is Werning's (2012) continuous-time specification:

$$\mathrm{d}(c_t/c_{ss}-1) = (i_t-r_t-\pi_t)\mathrm{d}t$$

and the natural rate reads $r_t \equiv \rho+\rho_d(d_t/d_{ss}-1))$.

To summarize, the equilibrium dynamics of the linearized system can be simplified in the version with full price indexation ($\chi=1$) to:

$$\begin{aligned} \mathrm{d}(c_t/c_{ss}-1) &= (i_t-\rho-\pi_t-\rho_d(d_t/d_{ss}-1))\mathrm{d}t \\ \mathrm{d}\pi_t &= (\rho(\pi_t-\pi_t^*)-\delta\,(\rho+\delta)\,(mc_t/mc_{ss}-1)+\delta(d_t/d_{ss}-1))\mathrm{d}t \\ \mathrm{d}i_t &= (\theta\phi_\pi(\pi_t-\pi_t^*)+\theta\phi_y(y_t/y_{ss}-1)-\theta(i_t-i_t^*))\mathrm{d}t \\ \mathrm{d}(v_t/v_{ss}-1) &= -\delta(v_t/v_{ss}-1)\mathrm{d}t \end{aligned}$$

After transitional dynamics and by assuming $s_g=0$, this system coincides with the model in Werning (2012) and Cochrane (2017b) :

$$\begin{aligned} \mathrm{d}x_t &= (i_t-r_t-\pi_t)\mathrm{d}t \\ \mathrm{d}\pi_t &= (\rho(\pi_t-\pi_t^*)-\delta\,(\rho+\delta)\,(1+\vartheta)x_t)\,\mathrm{d}t \\ \mathrm{d}i_t &= (\theta\phi_\pi(\pi_t-\pi_t^*)+\theta\phi_y x_t-\theta(i_t-i_t^*))\mathrm{d}t \end{aligned}$$

where $x_t \equiv c_t/c_{ss}-1$ defines the output gap. Solving forward finally yields:

$$\pi_t-\pi_t^* = \int_t^\infty e^{-\rho(s-t)}\delta(\rho+\delta)(1+\vartheta)x_s\mathrm{d}s \qquad \text{(A.1)}$$

A.1.1. Determinacy

While the simple NK model with a feedback rule introduces the interest rate as a control variable, the partial adjustment model makes the interest rate a state variable, which is given by past inflation. For the ease of presentation, we set $r_t = r_t^* = \rho$ in this section.

This simple NK model with a *feedback rule* has no relevant state variables. The system can be analyzed in terms of two equations (1) and (2) using (3a). A unique locally bounded solution requires two positive eigenvalues of the Jacobian matrix:[24]

$$A_1 = \begin{bmatrix} 0 & \phi - 1 \\ -\kappa & \rho \end{bmatrix}.$$

Hence, a necessary (and sufficient) condition for local determinacy is $\phi > 1$. So the unique locally bounded solution is $x_t = 0$ and $\pi_t = \pi_t^*$ such that $i_t = \rho + \pi_t^*$. In other words, a negative (short-run) response of inflation to raising interest rates is not possible as long as the monetary authority implements the Taylor principle. Any monetary policy shock, which affects the policy targets, would be permanent and operates instantaneously. The response of inflation is unambiguously *positive*. In this perfect-foresight model, interest rates can be expressed in terms of future output gaps. We would also need to include a serially correlated shock in order to generate transitional dynamics in the model.

In the simple NK model with *partial adjustment*, the only relevant state variable is the interest rate (historically given inflation rates). We thus obtain the equilibrium values for the output gap and the inflation rate as policy functions $x_t = x(i_t)$ and $\pi_t = \pi(i_t)$. The system can be analyzed in terms of three equations (1), (2) and (3b), where a unique locally bounded solution requires two positive eigenvalues of the Jacobian matrix[25]

$$A_2 = \begin{bmatrix} 0 & -1 & 1 \\ -\kappa & \rho & 0 \\ 0 & \phi\theta & -\theta \end{bmatrix}.$$

Again, a necessary (and sufficient) condition for local determinacy is $\phi > 1$. One caveat is that the model is linearized around zero inflation targets (or full indexation). It can be shown that the condition $\phi > 1$ remains necessary (for details see web appendix).

