Approximating time varying structural models with time invariant structures

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Abstract

The paper studies what parameter variations does to the decision rules of a DSGE model and to structural inference. We provide diagnostics to detect parameter variations and to ascertain whether variations are exogenous or endogenous. Identification and estimation distortions when a constant parameter model is incorrectly assumed are examined; likelihood-based and VAR-based estimates of the structural dynamics when the DGP features neglected parameter variations are compared. The features of time variations in the financial frictions of a Gertler and Karadi (2010) model are studied.

Key words: Structural model, time varying coefficients, endogenous variations, misspecification.

JEL Classification: C10, E27, E32.

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1 Introduction

In macroeconomics, it is standard to study models that are structural in the sense of Hurwicz (1962); that is, model where the parameters characterizing the preference and the constraints of the agents and the technologies to produce goods and services are invariant to changes in the parameters describing the policy actions by the government. Such a requirement is fundamental to distinguish structural models from reduced form ones, and it is of fundamental importance to conduct correctly designed policy counterfactuals in dynamic stochastic general equilibrium (DSGE) models.

Recent work by Dueker et al. (2007), Fernandez Villaverde and Rubio Ramirez (2007), Canova (2009), Rios Rull and Santaeularia Llopis (2010), Liu et al. (2011), Galvao, et al. (2014), Vavra (2014), Seoane (2014), Meier and Sprengler (forthcoming) among others, has convincingly indicated that the parameters of DSGE are not time invariant, and in many cases, smoothly evolve over time. The evidence these papers provide agrees with Stock and Watson' (1996) conclusion that reduced economic relationships show small but persistent time variations.

The presence of time variations in DSGE parameters can not be taken as direct evidence that these models are not structural. For example, Cogley and Yagihashi (2010), and Chang et al. (2013) have shown that parameter variations may result from the misspecification of a model with time invariant structure. On the other hand, parameter variations may be needed in certain small open economy models to insure the existence of a stationary equilibrium, see e.g. Schmitt Grohe and Uribe (2003).

The approach the DSGE literature has taken to model parameter variations follows the VAR literature, pioneered by Cogley and Sargent (2005) and Primiceri (2005), and makes parameters exogenously drifting over time as independent random walks, see e.g. Fernandez and Rubio (2007). Many economic questions, however, hint at the possibility that parameter variations may instead be endogenous. For example, is it reasonable to assume that the Federal Reserve reacts in the same way to inflation in an expansion or in a contraction? Davig and Leeper (2006) analyze a situation where the policy rule could be state dependent and describe how such a rule may impact on the dynamics induced by different structural shocks. Does the propagation of shocks depend on the state of private and government debt? Are the multipliers of fiscal expansions driven by the amount of inequality present in the economy, see e.g. Brinca et al. (2014)? Are household as risk averse when they are wealthy as when they are poor or as impatient in their consumption needs when the capital stock is high or low? Questions of this type are potentially numerous and the answer is crucial for policymakers' decisions. Clearly, counterfactual analyses and optimal monetary policy conclusions derived assuming time invariant parameters or an inappropriate forms of time variations may be incorrect; comparisons of the welfare costs of business cycles are likely to be biased; growth prescriptions may be invalid; and standard impulse responses, historical and variance decompositions tools may provide a distorted picture of the dynamics of an economy in real time.

This paper has a number of goals. First, we want to characterize the decision rules of a DSGE when parameter variations are exogenous or endogenous, and in the latter case, when agents internalize or not the effects that their optimal decisions may have on parameter variations. By characterizing their structure and their differences, we hope to shed light on the consequences that alternative modelling assumptions may have for our understanding of how the economy works. Second, we wish to measure how distorted are standard statistics when a researcher erroneously assumes a time invariant structure but the data generating process (DGP) features time varying parameters and provide diagnostics to detect the misspecification driven by parameter variations. Third, we are interested in studying the consequences of using time invariant models when the parameters are time varying in terms of identification, estimation, inference, and policy analyses. Finally, we want to compare likelihood-based and SVAR-based estimates of the dynamic responses to structural shocks when time variations in the parameters are neglected. In particular, in the same vein as Canova and Paustian (2011), we wish to ask whether in the presence of misspecification driven by parameter variations, approaches that take a less structural approach are as good as likelihood based methods in describing the dynamics induced by structural shocks.

The existing literature is generally silent on the issues of interest in this paper. Seoane (2014) is the closest, in the sense that parameter variations are used to gauge potential model misspecifications. Kulish and Pagan (2014) characterize the decision rules of a DSGE model when structural breaks which are partly predictable. Magnusson and Mavroedis (2014) and Huang (2014) examine how time variations in the certain parameters may affect the identification of other structural parameters, the asymptotic theory of maximum likelihood estimators, and standard break tests. Andreasen (2012) studies how time variations in the variance of the exogenous shocks affect risk premia in models approximated to the second and third order and Fernandez et al. (2013) investigate to what extent variations in shock volatility matter for real variables. Ireland (2007) assumes that trend inflation in a standard New Keynesian model is driven by structural shocks and estimates the model by likelihood techniques, while Ascari and Sbordone (2014) highlight that trend inflation may be a function of monetary policy decisions.

The next section characterizes the decision rules in a general DSGE setup where both exogenous and endogenous parameter variations are possible. We consider time variations in the parameters regulating preferences, technologies, and constraints and disregard variations in the auxiliary parameters regulating the persistence and the volatility of structural shocks and of the parameters. We consider both first order approximations and higher order solutions. We present a simple RBC example to illustrate our results and to provide some intuition for our modelling choices of time variations.

We show that if parameter variations are purely exogenous, the contemporaneous impact and the dynamics induced by structural shocks are the same as in a model with no parameter variations. Thus, if one considers a time invariant version of the model and correctly identifies the structural disturbances, she would make no mistakes in characterizing the dynamics in response to these shocks. Clearly, variance decompositions exercises will be distorted, since some sources of variations (the disturbances to the parameters) will be omitted.

On the other hand, if parameter variations are endogenous, the instantaneous impact and the dynamics in responses to structural shocks may be different from the one of a constant coefficient model. The extent of the differences in the two specifications depends on two matrices. We show situations when these matrices are zero (making the dynamics of time varying and time invariant models identical) and situations when they are not. We show that the conclusions obtained when the model solution is approximated with a first order expansion around the steady state carry over when a second order expansion is constructed.

Section 3 provides diagnostics to detect the misspecification induced by employing a time invariant model when the data has been generated by a time varying coefficient model. We present statistics constructed using the optimality wedges of Chari et al. (2007) and the forecast errors and apply them to the RBC example previously considered. We also describe a marginal likelihood diagnostics which can help us to recognize whether the time variations detected with these statistics are of exogenous or endogenous nature.

Section 4 deals with parameter identification. We are interested in examining whether ignoring time variations in the parameters may have important repercussions for the way time invariant parameters can be identified from the likelihood of the model. Since the likelihood is constructed using the forecast errors of the model, which are generally misspecified when a time invariant model is used, one expects the shape of the likelihood function to be altered in a somewhat unpredictable way. Once again, in the context of the RBC example, we show that the likelihood function can be strongly altered when a time invariant model is considered, but its curvature does not acquire pathological characteristics when time variations in the parameters are neglected. Overall, our conclusions broadly agree with Huang (2014) in the sense that weakly identified parameters do not become better identified when time variations are present.

Section 5 considers the structural estimation of a model with time invariant parameter when the data is generated by models with time varying coefficients. The forecast errors used to construct the likelihood generally differ from the structural shocks because the dynamics assumed by the constant coefficient model are incorrect and because shocks misaggregation is present. Thus, one should expect important difference between the parameters of the DGP and the estimated ones. Indeed, we show that distortions in parameter estimates are present, are more likely to be significant when parameter variations are exogenous, and are generally produced because the parameters regulating the income and substitution effects are poorly estimated. As a consequence, estimated impulse responses differ from the true ones in terms of impact magnitude, shape and persistence, and the contribution of structural shocks to the variability of the endogenous variables is generally biased, with technology shocks absorbing to a large extent the contribution of missing shocks.

Section 6 studies whether less structural time invariant VAR methods can help to recover the dynamics induced by structural shocks when time varying parameters are present. We endow the researcher with the population autoregressive matrices of the time invariant and of the time varying model and we ask whether it is possible to "invert" the information present in VAR errors using model based restrictions. When the model displays parameters with exogenous variations, SVAR methods are right on target, since VAR shocks contain the relevant information about the structural shocks. The picture still very good when there are endogenous time variation: most of the qualitative features (impact effect, shape and persistence) of the structural dynamics are well captured by the SVAR methods. The good performance of SVAR methods is due to two features of the DGP: the fact that the dynamics in response to shocks are not very different when the model has fixed or time varying parameters; and the fact that shocks to the parameters, which are disregarded when a constant coefficient model is used, have low persistence.

Section 7 estimates a version of the Gertler and Karadi (2010) model of unconventional monetary policy, applies the diagnostics we propose to detect time variations in the parameters and estimates a version of the baseline model where the parameter controlling the extent of moral hazard is allowed to vary over time. We show that there is evidence that a fixed coefficient model is misspecified, that making parameter variations endogenous is preferable, and that the conclusions regarding the dynamic effect capital quality shock are altered. Section 8 concludes.

2 The setup

The equilibrium conditions of a dynamic stochastic general equilibrium (DSGE) model can be represented as:

$$E_t\left[f(X_{t+1}, X_t, X_{t-1}, Z_{t+1}, Z_t, \Theta_{1t+1}, \Theta_{1t})\right] = 0 \tag{1}$$

where X_t is an $n_x \times 1$ vector of endogenous variables, Z_t is an $n_z \times 1$ vector of strictly exogenous variables, Θ_{1t} is an $n_{\theta 1} \times 1$ vector of possibly time varying structural parameters and f is a continuous function, differentiable up to order q, mapping onto a $\mathbb{R}^{n_x}$ space.

The law of motion of the strictly exogenous processes Z_{t+1} is given by

$$Z_{t+1} = \Psi(Z_t, \sigma \Sigma_\epsilon \epsilon^z_{t+1}) \tag{2}$$

where Ψ is a continuous function, differentiable up to order q, mapping onto a $\mathbb{R}^{n_z}$ space; ϵ^z_{t+1} is a $n_e \times 1$ vector of i.i.d. structural disturbances with mean zero and identity covariance matrix, $n_z \geq n_e$; $\sigma \geq 0$ is an auxiliary scalar, Σ_ϵ is a known $n_z \times n_e$ matrix.

Let $\Theta_t = [\Theta_{1t}, \Theta_{2t}]$, where Θ_{2t} is a $n_{\theta 1} n_{x1} \times 1$ vector of parameters, $n_x \geq n_{x1}$,appearing in the case agents internalize the effects that their decisions have on the parameters. The law of motion of the structural parameters Θ_t is given by

$$\Theta_{t+1} = \Phi(\Theta, X_t, U_{t+1}) \tag{3}$$

where Φ is a continuous function, differentiable up to order q, mapping onto the $\mathbb{R}^{n_\theta}$ space; U_t is a $n_u \times 1$ vector of exogenous disturbances, $n_\theta = n_{\theta 1}(1 + n_{x1}) \geq n_u$; Θ is a vector of constants.We assume that the law of motion of U_{t+1} is given by

$$U_{t+1} = \Omega(U_t, \sigma \Sigma_u \epsilon_{t+1}^u) \tag{4}$$

where Ω is continuous and differentiable up to order q,mapping onto the $\mathbb{R}^{n_u}$ space; ϵ_t^u is a $n_u \times 1$ vector of i.i.d. disturbances with mean zero and identity covariance matrix, uncorrelated with the ϵ_{t+1}^z,and Σ_u is a known $n_u \times n_u$ matrix.

Three features of the setup need some discussion. First, the vector of structural disturbances ϵ_{t+1}^z may be smaller than the vector of exogenous variables Z_{t+1}, in which case there are common sources of variations in the exogenous variables Similarly, the dimension of ϵ_{t+1}^u may be smaller than the dimension of the structural parameters. Second, we allow for time variations in the parameters regulating preferences, technologies and constraint but we not consider variations in the auxiliary parameters regulating the law of motion of the Z_t and the U_t as we are not interested in stochastic volatility, GARCH or rare events phenomena (as in e.g. Andreasan, 2012), nor in time variations driven by evolving persistence of the exogenous processes. Third, (1) makes no distinction between states and controls. Thus, the solution we derive has to be thought of as final form (endogenous variables as a function of the exogenous variables and the parameters) rather than a state space form (control variables as a function of the states and of the parameters).

2.1 The first order approximate decision rules

We start by studying the implications of structural parameters variation for agents' decision rules when a first order approximate solution is considered. Taking a linear expansion of (1) around the steady states leads to

$$0 = E_t \left[F x_{t+1} + G x_t + H x_{t-1} + L z_{t+1} + M z_t + N \theta_{t+1} + O \theta_t \right] \tag{5}$$

where $F = \partial f / \partial X_{t+1}$, $G = \partial f / \partial X_t$, $H = \partial f / \partial X_{t-1}$, $L = \partial f / \partial Z_{t+1}$, $M = \partial f / \partial Z_t$, $N = \partial f / \partial \Theta_{t+1}$ $O = \partial f / \partial \Theta_t$ all evaluated at the steady states values of (X_t, Z_t, Θ_t) and lower case letters indicate deviations from the steady states. Let $\phi_u = \partial \Phi / \partial U_{t+1}$, $\phi_x = \partial \Phi / \partial X_t$, $\psi_z = \partial \Psi / \partial Z_t$, $\omega_u = \partial \Omega / \partial U_t$ and assume that all the eigenvalues of ψ_z and of ω_u are strictly less than one in absolute value.

Proposition 2.1. *Assume that equations (2)-(4) hold and that there exists a recursive equilibrium law of motion solving* (5)*. Then, the unique recursive law of motion is*

$$x_t = Px_{t-1} + Qz_t + Ru_t \tag{6}$$

where

- P *solves* $FP^2 + (G + N\phi_x)P + (H + O\phi_x) = 0$.
- *Given* P*,* Q *solves* $VQ = -vec(L\psi_z + M)$ *and* $V = \psi_z' \otimes F + I_{n_z} \otimes (FP + G + N\phi_x)$ *where vec denotes the columnwise vectorization.*
- *Given* P*,* R *solves* $WR = -vec(N\phi_u\omega_u + O\phi_u)$ *where* $W = \omega_u' \otimes F + I_{n_\theta} \otimes (FP + G + N\phi_x)$

Proof. The proof is straightforward. Substituting (6) into (5), we obtain

$$\begin{aligned} 0 =& [FP^2 + (G + N\phi_x)P + (H + O\phi_x)]x_{t-1} + [(FP + G + N\phi_x)Q + FQ\psi_z + L\psi_z + M]z_t \\ &+ [(FP + G + N\phi_x)R + FR\omega_u + N\phi_u\omega_u + O\phi_u]u_t \end{aligned}$$

Since the solution must hold for every realization of x_{t-1}, z_t, u_t, we need to equate their coefficient to zero and the result obtains. □

Corollary 2.2. *Assume that the condition stated in Proposition 2.1 hold and* $\phi_x = 0$*. Then, variations in the j-th structural parameter,* θ^j*, have no impact on the dynamics induced by structural shocks. Their instantaneous effect on the endogenous variables will be non-zero if and only if the* j^{th} *column of* $N\phi_u\omega_u + O\phi_u$ *is non zero.*

Corollary 2.3. *Assume that the conditions stated in Proposition 2.1 hold and* $\phi_u = 0$*. If the matrices* $N\phi_x$ *and* $O\phi_x$ *are zero, variations in the j-th structural parameter* θ^j *have no impact on the endogenous variables* x_t*.*

Proposition 2.1 indicates that the format of the first order approximate solution will have, as in a constant coefficient setup, a VARMA structure. Disturbances to the parameters play the same role as additional structural shocks, making a model with m structural shocks and a model with m_1 structural shocks and m_2 disturbances to the parameters, m= $m_1 + m_2$, potentially indistinguishable.

