Conditioning and Updating under Cumulative Prospect Theory

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Abstract

Under expected utility theory, unconditional expected utility can be decomposed into a weighted sum of conditional expected utilities where the weights are marginal probabilities. We derive necessary and sufficient conditions for a similar decomposition in the framework of Cumulative Prospect Theory (CPT). The conditions also ensure that a decision maker's conditional preferences (given some event) remain within the CPT class. Our results are important for empirical analyses in which weighted marginal probabilities of events are used to explain a decision maker's choices. The use of such marginal probabilities is a practical necessity in non-experimental settings.

1   Introduction

This paper is motivated by the need for empirical modeling of risky choices under alternatives to expected utility theory. So far, most tests of alternative utility theories against expected utility have been based on experiments in which test subjects had to express preferences with respect to simple lotteries. Such lotteries are, however, extremely rare

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outside the laboratory. In real life, decision makers typically choose between compound lotteries, i.e. payoffs are tied to events that are non-degenerate sets of states and are defined in terms of random variables that correlate with other state variables.[1]

Under expected utility theory, compound lotteries can, without further ado, be reduced to simple lotteries and unconditional expected utility can be decomposed into marginal probabilities and conditional expected utilities. This paper specifies necessary and sufficient conditions for a similar decomposition in the framework of Cumulative Prospect Theory (CPT). The conditions also guarantee that conditional preferences remain within the class that has a sign and rank dependent CPT representation. These properties are important for empirical analyses of decision makers' behavior outside the laboratory, when marginal probabilities enter as explanatory variables through "probability weights".

To fix ideas, we consider a specific example. Suppose a decision maker (DM) wishes to insure himself against an event A and faces a choice between two different insurance policies Q and R. For insurance $m \in \{Q, R\}$, the DM pays a premium c_m and receives a gross payment of t_m from the insurance company if event A occurs. In the absence of insurance, the DM's income is given by the random variable z.

In real world situations, the insured event A as well as its complement will consist of a number of states of the world. For example, the DM could be a farmer who buys insurance against the event that it rains more than a specific amount. More of his crops will be ruined if the actual rainfall is well above this amount than if it is only just above it; likewise his crops will be affected by different amounts of rain below the amount that triggers the insurance payment. In such contexts, the probability of the insured event A must be interpreted as a marginal probability since the event is a set of states. We denote this probability by p_A.

Under expected utility, we can analyze the DM's behavior based on the standard decomposition of unconditional expected utility into marginal probabilities and conditional expected utility. That is, insurance Q is weakly preferred to insurance R if

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\begin{aligned}
p_A E[u(z - c_Q + t_Q)|A] + (1 - p_A)E[u(z - c_Q)|A^{\complement}] \geq \\
p_A E[u(z - c_R + t_R)|A] + (1 - p_A)E[u(z - c_R)|A^{\complement}].
\end{aligned}
\tag{1}
$$

This decomposition enables a description of the DM's choice with an econometric model in which the probability p_A is used as an explanatory variable. The decomposition is valid since, under the implicit assumption of dynamic consistency, Bayesian updating

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[1]In analyses of laboratory experiments, it is commonly assumed that lotteries are based on random variables that are independent of anything of "real life" relevance to the decision makers. It is in this sense that the experiments are based on simple lotteries.

is a corollary of the subjective expected utility theorem. That is, with a state space $S = \{1, \ldots, n\}$ and prior probabilities $\pi(s)$, the posterior probabilities after arrival of information that the true state is in $A \neq \emptyset$ are given by

$$\pi_A(s) = \pi(s|A) = \begin{cases} \frac{\pi(s)}{\pi(A)} & \text{if } s \in A \\ 0 & \text{if } s \notin A. \end{cases}$$

The present paper is concerned with preferences that can be represented by a sign and rank dependent CPT representation, as in Tversky and Kahneman (1992), Wakker and Tversky (1993), and Prelec (1998).[2] We establish conditions under which (i) a sign and rank dependent CPT representation is maintained for updated preferences *and* (ii) a decomposition similar to (1) holds under CPT. These conditions enable empirical analyses based on models in which marginal probabilities like p_A are used as explanatory variables that determine DMs' choices via probability weights $w(p_A)$. The use of weighted marginal probabilities is a practical necessity in non-experimental settings since it will rarely be possible to specify the joint distribution of all state variables that matter to DMs and correlate with the random variable that defines whether event A occurs.[3]

Our approach is to impose dynamic consistency in the sense that optimal contingent plans remain optimal contingently. Given the result in Karni and Schmeidler (1991), since a CPT-utility maximizer does not satisfy the independence axiom, another dynamic assumption has to be relaxed. We relax consequentialism and allow preferences conditional on an event to depend on the fact that its complement could have occurred. Machina (1989) provides an argument in favor of not imposing consequentialism for non-expected utility maximizers. In addition to dynamic consistency of preferences, which (by definition) is satisfied by our updating rule, Epstein and Le Breton (1993) advocate dynamic consistency of models, meaning that axioms imposed on the initial preference ordering should be satisfied also by subsequent updated orderings. Since our conditions ensure that updated preferences remain within the CPT class, our update rules satisfy dynamic consistency in this sense as well.

Gilboa and Schmeidler (1993) define a set of f-Bayesian update rules, where f can be interpreted as "what does the DM implicitly assume would have resulted if the event on which conditioning takes place had not occurred." We refer to this f as the DM's benchmark prospect. For expected utility, the benchmark can be any possible act, i.e.

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[2]Quiggin (1982) and Yaari (1984) provide models in which utility is rank dependent but not sign dependent.

[3]If it were possible to specify the joint distribution of all relevant state variables, the DMs' choices could be modeled in terms of preferences with respect to one-stage lotteries.

the update rule does not depend on the benchmark. For probability weights in CPT representations, however, the prospect used as a benchmark does matter for the update rule, just like the benchmark act matters for Gilboa and Schmeidler's updating of capacities when preferences can be represented by Choquet Expected Utility.

In the present paper, we provide necessary and sufficient restrictions on the benchmark prospect in order to obtain a decomposition of unconditional CPT utility into weighted marginal probabilities and conditional CPT utility, which parallels the standard decomposition of expected utility.[4] We derive the corresponding update rules for probability weighting functions when preferences can be represented by CPT. The update rules are based on benchmark prospects that either make the conditioning event seem more or less extreme. For this reason, we name our conditioning and update rules extremal and moderating conditioning and updating. Our update rules have in common with Hanany and Klibanoff's (2007) update rules for Max-Min Expected Utility preferences that the update rules depend on preferences, the conditioning event, and the feasible set of acts.

The paper is structured as follows: Section 2 presents notation and definitions. Section 3 contains our results while Section 4 contains a further discussion of the econometric issues.

2 Extremal and Moderating Conditioning and Updating

Let $S = \{1, \ldots, n\}$ be a finite state space and let X be a compact set of outcomes that describe changes with respect to the status quo or an appropriate reference level. The states in S occur with probabilities $(p_1, \ldots, p_n)$. Let $\mathscr{F}$ denote the set of prospects, i.e. functions from S to X. For an outcome $x \in X$, let $\bar{x} \in \mathscr{F}$ denote the constant prospect $(x, \ldots, x)$. The status quo is an element of the set of outcomes X, denoted by 0, and is assumed fixed. For any prospect $f \in \mathscr{F}$, $f(s)$ denotes the outcome that f returns in state s.

Let $\succsim$ denote a preference relation on $\mathscr{F}$, with $\succ$ and $\sim$ denoting the asymmetric and symmetric parts respectively. We use the same notation for preference relations on the set of outcomes X, i.e. for any $x, y \in X$, $x \succsim y$ if and only if $\bar{x} \succsim \bar{y}$. An outcome $x \succ 0$ is positive and an outcome $x \prec 0$ is negative. We refer to the set of non-positive outcomes

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[4]Such a decomposition is not generally valid for non-standard preferences. For example, it is not valid for the update rules of Chateauneuf, Eichberger, and Grant's (2007) neo-additive capacites considered in Eichberger, Grant, and Kelsey (2010). The general non-validity is also pointed out by Wang (2003).

$X_L \equiv \{x \in X \ : \ 0 \succsim x\}$ as the loss domain and the set of non-negative outcomes $X_G \equiv \{x \in X \ : \ x \succsim 0\}$ as the gain domain. Define $\sum_{j=a}^{b} z_j \equiv 0$, whenever $b < a$.

We assume throughout that preferences are represented by a Cumulative Prospect Theory (CPT) representation. With the notation from Prelec (1998) this means that in general, for $x_n \succsim \cdots \succsim x_{k+1} \succsim 0 \succsim x_k \succsim \cdots \succsim x_1$, preferences are represented by

$$V(f) = \sum_{i=1}^{k} \left[w^-\left(\sum_{j=1}^{i} p_j\right) - w^-\left(\sum_{i=1}^{i-1} p_j\right) \right] v(x_i) + \sum_{i=k+1}^{n} \left[w^+\left(\sum_{j=i}^{n} p_j\right) - w^+\left(\sum_{j=i+1}^{n} p_j\right) \right] v(x_i),$$
(2)

where $w^-(\cdot)$ and $w^+(\cdot)$ are unique nondecreasing weighting functions satisfying $w^-(0) = w^+(0) = 0$ and $w^-(1) = w^+(1) = 1$ and $v(x)$ is a continuous and increasing ratio scale. For each i, the expression in brackets is the decision weight associated with outcome x_i. Wakker and Tversky (1993) provide an axiomatization for the CPT representation. Since an axiomatization is provided elsewhere, we take the representation as given and provide the additional conditions needed to obtain our results.

