Optimal Agents[*]

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Abstract

This paper studies the investor's decision to provide financing for an entrepreneur in a standard risk-neutral principal-agent framework with contractual constraints when entrepreneurs have different production technologies. The investor (principal) has the choice between different types of entrepreneurs (agents) that generate different probability distributions of output subject to moral hazard. I provide a complete characterization of optimal incentive and financial contracts, agency rents, and pledgeable incomes, and characterize how the contracting problem biases and potentially distorts the investor's investment decision.

Keywords: Moral hazard, contractual constraints, financial constraints.
JEL Classifications: D86, G32.

1   Introduction

Agency problems between firm insiders and investors can lead to financial constraints such that investment projects are not undertaken even though they have a positive net present value. While most firms are likely to be financially constrained to some extent, an important part of the corporate finance literature (both empirical and theoretical) attempts to understand the heterogeneity in the intensity of financial constraints across firms and over time.

This paper revisits a standard financial contracting framework to study a potentially important determinant of firms' financial constraints: firms' production technologies. I consider an investor who can provide capital to a single entrepreneur. An entrepreneur's effort affects the output distribution and the investor designs a contract to induce the entrepreneur's effort, subject to standard contractual constraints: limited liability and monotonicity. The view taken in this paper is that entrepreneurs differ in *how* they affect the output distribution, which can reflect differences in entrepreneur characteristics such as human capital or differences in project characteristics such as assets. For example, consider two entrepreneurs who have different capabilities to develop a new drug. One entrepreneur has experience with cutting-edge technology to develop a valuable novel drug, which increases the probability of high profits. The other entrepreneur has experience with proven technology to develop a less novel and less valuable drug, which increases the probability of medium profits but is more certain.

The main contribution of the paper is to characterize optimal incentive and financial contracts, agency rents, and the investor's investment decision for a large class of entrepreneurs. Entrepreneurs require different optimal contracts, which imply different agency rents. An entrepreneur with a higher agency rent has a lower (expected) pledgeable income, defined as the maximum expected income that can be pledged to the investor without jeopardizing the entrepreneur's incentives. A higher agency rent and lower pledgeable income makes it less likely to receive financing from the investor. The central result is that the investor's investment decision depends on the productivity of effort, the ratio of the expected value of effort and the disutility of effort. If the productivity of effort is high (low), the investor prefers an entrepreneur who has the strongest impact on a higher (lower) region of the output distribution and provides more debt-like (equity-like) financing. In general, the results relate entrepreneurs' financial constraints measured by their

pledgeable income to their production technologies and financial contracts.

I consider the following principal-agent framework with a risk-neutral investor (the principal) and a set of penniless risk-neutral entrepreneurs (the agents). Each entrepreneur owns a project that requires financing from the investor and generates a random cash flow from a finite set of possible cash flows. Without an entrepreneur's effort, each cash flow has a positive probability, which is identical for all entrepreneurs and referred to as the baseline technology. Entrepreneurs differ in that they generate different probability distributions of cash flows when they exert effort, referred to as entrepreneurs' technologies. Specifically, I consider entrepreneurs with technologies that first-order stochastically dominate the baseline technology with a single-peaked likelihood ratio, a generalization of the monotone likelihood ratio property. Intuitively, the baseline technology captures the uncertainty that entrepreneurs cannot affect, whereas entrepreneurs' technologies capture the uncertainty that entrepreneurs can affect. For example, differences in entrepreneurs' technologies can capture differences in entrepreneur characteristics such as human capital or differences in project characteristics such as assets.

To isolate the paper's novel contribution, I assume that all entrepreneurs generate the same increase in their project's expected cash flow through effort (expected value of effort), incur the same disutility of effort, and have the same reservation utility equal to zero. In particular, all entrepreneurs generate the same net present value and are equally productive. In an extension, I consider entrepreneurs with different net present values as well as different technologies.