Corollaries 2.2 and 2.3 constitute the main results of this section and give conditions under which parameter variations have an impact on the dynamics induced by structural disturbances. If parameter variations are purely exogenous, $\phi_x = 0$, and the P and Q matrices will be the same as those of a constant coefficient model. The intuition for this result is simple: as long as the shocks to Z_t and Θ_t are uncorrelated, parameter variations add variations in the endogenous variables dynamics without altering the dynamics produced by existing disturbances. In other words, suppose an economy is perturbed by a technology shocks. Then, the dynamics induced by these shocks do not depend on whether the discount factor is constant or time varying, provided that its innovations exogenous and unrelated to the innovations in the technological process.

One implication of this results is that if one considers a time invariant version of the model and correctly identifies the structural disturbances ϵ_t^z, she would make no mistakes in characterizing the dynamics in response to structural shocks. Clearly, variance of historical decompositions exercises will be distorted, since certain sources of variations (the ϵ_t^u disturbances) are omitted from the model. One interesting question is whether standard procedures allow a researcher employing a time invariant setup to recover the correct ϵ_t^z from the data when the DGP is a model with time varying structural parameters. If this is not the case, one would like to know which structural disturbance will absorbs the effect of missing shocks.

On the other hand, if parameter variations are purely endogenous, $\phi_u = 0$, the dynamics in response to structural shocks may be altered. To know if distortions are present one needs to check whether the matrices $N\phi_x$ and $O\phi_x$are identically equal to zero or not. If they are not, a researcher employing a time invariant version of the model is likely to incorrectly characterize both the dynamics in response to structural shocks and the relative importance of different sources of disturbances for the variability of the endogenous variables.

Exogenous variations in the structural parameters typically have an instantaneous effect on the endogenous variables of the model, i.e. R$\neq 0$, but purely endogenous parameter variations have zero instantaneous effect on x_t. Thus, to avoid identification problems in the latter case, it is preferable to have the relationship between parameters and the endogenous variables disturbed by shocks.

2.2 The second order approximate decision rules

One may be curious as to whether the conclusions we have reached are affected when higher order approximations are considered. We know from Schmitt-Grohe and Uribe (2004) that the first order terms in first and higher order approximations will be the same. To examine whether quadratic terms will be affected by the presence of time variations we need to simplify the notation. Let $W_t = [Z_t', U_t']'$, $Y_{t+1} = [X_{t+1}', X_t']'$, $\epsilon_{t+1} = [\epsilon_{t+1}^{z\prime}, \epsilon_{t+1}^{u\prime}]', \Sigma = diag[\Sigma_z, \Sigma_u]$ so that (1) is

$$0 = E_t[F(Y_t, W_t, \sigma\Sigma\epsilon_{t+1}, \Theta)] \tag{7}$$

once the solution $Y_{t+1} = h(Y_t, W_t, \sigma\Sigma\epsilon_{t+1})$ is taken into account. The second order approximation of (7) is

$$\begin{aligned} E_t[(F_y y_t + F_w w_t + F_\sigma \sigma) + 0.5(F_{yy}(y_t \otimes y_t) + F_{ww}(w_t \otimes w_t) + F_{\sigma\sigma}\sigma^2) &\quad + \\ F_{yw}(y_t \otimes w_t) + F_{y\sigma} y_t \sigma + F_{w\sigma} w_t \sigma] &\quad = \quad 0 \end{aligned} \tag{8}$$

Note that $F_\sigma \sigma$, $F_{y\sigma} y_t \sigma, F_{w\sigma} w_t \sigma$ are all zero, see Schmitt Grohe and Uribe (2004). The second order expansion of the decision rule is

$$\begin{aligned} y_{t+1} &= g_y y_t + g_w w_{t+1} + 0.5(g_{yy}(y_t \otimes y_t) + g_{ww}(w_{t+1} \otimes w_{t+1}) + g_{\sigma\sigma}\sigma^2) \\ &+ g_{yw}(y_t \otimes w_{t+1}) + g_{y\sigma} y_t \sigma + g_{w\sigma} w_{t+1} \sigma \end{aligned} \tag{9}$$

Proposition 2.4. *Consider the second order approximation of the optimality condition (8). Time variations in the structural parameters affect g_{yy} and g_{yw} the if only if they affect g_y and g_w.*

Proof. The proof is strightforward. Matching coefficient and requiring, for example, $F_{yy} = 0$ implies that $g_{yy} = -(F_{y'})^{-1} J_1$ where

$$J_1 = 0.5F_{y'y'}(g_y \otimes g_y) + F_{y'y}g_y + F_{y'w'}g_y h_y + 0.5F_{yy} + F_{yw'}h_y + 0.5F_{w'w'}h_y {h_y}' \quad (10)$$

where primes indicate future values. Thus g_{yy} depends only on the first order term g_y, the matrix of derivatives of the optimality conditions with respect to the argument evaluated at the steady state ($F_{y'y'}, F_{w'w'}$, etc.) and h_y. Similarly $F_{ww} = 0$ implies that $g_{ww} = -(F_{y'})^{-1} J_2$ where

$$J_2 = 0.5F_{y'y'}(g_w \otimes g_w) + F_{y'w'}h_w g_w + F_{yw}g_w + 0.5F_{w'w'}h_w {h_w}' + F_{w'w}h_w + 0.5F_{ww} \quad (11)$$

which also depends on first order terms (g_w),the matrix of derivatives of the optimality conditions with respects to the argument evaluated at the steady state ($F_{y'y'}, F_{w'w'}$, etc.) and h_w.The proof for the other terms of the expression is analogous. □

The intuition for proposition 2.4 is simple: since second order terms are function of first order terms, of the parameters of the law of motion of the W$_t$, and of the gradient and the Hessian of the optimality conditions evaluated at the steady states, they do not feature independent variations. Thus, the second order dynamics will be distorted by time variations only when the first order dynamics are affected.

2.3 Discussion

The results derived in this section requires time variations to be continuous. This is in line with the evidence produced by Stock and Watson (1996) and with the standard practice employed in SVAR. Note that our framework is flexible and can accommodate once-and-for-all breaks (at a known date) as long as transition between the two states is smooth. For example, a smooth threshold switching specification such as $\theta_{t+1} = (1-\rho)\theta + \rho\theta_t + exp(t-T_0)/(1+exp(t-T_0))$, $t = 1, \ldots, T_0-1, T_0, T_0+1, \ldots T$ is an acceptable form of exogenous time variations. Similarly, endogenous forms of time variations of the type $\theta_{t+1} = (1-\rho)\theta + \rho\theta_t + a\exp(-(K_t - K + U_{\theta,t+1}))/(b + exp(-(K_t - K + U_{\theta,t+1}))$ where a and b are vectors of parameters can also be used. What the framework does not allow is for Markov switching variations in the parameters which occur at unknown dates or for abrupt changes, such as those considered in Davig and Leeper (2006), since the smoothness conditions on the f function may not hold.

Kulish and Pagan (2014) have developed solution and estimation procedures for models with abrupt breaks and learning between the states. Their solution for the pre and post break period is a constant coefficient VAR, while for the learning period is a time varying coefficient VAR. Since (6) is a constant coefficient

VAR with an extended set of shocks, a few words of comparison are needed. First, they are interested in characterizing the solution during the learning period, when the structure is unchanged but expectations move, while we are interested in the solution when parameters are continuously varying. Second, their modelling of time variations is abrupt and the solution is designed to deal with that situation, while in our case variations are continuous. Third, in our setup expectations are varying with the variations of the structure, while Kulish and Pagan have expectations varying only in anticipation of a (foreseeable) break.

While we are able to establish what parameter variations do to the decision rules in the first and second order approximate solutions, the conclusions for higher order approximations will generally depend on the details of the model specification. One should expected third and higher order dynamics to differ in time varying coefficient and time invariant models, regardless of whether time variations are exogenous or endogenous, because third order terms depend on the variance of the shocks to the parameters and thus they will be different from those in constant coefficient models which do not feature these additional shocks. For the parameterization employed in the example of the next subsection, the dynamics in exogenously varying and fixed coefficient model differ only in the fifth digit, but in general one can conceive examples where second and third order approximations will not be the same.

One way of thinking about the differences between exogenous and endogenous parameter variations is that in the former each parameter evolves independently and covariations can be modelled by selecting the matrix Σ_u to be of reduced rank. With endogenous variations, there is an observable factor (the X's) which drives the common parameter variations. Thus, Σ_u will be diagonal and will feature reduced rank if for some parameter variations are purely endogenous.

2.4 An example

To convey some intuition into the mechanics of corollaries 2.2-2.3, we use a simple, closed economy, RBC model. The representative agent maximizes the discounted stream of future utilities given by

$$\max E_0 \sum_{t=1}^{\infty} \beta_t \left(\frac{C_t^{1-\eta}}{1-\eta} - A\frac{N_t^{1+\gamma}}{1+\gamma}\right) \tag{12}$$

subject to the sequence of constraints

$$\begin{aligned} Y_t(1-g_t) &= C_t + K_t - (1-\delta_t)K_{t-1} \\ Y_t &= \zeta_t K_{t-1}^{\alpha} N_t^{1-\alpha} \end{aligned}$$

where Y_t is output, C_t consumption, K_t the stock of capital and N_t is hours worked and $g_t = \frac{G_t}{Y_t}$ is the share of government expenditure in output. The system is perturbed by two exogenous structural disturbances: one to the technology Z_t and one to the government spending share, g_t, assumed to follow time invariant AR(1) processes

$$\begin{aligned}\ln \zeta_t &= (1-\rho_\zeta)\ln \zeta + \rho_\zeta \ln \zeta_{t-1} + e_t^\zeta \\ \ln g_t &= (1-\rho_g)\ln g + \rho_g \ln g_{t-1} + e_t^g \end{aligned} \tag{13}$$

where variables without time subscript denote steady state quantities. There are 12 parameters in the model: 6 structural ones (α is the capital share, η the risk aversion coefficient, γ the inverse of the Frisch elasticity of labor supply, A the constant in front of labor in utility, β_t the time discount factor and δ_t the depreciation rate), and 6 auxiliary ones (the steady state values of the government expenditure share and of TFP, (ζ, g), their autoregressive parameters, (ρ_ζ, ρ_g), and their standard deviations (σ_ζ, σ_g)). We assume that all parameters but β_t and δ_t are time invariant. Dueker et al. (2007), Liu et al (2011) and Meier and Sprenger (forthcoming) provide evidence that these two parameters are indeed varying. Since Canova and Sala have shown that in this model they are only weakly identified, we can use the model to verify some of the claims of Magnusson and Mavroedis (2014) in a Likelihood context. The first order approximation to their law of motion is described below.

The optimality conditions of the problem are:

$$AC_t^\eta N_t^\gamma = (1-\alpha)(1-g_t)Y_t/N_t \tag{14}$$

$$\beta_t C_t^{-\eta} = E_t \frac{\partial \beta_{t+1}}{\partial K_t} u(C_{t+1}, N_{t+1}) + E_t \left(\beta_{t+1} C_{t+1}^{-\eta} \left(\frac{\alpha(1-g_{t+1})Y_{t+1}}{K_{t+1}} + (1-\delta_{t+1}) - \frac{\partial \delta_{t+1}}{\partial K_t} K_t \right) \right) \tag{15}$$

$$(1-g_t)Y_t = C_t + K_t - (1-\delta_t)K_{t-1} \tag{16}$$

$$Y_t = \zeta_t K_{t-1}^\alpha N_t^{1-\alpha} \tag{17}$$

Time variations in β_t and δ_t affect optimal choices in two ways. First, there is a direct effect in the Euler equation and in the resource constraint when β_t and δ_t are time varying. Second, if agents take into account the fact that their decisions may affect parameter variations, there will be a second (endogenous) effect coming from variations in the derivatives of β_{t+1} and δ_{t+1} with respect to the endogenous states - see the Euler equation (15).

We specialize this setup to consider various possibilities.

2.4.1 Model A: Constant coefficients.

As a benchmark, we let $\beta_t = \beta^t$ and $\delta_t = \delta$. The optimality conditions are

$$\begin{aligned} AC_t^\eta N_t^{\gamma+1} &= (1-\alpha)(1-g_t)Y_t \\ C_t^{-\eta} &= E_t \beta C_{t+1}^{-\eta} \left(\alpha(1-g_{t+1})Y_{t+1}/K_t + 1 - \delta\right) \\ (1-g_t)Y_t &= C_t + K_t - (1-\delta)K_{t-1} \\ Y_t &= \zeta_t K_{t-1}^\alpha N_t^{1-\alpha} \end{aligned}$$

In the steady state, it must be the case that:

$$\frac{K}{Y}=\frac{\alpha(1-g)}{\delta-1+1/\beta};\ \frac{C}{Y}=1-\delta\frac{K}{Y}-\frac{g}{Y};\ \frac{N}{Y}=\zeta^{\frac{1}{1-\alpha}}\left(\frac{K}{Y}\right)^{\frac{\alpha}{\alpha-1}};\ Y=\left[\frac{A}{(1-\alpha)(1-g)}\left(\frac{C}{Y}\right)^{\eta}\left(\frac{N}{Y}\right)^{1+\gamma}\right]^{-\frac{1}{\eta+\gamma}}. \tag{18}$$

2.4.2 Model B: Exogenous parameter variations

Here we set $d_t=\beta_{t+1}/\beta_t$ and let $\Theta_{t+1}-\Theta\equiv(d_{t+1}(1-\rho_\beta)\beta,\delta_{t+1}-(1-\rho_\delta)\delta)'=U_{t+1}$. We postulate

$$u_{d,t+1} = \rho_d u_{d,t}+e_{d,t+1} \tag{19}$$
$$u_{\delta,t+1} = \rho_\delta u_{\delta,t}+e_{\delta,t+1} \tag{20}$$

Since Θ_{t+1} is independent of the capital stock, $\partial\beta_{t+1}/\partial K_t=\partial\delta_{t+1}/\partial K_t=0$ and the f function becomes

$$E_t\left[f(X_{t+1},X_t,X_{t-1},Z_{t+1},Z_t,\Theta_{t+1},\Theta_t)\right] =$$
$$E_t\begin{pmatrix} AC_t^{\eta}N_t^{\gamma+1}-(1-\alpha)(1-g_t)Y_t \\ 1-d_tC_{t+1}^{-\eta}/C_t^{-\eta}\left(\alpha(1-g_{t+1})Y_{t+1}/K_t+1-\delta_{t+1}\right) \\ (1-g_t)Y_t-C_t-K_t+(1-\delta_t)K_{t-1} \\ Y_t-\zeta_tK_{t-1}^{\alpha}N_t^{1-\alpha}\end{pmatrix} = 0 \tag{21}$$

where $X_t=(K_t,Y_t,C_t,N_t)'$, $Z_t=(\zeta_t,g_t)'$ and $\Theta_t=\Theta_{1t}$.