[24] The Jacobian matrix has $\mathrm{tr}(A_1) = \lambda_1 + \lambda_2 = \rho > 0$ and $\det(A_1) = (\phi - 1)\kappa$ is positive for $\phi > 1$, thus both eigenvalues have positive real parts, $\lambda_1\lambda_2 = \det(A_1)$, such that $\lambda_{1,2} = \frac{1}{2}(\rho \pm \sqrt{\rho^2 - 4((\phi - 1)\kappa)})$.

[25] Note that $\det(A_2) = -\kappa\theta(\phi - 1)$ which is negative for $\phi > 1$. Further, we know that $\lambda_1 + \lambda_2 + \lambda_3 = \mathrm{tr}(A_2) = \rho - \theta$ and $\lambda_1\lambda_2\lambda_3 = \det(A_2) = -\kappa\theta(\phi - 1)$. Because a unique locally bounded solution requires two positive eigenvalues, $\phi > 1$ is necessary (and sufficient) to obtain determinacy in this model.

A.2. Alternative shock dynamics

Consider an alternative specification of a logistic growth process:

$$\mathrm{d}d_t = \rho_d d_t \left(1 - d_t\right) \mathrm{d}t \tag{A.2}$$

which is the logistic growth model with carrying capacity 1. The natural (lower) bound is zero and the turning point is 0.5. If the variable is near its carrying capacity, the dynamics are just like those of the process in (11), whereas if the variable is near its lower bound, the dynamics are similar to exponential growth. An extended version, such that it fits our needs to have a prolonged period of persistence of a shock at the beginning and later to revert back to the steady state level geometrically at rate ρ_d such that the higher ρ_d the lower persistence, the smaller ρ_d the more pronounced shocks are smeared out in time.

Now consider

$$\mathrm{d}d_t = \rho_d(d_t - \bar{d}) \left(1 - d_t\right) / (1 - \bar{d}) \,\mathrm{d}t \tag{A.3}$$

of which the solution is

$$d_t = \frac{d_{ss} - \bar{d}}{1 + \mathbb{C}e^{-\rho_d t}} + \bar{d}$$

The (unique) steady state value is the solution of

$$0 = \rho_d \left(d_t - \bar{d}\right) \left(1 - \left(d_t - \bar{d}\right) / (d_{ss} - \bar{d})\right) \,\mathrm{d}t$$

where we require that $d_t > \bar{d}$ for all time. Linearizing about d_{ss} yields

$$\mathrm{d}d_t / d_{ss} = -\rho_d (d_t / d_{ss} - 1) \,\mathrm{d}t$$

or

$$\mathrm{d}\hat{d}_t = -\rho_d \hat{d}_t \,\mathrm{d}t$$

It reflects that the logistic growth model for $d_t - \bar{d}$ such that d_t approaches d_{ss}. The variable $d_t - \bar{d}$ is defined on 0 and ∞ with carrying capacity $d_{ss} - \bar{d}$ and turning point at $(d_{ss} - \bar{d})/2$, such that the original variable d_t is defined between $\bar{d}$ and ∞ with turning point at $1 - (d_{ss} - \bar{d})/2$. For $\bar{d} = 0$ we assume logistic growth for d_t, whereas $\bar{d} \to d_{ss}$ squeezes the admissible region lower than the steady state level towards zero, such that $\bar{d}$ denotes the lower bound for d_t. Any (negative) shock larger than $1 - (d_{ss} - \bar{d})/2$ induces completely different dynamics, staying there for some time before returning to the steady state level (cf. Figure D.42). This effect only shows up in the nonlinear version of the model. While the logistic model looks very much like an exponential model in the beginning, around the steady state value, the linearized dynamics are the same as for the Ornstein-Uhlenbeck process. Hence, the linear model (9) would *not* capture those dynamics.

B. Tables and Figures

B.1. Tables

Table 1: Summary of the solution algorithm

Step 1	(Initialization)	Provide an initial guess for the unknown derivatives for a given set of collocation nodes and basis functions.
Step 2	(Solution)	Compute the optimal value of the controls for the set of nodal values for the state variables.
Step 3	(Update)	Update the consumption function derivatives.
Step 4	(Iteration)	Repeat Steps 2 and 3 until convergence.