We are concerned with how this preference relation is updated upon arrival of the information that the true state is in some event $A \subset S$. Our updating rules apply to prospects whose outcomes are comonotonic. A set of prospects H are comonotonic if for no $h, g \in H$ and no $s, s' \in S$, it holds that $h(s) \succ h(s')$ and $g(s') \succ g(s)$. We also restrict attention to prospects for which the conditioning events A and $A^{\complement}$ have $A^{\complement}$ being a dominating event for A: for all $s \in A$ and for all $s' \in A^{\complement}$, $h(s') \succsim h(s)$. That is, the outcomes associated with states in the event A are (weakly) worse than the outcomes associated with states in the complement of the event, $A^{\complement}$. Let $\mathscr{H}$ denote a generic class of comonotonic prospects for which $A^{\complement}$ is a dominating event for A.[5]

Let o be a permutation on $\{1, \ldots, n\}$ such that for the class of comonotonic prospects $\mathscr{H}$ under consideration, $h(o(n)) \succsim \ldots \succsim h(o(1))$ for all $h \in \mathscr{H}$. Hence, a prospect is non-positive if $0 \succsim h(o(n))$, while a prospect is non-negative if $h(o(1)) \succsim 0$. For ease of notation, we henceforth write $h_{o(i)}$ rather than $h(o(i))$, whenever this does not create any confusion.

We assume that the events A and $A^{\complement}$ are both assigned non-zero weights in the CPT-representation. Since for all $h \in \mathscr{H}$ we have that $\forall s \in A$, and $\forall s' \in A^{\complement}$, $h_{s'} \succsim h_s$, there exists $k \in \{1, \ldots, n\}$ such that $A = \{o(1), \ldots, o(k)\}$ and $A^{\complement} = \{o(k+1), \ldots, o(n)\}$.

Define $\mathscr{H}_L \subset \mathscr{H}$ to be the set of non-positive prospects in $\mathscr{H}$, and define $\mathscr{H}_G \subset \mathscr{H}$ to be the set of non-negative prospects in $\mathscr{H}$. Also, define $\mathscr{H}_P \subset \mathscr{H}$ to be the set of prospects in $\mathscr{H}$ that constitute a loss/gain partition, i.e. $\mathscr{H}_P = \{h \in \mathscr{H} | h_{o(k+1)} \succsim 0 \succsim h_{o(k)}\}$. Note

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[5]I.e. $\mathscr{H}$ is a class of all prospects that rank the states in a particular order and for which A is dominated by $A^{\complement}$.

that $\mathscr{H}_L \cap \mathscr{H}_G = \mathscr{H}_L \cap \mathscr{H}_P = \mathscr{H}_P \cap \mathscr{H}_G = \{\bar{0}\}$. For prospects whose outcomes are exclusively in the loss domain, i.e. for $h \in \mathscr{H}_L$, the representation of preferences in (2) simplifies to

$$V(h) = \sum_{i=1}^{n} \left[w^-\left(\sum_{j=1}^{i} p_{o(j)}\right) - w^-\left(\sum_{j=1}^{i-1} p_{o(j)}\right) \right] v(h_{o(i)}), \tag{3}$$

while for prospects whose outcomes are exclusively in the gain domain, i.e. for $h \in \mathscr{H}_G$, the representation in (2) simplifies to

$$V(h) = \sum_{i=1}^{n} \left[w^+\left(\sum_{j=i}^{n} p_{o(j)}\right) - w^+\left(\sum_{j=i+1}^{n} p_{o(j)}\right) \right] v(h_{o(i)}). \tag{4}$$

Let $x_* \in X$ denote the worst possible outcome in X and let $x^* \in X$ denote the best possible outcome in X. Note that since $0 \in X$, $x_* \in X_L$ and $x^* \in X_G$. Let $\hat{y}_i \in X_G$ for $i = k+1, \ldots, n$, i.e. non-negative outcomes, and let $\tilde{y}_i \in X_L$ for $i = 1, \ldots, k$, i.e. non-positive outcomes. Let f_* be given by

$$f_*(s) = \begin{cases} x_* & \text{if } s \in A \\ \hat{y}_{k+1} & \text{if } s = o(k+1) \\ \vdots & \vdots \\ \hat{y}_n & \text{if } s = o(n), \end{cases} \tag{5}$$

i.e. a prospect that returns the worst possible outcome if A occurs, and in each state in A^C, some outcome in the gain domain. Define

$$F_* \equiv \{f \in \mathscr{F} | f \text{ is of the form (5)}\}.$$

Also, let f^* be given by

$$f^*(s) = \begin{cases} \tilde{y}_1 & \text{if } s = o(1) \\ \vdots & \vdots \\ \tilde{y}_k & \text{if } s = o(k) \\ x^* & \text{if } s \notin A, \end{cases} \tag{6}$$

i.e. a prospect that in each state in A returns some outcome in the loss domain and the best possible outcome if A^C occurs. Define

$$F^* \equiv \{f \in \mathscr{F} | f \text{ is of the form (6)}\}.$$

Finally, let f_*^* be given by

$$f_*^*(s) = \begin{cases} \tilde{y}_1 & \text{if } s = o(1) \\ \vdots & \vdots \\ \tilde{y}_k & \text{if } s = o(k) \\ \hat{y}_{k+1} & \text{if } s = o(k+1) \\ \vdots & \vdots \\ \hat{y}_n & \text{if } s = o(n), \end{cases} \tag{7}$$

that is, a prospect that in each state in A returns some outcome in the loss domain and in each state in $A^\complement$, some outcome in the gain domain. Define

$$F_*^* \equiv \{f \in \mathscr{F} | f \text{ is of the form } (7)\}.$$

For any $h, f \in \mathscr{F}$, define conditional acts by

$$h_A f = \begin{cases} h(s) & \text{if } s \in A \\ f(s) & \text{if } s \notin A. \end{cases}$$

Define also conditional preference by

$$h \succsim_A g \text{ iff } h_A f \succsim g_A f, \tag{8}$$

i.e. $\succsim_A$ is the DM's preferences conditional on knowing that s is in A. The prospect f is the DM's benchmark for conditioning and updating. Defining conditional preferences as in (8) implies that the DM will be dynamically consistent.

We next provide definitions of extremal and moderating conditional CPT utility.

Definition 1 *The* **loss domain extremal conditional CPT utility** *of prospect h is given by*

$$E_L^e[h|s \in A] \equiv \sum_{i=1}^{k} \frac{w^-\left(\sum_{j=1}^{i} p_{o(j)}\right) - w^-\left(\sum_{j=1}^{i-1} p_{o(j)}\right)}{w^-\left(\sum_{j=1}^{k} p_{o(j)}\right)} v(h_{o(i)}).$$

Notice that the denominator is the sum of the decision weights in the numerator and equals the probability weight of the event A that consists of the states $o(1), \ldots, o(k)$, which in the loss domain give the most extreme outcomes relative to the status quo. The information that has arrived reveals that one of the extreme states will occur.

Definition 2 *The* **loss domain moderating conditional CPT utility** *of prospect h is given by*

$$E_L^m[h|s \in A^\complement] \equiv \sum_{i=k+1}^{n} \frac{w^-\left(\sum_{j=1}^{i} p_{o(j)}\right) - w^-\left(\sum_{j=1}^{i-1} p_{o(j)}\right)}{1 - w^-\left(\sum_{j=1}^{k} p_{o(j)}\right)} v(h_{o(i)}).$$

Again the denominator is the sum of the decision weights in the numerator, which now equals one minus the probability weight of the event $(A^{\complement})^{\complement} = A$. The information that has arrived reveals that one of the less extreme states will occur.

Definition 3 *The* **gain domain extremal conditional CPT utility** *of prospect h is given by*

$$E_G^e[h|s \in A^{\complement}] \equiv \sum_{i=k+1}^{n} \frac{w^+\left(\sum_{j=i}^{n} p_{o(j)}\right) - w^+\left(\sum_{j=i+1}^{n} p_{o(j)}\right)}{w^+\left(\sum_{j=k+1}^{n} p_{o(j)}\right)} v(h_{o(i)}).$$

Once again, the denominator is the sum of the decision weights in the numerator, which here equals the probability weight of the event $A^{\complement}$. Note that $A^{\complement}$ consists of the states $o(k+1), \ldots, o(n)$, which for the gain domain result in the most extreme outcomes relative to the status quo. The information that has arrived reveals that one of the extreme states will occur.

Definition 4 *The* **gain domain moderating conditional CPT utility** *of prospect h is given by*

$$E_G^m[h|s \in A] \equiv \sum_{i=1}^{k} \frac{w^+\left(\sum_{j=i}^{n} p_{o(j)}\right) - w^+\left(\sum_{j=i+1}^{n} p_{o(j)}\right)}{1 - w^+\left(\sum_{j=k+1}^{n} p_{o(j)}\right)} v(h_{o(i)}).$$

Here the denominator equals one minus the probability weight of the event $A^{\complement}$. The information that has arrived reveals that one of the less extreme states will occur.