The investor has enough capital to invest in a single entrepreneur. Alternatively, she can invest in an asset with an expected gross return of one. The investor chooses the type of entrepreneur to invest in and designs a contract to induce the entrepreneur's effort. As is standard in the financial contracting literature, contracts have to satisfy three contractual constraints. Entrepreneurs are protected by limited liability and both the entrepreneur's and the investor's contractual payoff has to be nondecreasing in cash flows.

As a first step, I derive optimal contracts for an arbitrary entrepreneur in closed form. An optimal incentive contract is a capped bonus contract, that is, the entrepreneur's contractual payoff is zero if the cash flow is below a first threshold, increases one-to-one if the cash flow is between the first and a second threshold, and remains flat if the cash flow is above the second threshold.

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Intuitively, if the entrepreneur has the strongest impact on a lower (higher) region of the cash flow distribution, his contractual payoff becomes more debt-like (equity-like). The investor's contractual payoff is a combination of debt and equity.[1]

Next, I characterize the investor's investment decision. The general insight is that, despite having identical net present values, different entrepreneurs require different optimal contracts, which imply different agency rents. I derive entrepreneurs' agency rents under optimal contracts in closed form. The higher an entrepreneur's rent, the lower his pledgeable income, which is a standard measure of financial constraints (see, e.g., Tirole, 2006). In addition, I show that an entrepreneur's agency rent depends only on the ratio of the expected value of effort and the disutility of effort. I refer to the ratio as the productivity of effort, which I have assumed is common to all entrepreneurs.

For illustration, consider an example with cash flows 0, 1, and 2, and two entrepreneurs. Entrepreneur 1's effort increases the probability of cash flow 1, entrepreneur 2's increases that of cash flow 2. The investor has to reward entrepreneur 1 in state 1 to induce effort but also in state 2 due to the entrepreneur's monotonicity constraint. If the productivity of effort is high, the investor has to reward entrepreneur 2 only in state 2 to induce effort and no monotonicity constraint binds. Hence, everything else equal, entrepreneur 2 requires a lower agency rent and has a higher pledgeable income if the productivity of effort is high. In particular, entrepreneur 2 requires equity incentives and receives debt financing. However, if the productivity of effort is low, the investor has to give entrepreneur 2 a high share of the cash flow in state 2 to induce effort, and also a high share of the cash flow in state 1 due to the investor's monotonicity constraint. In contrast, the investor must give entrepreneur 1 only the same level in state 2 as in state 1 but a lower share. Hence, entrepreneur 1 requires a lower agency rent and has a higher pledgeable income if the productivity of effort is sufficiently low. In particular, entrepreneur 1 requires debt incentives and receives equity financing.

I solve the investor's investment decision explicitly for two subsets of entrepreneurs, who receive debt financing and equity incentives (debt entrepreneurs) and equity financing and debt incentives (equity entrepreneurs), respectively. In each of these two subsets of entrepreneurs, there exists a unique entrepreneur who has the lowest agency rent, the highest pledgeable income, and is

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[1]Following the conventions of the literature, I refer to a contractual payoff as a function of the cash flow x of the form $\min\{x, \phi\}$ for some $\phi \geq 0$ as debt, and a contractual payoff of the form $\max\{0, x - \phi\}$ as equity.

thus least financially constrained. If the two optimal entrepreneurs have the same maximum likelihood ratio, the investor prefers the optimal debt entrepreneur if the productivity of effort is high, but prefers the optimal equity entrepreneur if the productivity of effort is low. As illustrated in the example, due to the contractual constraints, it is more costly to incentivize equity entrepreneurs compared to debt entrepreneurs if the productivity of effort is high but less if it is low.

The set of entrepreneurs who receive debt financing is equal to the set of entrepreneurs satisfying the monotone likelihood ratio property (MLRP). In particular, the optimal debt entrepreneur is also the optimal MLRP entrepreneur. This implies that the investor would not choose any MLRP entrepreneur if the productivity of effort is sufficiently low. Importantly, these are the environments in which agency rents are high and agency problems are thus likely to matter most for firms and investors. It is exactly in these environments that MLRP might not be a plausible assumption.