Table 2: Parameterization

ϑ	1	Frisch labor supply elasticity
ρ	0.03	subjective rate of time preference, $\rho = -4 \log 0.9925$
ψ	1	preference for leisure
δ	0.65	Calvo parameter for probability of firms receiving signal, $\delta = -4 \log 0.85$
ε	25	elasticity of substitution intermediate goods
s_g	0	share of government consumption
ρ_d	0.4214	autoregressive component preference shock, $\rho_d = -4 \log 0.9$
ρ_A	0.4214	autoregressive component technology shock, $\rho_A = -4 \log 0.9$
ρ_g	0.4214	autoregressive component government shock, $\rho_g = -4 \log 0.9$
σ_d	0.02	variance preference shock
σ_A	0.02	variance technology shock
σ_g	0	variance government shock
σ_i	0.02	variance monetary policy shock
ϕ_π	4	inflation response Taylor rule
ϕ_y	0	output response Taylor rule
θ	0.5	interest rate response Taylor rule
π_{ss}	0.02	inflation target rate
χ	1	indexation at steady-state inflation rate, $\chi = 1$ is full indexation

B.2. Figures

Figure 1: US federal funds rate, 10-year treasury rate and inflation rate
In this figure we show time series plots of the US Effective Federal Funds Rate (Fed Funds), the 10-Year Treasury Constant Maturity Rate (10Y Govt), the Consumer Price Index (Core CPI), seasonally adjusted, the 10-Year Treasury Inflation Protected Securities Rate (10Y TIPS), at the monthly frequency, and the Output gap (HP Filter) at the quarterly frequency. All series are obtained from the Federal Reserve Bank of St. Louis Economic Dataset (FRED). The sample runs from January, 1990, through June, 2017.

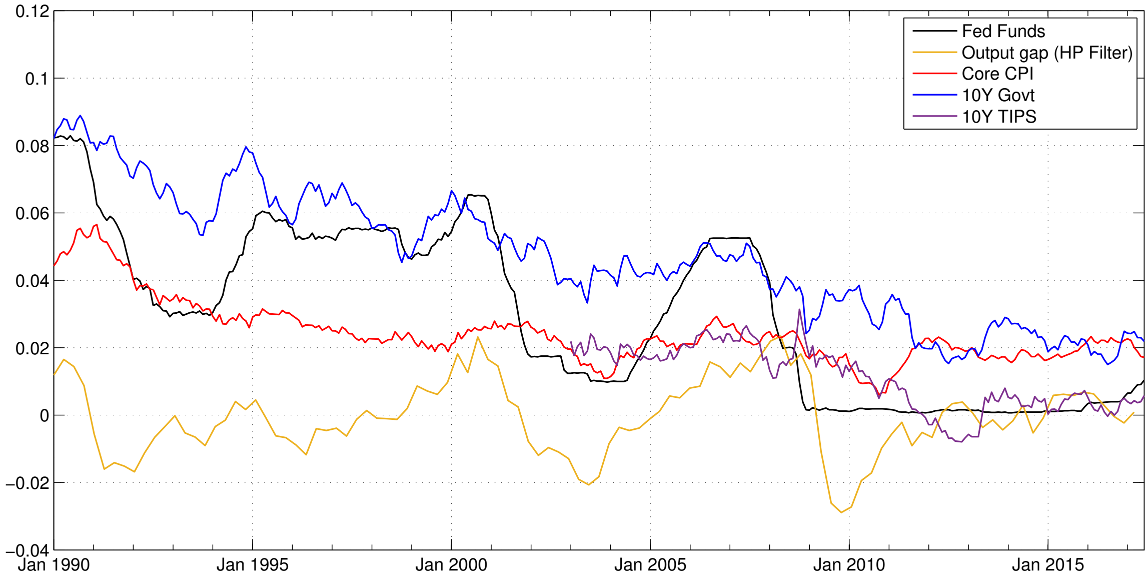

Figure 2: Implied natural rate
In this figure we show time series plots of the model-implied 'natural rate' using the simple NK model, allowing for temporary shocks to the natural rate, when matching the observed US Effective Federal Funds Rate (Fed Funds) and the Consumer Price Index (Core CPI), seasonally adjusted, at the monthly frequency, and both series together with the Output gap (HP Filter) at the quarterly frequency. The sample runs from January, 1990, through June, 2017.