3   Necessary and sufficient restrictions

For each of the domains $\mathscr{H}_L$, $\mathscr{H}_G$, and $\mathscr{H}_P$, Theorems 1, 2, and 3, respectively, provide necessary and sufficient restrictions on the benchmark prospect in order for unconditional CPT utility to be decomposed into weighted marginal probabilities and conditional CPT utilities, similar to how unconditional expected utility can be decomposed into marginal probabilities and conditional expected utility. The restrictions therefore also ensure that updated preferences remain within the class that can be represented by CPT. The proof of Theorem 1 is given in the appendix, while the proofs of Theorems 2 and 3 are omitted, since they follow the proof of Theorem 1 closely.

Theorem 1 *Suppose there are at least four outcomes in the loss domain and that A and $A^{\complement}$ each contains at least two states. The following two statements are equivalent:*

(a) *For any non-decreasing weighting functions $w^-(\cdot)$ and $w^+(\cdot)$ satisfying $w^-(0) = w^+(0) = 0$ and $w^-(1) = w^+(1) = 1$, for any continuous increasing ratio scale $v(x)$, and for any prospect $h \in \mathscr{H}_L$, preferences conditional on arrival of information that $s \in A$ are represented by $E_L^e[h|s \in A]$, preferences conditional of arrival of information that $s \in A^{\complement}$ are represented by $E_L^m[h|s \in A^{\complement}]$, and unconditional CPT utility is given by*

$$V(h) = w^-(p_A)E_L^e[h|s \in A] + (1 - w^-(p_A))E_L^m[h|s \in A^{\complement}].$$

(b) *The benchmark used for updating is $f_* \in F_*$.*

Proof: See the appendix.■

In the loss domain, A is the most extreme event relative to the reference point. According to Theorem 1, if information arrives that the true state is in A, the DM acts as if he implicitly assumes that outcomes in the gain domain would have resulted had A not occured, which makes A seem even more extreme when conditioning on it. Conditional preferences are then represented by the loss domain extremal conditional CPT utility. On the other hand, if information arrives that the true state is in the less extreme event $A^{\complement}$, the DM acts as if the most extreme outcome in the loss domain would have resulted had $A^{\complement}$ not occured, which makes $A^{\complement}$ seem even less extreme when conditioning on it. Conditional preferences are then represented by the loss domain moderating conditional CPT utility.

Theorem 2 *Suppose there are at least four outcomes in the gain domain and that A and $A^{\complement}$ each contains at least two states. The following two statements are equivalent:*

(a) *For any non-decreasing weighting functions $w^-(\cdot)$ and $w^+(\cdot)$ satisfying $w^-(0) = w^+(0) = 0$ and $w^-(1) = w^+(1) = 1$, for any continuous increasing ratio scale $v(x)$, and for any prospect $h \in \mathscr{H}_G$, preferences conditional on the arrival of information that $s \in A$ are represented by $E_G^m[h|s \in A]$, preferences conditional on the arrival of information that $s \in A^{\complement}$ are represented by $E_G^e[h|s \in A^{\complement}]$, and unconditional CPT utility is given by*

$$V(h) = (1 - w^+(p(A^{\complement})))E_G^m[h|s \in A] + w^+(p(A^{\complement}))E_G^e[h|s \in A^{\complement}].$$

(b) *The benchmark used for updating is $f^* \in F^*$.*

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Proof: The proof is omitted since it follows that of Theorem 1 closely.∎

In the gain domain, $A^\complement$ is the most extreme event relative to the reference point. According to Theorem 2, if information arrives that the state is in $A^\complement$, the DM acts as if he implicitly assumes that outcomes in the loss domain would have resulted had A not occured, which makes $A^\complement$ seem even more extreme when conditioning on it. On the other hand, if information arrives that the state is in the less extreme event A, the DM acts as if the most extreme outcome in the gain domain would have resulted had A not occured, which makes A seem even less extreme when conditioning on it.

Theorem 3 *Suppose there are at least four outcomes in each of the loss and gain domains and that A and $A^\complement$ each contains at least two states. The following two statements are equivalent:*

(a) For any non-decreasing weighting functions $w^-(\cdot)$ and $w^+(\cdot)$ satisfying $w^-(0) = w^+(0) = 0$ and $w^-(1) = w^+(1) = 1$, for any continuous increasing ratio scale $v(x)$, and for any prospect $h \in \mathscr{H}_P$, preferences conditional on the arrival of information that $s \in A$ are represented by $E_L^e[h|s \in A]$, preferences conditional on the arrival of information that $s \in A^\complement$ are represented by $E_G^e[h|s \in A^\complement]$, and unconditional CPT utility is given by

$$V(h) = w^-(p(A))E_L^e[h|s \in A] + w^+(p(A^\complement))E_G^e[h|s \in A^\complement].$$

(b) The benchmark used for updating is $f_*^* \in F_*^*$.

Proof: The proof is omitted since it follows that of Theorem 1 closely.∎

An immediate consequence of Theorems 1, 2, and 3 are the following corollaries, which describe updating of the probability weighting function for the loss domain, the gain domain, and the loss/gain partition, respectively. Let o^{-1} denote the inverse of the permutation function o.

Corollary 1 *Under the conditions of Theorem 1, if the DM uses $f_* \in F_*$ as a benchmark for updating and starts with a prior probability weighting function $w^-(\cdot)$ for the loss domain, his conditional probability weighting function $w_A^-(\cdot)$ given the information that $s \in A$ is*

$$w_A^-\left(\sum_{s' : o^{-1}(s') \leq o^{-1}(s)} p_{s'} \right) = \begin{cases} \dfrac{w^-\left(\sum_{s' : o^{-1}(s') \leq o^{-1}(s)} p_{s'} \right)}{w^-\left(\sum_{j=1}^{k} p_{o(j)} \right)} & \text{if } s \in A \\ 0 & \text{if } s \in A^\complement, \end{cases}$$

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while his conditional probability weighting function $w_{A^{\complement}}^-(\cdot)$ given the information that $s \in A^{\complement}$ is

$$w_{A^{\complement}}^- \left(\sum_{s' \,:\, o^{-1}(s') \leq o^{-1}(s)} p_{s'} \right) = \begin{cases} 0 & \text{if } s \in A \\ \dfrac{w^-\left(\sum_{s' \,:\, o^{-1}(s') \leq o^{-1}(s)} p_{s'}\right) - w^-\left(\sum_{j=1}^k p_{o(j)}\right)}{1 - w^-\left(\sum_{j=1}^k p_{o(j)}\right)} & \text{if } s \in A^{\complement}. \end{cases}$$

Corollary 2 *Under the conditions of Theorem 2, if the DM uses $f^* \in F^*$ as a benchmark for updating and starts with a prior probability weighting function $w^+(\cdot)$ for the gain domain, his conditional probability weighting function $w_A^+(\cdot)$ given the information that $s \in A$ is*

$$w_A^+ \left(\sum_{s' \,:\, o^{-1}(s') \geq o^{-1}(s)} p_{s'} \right) = \begin{cases} \dfrac{w^+\left(\sum_{s' \,:\, o^{-1}(s') \geq o^{-1}(s)} p_{s'}\right) - w^+\left(\sum_{j=k+1}^n p_{o(j)}\right)}{1 - w^+\left(\sum_{j=k+1}^n p_{o(j)}\right)} & \text{if } s \in A \\ 0 & \text{if } s \in A^{\complement}, \end{cases}$$

while his conditional probability weighting function $w_{A^{\complement}}^+(\cdot)$ given the information that $s \in A^{\complement}$ is

$$w_{A^{\complement}}^+ \left(\sum_{s' \,:\, o^{-1}(s') \geq o^{-1}(s)} p_{s'} \right) = \begin{cases} 0 & \text{if } s \in A \\ \dfrac{w^+\left(\sum_{s' \,:\, o^{-1}(s') \geq o^{-1}(s)} p_{s'}\right)}{w^+\left(\sum_{j=k+1}^n p_{o(j)}\right)} & \text{if } s \in A^{\complement}. \end{cases}$$

Note that, according to Corollaries 1 and 2, when conditioning on the most extreme event relative to the reference point, the DM uses an update rule that corresponds to Gilboa and Schmeidler's (1993) optimistic rule: $\nu_A(B) = \dfrac{\nu(B \cap A)}{\nu(A)}$. When conditioning on the less extreme event relative to the reference point, the update rule corresponds to Gilboa and Schmeidler's pessimistic rule $\nu_A(B) = \dfrac{\nu((B \cap A) \cup A^{\complement}) - \nu(A^{\complement})}{1 - \nu(A^{\complement})}$, which is the Dempster-Shafer rule. In light of the restrictions on the benchmark prospect and the comments following Theorems 1 and 2, we suggest referring to the former as the extremal update rule and the latter as the moderating update rule in the context of CPT.