It is easy to verify that in the steady state $(\frac{K}{Y},\frac{C}{Y},\frac{N}{Y},Y)$ will coincide with those of the constant coefficient model. In addition, since $\phi_x=0$, time variations in (d_{t+1},δ_{t+1}) leave the decision rule matrices P and Q as in model A. Thus, as far as impulse responses to structural shocks are concerned, models A and B are observationally equivalent.

To examine whether variations in Θ_t have an instantaneous impact on the endogenous variables, we need to check whether the columns of $N\phi_u\omega_u+O\phi_u$ are zero. The relevant matrices of partial derivative evaluated at the steady state are

$$N=\frac{\partial f}{\partial\Theta_{t+1}}=\begin{pmatrix}0&0\\0&1/\beta\\0&0\\0&0\end{pmatrix},\quad O=\frac{\partial f}{\partial\Theta_t}=\begin{pmatrix}0&0\\-1/\beta&0\\0&-K\\0&0\end{pmatrix},\ \omega_u=\begin{pmatrix}\rho_d&0\\0&\rho_\delta\end{pmatrix}\ \phi_u=\frac{\partial\Phi}{\partial U_{t+1}}=\begin{pmatrix}1&0\\0&1\end{pmatrix} \tag{22}$$

Thus:

$$N\phi_u\rho_u+O\phi_u=\begin{pmatrix}0&0\\-1/\beta&-\rho_\delta/\beta\\0&-K\\0&0\end{pmatrix} \tag{23}$$

Hence, time variations in (d_{t+1},δ_{t+1}) have an impact effect on X_t - at least the entry of $N\phi_u\omega_u+O\phi_u$ is non zero. Note that if d_t is a fast moving variable, the impact effect on X_t depends on the persistence of shocks to the growth rate of the discount factor. For example, if $\rho_d = 0$, shocks to the growth rate of the time discount factor have no effects on X_t. In this case if only the discount factor is time varying and variations in its growth rate are i.i.d., the decision rules of the models A and B will be identical.

2.4.3 Model C: State dependent parameter variations without internalization

Assume that the time variations in the growth rate of the discount factor and in the depreciation rate are driven by the aggregate capital stock. We specify

$$\Theta_{t+1} = [\Theta_u - (\Theta_u - \Theta_l)e^{-\phi_a(K_t - K + U_{\theta,t+1})}] + [\Theta_u - (\Theta_u - \Theta_l)e^{\phi_b(K_t - K + U_{\theta,t+1})}] \quad (24)$$

where $\phi_a, \phi_b, \Theta_u, \Theta_l$ are vectors of parameters and $lu_{\theta,t+1}$ be a zero mean, i.i.d. vector of shocks. To insure that model C has the same steady state as model A, we let $\Theta_l = (\beta/2,\ \delta/2)$, so that

$$\begin{pmatrix} d_{t+1} \\ \delta_{t+1} \end{pmatrix} = \Phi(\Theta, K_t, U_{t+1}) = \begin{pmatrix} 2d_u - (d_u - \beta/2)[e^{-\phi_1(K_t - K + U_{\beta,t+1})} + e^{\phi_2(K_t - K + U_{\beta,t+1})}] \\ 2\delta_u - (\delta_u - \delta/2)[e^{-\phi_3(K_t - K + U_{\delta,t+1})} + e^{\phi_4(K_t - K + U_{\delta,t+1})}] \end{pmatrix} \quad (25)$$

This specification is quite flexible: depending on the choice of $\phi' s$,we can accommodate linear or quadratic relationships, which are symmetric or asymmetric. We assume that agents treat the capital stock appearing in (25) as an aggregate variable. This assumption is similar to the 'small k -big k' situation encountered in standard rational expectations models or to the distinction between internal and external habit formation. Thus, agents' first order conditions do not take into account the fact that their optimal capital choice of K_t changes d_t and δ_t and $\partial\beta_{t+1}/\partial K_t = \partial\delta_{t+1}/\partial K_t = 0$. Hence, the equilibrium conditions are then as in (21) Since the f function is the same as in model B, the matrices N and O are unchanged.

We examine first whether parameter variations affect the matrices P and Q regulating the dynamics induced by the structural shocks. We have,

$$N\phi_x = \begin{pmatrix} 0 & 0 \\ 0 & 1/\beta \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} (d_u - \beta/2)(\phi_1 - \phi_2) & 0 & 0 & 0 \\ (\delta_u - \delta/2)(\phi_3 - \phi_4) & 0 & 0 & 0 \end{pmatrix} \quad (26)$$

$$O\phi_x = \begin{pmatrix} 0 & 0 \\ -1/\beta & 0 \\ 0 & -K \\ 0 & 0 \end{pmatrix} \begin{pmatrix} (d_u - \beta/2)(\phi_1 - \phi_2) & 0 & 0 & 0 \\ (\delta_u - \delta/2)(\phi_3 - \phi_4) & 0 & 0 & 0 \end{pmatrix} \quad (27)$$

Thus, unless $\phi_1 \neq \phi_2$ and/or $\phi_3 \neq \phi_4$, endogenous variations in d_t, δ_t leave P and Q unaffected, i.e. symmetries in the law of motion for time variations in d_t, δ_t are needed to produce structural dynamics which are different from those of a constant coefficient model.

To verify whether parameter variations have an impact effect on X_t, note that $\omega_u = 0_{2\times 2}$, $\rho_d = \rho_\delta = 0$ and that

$$\phi_u = \frac{\partial \Phi}{\partial U_{t+1}} \begin{pmatrix} (\beta_u - \beta/2)(\phi_1 - \phi_2) & 0 \\ 0 & (\delta_u - \delta/2)(\phi_3 - \phi_4) \end{pmatrix}, \tag{28}$$

$$\phi_x = \frac{\partial \Phi}{\partial X_t} \begin{pmatrix} (\beta_u - \beta/2)(\phi_1 - \phi_2) & 0 & 0 & 0 \\ (\delta_u - \delta/2)(\phi_3 - \phi_4) & 0 & 0 & 0 \end{pmatrix}. \tag{29}$$

Thus,

$$N\phi_u\omega_u + O\phi_u = \begin{pmatrix} 0 & 0 \\ 1/\beta(d_u - \beta/2)(-\phi_1 + \phi_2) & 0 \\ 0 & K(\delta_u - \delta/2)(-\phi_3 + \phi_4) \\ 0 & 0 \end{pmatrix} \tag{30}$$

As long as $\phi_1 \neq \phi_2$or $\phi_3 \neq \phi_4$, endogenous parameter variations have an instantaneous impact on X_t. Note that this is true regardless of whether shocks to the parameters are i.i.d. or persistent.

2.4.4 Model D: State dependent parameter variations with internalization.

We still assume that time variations in the discount factor and in the depreciation rate are driven by the aggregate capital stock and by an exogenous shock, as in equation (24). Contrary to case C, we assume that agent internalize the effects that their capital decisions have on parameter variations. The relevant derivatives are

$$d'_{t+1} \equiv \partial d_{t+1}/\partial K_t = -(\beta_u - \beta/2)[-\phi_1 e^{-\phi_1(K_t - K + u_{\beta,t+1})} + \phi_2 e^{\phi_2(K_t - K + u_{\beta,t+1})}] \tag{31}$$

$$\delta'_{t+1} \equiv \partial \delta_{t+1}/\partial K_t = -(\delta_u - \delta/2)[-\phi_1 e^{-\phi_1(K_t - K + u_{\delta,t+1})} + \phi_2 e^{\phi_2(K_t - K + u_{\delta,t+1})}] \tag{32}$$

and the Euler equation becomes

$$C_t^{-\eta} = E_t d'_t \; u(C_{t+1}, N_{t+1}) + E_t \left(d_t \; C_{t+1}^{-\eta}(\alpha Y_{t+1}/K_{t+1} + 1 - \delta_{t+1} - \delta'_{t+1} K_t) \right) \tag{33}$$

In order for the steady states of model D to equal to those of model A, we restrict $\phi_1 = \phi_2 = \phi_1$, $\phi_3 = \phi_4 = \phi_3$. Note that in model D, the $f(.)$ function differs from the one obtained in model C. In particular, we have

$$0 = E_t\left[f(X_{t+1}, X_t, X_{t-1}, Z_{t+1}, Z_t, \Theta_{t+1}, \Theta_t)\right] =$$

$$E_t \begin{pmatrix} AC_t^{\eta} N_t^{\gamma+1} - (1-\alpha)(1-g_t)Y_t \\ 1 - d'_t \; u(C_{t+1}, N_{t+1})/C_t^{-\eta} - d_t \; C_{t+1}^{-\eta}/C_t^{-\eta}(\alpha(1-g_{t+1})Y_{t+1}/K_{t+1} + 1 - \delta_{t+1} + \delta'_{t+1}K_t) \\ (1-g_t)Y_t - C_t - K_t + (1-\delta_t)K_{t-1} \\ Y_t - \zeta_t K_{t-1}^{\alpha} N_t^{1-\alpha} \end{pmatrix} \tag{34}$$

where as before $X_t = (K_t, Y_t, C_t, N_t)'$, $Z_t = (\zeta_t, g_t)'$ but now $\Theta_t = (d_t, \delta_t, d'_t, \delta'_t)'$ and its law of motion is

$$\begin{pmatrix} d_{t+1} \\ \delta_{t+1} \\ d'_{t+1} \\ \delta'_{t+1} \end{pmatrix} = \Phi(\Theta, k_t, u_{t+1}) = \begin{pmatrix} 2d_u - (d_u - \beta/2)[e^{-\phi_1(K_t - K + u_{\beta,t+1})} + e^{\phi_1(K_t - K + u_{\beta,t+1})}] \\ 2\delta_u - (\delta_u - \delta/2)[e^{-\phi_3(K_t - K + u_{\delta,t+1})} + e^{\phi_3(K_t - K + u_{\delta,t+1})}] \\ -(d_u - \beta/2)\phi[-e^{-\phi_1(K_t - K + u_{\beta,t+1})} + e^{\phi_1(K_t - K + u_{\beta,t+1})}] \\ -(\delta_u - \delta/2)\phi[-e^{-\phi_3(K_t - K + u_{\delta,t+1})} + e^{\phi_3(K_t - K + u_{\delta,t+1})}] \end{pmatrix} \tag{35}$$

The relevant matrices of derivatives evaluated at the steady states are $\omega_u = 0_{2\times 2}$

$$N = \frac{\partial f}{\partial \Theta_{t+1}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/\beta & -u(C,N)/C^{-\eta} & -\beta K \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad O = \frac{\partial f}{\partial \Theta_t} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1/\beta & 0 & 0 & 0 \\ 0 & -K & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\phi_x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2(\beta_u - \beta/2)\phi_1^2 & 0 & 0 & 0 \\ -2(\delta_u - \delta/2)\phi_3^2 & 0 & 0 & 0 \end{pmatrix}, \quad \phi_u = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -2(\beta_u - \beta/2)\phi_1^2 & 0 \\ 0 & -2(\delta_u - \delta/2)\phi_3^2 \end{pmatrix}$$

Clearly, $N\phi_x \neq 0$, $O\phi_x = 0$ and $N\phi_u\omega_u + O\phi_u = 0$. Thus, a shock to the law of motion of the parameters alters the dynamics produced by structural shocks but has no instantaneous effect on the x_t, i.e. $R = 0$.

In sum, to have time variations mattering for the dynamics to structural shocks we need endogenous parameter variations and either that the relationship between parameters variations and the states is asymmetric; or that agents internalize the consequences their decisions have on the law of motion of the parameters or both.

2.4.5 Impulse responses

Why are the decision rule matrices P and Q different in models C and D? We explore the shape of impulse responses to try to understand the economic differences present in the two setups. To compute impulse responses we need to select values for the parameters. For those common to all models we choose $\alpha = 0.30$, $\beta = 0.99$, $\delta = 0.025$, $\gamma = 2$, $\eta = 2$, $A = 4.50$, ζ=1;$\rho_\zeta = 0.90$, $\sigma_\zeta = 0.00712$, $g = 0.18$, $\rho_g = 0.50$ and $\sigma_g = 0.01$. For the parameter specific to the time varying parameters models we choose:

- Model B: $\rho_\beta = 0.985, \rho_\delta = 0.95$ and $\sigma_\beta = 0.002$ $\sigma_\delta = 0.07$.
- Model C : $\phi_{1\beta} = 0.01, \phi_{2\beta} = 0.03, \phi_{1\delta} = 0.2, \phi_{2\delta} = 0.1$, $\sigma_d = \sigma_\delta = 0.5$, $\beta_u = 0.999, \delta_u = 0.025$.
- Model D : $\phi_{1\beta} = 0.0001, \phi_{2\beta} = 0.016, \phi_{1\delta} = 0.2, \phi_{2\delta} = 0.1$, $\sigma_d = 0.0001$;$\sigma_\delta = 0.1$, $\beta_u = 0.999, \delta_u = 0.025$.