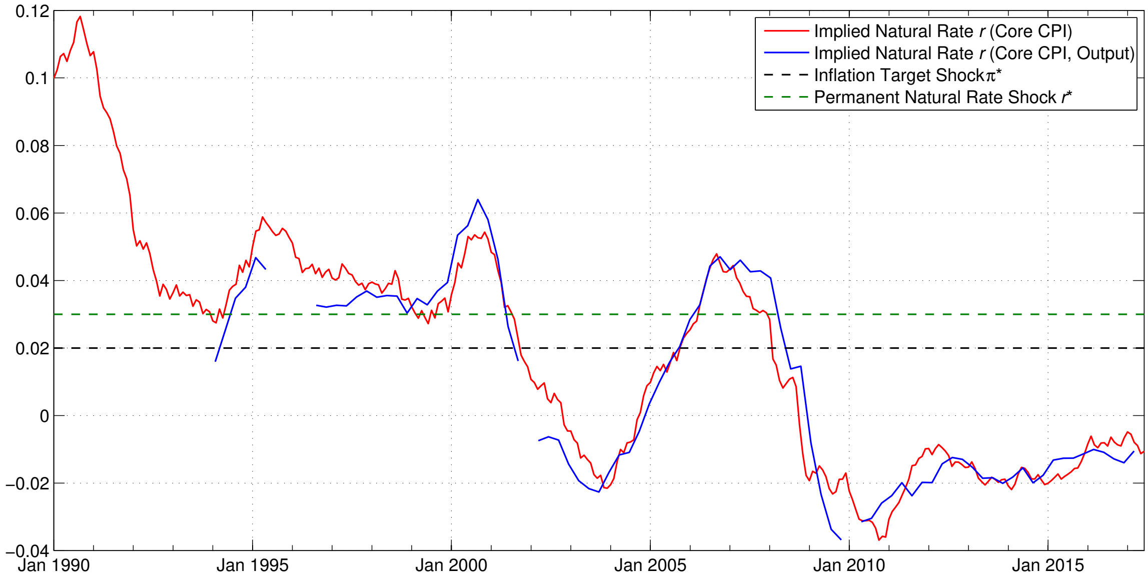

Figure 3: Implied inflation rates and 10-year treasury rates
In this figure we show time series plots of the model-implied inflation and the 10-year treasury rates using the simple NK model, allowing for temporary shocks to the natural rate, when matching the observed US Effective Federal Funds Rate (Fed Funds) and the Consumer Price Index (Core CPI), seasonally adjusted, at the monthly frequency. The sample runs from January, 1990, through June, 2017.