These insights extend to the general case in which entrepreneurs are financed by a combination of debt and equity and retain a capped bonus for incentive reasons. Given their optimal contracts, I characterize entrepreneurs as more debt-like (equity-like) if debt (equity) has a higher weight in their financing. I show that if the investor decides between a more debt- and a more equity-like entrepreneur, she prefers the more equity-like entrepreneur if the productivity of effort is sufficiently low.

If entrepreneurs also differ in their net present values (e.g., in their expected value of effort), in addition to their technologies, the investor may prefer an inefficient entrepreneur from a set of potential entrepreneurs if this entrepreneur requires a sufficiently lower agency rent relative to the otherwise efficient choice. Thus, the investor's investment decision can be distorted due to the contracting problem.

The model applies to environments in which investors decide between different types of entrepreneurs that require tailored financial contracts in an environment that is plagued by agency problems, which resembles the problem faced by venture capitalists (see, e.g., Kaplan and Strömberg, 2001). Indeed, venture capitalists (VCs) choose from a number of different investment opportunities and design the financial contracts, in particular to mitigate agency problems (Kaplan and Strömberg, 2004). Consistent with the theory presented in this paper, the investment decision of VCs interacts with the design of financial contracts in the sense that VCs tailor financial

contracts to the characteristics of individual entrepreneurs, which in turn affect VCs' investment decisions (Kaplan and Strömberg, 2000). Moreover, the contractual payoff of the VC corresponds in most cases to a combination of debt and equity (Kaplan and Strömberg, 2003), in line with the financial contracts in the model. My theory therefore suggests that an entrepreneur's ability to raise financing is a function of his production technology and varies systematically with factors that affect the productivity of effort.

Related Literature. There is a significant literature on optimal incentive and financial contracting, which takes the nature of the production technology as given. As such, this literature does not address the choice regarding entrepreneurs. For example, Innes (1990) studies a single entrepreneur with a fixed technology and shows that equity is the optimal incentive contract. Poblete and Spulber (2012) also consider a fixed entrepreneur but allow for more general technologies. In contrast, I study the investor's choice between entrepreneurs. Specifically, I derive the investor's and entrepreneur's expected utilities under optimal contracts in closed form, characterize the investor's investment decision and how it varies across productivity levels. Further, Hébert (2018) considers a model with both effort and risk shifting. In my model, the only dimension of moral hazard is the entrepreneur's effort and I allow the investor to choose between different types of entrepreneurs.

This paper is related to the literature on information systems, in particular Blackwell (1951, 1953), Holmstrom (1979), Grossman and Hart (1983), and Kim (1995). While the authors study the principal's response to changes in the agent's information system, they study the risk/incentive trade-off, which is absent in my model. In contrast, I study the choice of an output distribution in the presence of standard contractual constraints: limited liability and monotonicity.

In the context of principal-agent models, a number of papers study agent selection. For example, Legros and Newman (1996), Thiele and Wambach (1999), Newman (2007), and Chade and Vera de Serio (2014) study agent selection based on agents' wealth, and Sobel (1993), Silvers (2012), and von Thadden and Zhao (2012) consider the agent's information set. Other papers study the principal's choice between agents where the main friction is adverse selection (see, e.g., Faynzilberg and Kumar, 1997; Lewis and Sappington, 2000, 2001), which is absent in my setting. I contribute to the literature by studying agent selection based on differences in agents' technologies,

in the presence of standard contractual constraints.

The paper proceeds as follows: Section 2 introduces the theoretical framework. Section 3 studies optimal contracts. The main contribution of the paper is to study the resulting agency rents in Section 4 and the investor's investment decision in Section 5. Section 6 discusses the implications of the results. All proofs and additional material can be found in the appendix.