Corollary 3 *Under the conditions of Theorem 3, if the DM uses $f_*^* \in F_*^*$ as a benchmark for updating and starts with prior probability weighting functions $w^-(\cdot)$ and $w^+(\cdot)$, his conditional probability weighting function $w_A^-(\cdot)$ given the information that $s \in A$ is*

$$w_A^- \left(\sum_{s' \,:\, o^{-1}(s') \leq o^{-1}(s)} p_{s'} \right) = \begin{cases} \dfrac{w^-\left(\sum_{s' \,:\, o^{-1}(s') \leq o^{-1}(s)} p_{s'}\right)}{w^-\left(\sum_{j=1}^k p_{o(j)}\right)} & \text{if } s \in A \\ 0 & \text{if } s \in A^{\complement}, \end{cases}$$

while his conditional probability weighting function $w_{A^{\complement}}^+(\cdot)$ given the information that $s \in A^{\complement}$ is

$$w_{A^{\complement}}^+ \left(\sum_{s' \,:\, o^{-1}(s') \geq o^{-1}(s)} p_{s'} \right) = \begin{cases} 0 & \text{if } s \in A \\ \dfrac{w^+\left(\sum_{s' \,:\, o^{-1}(s') \geq o^{-1}(s)} p_{s'}\right)}{w^+\left(\sum_{j=k+1}^n p_{o(j)}\right)} & \text{if } s \in A^{\complement}. \end{cases}$$

The benchmark for updating differs between Theorems 1 through 3. If we want the benchmark to be the same across all three domains, it requires further restrictions. Theorem 4 below shows that there is exactly one benchmark prospect that works in all three domains.

Theorem 4 *Suppose there are at least four outcomes in each of the loss and gain domains and that A and $A^{\complement}$ each contains at least two states. The following two statements are equivalent:*

(a) *For any non-decreasing weighting functions $w^-(\cdot)$ and $w^+(\cdot)$ satisfying $w^-(0) = w^+(0) = 0$ and $w^-(1) = w^+(1) = 1$, and for any continuous increasing ratio scale $v(x)$, we have that*

    (i) *for any prospect $h \in \mathcal{H}_L$, preferences conditional on arrival of information that $s \in A$ are represented by $E^e_L[h|s \in A]$, preferences conditional of arrival of information that $s \in A^{\complement}$ are represented by $E^m_L[h|s \in A^{\complement}]$, and unconditional CPT utility is given by*

$$V(h) = w^-(p_A)E^e_L[h|s \in A] + (1 - w^-(p_A))E^m_L[h|s \in A^{\complement}],$$

    and

    (ii) *for any prospect $h \in \mathcal{H}_G$, preferences conditional on the arrival of information that $s \in A$ are represented by $E^m_G[h|s \in A]$, preferences conditional on the arrival of information that $s \in A^{\complement}$ are represented by $E^e_G[h|s \in A^{\complement}]$, and unconditional CPT utility is given by*

$$V(h) = (1 - w^+(p(A^{\complement})))E^m_G[h|s \in A] + w^+(p(A^{\complement}))E^e_G[h|s \in A^{\complement}],$$

    and

    (iii) *for any prospect $h \in \mathcal{H}_P$, preferences conditional on the arrival of information that $s \in A$ are represented by $E^e_L[h|s \in A]$, preferences conditional on the arrival of information that $s \in A^{\complement}$ are represented by $E^e_G[h|s \in A^{\complement}]$, and unconditional CPT utility is given by*

$$V(h) = w^-(p(A))E^e_L[h|s \in A] + w^+(p(A^{\complement}))E^e_G[h|s \in A^{\complement}].$$

(b) *The benchmark used for updating is given by $\hat{f} = x_{*A}x^*$.*

Proof: The proof follows from the fact that $F_* \cup F^* \cup F_*^* = \{\hat{f}\}$. Therefore, given the results in Theorems 1 through 3, clearly $\hat{f}$ is sufficient for (i) through (iii). If $f \neq \hat{f}$, then $f \notin F_*$ or $f \notin F^*$, so (i) or (ii) will fail to hold. Hence, $\hat{f}$ is necessary as well.■

Gilboa and Schmeidler's (1993) general benchmark act is of the form $x_{*T}x^*$ for some event T. For $T = \emptyset$ and $T = S$, they get, respectively, their pessimistic and optimistic rules. Theorem 4 shows that for CPT preferences we get the extremal and moderating rules and the desired decomposition for all three domains when T is exactly equal to the event A.

Sarin and Wakker (1998) suggest the following update rule of their revealed conditional likelihood for purely rank-dependent expected utility models:

$$\pi(Q|R) = \pi(Q, D|R, D') = \frac{\pi(Q \cap R, D)}{\pi(R, D')} = \frac{\nu((Q \cap R) \cup D) - \nu(D)}{\nu(R \cup D') - \nu(D')},$$

where D is a dominating event for $Q \cap R$, D' is a dominating event for R, π is the revealed likelihood, and ν is a capacity. Sarin and Wakker's (1998) definition is consistent with our update rules in the gain domain. If, in the sign dependent CPT model, the dominating events in Sarin and Wakker's definition are replaced with the more extreme events relative to the reference point, their update rule is consistent with ours in the loss domain as well.[6] Our necessary and sufficient restrictions on the benchmark prospect in order for conditioning to be extremal or moderating for sign and rank dependent CPT preferences show exactly which events will be dominating or more extreme.

4 Discussion

To illustrate the relevance of our results for empirical work, consider again the insurance example from the introduction, i.e. a comparison of two insurance policies Q and R that represent insurance against the same event A, but with different payouts and at different costs. Suppose that the two insurance options rank the states such that A is clearly dominated by $A^\complement$, and that the labels Q and R are assigned such that $t_Q - c_Q > t_R - c_R$. Suppose further that the DM uses bankruptcy as his reference point, and thus regards all outcomes as being in the gain domain. We can then use the result in Theorem 2 (or 4)

---

[6]Otherwise, Sarin and Wakker's definition only coincides with our update rules for the loss domain, if $w^-(1 - p) = 1 - w^-(p)$ for all $p \in [0, 1]$, which, of course, does not hold for probability weighting functions that exhibit subcertainty, and e.g. does not hold for the parameterizations proposed in Tversky and Kahneman (1992) and Prelec (1998).

to evaluate the insurance options:

$$
\begin{aligned}
Q \succeq R \quad \Leftrightarrow \quad & (1 - w^+(p_{A^{\complement}})) E_G^P[Q|s \in A] + w^+(p_{A^{\complement}}) E_G^O[Q|s \in A^{\complement}] \\
& \geq (1 - w^+(p_{A^{\complement}})) E_G^P[R|s \in A] + w^+(p_{A^{\complement}}) E_G^O[R|s \in A^{\complement}] \\
\Leftrightarrow \quad & \frac{w^+(p_{A^{\complement}})}{1 - w^+(p_{A^{\complement}})} \left(-\frac{E_G^O[Q|s \in A^{\complement}] - E_G^O[R|s \in A^{\complement}]}{E_G^P[Q|s \in A] - E_G^P[R|s \in A]} \right) \leq 1
\end{aligned}
$$

The last equation suggests an econometric specification in which the probability $p_{A^{\complement}}$ is used as an explanatory variable, based on a parametric specification for the "odds ratio" $w^+(p_{A^{\complement}})/(1 - w^+(p_{A^{\complement}}))$. Similar specifications could be obtained based on Theorems 1 or 3 for different sets of assumptions.

Appendix: Proof of Theorem 1.

Sufficiency: We first prove that $f_* \in F_*$ is sufficient for the decomposition. First consider conditioning on $s \in A$. Let α be a function defined on $\{1, \ldots, n-k\}$ that orders the states in $A^{\complement}$ such that $f_*(\alpha(n-k)) \succsim \cdots \succsim f_*(\alpha(1))$. With the benchmark prospect $f_* \in F_*$, we have

$$
\begin{aligned}
V(h_A f_*) &= V(h_{o(1)}, \ldots, h_{o(k)}, f_*(\alpha(1)), \ldots, f_*(\alpha(n-k))) \\
&= \sum_{i=1}^{k} \left(w^- \left(\sum_{j=1}^{i} p_{o(j)} \right) - w^- \left(\sum_{j=1}^{i-1} p_{o(j)} \right) \right) v(h_{o(i)}) \\
&\quad + \sum_{i=1}^{n-k} \left(w^+ \left(\sum_{j=i}^{n-k} p_{\alpha(j)} \right) - w^+ \left(\sum_{j=i+1}^{n-k} p_{\alpha(j)} \right) \right) v(f_*(\alpha(i))).
\end{aligned}
$$

and

$$
\begin{aligned}
V(g_A f_*) &= V(g_{o(1)}, \ldots, g_{o(k)}, f_*(\alpha(1)), \ldots, f_*(\alpha(n-k))) \\
&= \sum_{i=1}^{k} \left(w^- \left(\sum_{j=1}^{i} p_{o(j)} \right) - w^- \left(\sum_{j=1}^{i-1} p_{o(j)} \right) \right) v(g_{o(i)}) \\
&\quad + \sum_{i=1}^{n-k} \left(w^+ \left(\sum_{j=i}^{n-k} p_{\alpha(j)} \right) - w^+ \left(\sum_{j=i+1}^{n-k} p_{\alpha(j)} \right) \right) v(f_*(\alpha(i))).
\end{aligned}
$$