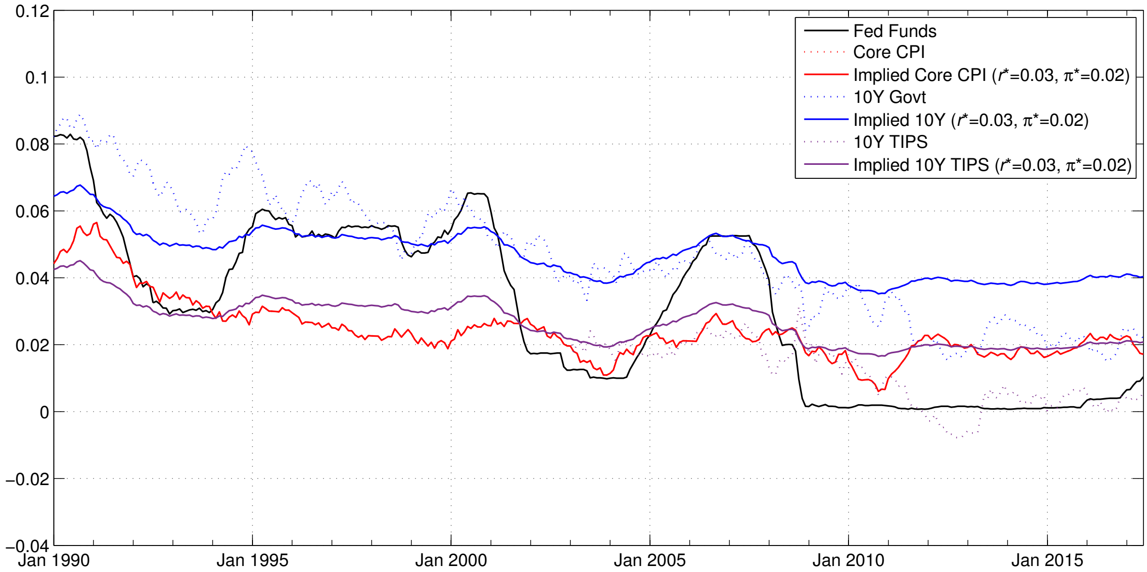

Figure 4: Implied inflation rates, 10-year treasury rates and output gap
In this figure we show time series plots of the model-implied inflation, 10-year treasury rates, and the output gap using the simple NK model, allowing for temporary shocks to the natural rate, when matching the observed US Effective Federal Funds Rate (Fed Funds) and the Consumer Price Index (Core CPI), seasonally adjusted, and the Output gap (HP Filter) at the quarterly frequency (1990Q1-2017Q2).

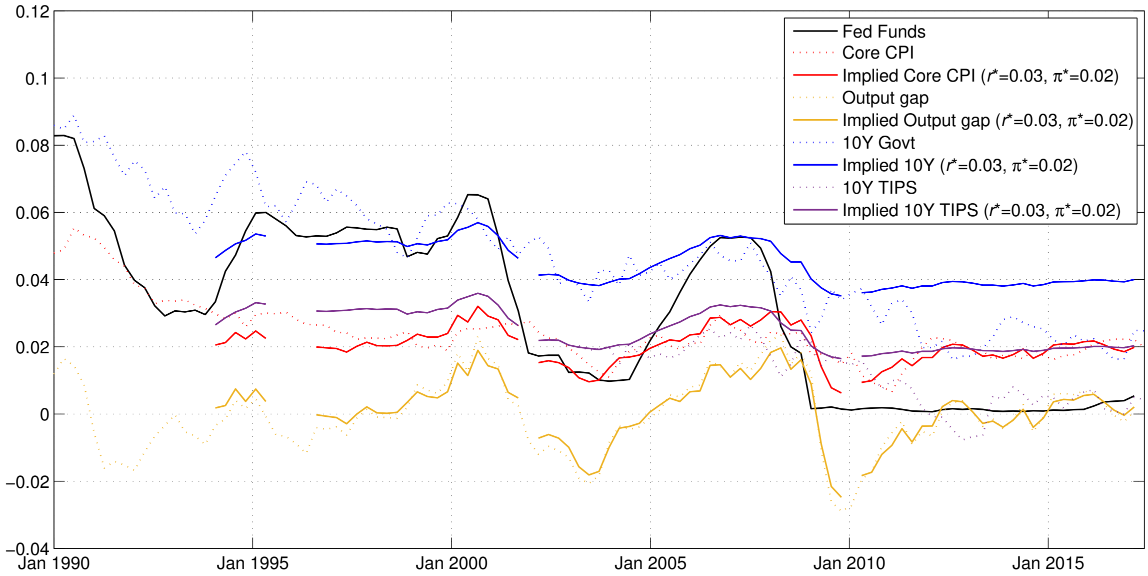

Figure 5: Implied natural rate

In this figure we show time series plots of the model-implied 'natural rate' using the simple NK model, allowing for temporary and permanent shocks to the natural rate and the inflation target, when matching the observed US Effective Federal Funds Rate (Fed Funds), the 10-Year Treasury Constant Maturity Rate (10Y Govt), the 10-Year Treasury Inflation Protected Securities Rate (10Y TIPS), and the Consumer Price Index (Core CPI), seasonally adjusted, at the monthly frequency, and together with the Output gap (HP Filter) at the quarterly frequency. The sample runs from January, 1990, through June, 2017.