2 Model

There are three dates $t \in \{0,1,2\}$ and no time discounting. There is a risk-neutral investor (the principal) and a set of penniless risk-neutral entrepreneurs (the agents).[2] Each entrepreneur owns a project, which costs $I > 0$ at $t = 0$ and generates a cash flow $x_i \in \mathbb{R}_+ := [0,\infty)$ in state $i \in \Omega := \{0,\ldots,n\}$ at $t = 2$, where $0 = x_0 < x_1 < \cdots < x_n$ and $n \geq 2$. The investor has I units of capital, which allows her to invest in a single entrepreneur. Alternatively, she can invest in an asset with an expected gross return of one.

If an entrepreneur receives financing at $t = 0$, he chooses effort $e \in \{0,1\}$ at $t = 1$, which is not verifiable. Entrepreneurs have a disutility of effort $c \geq 0$, which is noncontractible, and a reservation utility of zero. If an entrepreneur does not exert effort ($e = 0$), the cash flow is drawn according to the probability distribution q, where $q_i = \mathbb{P}(x = x_i | e = 0) > 0$, $i \in \Omega$. In particular, all entrepreneurs have the same q, also referred to as the baseline technology.[3] Intuitively, the baseline technology q captures the uncertainty that the entrepreneur cannot affect such as demand fluctuations. Entrepreneurs differ in their cash flow distributions when they exert effort ($e = 1$). Denote the set of entrepreneur types by $\mathscr{P}$. If an entrepreneur $p \in \mathscr{P}$ exerts effort, the cash flow is drawn according to the probability distribution p, where $p_i = \mathbb{P}(x = x_i | e = 1)$, $i \in \Omega$. Intuitively, the difference between an entrepreneur's technology $p \in \mathscr{P}$ and the baseline technology q captures the uncertainty that the entrepreneur can affect. For example, differences in $p \in \mathscr{P}$ can capture differences in entrepreneur characteristics such as human capital or differences in project characteristics

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[2]Appendix C discusses the case of risk-averse entrepreneurs.

[3]I consider a general q, such that the results can be used to study differences in the no-effort distributions in addition to the differences in the effort distributions.

such as assets.[4]

Assumption 1. *Each entrepreneur $p \in \mathscr{P}$ generates the same expected value of effort $\pi \geq c$, that is, for all $p \in \mathscr{P}$, $\mathbb{E}_p[x] - \mathbb{E}_q[x] = \pi$.*

Since all entrepreneurs have the same baseline technology q, they generate the same net present value $\mathbb{E}_q[x] - I$ when they do not exert effort. Assumption 1 implies that all entrepreneurs generate the same net present value

$$\mathbb{E}_p[x] - c - I = \mathbb{E}_q[x] + \pi - c - I$$

when they exert effort. Section 5.3 relaxes this assumption and considers entrepreneurs with different net present values under effort.

Assumption 2. $\mathbb{E}_q[x] + \pi - c - I \geq 0 \geq \mathbb{E}_q[x] - I.$

Assumption 2 implies that effort is a necessary condition for a positive net present value.

Definition 1. *Consider a probability measure p. The likelihood ratio $l(p) = (l_i(p))_{i \in \Omega} \in \mathbb{R}^{n+1}$ is defined as follows: $l_i(p) := \frac{p_i - q_i}{q_i}$, $i \in \Omega$. Denote the maximum likelihood ratio by $l^*(p) := \max_{i \in \Omega} l_i(p)$.*

Definition 2. *Consider a probability measure p. The likelihood ratio $l(p)$ is called single-peaked if there exists a state $m \in \Omega$, such that for all $i \leq m$, $l(p)$ is nondecreasing in $i \in \Omega$, and for all $i \geq m$, $l(p)$ is nonincreasing in $i \in \Omega$.*

Assumption 3. *For all $p \in \mathscr{P}$, p first-order stochastically dominates q.*

Assumption 4. *For all $p \in \mathscr{P}$, the likelihood ratio $l(p)$ is single-peaked.*

Assumptions 3 and 4 propose a generalization of the monotone likelihood ratio property. A monotone likelihood ratio peaks in the highest state n, which implies first-order stochastic dominance. I allow the likelihood ratio to peak in any state $m \in \Omega$ while maintaining first-order stochastic dominance.