Figures 1 and 2 reports the responses to the two structural shocks for each of the four model: the first column has the responses to technology shocks;

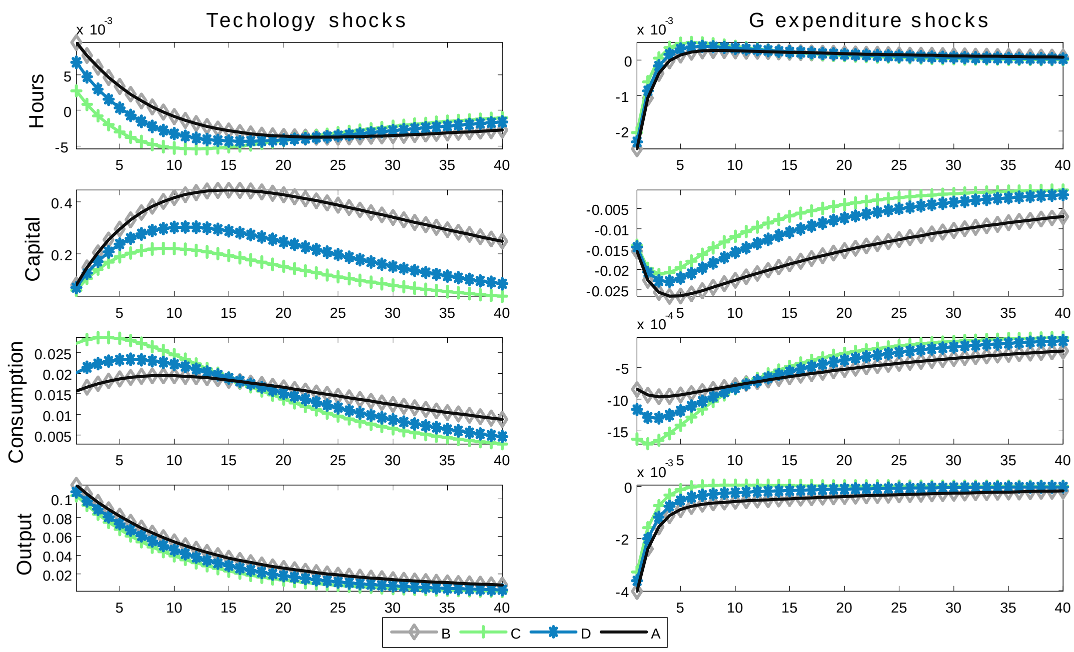


Figure 1: Impulse responses, first order approximation

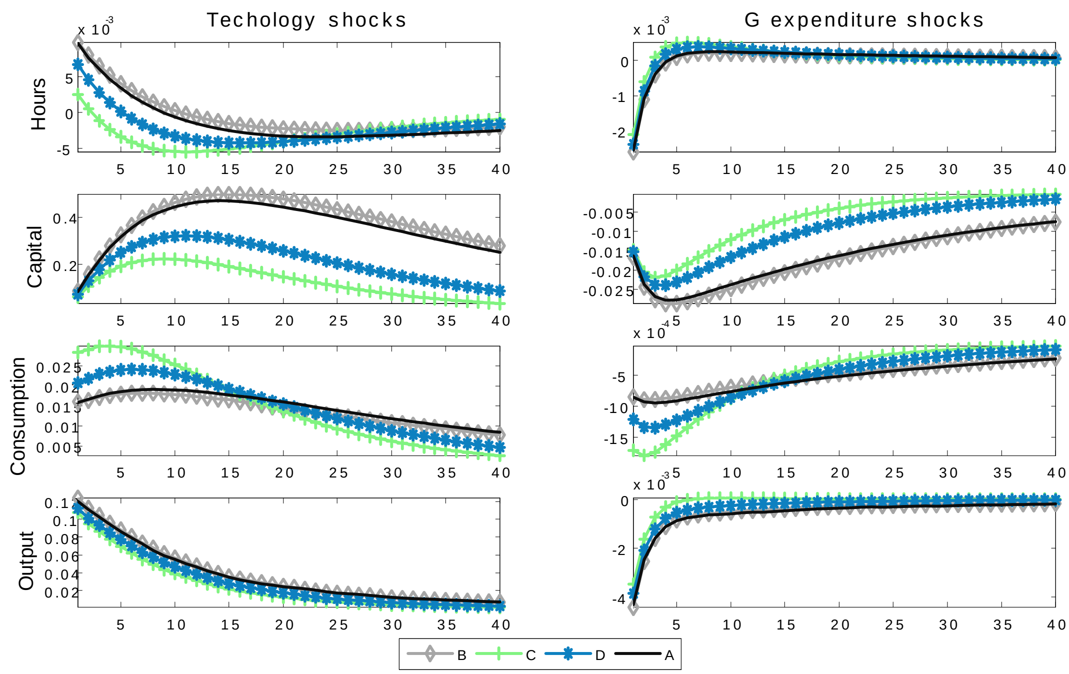


Figure 2: Impulse responses, second order approximation

the second the responses to government expenditure shocks. Figure 1 reports responses using a first order approximation and Figure 2 responses obtained with a second order approximation [1].

As expected, models A and B have identical first order dynamics dynamics. Thus, if technology and government expenditure shocks could correctly be identified, one would be able recover the true dynamics to structural disturbances using a constant coefficient model even when the DGP displays exogenous parameter variations. The responses of models C and D differ from those of a constant coefficient model. With both shocks, it is primarily the shape and the persistence of consumption and capital responses which is altered primarily because income and substitution effects are different from those in model A. For instance, in responses to technology shocks, agents work less, save less and consume more in models C and D than in the constant coefficients model, while in response to government expenditure shocks consumption fall more and capital fall less relative to the constant coefficients case. Finally confirming proposition 2.4, second order approximation deliver responses that differ across models only if they different in a first order approximation. The magnitude of the responses and the direction of the differences in the two systems is similar, indicating that the magnitude of second order terms is small - nonlinearities in RBC models are known to become important only for extreme values of certain parameters - see e.g. Fernandez Villaverde and Rubio Ramirez (2004).

3 Characterizing misspecification due to time variations

One way to characterize the misspecification produced by a constant coefficient model when the true DGP has time varying coefficients is to use "wedges" (see Chari et al., 2008). In our case, wedges are calculated using the optimality conditions of the model with constant coefficients and the available data for X_t. Thus, while $\mathrm{E}f(X_{t+1}, X_t, X_{t-1}, Z_{t+1}, Z_t, \Theta_{t+1}, \Theta_t) = 0$ when (Z_t, X_t) are generated by f, it will have expected value different from zero for any $f^* \neq f$. Since $f(X_{t+1}, X_t, X_{t-1}, Z_{t+1}, Z_t, \Theta_{t+1}, \Theta_t)$ has a different format depending on the time varying specification one considers, wedges will be non-zero in general. In the RBC example previously considered, the components of f corresponding to the Euler equation and the resource constraint differ in models with fixed and time varying coefficients. Furthermore, additional terms appear in the Euler equation when agents internalize the effects their capital decisions

[1] Since the responses of hours and output to government expenditure shocks are somewhat different from what the conventional wisdom indicates, a few words of explanations are needed. In a standard RBC in response to government expenditure shocks hours and output typically increase because of the wealth effect that the shocks generates. However, the shock here affects the share of government expenditure in GDP. Thus, the positive wealth effect on labor supply is absent because government expenditure will increase exactly in the same proportion as output thus disincentivating agents to try to increase private output.

have on the parameters.

Because of the relevance of the Euler equation in determining income and substitution effects and the dynamics in response to structural shocks, we focus attention on the Euler wedge. Clearly, any component of the optimality conditions can used for diagnostic purpose. The expression for the Euler wedge is

$$E_t w_{t+1}^j = E_t[\beta(\frac{c_{t+1}^j}{c_t^j})^{-\eta} r_{t+1}^j] \qquad (36)$$

where $r_{t+1}^j = (\frac{\alpha y_{t+1}^j}{k_{t+1}^j} + 1 - \delta)$, and the superscript refers to data generated by model j. Let $PK_{t+1}^i = \beta(\frac{c_{t+1}^i}{c_t^i})^{-\eta}$ and $r_{t+1}^i = (\frac{\alpha y_{t+1}^i}{k_{t+1}^i} + 1 - \delta)$ be the pricing kernel and the real rate obtained with the constant coefficients DGP. Equation (36) can be rewritten as

$$E_t w_{t+1}^j = E_t[PK_{t+1}^i r_{t+1}^i][(\frac{c_{t+1}^j}{c_t^j})(\frac{c_t^i}{c_{t+1}^i})]^{-\eta}[\frac{r_{t+1}^j}{r_{t+1}^i}] \qquad (37)$$

The first term in (37) has expected value of 1; the other two terms are the wedges produced because consumption growth and the real rate may differ from those of the constant coefficient model. Because of these wedges, $E_t w_{t+1}^j \neq 1$, unless i=j.

In the RBC economy and using a long realization (T=1000) of (Z_t, Θ_t), we find that $E_t w_{t+1}^j$ is equal to 1.012 (DGP model B), to 1.008 (DGP model C) and to 1.011 (DGP model D). The time path of the Euler wedges and of their components are plotted in figure 3.

The means of the consumption growth wedge are 1.0015, 1.0021 and 1.0013. Since the volatilities are substantial the consumption growth profile generated by the constant coefficients and the time varying coefficients models are different. The means of the real rate wedge are 0.9995, 0.9998, 1.0001, respectively and the volatilities are small. Furthermore, a scatterplot of w_{t+1}^j against the consumption growth and the real rate wedges suggests that there is a tight positive relationship between w_{t+1}^j and the consumption growth wedge $[(\frac{c_{t+1}^j}{c_t^j})(\frac{c_t^A}{c_{t+1}^A})]^{-\eta}$ and a more blurred negative relationship between w_{t+1}^j and the real rate wedge $[\frac{r_{t+1}^j}{r_{t+1}^A}]$. Hence, neglecting covariance terms (which are small in this case), deviations of $E_t w_{t+1}^j$ from unity are mainly due to the fact that the consumption growth series produced by a constant coefficient model and by time varying coefficient models are different.

These arguments suggest a simple way to assess the misspecification due to time variations. If the DGP is a model with constant coefficients, $E_t w_{t+1}^j - 1$ should be a martingale difference and thus unpredictable given the information available at time t. However, if the data has been generated by a model with time varying coefficients and a researcher employs a constant coefficient model in the analysis, $E_t w_{t+1}^j - 1$ will not be a martingale difference and variables contained in the information set may help to predict it. Thus, a regression of w_{t+1}^j on a constant and on the lags of the observable variables should have non-zero and significant coefficients.

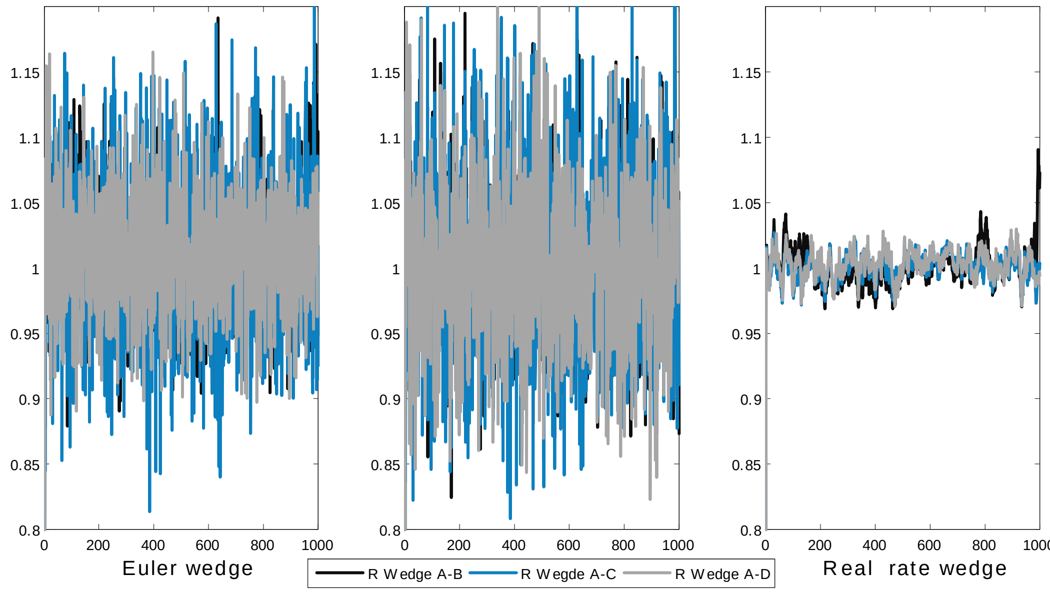


Figure 3: Simulated wedges

Applying this reasoning to the data generated by the RBC model, we find that the coefficients on the first lag of the real rate are 1.06 (DGP model B) 1.81 (DGP model C) and -0.59 (DGP model D) all significant different from zero, while the coefficients on the first lag of consumption growth are -0.0008 (DGP model B), 0.06 (DGP model C) and -0.05 (DGP model D) and the second is significantly different from zero.

Note that here we are jointly examining the hypothesis that the model is correctly specified under the null and that the coefficients are constant. To check what would happen to our diagnostic when the model is incorrectly specified but coefficients are constants, we simulate data from an RBC model with constant coefficients and one period time to build and consider our baseline model with no time to build and constant coefficients. We find that the coefficients on the first lag of the real rate and of consumption growth are -0.03 and 0.06, both insignificantly different from zero.

An alternative way to characterize the misspecification generated by a constant coefficients specification when the DGP has time varying coefficients is via forecast errors. The linearized decision rule in a constant coefficients model is $x_t^i = P^i x_{t-1}^i + Q^i z_t$, and in a time varying coefficients model is $x_t^j = P^j x_{t-1}^j + Q^j z_t + R^j u_t$. Let v_t^j be the forecast error in predicting x_t^j using the decision rules of the constant coefficient model and the data from model j. The forecast error can be decomposed into

$$v_t^j = x_t^j - P^i x_{t-1}^j = Q^j z_t + R^j u_t + (P^j - P^i) x_{t-1}^j \tag{38}$$

Thus, the forecast errors are functions of the lags of the observables x_{t-1}^j. This is obvious when P^j is different from P^i. However, even if $P^j = P^i$, the forecast error could be function of the lags of the observables as long as u_t is serially correlated. Thus, a simple diagnostic for model misspecification involves regressing the forecast errors v_t^j on lagged values of the observables and checking the significance of the regression coefficients.

For the RBC model, and focusing for illustration on the forecast error produced in the hour equation, we find that lagged values of the endogenous variables are significant regardless of the process generating time variations (see table 1) and that an F-test strongly rejects the null hypothesis that all the coefficient are jointly zero.

DGP	n_{t-1}	k_{t-1}	y_{t-1}	c_{t-1}	Ftest, P-value
B	0.08	-0.40	0.05	0.51	0.00
	(0.004)	(0.006)	(0.007)	(0.002)	
C	0.08	-0.28	-0.15	0.43	0.00
	(0.002)	(0.007)	(0.003)	(0.29)	
D	0.27	0.09	0.33	-1.93	0.00
	(0.06)	(0.01)	(0.02)	(0.21)	

Table 1: Regression coefficients, hours equation; the dependent variable is the forecast error obtained using the decisions rules of model A and data from the model in the first column; the indipendent variables are in the first row. In parenthesis standard errors.

3.1 Exogenous vs. endogenous parameter variations

If the wedge and the forecast error diagnostics indicate the presence of parameter variations, one may interested in knowing whether they are of exogenous or of endogenous type. One way to distinguish what type of time variations is present is to use the DGSE-VAR methodology of Del Negro and Schorfheide (2004). In a DSGE-VARs one uses the DSGE model as a prior for the VAR of the observable data and employs the marginal likelihood to measure the value of the additional information the DSGE provides. Intuitively, a DSGE prior can be thought as a set additional observations added to the VAR model. If the additional observations come from the DGP, the quality of the estimates will be improved (standard errors will be reduced), and the marginal likelihood, which measures the fit of the specification, increased. On the other hand, if the additional observations come from a DGP different from the one generating the data, noise will be added, and the precision of the estimates and the fit of the model reduced. Thus, for a given data set, a researcher comparing the marginal likelihood produced by adding data from the exogenous and the endogenous specifications, should be able to detect whether the observable sample is more likely to be generated by one of the two models.

Let $L(\alpha|y)$ be the likelihood of the VAR model for data y and let $g_j(\alpha|\gamma_j, M_j)$ be the prior induced by the DSGE model M_j using parameters γ_j on the VAR parameters α .The marginal likelihood is $h_j(y|\gamma_j, M_j) = \int L(\alpha|y)g_j(\alpha|\gamma_j, M_j)d\alpha$ which, for given y,is a function of M_j. Since $L(\alpha|y)$ is fixed, $h_j(y|\gamma_j, M_j)$ reflects the plausibility of $g_j(\alpha|\gamma_j, M_j)$ in the data. Thus, if g_1 and g_2 are two DSGE-based priors and $h_1(y|\gamma_1, M_1) > h_2(y|\gamma_2, M_2)$, there is better support for in the data for g_1.

Applying this technology to the RBC example, for a sample size of $T = 150$, we report in table 2 the log marginal likelihood when $T_1 = 150, 750$ simulated data from the DSGE listed in the first row are added to the actual data. Both when T_1 is large relative to T, and when it is of the same magnitude, the marginal likelihood is able to pick the correct DGP in all experiments. The differences within columns are quite large, even when $T_1 = T$, indicating that the diagnostic is quite informative.