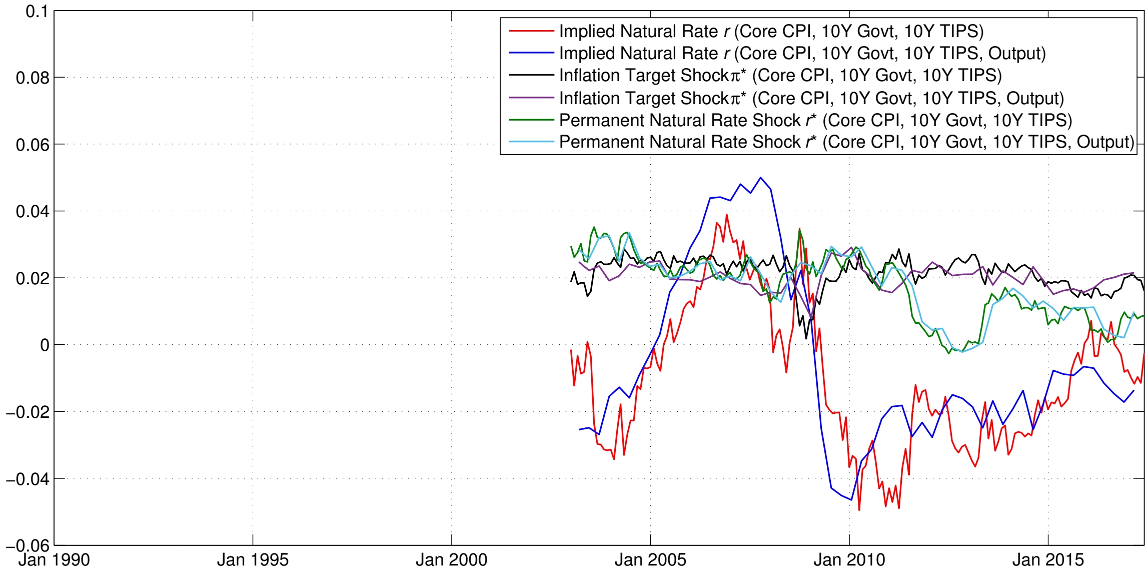

Figure 6: Implied inflation rates and 10-year treasury rates

In this figure we show time series plots of the model-implied inflation and the 10-year treasury rates using the simple NK model, allowing for temporary and permanent shocks to the natural rate and the inflation target, when matching the observed US Effective Federal Funds Rate (Fed Funds), the 10-Year Treasury Constant Maturity Rate (10Y Govt), the 10-Year Treasury Inflation Protected Securities Rate (10Y TIPS), and the Consumer Price Index (Core CPI), seasonally adjusted, at the monthly frequency. The sample runs from January, 1990, through June, 2017.

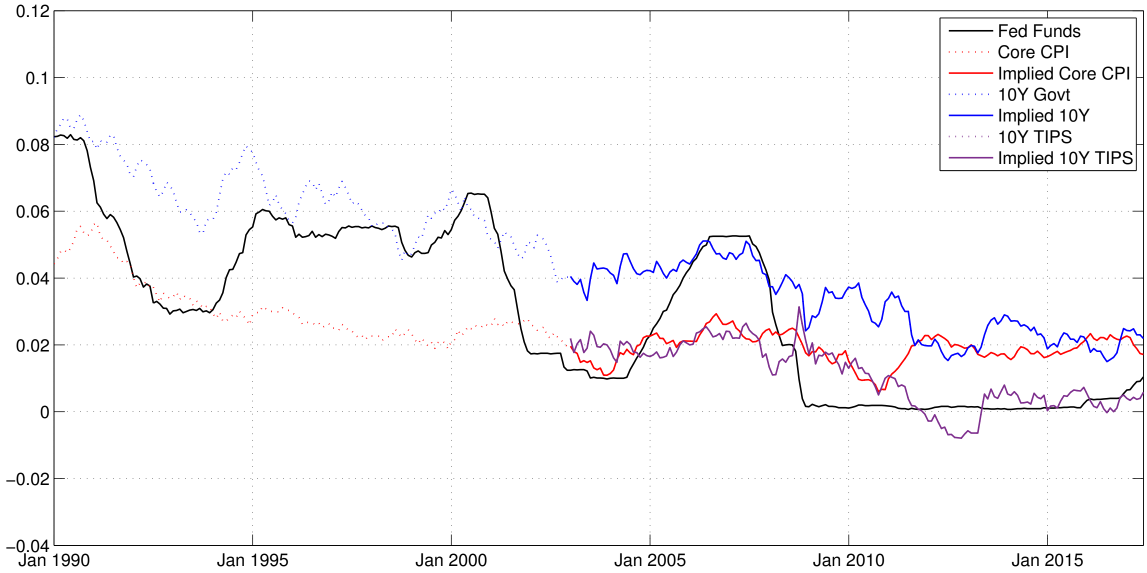

Figure 7: Implied inflation rates, 10-year treasury rates and output gap

In this figure we show time series plots of the model-implied inflation, 10-year treasury rates, and the output gap using the simple NK model, allowing for temporary and permanent shocks to the natural rate and the inflation target, when matching the observed US Effective Federal Funds Rate (Fed Funds), the 10-Year Treasury Constant Maturity Rate (10Y Govt), the 10-Year Treasury Inflation Protected Securities Rate (10Y TIPS), and the Consumer Price Index (Core CPI), seasonally adjusted, and the Output gap (HP Filter) at the quarterly frequency (1990Q1-2017Q2).