Thus,

$$h_A f_* \succsim g_A f_*$$

$$\Leftrightarrow \quad V(h_A f_*) \geq V(g_A f_*)$$

$$\Leftrightarrow \quad \sum_{i=1}^{k} \left(w^-\Big(\sum_{j=1}^{i} p_{o(j)}\Big) - w^-\Big(\sum_{j=1}^{i-1} p_{o(j)}\Big) \right) v(h_{o(i)}) + \sum_{i=1}^{n-k} \left(w^+\Big(\sum_{j=i}^{n-k} p_{\alpha(j)}\Big) - w^+\Big(\sum_{j=i+1}^{n-k} p_{\alpha(j)}\Big) \right) v(f_*(\alpha(i)))$$

$$\geq \sum_{i=1}^{k} \left(w^-\Big(\sum_{j=1}^{i} p_{o(j)}\Big) - w^-\Big(\sum_{j=1}^{i-1} p_{o(j)}\Big) \right) v(g_{o(i)}) + \sum_{i=1}^{n-k} \left(w^+\Big(\sum_{j=i}^{n-k} p_{\alpha(j)}\Big) - w^+\Big(\sum_{j=i+1}^{n-k} p_{\alpha(j)}\Big) \right) v(f_*(\alpha(i)))$$

$$\Leftrightarrow \quad \sum_{i=1}^{k} \left(w^-\Big(\sum_{j=1}^{i} p_{o(j)}\Big) - w^-\Big(\sum_{j=1}^{i-1} p_{o(j)}\Big) \right) v(h_{o(i)}) \geq \sum_{i=1}^{k} \left(w^-\Big(\sum_{j=1}^{i} p_{o(j)}\Big) - w^-\Big(\sum_{j=1}^{i-1} p_{o(j)}\Big) \right) v(g_{o(i)})$$

$$\Leftrightarrow \quad \sum_{i=1}^{k} \frac{w^-\big(\sum_{j=1}^{i} p_{o(j)}\big) - w^-\big(\sum_{j=1}^{i-1} p_{o(j)}\big)}{w^-\big(\sum_{j=1}^{k} p_{o(j)}\big)} v(h_{o(i)}) \geq \sum_{i=1}^{k} \frac{w^-\big(\sum_{j=1}^{i} p_{o(j)}\big) - w^-\big(\sum_{j=1}^{i-1} p_{o(j)}\big)}{w^-\big(\sum_{j=1}^{k} p_{o(j)}\big)} v(g_{o(i)}),$$

so preferences conditional on the information that $s \in A$ are represented by $E_L^e[h|s \in A]$.

Now consider conditioning on the complement $A^{\complement} = \{o(k+1), \dots, o(n)\}$. We have

$$V(h_{A^{\complement}} f_*) = V(x_*, \dots, x_*, h_{o(k+1)}, \dots, h_{o(n)})$$

$$= w^-\Big(\sum_{j=1}^{k} p_{o(j)}\Big) v(x_*) + \sum_{i=k+1}^{n} \left(w^-\Big(\sum_{j=1}^{i} p_{o(j)}\Big) - w^-\Big(\sum_{j=1}^{i-1} p_{o(j)}\Big) \right) v(h_{o(i)})$$

and

$$V(g_{A^{\complement}} f_*) = V(x_*, \dots, x_*, g_{o(k+1)}, \dots, g_{o(n)})$$

$$= w^-\Big(\sum_{j=1}^{k} p_{o(j)}\Big) v(x_*) + \sum_{i=k+1}^{n} \left(w^-\Big(\sum_{j=1}^{i} p_{o(j)}\Big) - w^-\Big(\sum_{j=1}^{i-1} p_{o(j)}\Big) \right) v(g_{o(i)}).$$

Thus,

$$h_{A^{\complement}} f_* \succsim g_{A^{\complement}} f_*$$

$$\Leftrightarrow \quad V(h_{A^{\complement}} f_*) \geq V(g_{A^{\complement}} f_*)$$

$$\Leftrightarrow \quad w^-\Big(\sum_{j=1}^{k} p_{o(j)}\Big) v(x_*) + \sum_{i=k+1}^{n} \left(w^-\Big(\sum_{j=1}^{i} p_{o(j)}\Big) - w^-\Big(\sum_{j=1}^{i-1} p_{o(j)}\Big) \right) v(h_{o(i)})$$

$$\geq w^-\Big(\sum_{j=1}^{k} p_{o(j)}\Big) v(x_*) + \sum_{i=k+1}^{n} \left(w^-\Big(\sum_{j=1}^{i} p_{o(j)}\Big) - w^-\Big(\sum_{j=1}^{i-1} p_{o(j)}\Big) \right) v(g_{o(i)})$$

$$\Leftrightarrow \quad \sum_{i=k+1}^{n} \left(w^-\Big(\sum_{j=1}^{i} p_{o(j)}\Big) - w^-\Big(\sum_{j=1}^{i-1} p_{o(j)}\Big) \right) v(h_{o(i)}) \geq \sum_{i=k+1}^{n} \left(w^-\Big(\sum_{j=1}^{i} p_{o(j)}\Big) - w^-\Big(\sum_{j=1}^{i-1} p_{o(j)}\Big) \right) v(g_{o(i)})$$

$$\Leftrightarrow \quad \sum_{i=k+1}^{n} \frac{w^-\big(\sum_{j=1}^{i} p_{o(j)}\big) - w^-\big(\sum_{j=1}^{i-1} p_{o(j)}\big)}{1 - w^-\big(\sum_{j=1}^{k} p_{o(j)}\big)} v(h_{o(i)}) \geq \sum_{i=k+1}^{n} \frac{w^-\big(\sum_{j=1}^{i} p_{o(j)}\big) - w^-\big(\sum_{j=1}^{i-1} p_{o(j)}\big)}{1 - w^-\big(\sum_{j=1}^{k} p_{o(j)}\big)} v(g_{o(i)}).$$

so preferences conditional on the information that $s \in A^\complement$ are represented by $E_L^m[h|s \in A^\complement]$.

Since $p_A = \sum_{j=1}^k p_{o(j)}$, it now follows that in the loss domain

$$w^-(p_A)E_L^e[h|s \in A] + (1 - w^-(p_A))E_L^m[h|s \in A^\complement]$$

$$= w^-\left(\sum_{j=1}^k p_{o(j)}\right)\sum_{i=1}^k \frac{w^-\left(\sum_{j=1}^i p_{o(j)}\right) - w^-\left(\sum_{j=1}^{i-1} p_{o(j)}\right)}{w^-\left(\sum_{j=1}^k p_{o(j)}\right)}v(h_{o(i)})$$

$$+ (1 - w^-\left(\sum_{j=1}^k p_{o(j)}\right))\sum_{i=k+1}^n \frac{w^-\left(\sum_{j=1}^i p_{o(j)}\right) - w^-\left(\sum_{j=1}^{i-1} p_{o(j)}\right)}{1 - w^-\left(\sum_{j=1}^k p_{o(j)}\right)}v(h_{o(i)})$$

$$= \sum_{i=1}^n \left(w^-\left(\sum_{j=1}^i p_{o(j)}\right) - w^-\left(\sum_{j=1}^{i-1} p_{o(j)}\right)\right)v(h_{o(i)}),$$

which equals the unconditional CPT utility in (3). Hence, we have the desired decomposition.

Necessity: We now turn to showing necessity. We proceed by showing that if $f \notin F_*$, then we can choose $w^-(\cdot)$ and $v(\cdot)$ such that conditional preferences are not represented as stated in part (a) of the theorem. That is, we can find functions $w^-(\cdot)$ and $v(\cdot)$ satisfying the stated conditions such that $g_{A^\complement}f \succ h_{A^\complement}f$ but $E_L^m[h|s \in A^\complement] \geq E_L^m[g|s \in A^\complement]$, or $g_A f \succ h_A f$ but $E_L^e[h|s \in A] \geq E_L^e[g|s \in A]$, for some acts $h, g \in \mathscr{H}_L$. A prospect $f \notin F_*$ if $f_s \succ x_*$ for some $s \in A$ or $0 \succ f_s$ for some $s \in A^\complement$.

We first show that if $f_s \succ x_*$ for some $s \in A$, then we can have $g_{A^\complement}f \succ h_{A^\complement}f$ but $E_L^m[h|s \in A^\complement] \geq E_L^m[g|s \in A^\complement]$. So suppose that there indeed exists $s \in A$ such that $f_s \succ x_*$. Since there are at least four outcomes in the loss domain and at least two states in $A^\complement$, we can choose prospects $h, g \in \mathscr{H}_L$ such that

$$0 \succsim h_{o(k+2)} \succ g_{o(k+2)} \succ g_{o(k+1)} \succ h_{o(k+1)} \succsim x_*$$

and $h_{o(i)} = g_{o(i)} = h_{o(k+2)}$ for all $i > k + 2$. Depending on f, there are different cases we need to consider.

The different cases arise because the details of the proof depend on how f relates to the outcomes in the loss domain (recall that all we require is that there are at least four outcomes in X_L). When there are more than four outcomes in X_L, we may simultaneously be able to use the approach of two different cases. The procedure is the same in each case: We calculate the utility of $h_{A^\complement}f$ and of $g_{A^\complement}f$ and derive conditions for $g_{A^\complement}f \succ h_{A^\complement}f$ and for $E_L^m[h|s \in A^\complement] \geq E_L^m[g|s \in A^\complement]$. We then show that there exist $w^-(\cdot)$ and $v(\cdot)$ such that both of these conditions are satisfied. Since such $w^-(\cdot)$ and $v(\cdot)$ exist, preferences are not represented as in statement (a) of Theorem 1.