	T_1=150			T_1=750		
DGP	Model B	Model C	Model D	Model B	Model C	Model D
Simulated from B	1586	-6709	-5108	9714	-3478	-12597
Simulated from C	1421	2005	-855	7480	4828	-409
Simulated from D	697	-2649	1864	6083	622	11397

Table 2: Log marginal likelihood obtained using T data points produced by the models listed in the first row and T_1 simulated data from the model listed in the first column.

4 Parameter identification

Since forecast errors are typically used to construct the likelihood function via the Kalman filter, one should expect the misspecification present in the forecast errors to spread to the likelihood function, making parameter estimate problematic. In this section we examine whether the time invariant parameters of a model can be identified from a potentially misspecified likelihood function. Canova and Sala (2009) have shown that standard DSGE models feature several population identification problems, intrinsic to the models and to the solution method employed. These problems typically show up because several structural parameters turn out to be weakly or partially identified and others close to be underidentified. The issue we are concerned with here is whether parameters which could be identified if the likelihood function would be correctly specified became weakly or underidentified because the likelihood function is constructed using the wrong forecast errors. In other words, we ask whether weak or underidentification of time invariant parameters may emerge as a by-product of the assuming that the model is structural when it is not. Magnusson and Mavroedis (2014) have shown that when GMM is used, time variations in certain parameters help rather than hurt in the identification of time invariant parameters. Huang (2014) qualifies the result by showing that time variations in weakly identified parameters have no effect on the asymptotic distribution of strongly identified parameters.

Figures 4 and 5 plot the likelihood function of the RBC model in the risk aversion coefficient γ and the share parameter η; and in the labor share α and the autoregressive parameter of the technology ρ_ζ when the forecast errors of the correct model (top row) and of the constant coefficient model (bottom row) are used to construct the likelihood function. The first column considers data generated by the model B, the second data generated by model C and the third data generated by model D.

While the curvature of the likelihood constructed using the correct model in not large, it is easy to verify that the maximum occurs at $\gamma = 2, \eta = 2, \alpha = 0.30, \rho_\zeta = 0.9$ for all three specifications displaying time varying discount factor and time varying depreciation rate. When the decision rules of the constant coefficients model are used to construct the likelihood function and the true DGP is model B, distortions are present and the risk aversion coefficient γ become very weakly identified. Thus at least in the (γ, η) dimensions the curvature of the likelihood is altered and estimation of γ becomes difficult.

When the true model features endogenous time variations, distortions are larger - likelihood function become convex in ρ_ζ, γ and α become very weakly identified - and the maximum in the ρ_ζ is shifted away from the true value.

Hence, at least in the example we consider, weak identification problems could be the result of neglected time variations in certain parameters. Nevertheless, one should note that the likelihood has generally more curvature when parameter variations are neglected, even though the increased curvature comes at the cost of incorrectly centering the likelihood, making it locally convex, and introducing ridges that may complicate inference about the parameters.

These observations are confirmed when one uses the Koop et al. (2013) statistic, which we report in table 4. Koop et al. show that asymptotically the precision matrix of the estimates grows at the rate T for identified parameters and at rate less than T for underidentified parameters. Thus, the precision of the estimates scaled by the sample size should converge to a constant for identified parameters and to zero for underidentified parameters. Furthermore, the magnitude of the constant can be used to assess identification strength: a large value indicates a strongly identified parameter; a small value a weakly identified one.

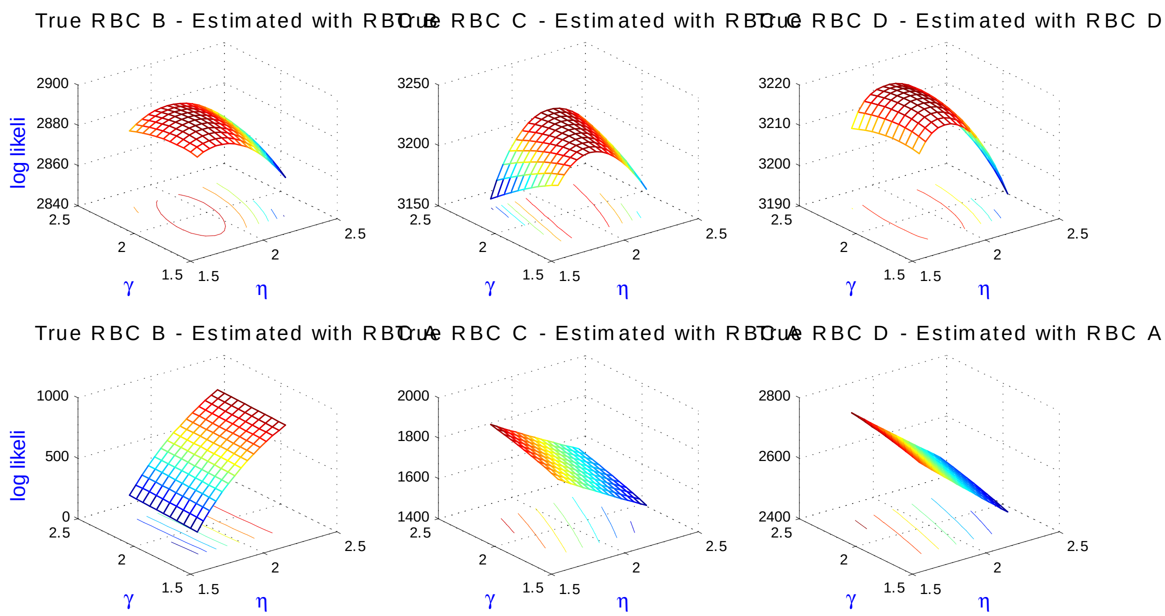


Figure 4: Likelihood surfaces

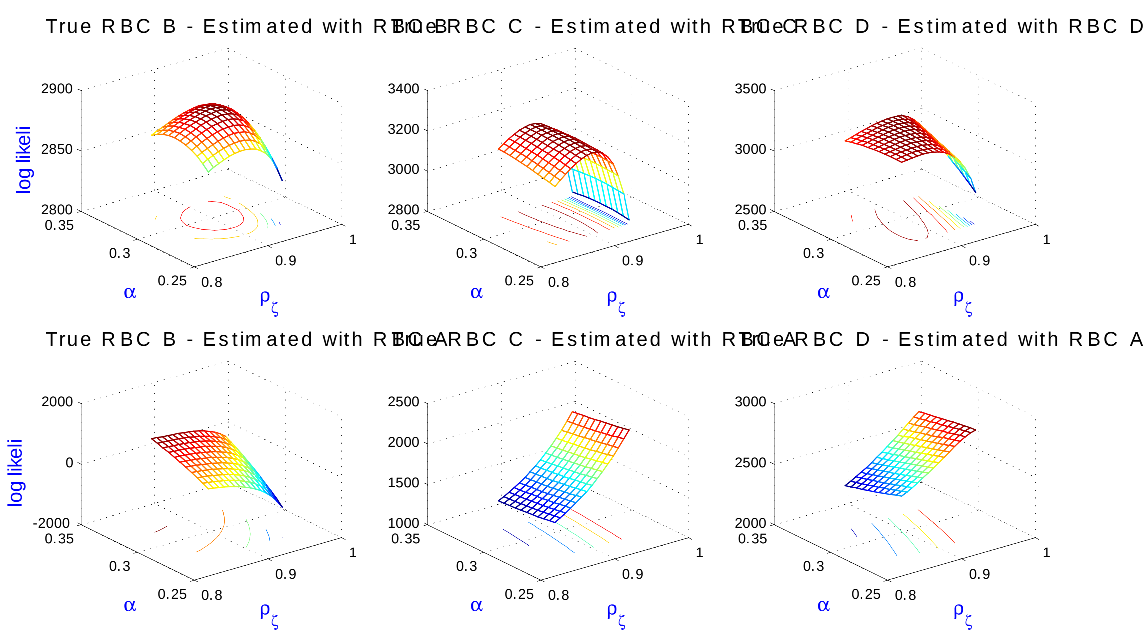


Figure 5: Likelihood surfaces

When the DGP is model B and a fixed coefficients model is estimated, all parameters are identified, even though some weak identification issues for A and η exist. When the DGP are models C and D, all parameters but ρ_g seem identifiable even for a small sample size. The curvature of the likelihood in the ρ_g dimension is small but the maximum can be found without numerical difficulties.

It is interesting to note, that when the DGP are models B and C, ρ_g is weakly identified, even when the correct likelihood is used. Thus, the presence of time variations in β_t and δ_t does not help in the identification of ρ_g, a result which is in line with the conclusions of Huang (2014), since both β and δ are weakly identified in the constant coefficient version of the model, see Canova and Sala, (2009).

Koop, Pesaran, Smith diagnostic							
Parameter	T=150	T=300	T=500	T=750	T=1000	T=1500	T=2500
	DGP Model B, Estimated model A						
η	15.9	17.8	17.2	18.8	18.4	19.3	17.9
γ	28.5	45.7	108.4	81.4	93.6	104.2	90.17
ρ_z	1.8e+4	2.6e+4	4.2e+4	4.2e+4	4.5e+4	4.9e+4	4.37e+4
ρ_g	209.2	655.5	2741	2190	2860	3417	2802
δ	927.3	973.8	1.7e+4	1.7e+4	2.4e+4	2.3e+4	2.5e+4
α	140.2	156.2	264.2	215.5	239.1	252.1	229.3
A	28.42	30.67	7.99	10.99	9.15	7.83	9.83
	DGP Model C, Estimated model A						
η	822	1033	743	785	759	746	752
γ	2261	3147	2682	2809	2720	2579	2566
ρ_z	3073	2673	2952	2909	2799	2806	2877
ρ_g	1.74	2.23	2.44	2.96	3.17	2.82	2.90
δ	4.6e+5	4.4e+5	4.3e+5	4.0e+5	3.8e+5	4.4e+5	4.3e+5
α	1.8e+4	1.1e+4	1.4e+4	1.2e+4	1.1e+4	1.6e+4	1.5e+4
A	351	493	441	505	500	449	444
	DGP Model D, Estimated model A						
η	550	575	592	610	545	542	494
γ	3577	2442	2660	2870	2564	2711	2430
ρ_z	1613	1243	1120	1162	1068	1189	1074
ρ_g	1.22	1.28	1.44	1.53	1.60	1.62	1.67
δ	5.2e+5	6.7e+5	6.5e+5	6.0e+5	5.7e+5	5.8e+5	5.7e+5
α	1.1e+4	2.5e+4	2.4e+4	1.9e+4	2.1e+4	2.0e+4	2.1e+4
A	488	276	340	382	349	395	334

Table 3: Koop et al diagnostic. Different sample sizes

5 Likelihood estimation with a misspecified model

To study the properties of likelihood based estimates of a misspecified fixed coefficients model, we conduct a Monte Carlo exercise. We generate either 150 or 1000 data points from versions B, C, D of the RBC model previously considered, estimate the structural parameters using the likelihood function constructed with the decision rules of the time invariant model A, and repeat the exercise 150 times using different realizations of the shocks. We also estimate the structural parameters using the likelihood constructed using the decision rules of the correct model (i.e. model B rules if the data has been generated by model B, and so on), to account for numerical difficulties one may encounter in estimating the model.

We consider two setups, one where time variations in the parameters are small (2-5 percent of the variance of output is explained by shocks to the parameters, henceforth DGP1) and one where time variations in the parameters are substantial (around 20 percent of the variance of output is explained by shocks to the parameters, henceforth DGP2). Table 4 has the results for DGP1: it reports the fixed parameters used to generate the data (column 1), the mean posterior estimate (across replications) obtained when the likelihood is constructed using the correct decision rules (column 2), and the mean posterior estimate, the 5th and the 95 percentile of the distribution of estimates obtained when the likelihood function is constructed with the decision rules of the time invariant model when T=150 (columns 3-5) and when T=1000 (columns 6-8). Table A1 in the appendix has the results for DGP2. Figure 6 presents the distributions of estimates for DGP1: the vertical line represents the true parameter value; in solid black lines we have distributions obtained with the correct model; in solid blue lines (solid red lines) the distributions obtained with the incorrect constant coefficient model when T=150 (T=1000). Figure A1 in the appendix has the same information for DGP2. When the model is correctly specified, the distribution of estimates should collapse around the true value. Thus, if the mean is away from the true parameter value and/or the spread of the distribution is large, one can conclude that likelihood based methods have difficulties in recovering the constant parameters of the data generating process. Figure 7 presents the impulse responses for DGP1: in the first three columns are the responses to technology shocks and in the last three the responses to government spending shocks. In each box we report the impulse response obtained using mean value of the distribution of estimates produced with the correct model, and the 16th and 84th percentiles of the distribution of impulse responses obtained using the estimated distribution of parameters produced by the time invariant model. Figure A2 in the appendix has the same information for DGP2. Finally, table 6 presents the long run variance decomposition for DGP1 (first four columns) and DGP2 (last four columns) when T=150 and the mean posterior estimate is used in the computations In each block, the first two columns have the variance decomposition to technology and government spending shocks in the correct model; the last two columns have the variance decomposition to technology and government spending shocks when the incorrect time invariant model is used.

For the two time varying parameters, we set $d_t = \beta_{t+1}/\beta_t$, and assume that in model B, $\Theta_{t+1} - \Theta \equiv (d_{t+1}(1-\rho_\beta)\beta, \delta_{t+1} - (1-\rho_\delta)\delta)' = U_{t+1}$, where $\beta = 0.99$,the two components of $U_{t+1} = (u_{d,t+1}, u_{\delta,t+1})'$ are independent AR(1) process with persistence $\rho_d = 0.9, \rho_\delta = 0.8$, and standard deviations $\sigma_d = 0.002, \sigma_\delta = 0.07$. For models C and D, the law of motion of the time varying parameters is $\Theta_{t+1} = [\Theta_u - (\Theta_u - \Theta_l)e^{-\phi_a(K_t-K)}] + [\Theta_u - (\Theta_u - \Theta_l)e^{\phi_b(K_t-K)}] + U_{t+1}$, where $\Theta_u' = (0.9999, 0.03)$, $\phi_a' = (0.03, 0.2)$, $\phi_b' = (0.031, 0.1)$, U_{t+1} is iid with Σ_u diagonal and $\sigma_d = 0.03, \sigma_\delta = 0.008$.