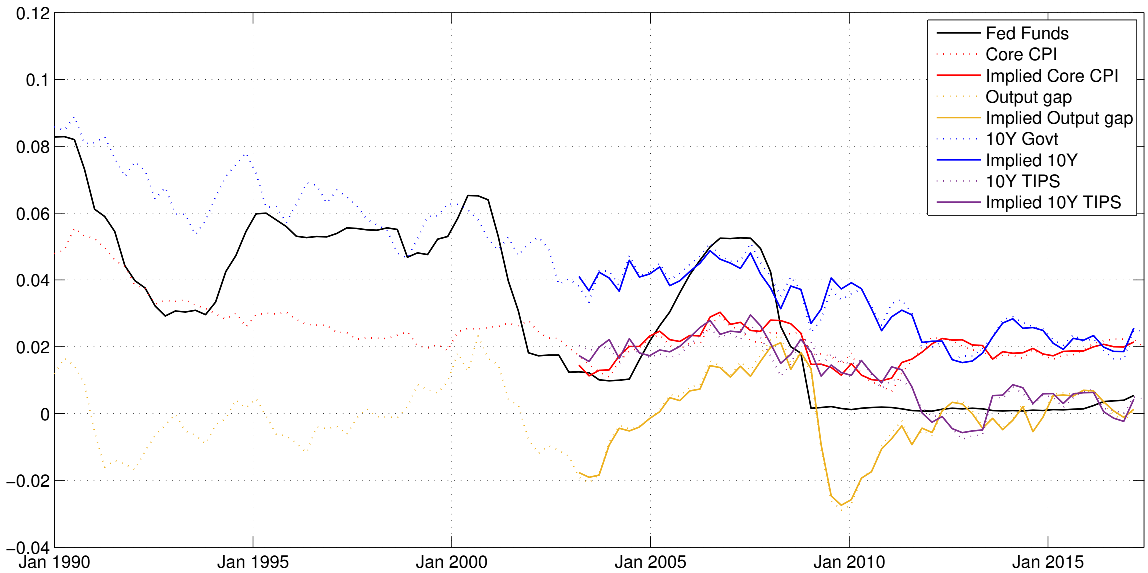

Figure 8: Simulated shock to interest rate and target rate (2007-2011)

In this figure we show (from left to right, top to bottom) the simulated responses for unexpected shocks to the (initial) interest rate (-0.0475), the inflation target rate (-0.02), and preferences (-0.1), with effects for the output gap, the inflation rate, and the level and the slope of the interest rate (blue solid), the no-target rate shock scenario (black dashed, $\pi_{ss} = 0.02, \chi = 0$), and the pre-shock scenario (dotted).