To this end, we introduce the following notation: Let $\hat{A} = \{s \in A | f_s \succ x_*\}$ and let $\hat{A}_L = \{s \in \hat{A} | 0 \succ f_s\}$. Note that $\hat{A}$ is non-empty by the supposition above. Let $|\hat{A}|$ and $|\hat{A}_L|$ denote the number of states in $\hat{A}$ and $\hat{A}_L$, respectively, and let τ be a function defined on $\{1, \ldots, |\hat{A}|\}$ that orders the states in $\hat{A}$ such that $f_{\tau(|\hat{A}|)} \succsim \cdots \succsim f_{\tau(1)}$.

Suppose first that $f_{\tau(1)} \succsim 0$. Then

$$
\begin{aligned}
V[h_{A^{\complement}}f] = {} & w^-\Big(\sum_{\substack{j=1 \\ o(j)\notin\hat{A}}}^{k} p_{o(j)}\Big)v(x_*) + \sum_{i=k+1}^{n}\Big(w^-\Big(\sum_{\substack{j=1 \\ o(j)\notin\hat{A}}}^{i} p_{o(j)}\Big) - w^-\Big(\sum_{\substack{j=1 \\ o(j)\notin\hat{A}}}^{i-1} p_{o(j)}\Big)\Big)v(h_{o(i)}) \\
& + \sum_{i=1}^{|\hat{A}|}\Big(w^+\Big(\sum_{j=i}^{|\hat{A}|} p_{\tau(j)}\Big) - w^+\Big(\sum_{j=i+1}^{|\hat{A}|} p_{\tau(j)}\Big)\Big)v(f_{\tau(i)})
\end{aligned}
$$

and

$$
\begin{aligned}
V[g_{A^{\complement}}f] = {} & w^-\Big(\sum_{\substack{j=1 \\ o(j)\notin\hat{A}}}^{k} p_{o(j)}\Big)v(x_*) + \sum_{i=k+1}^{n}\Big(w^-\Big(\sum_{\substack{j=1 \\ o(j)\notin\hat{A}}}^{i} p_{o(j)}\Big) - w^-\Big(\sum_{\substack{j=1 \\ o(j)\notin\hat{A}}}^{i-1} p_{o(j)}\Big)\Big)v(g_{o(i)}) \\
& + \sum_{i=1}^{|\hat{A}|}\Big(w^+\Big(\sum_{j=i}^{|\hat{A}|} p_{\tau(j)}\Big) - w^+\Big(\sum_{j=i+1}^{|\hat{A}|} p_{\tau(j)}\Big)\Big)v(f_{\tau(i)}).
\end{aligned}
$$

Since $h_{o(i)} = g_{o(i)} = h_{o(k+2)}$ for all $i > k+2$, we have

$$
\begin{aligned}
g_{A^{\complement}}f \succ h_{A^{\complement}}f \;\Leftrightarrow\; & V[g_{A^{\complement}}f] > V[h_{A^{\complement}}f] \\
\Leftrightarrow\; & \Big(w^-\Big(\sum_{\substack{j=1 \\ o(j)\notin\hat{A}}}^{k+1} p_{o(j)}\Big) - w^-\Big(\sum_{\substack{j=1 \\ o(j)\notin\hat{A}}}^{k} p_{o(j)}\Big)\Big)\big(v(h_{o(k+1)}) - v(g_{o(k+1)})\big) \\
& + \Big(w^-\Big(\sum_{\substack{j=1 \\ o(j)\notin\hat{A}}}^{k+2} p_{o(j)}\Big) - w^-\Big(\sum_{\substack{j=1 \\ o(j)\notin\hat{A}}}^{k+1} p_{o(j)}\Big)\Big)\big(v(h_{o(k+2)}) - v(g_{o(k+2)})\big) < 0. \quad (9)
\end{aligned}
$$

Conditional preferences are not represented by $E_L^m[\cdot|s \in A^{\complement}]$ if we also have that $E_L^m[h|s \in A^{\complement}] \geq E_L^m[g|s \in A^{\complement}]$, which with $h_{o(i)} = g_{o(i)} = h_{o(k+2)}$ for all $i > k+2$ reduces to

$$
\begin{aligned}
& \Big(w^-\Big(\sum_{j=1}^{k+1} p_{o(j)}\Big) - w^-\Big(\sum_{j=1}^{k} p_{o(j)}\Big)\Big)\big(v(h_{o(k+1)}) - v(g_{o(k+1)})\big) \\
& + \Big(w^-\Big(\sum_{j=1}^{k+2} p_{o(j)}\Big) - w^-\Big(\sum_{j=1}^{k+1} p_{o(j)}\Big)\Big)\big(v(h_{o(k+2)}) - v(g_{o(k+2)})\big) \geq 0. \quad (10)
\end{aligned}
$$

Rearranging (9) and (10), conditional preferences are not represented by $E_L^m[\cdot|s \in A^\complement]$ if

$$\frac{w^-\left(\sum_{\substack{j=1 \\ o(j)\notin \hat{A}}}^{k+2} p_{o(j)}\right) - w^-\left(\sum_{\substack{j=1 \\ o(j)\notin \hat{A}}}^{k+1} p_{o(j)}\right)}{w^-\left(\sum_{\substack{j=1 \\ o(j)\notin \hat{A}}}^{k+1} p_{o(j)}\right) - w^-\left(\sum_{\substack{j=1 \\ o(j)\notin \hat{A}}}^{k} p_{o(j)}\right)} < \frac{v(g_{o(k+1)}) - v(h_{o(k+1)})}{v(h_{o(k+2)}) - v(g_{o(k+2)})}$$

$$\leq \frac{w^-\left(\sum_{j=1}^{k+2} p_{o(j)}\right) - w^-\left(\sum_{j=1}^{k+1} p_{o(j)}\right)}{w^-\left(\sum_{j=1}^{k+1} p_{o(j)}\right) - w^-\left(\sum_{j=1}^{k} p_{o(j)}\right)}. \quad (11)$$

The functions $w^-(\cdot)$ and $v(\cdot)$ can be chosen such that both inequalities in (11) are satisfied. Therefore, preferences are not represented as in statement (a) of Theorem 1.

Suppose second that $0 \succ f_{\tau(1)}$. Define

$$\mu \equiv \begin{cases} \min\left\{j \in \{1,\ldots,|\hat{A}_L|\} \ : \ f_{\tau(j)} \succ f_{\tau(1)}\right\} & \text{if there exists } s \in \hat{A}_L \text{ such that } f_s \succ f_{\tau(1)} \\ |\hat{A}_L| + 1 & \text{if } f_s \sim f_{\tau(1)} \text{ for all } s \in \hat{A}_L. \end{cases}$$

Since there are at least four outcomes in the loss domain, there exists $\hat{x}$ such that $f_{\tau(1)} \succ \hat{x} \succ x_*$ and/or there exists x' such that $0 \succ x' \succ f_{\tau(1)}$. In the former case, since $0 \succ f_{\tau(1)}$, we can let $f_{\tau(\mu)} \succsim h_{o(k+2)} \succ g_{o(k+2)} = f_{\tau(1)} \succ g_{o(k+1)} \succ h_{o(k+1)} = x_*$. In the latter case, if it is also the case that $f_{\tau(\mu)} \succ \tilde{x} \succ f_{\tau(1)}$ for some $\tilde{x} \in X_L$, we can let $f_{\tau(\mu)} \succsim h_{o(k+2)} \succ g_{o(k+2)} \succ f_{\tau(1)} = g_{o(k+1)} \succ h_{o(k+1)} = x_*$. In either of these two cases,

$$\begin{aligned}
V[h_{A^\complement} f] &= w^-\left(\sum_{\substack{j=1 \\ o(j)\notin\hat{A}}}^{k} p_{o(j)}\right) v(x_*) + \left(w^-\left(\sum_{\substack{j=1 \\ o(j)\notin\hat{A}}}^{k+1} p_{o(j)}\right) - w^-\left(\sum_{\substack{j=1 \\ o(j)\notin\hat{A}}}^{k} p_{o(j)}\right)\right) v(h_{o(k+1)}) \\
&+ \sum_{i=1}^{\mu-1}\left(w^-\left(\sum_{\substack{j=1 \\ o(j)\notin\hat{A}}}^{k+1} p_{o(j)} + \sum_{j=1}^{i} p_{\tau(j)}\right) - w^-\left(\sum_{\substack{j=1 \\ o(j)\notin\hat{A}}}^{k+1} p_{o(j)} + \sum_{j=1}^{i-1} p_{\tau(j)}\right)\right) v(f_{\tau(i)}) \\
&+ \sum_{i=k+2}^{n}\left(w^-\left(\sum_{\substack{j=1 \\ o(j)\notin\hat{A}}}^{i} p_{o(j)} + \sum_{j=1}^{\mu-1} p_{\tau(j)}\right) - w^-\left(\sum_{\substack{j=1 \\ o(j)\notin\hat{A}}}^{i-1} p_{o(j)} + \sum_{j=1}^{\mu-1} p_{\tau(j)}\right)\right) v(h_{o(i)}) \\
&+ \sum_{i=\mu}^{|\hat{A}_L|}\left(w^-\left(\sum_{\substack{j=1 \\ o(j)\notin\hat{A}}}^{n} p_{o(j)} + \sum_{j=1}^{i} p_{\tau(j)}\right) - w^-\left(\sum_{\substack{j=1 \\ o(j)\notin\hat{A}}}^{n} p_{o(j)} + \sum_{j=1}^{i-1} p_{\tau(j)}\right)\right) v(f_{\tau(i)}) \\
&+ \sum_{i=|\hat{A}_L|+1}^{|\hat{A}|}\left(w^+\left(\sum_{j=i}^{|\hat{A}|} p_{\tau(j)}\right) - w^+\left(\sum_{j=i+1}^{|\hat{A}|} p_{\tau(j)}\right)\right) v(f_{\tau(i)}) \qquad (12)
\end{aligned}$$