True Parameter	Correct Mean T=150	Time invariant Mean T=150	5th percentile	95th percentile	Time invariant Mean T=1000	5th percentile	95th percentile
DGP Model B							
$\eta = 2.0$	2.00	2.03	1.47	2.88	2.32	1.55	3.37
$\gamma = 2.0$	2.02	1.23	-0.14	2.07	0.96	-0.38	2.04
$\rho_z = 0.98$	0.97	0.99	0.97	1.00	0.99	0.96	1.00
$\rho_g = 0.5$	0.47	0.74	0.60	0.96	0.87	0.77	0.98
$\delta = 0.025$	0.03	0.01	0.01	0.02	0.01	0.01	0.05
$\alpha = 0.3$	0.30	0.19	0.11	0.28	0.23	0.15	0.40
$A = 4.5$	4.55	2.79	1.33	4.12	2.68	1.23	4.06
DGP Model C							
$\eta = 2.0$	2.00	2.42	1.63	3.85	2.85	1.73	6.14
$\gamma = 2.0$	2.00	0.64	-0.26	1.77	0.60	-0.50	1.79
$\rho_z = 0.98$	0.98	0.99	0.97	1.00	0.97	0.85	1.00
$\rho_g = 0.5$	0.48	0.43	-0.10	0.96	0.65	0.27	0.98
$\delta = 0.025$	0.03	0.01	0.01	0.02	0.02	0.01	0.09
$\alpha = 0.3$	0.30	0.22	0.13	0.34	0.29	0.18	0.47
$A = 4.5$	4.49	2.14	1.18	3.47	2.37	1.18	3.66
DGP Model D							
$\eta = 2.0$	2.00	2.58	1.69	3.34	2.40	1.74	3.26
$\gamma = 2.0$	2.01	0.29	-0.28	1.54	1.09	-0.30	1.99
$\rho_z = 0.97$	0.96	0.99	0.94	1.00	0.96	0.91	1.00
$\rho_g = 0.5$	0.48	0.51	-0.26	0.96	0.66	0.39	0.98
$\delta = 0.025$	0.02	0.01	0.01	0.03	0.01	0.01	0.02
$\alpha = 0.3$	0.30	0.22	0.14	0.35	0.22	0.15	0.30
$A = 4.5$	4.52	2.32	1.42	3.68	3.45	1.37	4.51

Table 4: Distributions of posterior estimates whe parameter variations explain a small amount of output variance.

A few features of the results are worth discussing. First, notice that when the correct model is employed, estimation is very successful even when T=150, regardless of the DGP employed and of whether time variations are exogenous or endogenous. Thus, there seems to be no numerical distortions one should worry about. Second, with the first DGP, a number of distortions occur when a time invariant model is used in estimation. For example, when exogenous variations are present, the persistence of government spending shock is poorly estimated (mean persistence is about 50 percent larger than the true one), while estimates of δ, α and A are severely biased downward. The distortions are smaller when the time variations are endogenous (models C and D): nevertheless significant downward bias exists for the inverse of the Frish elasticity γ, for δ and α. Notice that the performance of the time invariant model is similar if we have externalized or internalized endogenous time variations. Third, the performance of the time invariant model does not improve, and if anything worsens, when T=1000 for all three specifications. Thus, as sample size increases, failure to converge to the true DGP becomes even more obvious.

When the model features time variations which explain a significant portion of the variability of output, the above features become more striking. For example, when parameter variations are exogenous, estimating a time invariant model leads to an overestimation of the persistence of the structural shocks. Thus, the only way a time invariant model can accommodate the additional serial dynamics and variability present in the endogenous variables is by increasing the persistence of both technology and government spending shocks. In models C and D the distortions become considerably larger and for example, the mean posterior estimate of inverse of the Frish elasticity is now negative. Furthermore, the distribution of estimates are typically skewed and multimodal. Thus, neglecting parameter variations is more detrimental when the variations account for a significant portion of the variability of the endogenous variables.

The impulse responses we obtain present confirm these conclusions. When parameter variations explain a small fraction of the variability of output, we find that responses to technology shocks are off in terms of impact magnitude, in particular for output, the response produced with the parameters estimates obtained with the true model tend to be on the upper limit of the estimated 68 percent band, even though their shape is consistent with the shape of the bands produced with the time invariant model. Interestingly, output responses are those which more poorly characterized with the time invariant model and, consistent with previous findings, the misspecification obtained when the true model features exogenous time variations is larger. On the contrary, the responses to government expenditure shocks obtained with a time invariant model are different from those obtained estimating the correct model in terms of magnitude, shape and persistence. Since the signal that government expenditure produces in the model is weak, it is not surprising that it is obscured by the presence of time variations.

The distortions obtained when parameter variations are important for the variance of output are generally larger, but the pattern of results is similar. To be noticed is the fact that, the persistence of the responses to technology shocks is poorly estimated: while responses obtained estimating the true model tend to zero after 10 years, the bands obtained estimating a time invariant model do not include zero after 10 years.

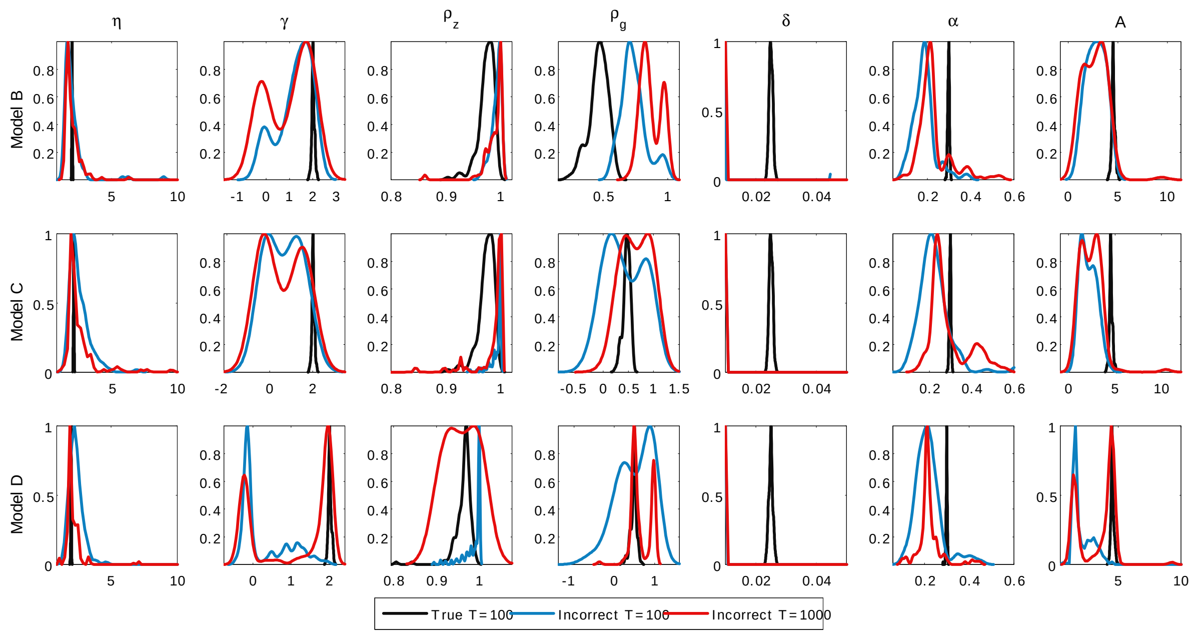


Figure 6:Density of estimates; DGP1 (time variations explain little of output variance).

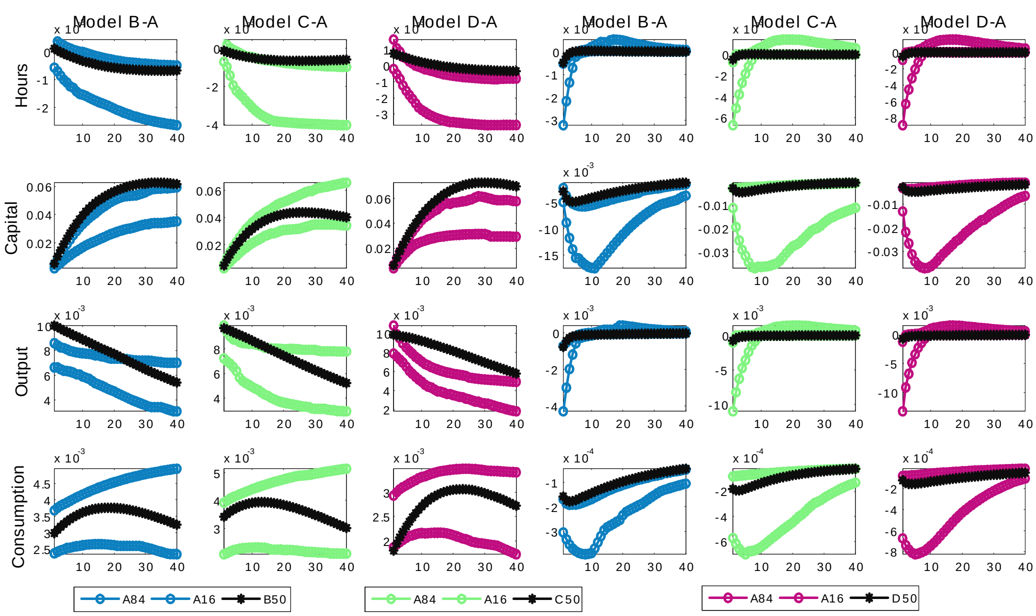


Figure 7:Impulse responses, DGP1

	Variance decomposition							
	DGP: Small time variations				DGP: Large time variations			
Variable	Technology	Government	Technology	Goverment	Technology	Government	Technology	Goverment
	Model B		Time invariant		Model B		Time invariant	
Y	94.1	0.3	99.7	0.4	81.3	0.1	99.8	0.6
C	89.5	0.2	99.9	0.1	55.3	0.1	99.8	0.2
N	60.2	0.5	98.6	1.4	15.6	0.4	97.8	2.3
K	70.2	0.4	99.5	0.6	40.6	0.1	99.4	0.8
	Model C		Time invariant		Model C		Time invariant	
Y	97.2	0.3	98.8	1.6	81.9	0.1	92.7	8.2
C	88.1	0.3	99.9	0.1	26.5	0.1	99.9	0.1
N	44.6	0.6	99.0	1.2	5.4	0.4	96.6	3.9
K	84.4	0.2	99.0	1.4	37.4	0.1	97.4	3.0
	Model C		Time invariant		Model C		Time invariant	
Y	98.0	0.1	99.3	1.5	82.2	0.1	93.6	7.2
C	92.2	0.2	99.8	0.3	32.8	0.1	99.6	0.8
N	35.9	0.5	97.3	3.4	10.2	0.5	92.8	7.9
K	96.6	0.3	99.2	1.2	60.0	0.4	97.9	2.8

Table 5: Long run variance decomposition

What is the contribution of structural shocks to the variability of the endogenous variables when the forecast errors of the time invariant model are used to construct the likelihood function? In general, one should expect the structural shocks of the time invariant model to be a contaminated version of the structural shocks of the time varying DGP for two reasons. First, the wrong P matrix is used to compute forecast errors. As we have previously seen this creates a correlation between the forecast errors and the lags of the endogenous variables and thus changes the timing of the innovations. Second, we are aggregating m (structural and parameter) shocks into $n < m$ (structural) shocks. This aggregation is known to generate complicated VARMA structures where the n structural shocks are functions of the leads and lags of the original disturbances (see e.g. Canova and Paustian, 2011). Thus, even if the P matrix were correctly specified, one should expect distortions to occur, unless the omitted shocks (those to the parameters) are unimportant and feature low persistence.

When parameter variations do not explain a large portion of the variance of output, technology shocks absorb the variability that is missing in the time invariant model. This is true regardless of whether time variations are exogenous or endogenous and the effect seems particularly strong for hours worked. When parameter variations explain a larger portion of the variance of output, technology shock still absorb a large amount of the missing variability. However, in some cases, the missing variability is also captured by government spending shocks. For example, while government spending shocks explain only 0.1 percent of the variability of output in the long run in the true model when parameter variations are endogenous, they explain 7-8 percent when a time invariant model is used.

In sum, for the DGP we consider and the parameterization employed, we find that estimating a constant parameter model when the DGP features time varying parameters leads to parameter distortions, regardless of the sample size, of whether variations are exogenous or endogenous, and of whether parameter variations matter for output variability or not. The parameters that are mostly distorted are those regulating the estimated persistence of the exogenous shocks and those controlling income and substitution effects. Since the Euler equation and the resource constraints are the optimality conditions which are different in the constant and time varying coefficient models, it is clear where these distortions come from [2]

6 Recovering structural dynamics with SVAR methods

Our results so far suggest that if the DGP features parameter variations, the decision rules derived with a time invariant model are distorted, and that these distortions lead to an incorrectly specified likelihood function, to biased structural parameter estimates, and to impulse responses that fail to capture the dynamics induced by structural shocks. Because of these problems, one may wonder whether less structural and computationally less demanding methods are competitive with likelihood based methods if structural dynamics is all that matters to the investigator. Canova and Paustian (2011) have shown that when the model is misspecified VAR methods which employ robust sign restrictions can be effective in capturing the dynamics induces by the structural disturbances. The misspecification they consider however is different - the models they employ omit certain features of the DGP; misspecification here comes from the presence of parameter variations.

The exercise we conduct in this section is as follows. Using the illustrative RBC model, we simulate data from the decision rules of models B, C, and D when parameters variations generate small output volatility (DGP1 in the previous section). We then compute VAR residuals using the population P matrix of the correct model and of the constant coefficient model, rotate the resulting residuals using an orthonormal matrix, and keep the responses if technology shocks generate a positive response of hours, capital, output and consumption on impact and if government expenditure shocks generate a negative response of hours, output,

[2] We have also performed a Monte Carlo exercise where the labor share is also time varying. Variations in the labor share have been documented in the literature (see e.g. Rios Rull and Santaeularia Llopis, 2010) and there is evidence that variations in this parameter are strongly countercyclical. This is relevant for our exercise because, in this case, all four optimatility conditions are affected by time variations. Thus, the strength of the income and substitution effect distortions are likely to be larger. Indeed, we do find that distortions in this case become quite large and it many cases it becomes difficult to estimate the time invariant model regardless of the DGP (results for this setup are available on request).

consumption and capital - these signs are those present in figure 1 and hold for variations of the (constant) structural parameters within a reasonable range. We repeat the exercise 150 times and collect the distribution of impulse responses to technology and government expenditure shocks for the correct and the time invariant specification and plot in figures 8-10 the median responses obtained with the correct model (red line in each box) and the 16 and 84 percentile of the distribution of responses obtained with the incorrect model. Figure 8 has the responses when the DGP is model B, figure 9 when the DGP is model C and figure 10 when the DGP is model D.

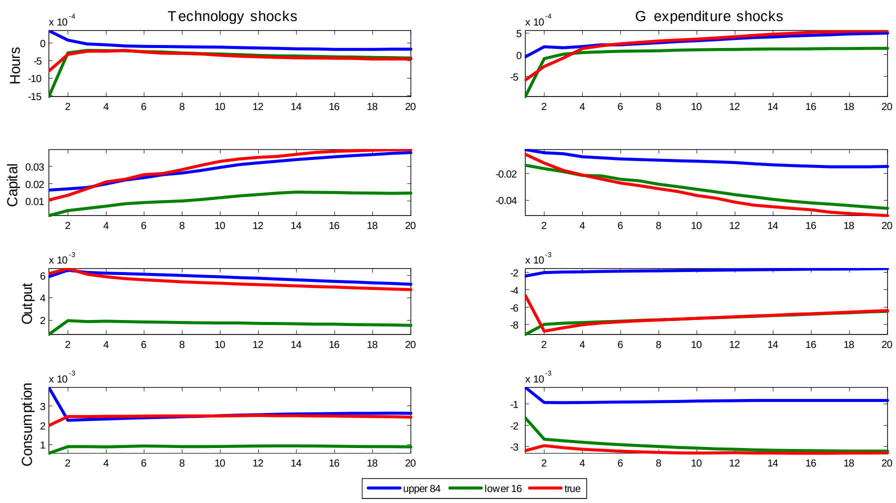


Figure 8: Impulse responses, exogenously varying and constant coefficient RBC models

Overall, SVAR methods seem to be competitive with likelihood methods when parameter variations are neglected. When the DGP is model B, the sign and the shape of the responses is correctly captured. Note that the although the responses to technology shocks obtained with the true model tend to be on the upper bound of the band obtained with the incorrect model and the responses to government spending shocks obtained with the true model tend to be on the lower bound of the bands obtained with the incorrect model, the performance of SVARs is as least as good as the one likelihood methods.