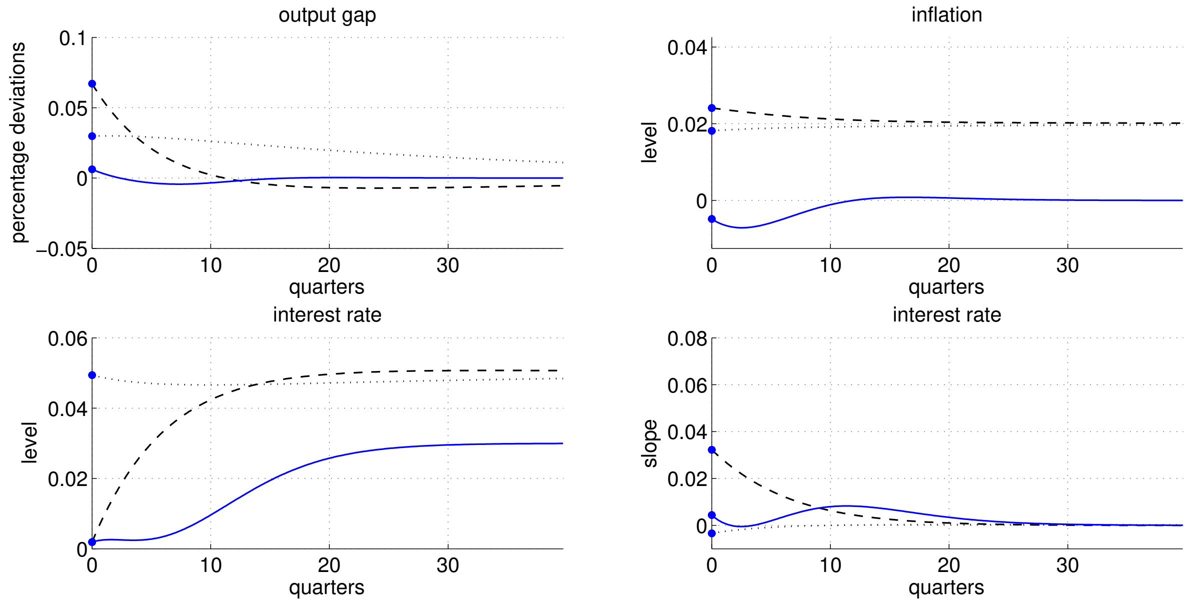

Figure 9: Simulated shock to interest rate and target rate (2007-2011), yields

In this figure we show (from left to right) the yield curve response to unexpected shocks to the (initial) interest rate (-0.0475), the inflation target rate (-0.02), and preferences (-0.1), with effects for the nominal and real yields (blue solid), the no-target rate shock scenario (black dashed, $\pi_{ss} = 0.02$, $\chi = 0$), and the pre-shock scenario (dotted).

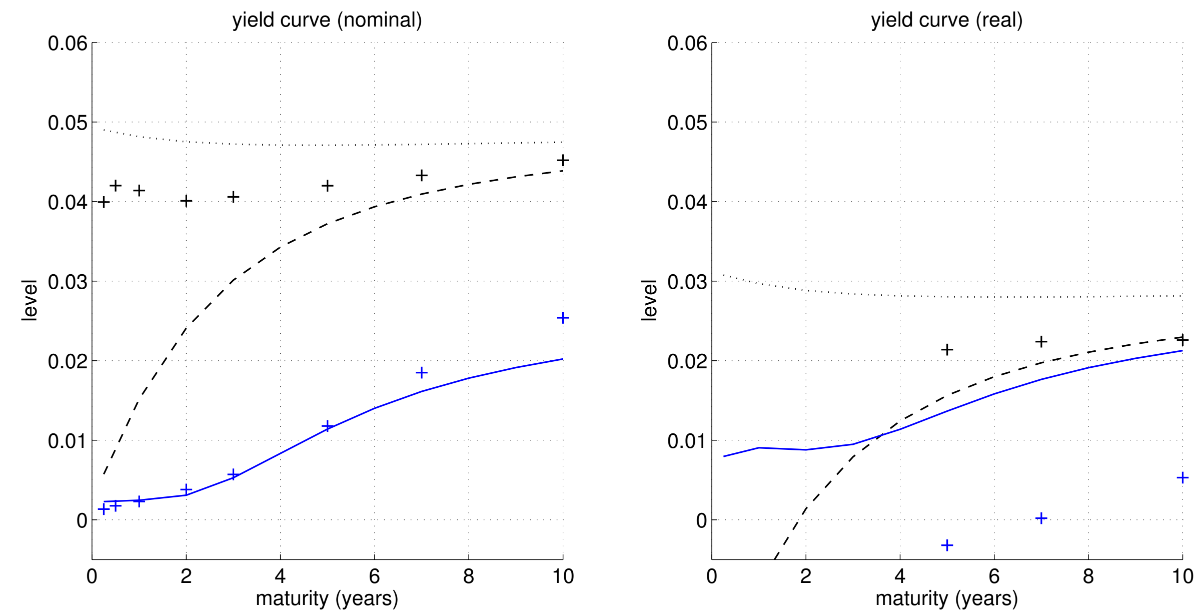

Figure 10: Term structure of interest rates in the stochastic NK model
In this figure we show (from left to right, top to bottom) the nominal and the real term structure of interest rates in the partial adjustment model (blue), and with a feedback rule value (red) after an unexpected shock to preferences (−0.1). The dashed lines show the risk-neutral term structure.