and

$$
\begin{aligned}
V[g_{A^\complement} f] \;=\;& w^-\Big(\sum_{\substack{j=1\\o(j)\notin\hat A}}^{k} p_{o(j)}\Big)v(x_*) + \Big(w^-\Big(\sum_{\substack{j=1\\o(j)\notin\hat A}}^{k+1} p_{o(j)}\Big) - w^-\Big(\sum_{\substack{j=1\\o(j)\notin\hat A}}^{k} p_{o(j)}\Big)\Big)v(g_{o(k+1)})\\
&+\sum_{i=1}^{\mu-1}\Big(w^-\Big(\sum_{\substack{j=1\\o(j)\notin\hat A}}^{k+1} p_{o(j)} + \sum_{j=1}^{i} p_{\tau(j)}\Big) - w^-\Big(\sum_{\substack{j=1\\o(j)\notin\hat A}}^{k+1} p_{o(j)} + \sum_{j=1}^{i-1} p_{\tau(j)}\Big)\Big)v(f_{\tau(i)})\\
&+\sum_{i=k+2}^{n}\Big(w^-\Big(\sum_{\substack{j=1\\o(j)\notin\hat A}}^{i} p_{o(j)} + \sum_{j=1}^{\mu-1} p_{\tau(j)}\Big) - w^-\Big(\sum_{\substack{j=1\\o(j)\notin\hat A}}^{i-1} p_{o(j)} + \sum_{j=1}^{\mu-1} p_{\tau(j)}\Big)\Big)v(g_{o(i)})\\
&+\sum_{i=\mu}^{|\hat A_L|}\Big(w^-\Big(\sum_{\substack{j=1\\o(j)\notin\hat A}}^{n} p_{o(j)} + \sum_{j=1}^{i} p_{\tau(j)}\Big) - w^-\Big(\sum_{\substack{j=1\\o(j)\notin\hat A}}^{n} p_{o(j)} + \sum_{j=1}^{i-1} p_{\tau(j)}\Big)\Big)v(f_{\tau(i)})\\
&+\sum_{i=|\hat A_L|+1}^{|\hat A|}\Big(w^+\Big(\sum_{j=i}^{|\hat A|} p_{\tau(j)}\Big) - w^+\Big(\sum_{j=i+1}^{|\hat A|} p_{\tau(j)}\Big)\Big)v(f_{\tau(i)}).
\end{aligned}
\tag{13}
$$

Since $h_{o(i)} = g_{o(i)} = h_{o(k+2)}$ for all $i > k+2$, we have

$$
\begin{aligned}
& g_{A^\complement} f \succ h_{A^\complement} f\\
\Leftrightarrow\quad & V[g_{A^\complement} f] > V[h_{A^\complement} f]\\
\Leftrightarrow\quad & \Big(w^-\Big(\sum_{\substack{j=1\\o(j)\notin\hat A}}^{k+2} p_{o(j)} + \sum_{j=1}^{\mu-1} p_{\tau(j)}\Big) - w^-\Big(\sum_{\substack{j=1\\o(j)\notin\hat A}}^{k+1} p_{o(j)} + \sum_{j=1}^{\mu-1} p_{\tau(j)}\Big)\Big)\big(v(h_{o(k+2)}) - v(g_{o(k+2)})\big)\\
& +\Big(w^-\Big(\sum_{\substack{j=1\\o(j)\notin\hat A}}^{k+1} p_{o(j)}\Big) - w^-\Big(\sum_{\substack{j=1\\o(j)\notin\hat A}}^{k} p_{o(j)}\Big)\Big)\big(v(h_{o(k+1)}) - v(g_{o(k+1)})\big) < 0.
\end{aligned}
\tag{14}
$$

When $g_{A^\complement} f \succ h_{A^\complement} f$, conditional preferences are not represented by $E_L^m[\cdot | s \in A^\complement]$ if (10) also holds. Hence, rearranging (10) and (14), conditional preferences are not represented by $E_L^m[\cdot | s \in A^\complement]$ if

$$
\begin{aligned}
& \frac{w^-\big(\sum_{\substack{j=1\\o(j)\notin\hat A}}^{k+2} p_{o(j)} + \sum_{j=1}^{\mu-1} p_{\tau(j)}\big) - w^-\big(\sum_{\substack{j=1\\o(j)\notin\hat A}}^{k+1} p_{o(j)} + \sum_{j=1}^{\mu-1} p_{\tau(j)}\big)}{w^-\big(\sum_{\substack{j=1\\o(j)\notin\hat A}}^{k+1} p_{o(j)}\big) - w^-\big(\sum_{\substack{j=1\\o(j)\notin\hat A}}^{k} p_{o(j)}\big)}\\
& < \frac{v(g_{o(k+1)}) - v(h_{o(k+1)})}{v(h_{o(k+2)}) - v(g_{o(k+2)})} \le \frac{w^-\big(\sum_{j=1}^{k+2} p_{o(j)}\big) - w^-\big(\sum_{j=1}^{k+1} p_{o(j)}\big)}{w^-\big(\sum_{j=1}^{k+1} p_{o(j)}\big) - w^-\big(\sum_{j=1}^{k} p_{o(j)}\big)}.
\end{aligned}
\tag{15}
$$

Again, the functions $w^-(\cdot)$ and $v(\cdot)$ can be chosen such that both inequalities in (15) are satisfied. Therefore, preferences are not represented as in statement (a) of Theorem 1.

If there does not exist $\hat{x} \in X_L$ such that $f_{\tau(1)} \succ \hat{x} \succ x_*$ and there does not exist $\tilde{x} \in X_L$ such that $f_{\tau(\mu)} \succ \tilde{x} \succ f_{\tau(1)}$, we can let $h_{o(k+2)} \succ g_{o(k+2)} = f_{\tau(\mu)} \succ f_{\tau(1)} = g_{o(k+1)} \succ h_{o(k+1)} = x_*$, which is possible since there are at least four outcomes in the loss domain. Define

$$
\xi \equiv \begin{cases} \min\left\{j \in \{1,\ldots,|\hat{A}_L|\} \;:\; f_{\tau(j)} \succ f_{\tau(\mu)}\right\} & \text{if there exists } s \in \hat{A}_L \text{ such that } f_s \succ f_{\tau(\mu)} \\ |\hat{A}_L| + 1 & \text{if } f_{\tau(\mu)} \succsim f_s \text{ for all } s \in \hat{A}_L. \end{cases}
$$

Then,

$$
\begin{aligned}
V[h_{A^{\complement}}f] \;=\;& w^-\Big(\sum_{\substack{j=1 \\ o(j)\notin\hat{A}}}^{k} p_{o(j)}\Big)v(x_*) + \Big(w^-\Big(\sum_{\substack{j=1 \\ o(j)\notin\hat{A}}}^{k+1} p_{o(j)}\Big) - w^-\Big(\sum_{\substack{j=1 \\ o(j)\notin\hat{A}}}^{k} p_{o(j)}\Big)\Big)v(h_{o(k+1)}) \\
&+ \sum_{i=1}^{\xi-1}\Big(w^-\Big(\sum_{\substack{j=1 \\ o(j)\notin\hat{A}}}^{k+1} p_{o(j)} + \sum_{j=1}^{i} p_{\tau(j)}\Big) - w^-\Big(\sum_{\substack{j=1 \\ o(j)\notin\hat{A}}}^{k+1} p_{o(j)} + \sum_{j=1}^{i-1} p_{\tau(j)}\Big)\Big)v(f_{\tau(i)}) \\
&+ \sum_{i=k+2}^{n}\Big(w^-\Big(\sum_{\substack{j=1 \\ o(j)\notin\hat{A}}}^{i} p_{o(j)} + \sum_{j=1}^{\xi-1} p_{\tau(j)}\Big) - w^-\Big(\sum_{\substack{j=1 \\ o(j)\notin\hat{A}}}^{i-1} p_{o(j)} + \sum_{j=1}^{\xi-1} p_{\tau(j)}\Big)\Big)v(h_{o(i)}) \\
&+ \sum_{i=\xi}^{|\hat{A}_L|}\Big(w^-\Big(\sum_{\substack{j=1 \\ o(j)\notin\hat{A}}}^{n} p_{o(j)} + \sum_{j=1}^{i} p_{\tau(j)}\Big) - w^-\Big(\sum_{\substack{j=1 \\ o(j)\notin\hat{A}}}^{n} p_{o(j)} + \sum_{j=1}^{i-1} p_{\tau(j)}\Big)\Big)v(f_{\tau(i)}) \\
&+ \sum_{i=|\hat{A}_L|+1}^{|\hat{A}|}\Big(w^+\Big(\sum_{j=i}^{|\hat{A}|} p_{\tau(j)}\Big) - w^+\Big(\sum_{j=i+1}^{|\hat{A}|} p_{\tau(j)}\Big)\Big)v(f_{\tau(i)})
\end{aligned}
$$