The performance with the other two DGPs is similar even though the details are slightly different. With model C it is the magnitude of the dynamic response of consumption which is misrepresented for both shocks, while with model D it is primarily the true persistence of certain responses to the two shocks that is somewhat underestimated.

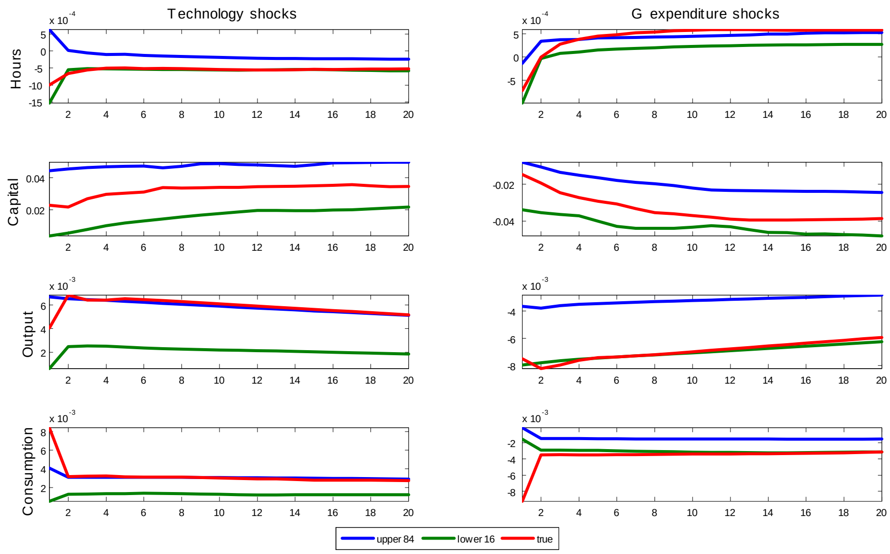


Figure 9: Impulse responses, endogenously varying (no internalization) and constant coefficients RBC models

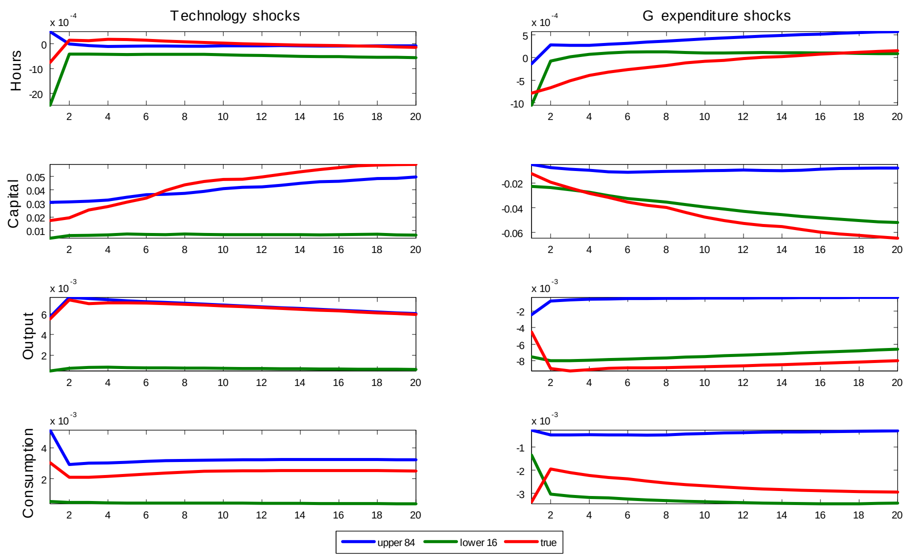


Figure 10: Impulse responses, endogenously varying (with internalization) and constant coefficients RBC models

Recall that here, two sources of misspecification are present: the P matrix is incorrect; aggregation problems are present. What our analysis indicates is that with the DGP we use i) the distortion in the P matrix are small; ii) the Q matrix is not very strongly affected by the misspecification; iii) shock misaggregation is minor. Because shocks to the parameters are i.i.d., timing distortions in the VAR innovations are also likely to be small.

In sum, when the parameters of the model, SVAR analyses with a time invariant model will be as or more successful than likelihood based inference in particular when examining the dynamics of government spending shocks. But to reach this conclusions, robust identification methods need to be used.

7 Time varying financial frictions?

We apply the technology developed in the paper to the unconventional monetary policy model of Gertler and Karadi (GK) (2010). Our contribution is three fold. First, we provide likelihood based estimates of the parameters specific to the model (the fraction of capital that can be diverted by banks λ, the proportional transfer to entering bankers ω, and the survival probability of bankers θ), that the authors have informally calibrated to match a steady state spread, a steady state leverage and a notional length of bank activity. Second, we use the diagnostic developed in the paper to gauge the extent of time variations in the parameters of model. Third, we provide estimates of the time variations present in tλ and compare responses to capital quality shocks in the fixed coefficient and the time varying coefficient models.

The equations of the GK model are summarized in the appendix B. We use US data from 1985Q2 to 2014Q3 on the growth rate of output, growth rate of consumption, growth rate of leverage, and growth intermediary demand for assets (credit) and the spread. The spread is measured by the difference between BAA 10 years corporate bond yields and a 10 year treasury constant maturity and it is from the FRED as are real personal consumption expenditures and GDP data. Leverage is from Haver and measures Tier 1 (core) capital as a percent of average total assets. Credit is measured as total loans (from Haver), scaled by size of US population.

Using Bayesian methods with a uniform prior, the estimates we obtain are $\lambda = 0.245$, $\theta = 0.464$, $\omega = 0.012$. The standard error are very tight in all cases (0.0182, 0.0008, 0.0098) making the estimates highly significant. For comparison, GK calibrated these three parameters to $\lambda = 0.318$, $\theta = 0.972$, $\omega = 0.002$. In the GK model λ regulates private leverage: our estimate implies a higher steady state leverage than the one implied by the authors (our estimate is 3.32, their is 1.38) and closer to the leverage found in the US in corporate and non-corporate business sectors over the sample. Our estimates also suggest that the survival probability of bankers is much lower than the one assumed by GK (about 10 years).

With these parameter estimates, we first perform forecast error mispecification diagnostics. Table 6 indicates that the forecast errors of all equations but consumption are highly predictable and typically lagged consumption and lagged spread are the variables which are most significant.

Equation	T-stat					F-stat
	Y_{t-1}	C_{t-1}	Credit$_{t-1}$	Leverage$_{t-1}$	Spread$_{t-1}$	
Y	0.84	2.61	0.24	0.52	10.00	4.39
C	-0.85	1.11	0.85	-0.65	0.33	1.26
Credit	1.06	2.61	1.65	-0.58	8.49	7.11
Leverage	-1.11	-2.50	-1.63	0.63	-8.25	7.04
Spread	-1.26	-3.06	-1.10	0.81	-8.46	8.16

Table 6: Regression diganostic for time variation. The left hand side of the regression is the forecast in the equation listed in the first column; the right hand side the variables listed in the second to the fifth column. Sample size T=116. Critical level of the F-stat(5,112)=2.56.

We have also computed the Euler wedge and checked whether lagged values of consumption and the investment to output ratio explain its dynamics. The mean value of the wedge is 0.02 with a standard error of 0.03; but both lag consumption and lag investment to output ratios significantly explain its movements (coefficients are respectively -0.10 and 0.72, with standard errors of 0.01 and 0.13). Thus, there seem to be evidence of misspecification and parameter variation could be the reason for it.

Armed with this preliminary evidence we estimate models where wee allow λ to be time varying.

Time variations are specified as

$$\lambda_t = (1-\rho_\lambda)\lambda + \rho_\lambda \lambda_{t-1} + e_{t,\lambda} \qquad \text{Exogenous variations} \tag{39}$$

$$\lambda_t = (2 * \lambda_u - (\lambda_u - \frac{\lambda}{2}) * (\exp(-\phi_1 * (X_{t-1} - X^s)) + \exp(\phi_2 * (X_{t-1} - X^s))) + e_{t,\lambda} \qquad \text{Endogenous variations} \tag{40}$$

where X net bank wealth N. Table 7 reports estimates of selected parameters

Note that, in the model with exogenously varying parameters, there are very persistent variations in λ_t. Furthermore, the estimates of λ, ω, θ are now larger making steady state leverage drop to about 2.9 and the lifetime of bankers to increase. When the endogenous specification is used, estimates of λ and ω further increase making steady state leverage fall to 1.9, but bankers survival probability is roughly unchanged. The data seem to require a very strong asymmetric specification ($\phi_1 < \phi_2$) implying a very strong negative relationship between the fraction of funds that bankers will steal and their net worth. Finally, note that the endogenous specification is highly preferable in marginal likelihood sense to the specification with exogenous time variations and to the fixed ceofficient specification.

To investigate how inference would differ in the three estimated model we plot in figure xx the responses of output, inflation, investment, net worth, leverage and the spread to a one percent capital quality shock. The constant coefficient specification

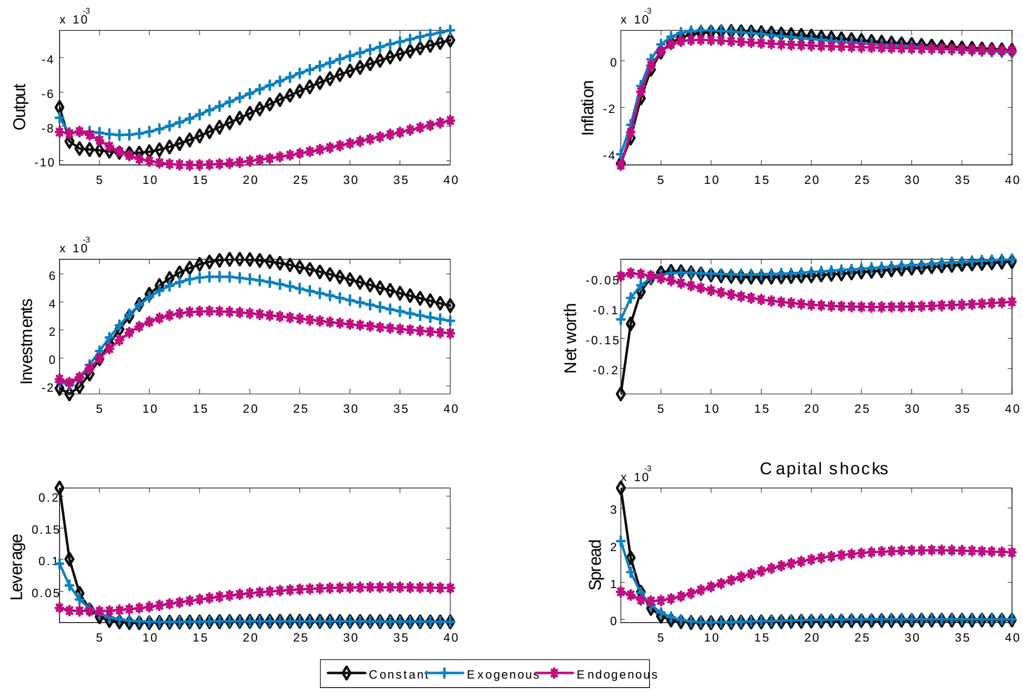
Parameter	Time Invariant		Exogenous TVC		Endogenous TVC	
					Function of net worth	
h	0.43	(0.006)	0.19	(0.03)	0.09	(0.02)
λ	0.24	(0.01)	0.37	(0.03)	0.55	(0.03)
ω	0.01	(0.008)	0.02	(0.002)	0.11	(0.008)
θ	0.46	(0.009)	0.54	(0.01)	0.52	(0.02)
ρ_λ			0.99	(0.004)		
σ_λ			0.02	(0.002)	0.03	(0.003)
λ_u					0.98	(0.008)
ϕ_1					0.02	(0.007)
ϕ_2					0.15	(0.009)
Log ML	-167.97		1546.18		1628.69	

Table 7: Parameter estimates.

closely replicate the dynamics presented by GK in their figure 3. There is a persistent decline in output a temporary and strong decline in inflation. Investment falls temporarily but it then increase because capital is below its steady state. Bankers net worth falls and there is a sharp increase in the spread between the expected return to capital and the riskless rate.

When we allow λ to be exogenously varying the qualitative features of the responses are very similar. Quantitatively, output falls more on impact but less in the short run, net worth falls less and the spread increases less in the short run. Thus, making λ exogenosuly time varying, reduces the ability of the model to capture the impact magnitude of the recession.

When variations are instead endogenous, the model possesses an additional mechanism of propagation of shocks since lower net worth implies higher share of funds diverted by banks and generally stronger accelerator dynamics. Since the dynamic responses of net worth are highly persistent, the spread persistently increases making investment increase less relative to the previous two cases and output to fall more and more presistently after a capital quality shock. Thus, neglecting that λ could be endogenously varying impair our ability to assess the consequences of shocks in the model.



Capital quality shock

8 Conclusions

This paper has a number of goals. It is interrested in i) characterizing the decision rules of a DSGE when parameter variations are exogenous or endogenous, and in the latter case, when agents internalize or not the effects that their optimal decisions may have on parameter variations; ii) measuring how distorted are standard statistics when a researcher erroneously assumes a time invariant structure but the data generating process (DGP) features time varying parameters; iii) providing diagnostics to detect the misspecification driven by parameter variations; iv) studying the consequences of using time invariant models when the parameters are time varying in terms of identification, estimation, inference, and policy analyses; v) comparing likelihood-based and SVAR-based estimates of the dynamic responses to structural shocks when time variations in the parameters are neglected.

In term of decision rules, we show that if parameter variations are purely exogenous, the contemporaneous impact and the dynamics induced by structural shocks are the same as in a model with no parameter variations. However, if parameter variations are endogenous, the instantaneous impact and the dynamics in responses to structural shocks may be different from the one of a constant coefficient model and the extent of the differences in the two specifications depends on the detail of the model.

We provide simple and powerful diagnostics to detect the misspecification induced by employing a time invariant model when the data has been generated by a time varying coefficient model using the optimality wedges of Chari et al. (2007) and the forecast errors of the model. We also describe a marginal likelihood diagnostics which can help us to recognize whether the time variations detected with these statistics are of exogenous or endogenous nature.

We show certain parameter identification problem noted in the literature may be the results of misspecification due to neglected time variations, even though the likelihood distortions that neglected time variations produce are probably more important. Our Monte Carlo study confirms that parameter and impulse response distortions may be large even for modest time variations in the parameters and that they tend to be stronger when variations are truely endogenous.It also demonstrates that, when parameter variations are neglected, SVAR methods are competitive with more structural likelihood based methods as far are characterizing the responses to structural disturbances. Thus, the hedge that likehood based methods have when the model is correctly specified vanishes when the model is an incorrect description of the data generating process.

In the context of the Gertler and Karadi (2010) model, we show that there is evidence that the parameter regulating the amount of moral hazard present in the model is indeed time varying taht that time variations are possibly linked to the amount of net worth bankers have. When we allow for this link, the fit of the model to the data dramatically improves, primarily because the model acquires an additional propagation channel which makes spread and thus output responses stronger and much more persistent.

Our analysis provides researcher with a new set of tools that can help them to respecify assess the quality of their models and respecify certain problematic features. There are a few interesting questions we do not address in this paper. How do we distinguish a model with time varying parameters from a model measurement errors are present? Is a model with m structrual shocks observationally equivalent to a model with m_1 structrual shocks and m_2 time varying parameters where $m_1 + m_2 = m$? To what extent parameter variations capture variations in the variances (or in higher moments) of the structrual shocks? We leave the answer to these questions for future research.

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Appendix A: Additional Monte Carlo figures and tables

True Parameter	Correct Mean	Time invariant Mean	5th percentile	95th percentile	Time invariant Mean	5th percentile	95th percentile
	T=150	T=150			T=1000		
DGP Model B							
$\eta = 2.0$	2.00	2.29	1.53	3.87	2.45	1.61	3.09
$\gamma = 2.0$	2.01	1.11	-0.33	2.06	0.25	-0.27	1.95
$\rho_z = 0.9$	0.94	0.99	0.96	1.00	0.99	0.97	1.00
$\rho_g = 0.5$	0.47	0.76	0.62	0.96	0.91	0.79	0.98
$\delta = 0.025$	0.03	0.01	0.01	0.03	0.01	0.01	0.01
$\alpha = 0.3$	0.30	0.19	0.11	0.41	0.21	0.10	0.34
$A = 4.5$	4.53	2.73	1.33	4.14	1.80	1.14	4.16
DGP Model C							
$\eta = 2.0$	2.00	3.40	1.56	7.51	5.19	1.77	22.90
$\gamma = 2.0$	2.00	-0.08	-0.32	0.73	-0.19	-0.35	0.35
$\rho_z = 0.9$	0.88	0.99	0.93	1.00	0.99	0.90	1.00
$\rho_g = 0.5$	0.48	0.56	0.08	0.97	0.91	0.59	0.98
$\delta = 0.025$	0.02	0.02	0.01	0.07	0.02	0.01	0.07
$\alpha = 0.3$	0.30	0.26	0.15	0.34	0.26	0.19	0.35
$A = 4.5$	4.50	1.71	1.25	2.77	2.27	1.24	8.17
DGP Model D							
$\eta = 2.0$	2.00	3.05	1.68	4.59	2.40	1.98	4.81
$\gamma = 2.0$	2.00	-0.06	-0.28	0.54	1.63	-0.27	1.98
$\rho_z = 0.9$	0.88	0.98	0.90	1.00	0.92	0.91	1.00
$\rho_g = 0.5$	0.47	0.42	-0.46	0.96	0.50	0.32	0.97
$\delta = 0.025$	0.02	0.01	0.01	0.03	0.01	0.01	0.01
$\alpha = 0.3$	0.30	0.23	0.15	0.32	0.21	0.13	0.27
$A = 4.5$	4.49	1.91	1.45	3.57	4.10	1.65	4.51

Table 8: Distributions of estimates, Parameter variations explain 20 percent of output variance.

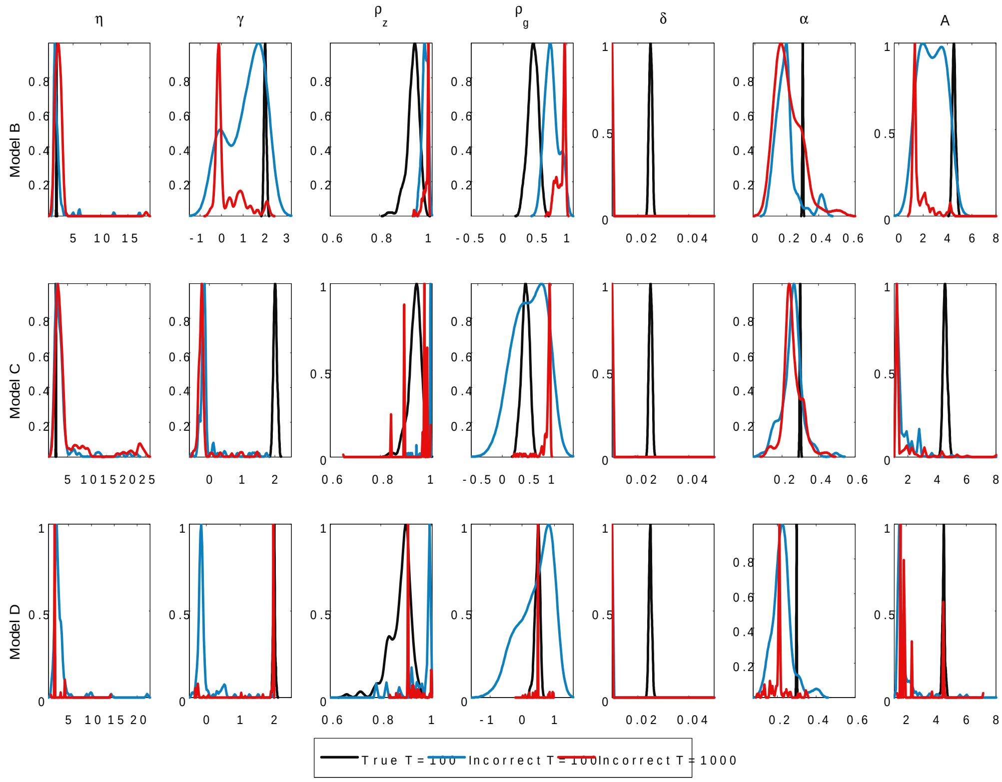

Figure A1:Density of estimates; parameter variations explain 20 percent of output variance

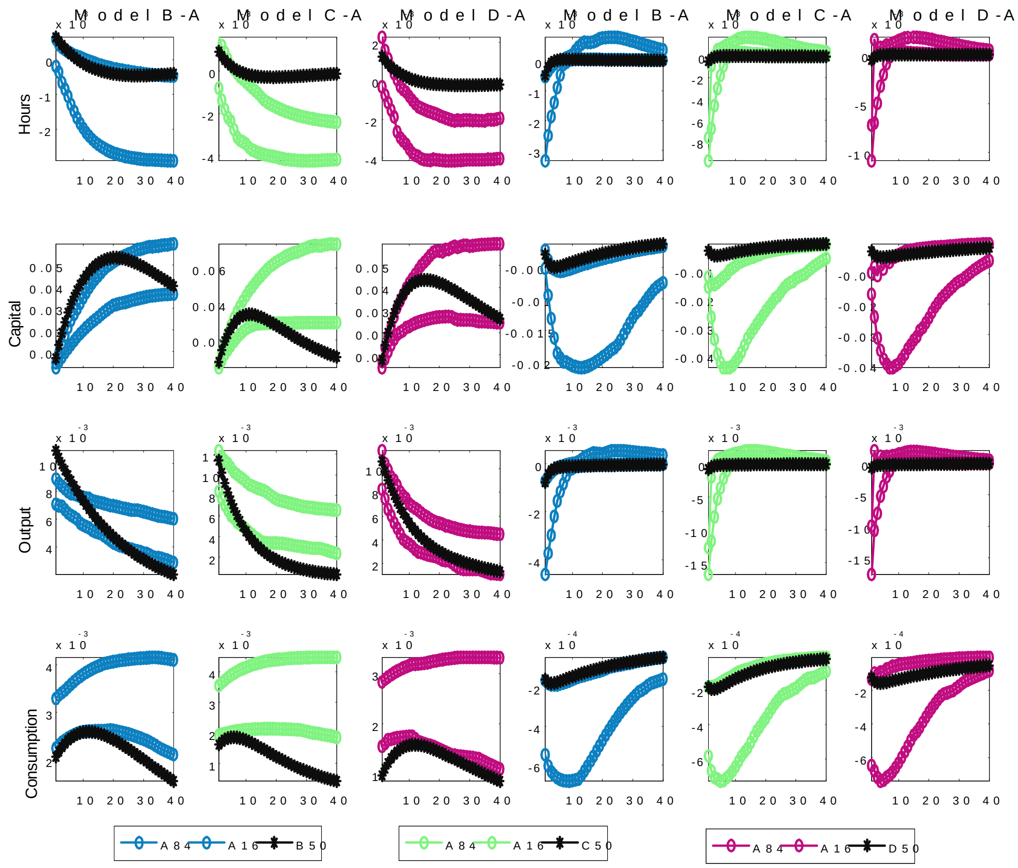

Figure A2:Impulse responses; parameter variations explain 20 percent of output variance

Appendix B : The equations of Gertler and Karadi model

$$
\begin{aligned}
\exp(\varrho_t) &= (\exp(C_t) - h\exp(C_{t-1}))^{-\sigma} - \beta h(\exp(C_{t+1}) - h\exp(C_t))^{-\sigma} && (41)\\
1 &= \beta\exp(R_t)\exp(\Lambda_{t+1}) && (42)\\
\exp(\Lambda_t) &= \frac{\exp(\varrho_t)}{\exp(\varrho_{t-1})} && (43)\\
\chi * \exp(L_t)^{\varphi} &= \exp(\varrho_t)\exp(P_{m,t})(1-\alpha)\frac{\exp(Y_t)}{\exp(L_t)} && (44)\\
\exp(\nu_t) &= (1-\theta)\beta\exp(\Lambda_{t+1})(\exp(R_{k,t+1}) - \exp(R_t)) + \beta\exp(\Lambda_{t+1})\theta\exp(x_{t+1})\exp(\nu_{t+1}) && (45)\\
\exp(\eta_t) &= (1-\theta) + \beta\exp(\Lambda_{t+1})\theta\exp(z_{t+1})\exp(\eta_{t+1}) && (46)\\
\exp(\phi_t) &= \frac{1}{(1-\psi_t)}\frac{\exp(\eta_t)}{\lambda - \exp(\nu_t)} && (47)\\
\exp(z_t) &= (\exp(R_{k,t}) - \exp(R_{t-1}))(1-\psi_{t-1})\exp(\phi_{t-1}) + \exp(R_{t-1}) && (48)\\
\exp(x_t) &= \frac{\exp(\phi_t)(1-\psi_t)}{(\exp(\phi_{t-1})(1-\psi_{t-1}))}\exp(z_t) && (49)\\
\exp(K_t) &= \frac{\exp(\phi_t)\exp(N_t)}{\exp(Q_t)} && (50)\\
\exp(N_t) &= \exp(Ne_t) + \exp(Nn_t) && (51)\\
\exp(Ne_t) &= \theta\exp(z_t)\exp(N_{t-1})\exp(-e_{Ne,t}) && (52)\\
\exp(Nn_t) &= \omega(1-\psi_{t-1})\exp(Q_t)\exp(\xi_t)\exp(K_{t-1}) && (53)\\
\exp(R_{k,t}) &= (\exp(P_{m,t})\alpha\frac{\exp(Y_{m,t})}{\exp(K_{t-1})} + \exp(\xi_t) * (\exp(Q_t) - \frac{\exp(\delta))}{\exp(Q_{t-1})} && (54)\\
\exp(Y_{m,t}) &= \exp(a_t) * (\exp(\xi_t) * \exp(U_t) * \exp(K_{t-1}))^{\alpha} * \exp(L_t)^{1-\alpha} && (55)\\
\exp(Q_t) &= 1 + 0.5\eta_i(\frac{(In_t + I^s)}{(In_{t-1} + I^s} - 1)^2 + \eta_i(\frac{(In_t + I^s)}{(In_{t-1} + I^s} - 1)\frac{(In_t + I^s)}{(In_{t-1} + I^s s)}\\
&- \beta\exp(\Lambda_{t+1})\eta_i(\frac{(In_{t+1} + I^s)}{(In_t + I^s} - 1)(\frac{(In_{t+1} + I^s)}{(In_t + I^s)})^2 && (56)\\
\exp(\delta) &= \delta_c + b/(1+\zeta) * \exp(U_t)^{1+\zeta} && (57)\\
\alpha\frac{\exp(Y_m)}{\exp(U_t)} &= \frac{b\exp(U_t)^{\zeta}\exp(\xi_t) * \exp(K_{t-1})}{\exp(P_{m,t})} && (58)\\
In_t &= \exp(I_t) - \exp(\delta) * \exp(\xi_t) * \exp(K_{t-1}) && (59)\\
\exp(K_t) &= \exp(\xi_t) * \exp(K_{t-1}) + In_t && (60)\\
\exp(G_t) &= G^s * \exp(g_t) && (61)\\
\exp(Y_t) &= \exp(C_t) + \exp(G_t) + \exp(I_t) + 0.5\eta_i(\frac{(In_t + I^s)}{(In_{t-1} + I^s} - 1)^2(In_t + I^s) + \tau\psi\exp(K_t) && (62)\\
\exp(Y_{m,t}) &= \exp(Y_t) * \exp(D_t) && (63)
\end{aligned}
$$

$$
\begin{aligned}
\exp(D_t) &= \gamma * \exp(D_{t-1}) * \exp(infl_{t-1})^{-\gamma_P * \epsilon} \exp(infl_t)^{\epsilon} \\
&+ (1-\gamma)((1-\gamma \exp(infl_{t-1})^{\gamma_P(1-\gamma)} \exp(infl_t)^{\gamma-1})/(1-\gamma))^{-\epsilon/(1-\gamma)} && (64) \\
\exp(X_t) &= 1/\exp(P_{m,t}) && (65) \\
\exp(F_t) &= \exp(Y_t) * \exp(P_{m,t}) + \beta\gamma \exp(\Lambda_{t+1}) \exp(infl_{t+1})^{\epsilon} (\exp(infl_t))^{-\epsilon\gamma_P} \exp(F_{t+1}) && (66) \\
\exp(Z_t) &= \exp(Y_t) + \beta\gamma \exp(\Lambda_{t+1}) \exp(infl_{t+1})^{\epsilon-1} \exp(infl_t)^{\gamma_P*(1-\epsilon)} \exp(Z_{t+1}) && (67) \\
\exp(infl_t^*) &= \frac{\epsilon}{\epsilon-1} \frac{\exp(F_t)}{\exp(Z_t)} \exp(infl_t) && (68) \\
(\exp(infl_t))^{1-\epsilon} &= \gamma \exp(infl_{t-1})^{\gamma_P(1-\epsilon)} + (1-\gamma)(\exp(infl_t^*))^{1-\epsilon} && (69) \\
\exp(i_t) &= \exp(R_t) * \exp(infl_{t+1}) && (70) \\
\exp(i_t) &= \exp(i_{t-1})^{\rho_i} (\beta^{-1} \exp(infl_t)^{\kappa_\pi} * (\exp(X_t)/(\epsilon/(\epsilon-1)))^{\kappa_y})^{1-\rho_i} \exp(e_{i,t}) && (71) \\
\psi_t &= \kappa * (R_{k,t+1} - R_t - R_k^s + R^s) + e_{\psi,t} && (72) \\
a_t &= \rho_a * a_{t-1} - \sigma_a * e_{a,t} && (73) \\
\xi_t &= \rho_\xi * \xi_{t-1} - \sigma_\xi * e_{\xi,t} && (74) \\
g_t &= \rho_g * g_{t-1} - e_{g,t} && (75) \\
e_{\psi,t} &= \rho_\psi * e_{\psi,t-1} + e_{\psi,t}; && (76)
\end{aligned}
$$