and

$$
\begin{aligned}
V[g_{A^\complement} f] \;=\; & w^-\Big(\sum_{\substack{j=1\\ o(j)\notin\hat{A}}}^{k} p_{o(j)}\Big)v(x_*) + \Big(w^-\Big(\sum_{\substack{j=1\\ o(j)\notin\hat{A}}}^{k+1} p_{o(j)}\Big) - w^-\Big(\sum_{\substack{j=1\\ o(j)\notin\hat{A}}}^{k} p_{o(j)}\Big)\Big)v(g_{o(k+1)}) \\
& + \sum_{i=1}^{\xi-1}\Big(w^-\Big(\sum_{\substack{j=1\\ o(j)\notin\hat{A}}}^{k+1} p_{o(j)} + \sum_{j=1}^{i} p_{\tau(j)}\Big) - w^-\Big(\sum_{\substack{j=1\\ o(j)\notin\hat{A}}}^{k+1} p_{o(j)} + \sum_{j=1}^{i-1} p_{\tau(j)}\Big)\Big)v(f_{\tau(i)}) \\
& + \sum_{i=k+2}^{n}\Big(w^-\Big(\sum_{\substack{j=1\\ o(j)\notin\hat{A}}}^{i} p_{o(j)} + \sum_{j=1}^{\xi-1} p_{\tau(j)}\Big) - w^-\Big(\sum_{\substack{j=1\\ o(j)\notin\hat{A}}}^{i-1} p_{o(j)} + \sum_{j=1}^{\xi-1} p_{\tau(j)}\Big)\Big)v(g_{o(i)}) \\
& + \sum_{i=\xi}^{|\hat{A_L}|}\Big(w^-\Big(\sum_{\substack{j=1\\ o(j)\notin\hat{A}}}^{n} p_{o(j)} + \sum_{j=1}^{i} p_{\tau(j)}\Big) - w^-\Big(\sum_{\substack{j=1\\ o(j)\notin\hat{A}}}^{n} p_{o(j)} + \sum_{j=1}^{i-1} p_{\tau(j)}\Big)\Big)v(f_{\tau(i)}) \\
& + \sum_{i=|\hat{A_L}|+1}^{|\hat{A}|}\Big(w^+\Big(\sum_{j=i}^{|\hat{A}|} p_{\tau(j)}\Big) - w^+\Big(\sum_{j=i+1}^{|\hat{A}|} p_{\tau(j)}\Big)\Big)v(f_{\tau(i)}).
\end{aligned}
$$

Since $h_{o(i)} = g_{o(i)} = h_{o(k+2)}$ for all $i > k+2$, we have

$$
\begin{aligned}
& V[g_{A^\complement} f] > V[h_{A^\complement} f] \\
\Leftrightarrow \; & \Big(w^-\Big(\sum_{\substack{j=1\\ o(j)\notin\hat{A}}}^{k+2} p_{o(j)} + \sum_{j=1}^{\xi-1} p_{\tau(j)}\Big) - w^-\Big(\sum_{\substack{j=1\\ o(j)\notin\hat{A}}}^{k+1} p_{o(j)} + \sum_{j=1}^{\xi-1} p_{\tau(j)}\Big)\Big)\big(v(h_{o(k+2)}) - v(g_{o(k+2)})\big) \\
& + \Big(w^-\Big(\sum_{\substack{j=1\\ o(j)\notin\hat{A}}}^{k+1} p_{o(j)}\Big) - w^-\Big(\sum_{\substack{j=1\\ o(j)\notin\hat{A}}}^{k} p_{o(j)}\Big)\Big)\big(v(h_{o(k+1)}) - v(g_{o(k+1)})\big) < 0. \quad (16)
\end{aligned}
$$

Combining (16) with (10), conditional preferences are not represented by $E_L^m[\cdot|s\in A^\complement]$ if

$$
\frac{w^-\Big(\sum_{\substack{j=1\\ o(j)\notin\hat{A}}}^{k+2} p_{o(j)} + \sum_{j=1}^{\xi-1} p_{\tau(j)}\Big) - w^-\Big(\sum_{\substack{j=1\\ o(j)\notin\hat{A}}}^{k+1} p_{o(j)} + \sum_{j=1}^{\xi-1} p_{\tau(j)}\Big)}{w^-\Big(\sum_{\substack{j=1\\ o(j)\notin\hat{A}}}^{k+1} p_{o(j)}\Big) - w^-\Big(\sum_{\substack{j=1\\ o(j)\notin\hat{A}}}^{k} p_{o(j)}\Big)}
$$
$$
< \frac{v(g_{o(k+1)}) - v(h_{o(k+1)})}{v(h_{o(k+2)}) - v(g_{o(k+2)})} \leq \frac{w^-\Big(\sum_{j=1}^{k+2} p_{o(j)}\Big) - w^-\Big(\sum_{j=1}^{k+1} p_{o(j)}\Big)}{w^-\Big(\sum_{j=1}^{k+1} p_{o(j)}\Big) - w^-\Big(\sum_{j=1}^{k} p_{o(j)}\Big)}. \quad (17)
$$

The functions $w^-(\cdot)$ and $v(\cdot)$ can be chosen such that both inequalities in (17) are satisfied. Therefore, preferences are not represented as in statement (a) of Theorem 1. The cases considered exhaust all possibilities for $f_s \succ x_*$ for some $s \in A$.

The second part of the necessity proof is to show that if $0 \succ f_s$ for some $s \in A^\complement$, then we can have $g_A f \succ h_A f$ but $E_L^e[h|s \in A] \geq E_L^e[g|s \in A]$. Since there are at least four outcomes in the loss domain, we can choose $h, g \in \mathscr{H}_L$ such that

$$0 \succsim h_{o(k)} \succ g_{o(k)} \succ g_{o(k-1)} \succ h_{o(k-1)} \succsim x_*$$

and $h_{o(i)} = g_{o(i)} = h_{o(k-1)}$ for all $i < k - 1$. Again, depending on how f relates to the outcomes in the loss domain, there are different cases we need to consider.

The procedure is the same as above: In each case, we calculate the utility of $h_A f$ and $g_A f$ and derive the conditions for $g_A f \succ h_A f$ and $E_L^e[h|s \in A] \geq E_L^e[g|s \in A]$. We then show existence of a $w^-(\cdot)$ and $v(\cdot)$ such that both conditions are satisfied. Since such $w^-(\cdot)$ and $v(\cdot)$ exist, preferences are not represented as in statement (a) of Theorem 1. Because the procedure resembles that above and the details are notationally cumbersome, we relegate the details to an online appendix.[7]

References

[1] Chateauneuf, A., J. Eichberger, and S. Grant (2007): "Choice under uncertainty with the best and worst in mind: neo-additive capacities," *Journal of Economic Theory* 137, 538-567.

[2] Eichberger, J., S. Grant, and D. Kelsey (2010): "Comparing three ways to update Choquet beliefs," *Economic Letters* 107, 91-94.

[3] Epstein, L. G. and M. Le Breton (1993): "Dynamically consistent beliefs must be Bayesian," *Journal of Economic Theory* 61, 1-22.

[4] Gilboa, I. and D. Schmeidler (1993): "Updating ambiguous beliefs," *Journal of Economic Theory* 59, 33-49.

[5] Hanany, E. and P. Klibanoff (2007): "Updating preferences with multiple priors," *Theoretical Economics* 2, 261-298.

[6] Karni, E. and D. Schmeidler (1991): "Atemporal dynamic consistency and expected utility theory," *Journal of Economic Theory* 54, 401-408.

[7] Machina, M. J. (1989): "Dynamic consistency and non-expected utility models of choice under uncertainty," *Journal of Economic Literature* 27, 1622-1668.

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[7]See http://web.mit.edu/astomper/www/r_working.html

[8] Prelec, D. (1998): "The probability weighting function," *Econometrica* 66, 497-527.

[9] Quiggin, J. (1982): "A theory of anticipated utility," *Journal of Economic Behavior and Organization* 3, 323-343.

[10] Sarin, R. and P. Wakker (1998): "Revealed likelihood and Knightian uncertainty," *Journal of Risk and Uncertainty* 16, 223-250.

[11] Tversky, A. and D. Kahneman (1992): "Advances in prospect theory: cumulative representation of uncertainty," *Journal of Risk and Uncertainty* 5, 297-323.

[12] Wakker, P. and A. Tversky (1993): "An axiomatization of cumulative prospect theory," *Journal of Risk and Uncertainty* 7, 147-176.

[13] Wang, T. (2003): "Conditional preferences and updating," *Journal of Economic Theory* 108, 286-321.

[14] Yaari, M. E. (1987): "The dual theory of choice under risk," *Econometrica* 55, 95-115.