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[4]The idea is consistent with the view in the production-based asset pricing literature that firms can substitute output across states of nature (see, e.g., Cochrane, 1993; Belo, 2010).

At $t = 0$, the investor can offer a contract $s = (s_i)_{i \in \Omega} \in \mathbb{R}^{n+1}$ to an entrepreneur $p \in \mathscr{P}$ in exchange for providing capital I to the entrepreneur, where s_i is the entrepreneur's and $x_i - s_i$ is the investor's contractual payoff in state $i \in \Omega$ at $t = 2$. The following standard contractual constraints define the set of feasible contracts.[5]

Assumption 5. *For all $i \in \Omega$, the contract s satisfies $s_i \geq 0$.*

Assumption 6. *For all $i \in \{1, \ldots, n\}$, the contract s satisfies $s_i \geq s_{i-1}$.*

Assumption 7. *For all $i \in \{1, \ldots, n\}$, the contract s satisfies $x_i - s_i \geq x_{i-1} - s_{i-1}$.*

Assumption 5 corresponds to a limited liability constraint for the entrepreneur.[6] Assumptions 6 and 7 require that both the entrepreneur's and the investor's contractual payoff is nondecreasing in the cash flow, and I refer to the two constraints as the entrepreneur's and the investor's monotonicity constraint, respectively.[7]

Example 1. *Consider two entrepreneurs with projects generating cash flows $x_i = i$, $i \in \{0, 1, 2\}$, with project cost $I = \frac{1}{5}$, disutility of effort $c = \frac{1}{2}$, and the baseline technology $q = \left(\frac{17}{20}, \frac{2}{20}, \frac{1}{20}\right)$. Entrepreneur $p' = \left(\frac{1}{20}, \frac{18}{20}, \frac{1}{20}\right)$ implies the likelihood ratio $l(p') = \left(-\frac{16}{17}, 8, 0\right)$, and entrepreneur $p'' = \left(\frac{9}{20}, \frac{2}{20}, \frac{9}{20}\right)$ implies the likelihood ratio $l(p'') = \left(-\frac{8}{17}, 0, 8\right)$, with the expected value of effort $\pi = \frac{4}{5}$.*

In Example 1, both entrepreneurs generate the same net present value $\mathbb{E}_q[x] + \pi - c - I = \frac{3}{10}$ when they exert effort, and the same net present value $\mathbb{E}_q[x] - I = 0$ when they do not exert effort. Entrepreneur p' has a likelihood ratio that peaks in state $i = 1$, whereas entrepreneur p'' implies a monotone likelihood ratio that peaks in state $i = 2$.

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[5]See, e.g., Harris and Raviv (1989); Innes (1990); Nachman and Noe (1994); Hart and Moore (1995); DeMarzo and Duffie (1999); Biais and Mariotti (2005); DeMarzo (2005); DeMarzo et al. (2005).

[6]Note that an optimal contract in Proposition 1 satisfies the investor's limited liability constraint: $\forall i \in \Omega : s_i \leq x_i$.

[7]A standard motivation for the two monotonicity constraints is that, if the contract violated one of the constraints, then the investor or the entrepreneur has an incentive to "sabotage" the project and "destroy" output. The two constraints can also be motivated if the entrepreneur can "destroy" output and "secretly borrow" at zero cost to inflate output. In general, these arguments correspond to the important concern of performance manipulation by firms (see, e.g., Frydman and Jenter, 2010; Murphy, 2013).

3 Optimal Contracts

As a first step, I characterize optimal contracts for an arbitrary entrepreneur $p \in \mathscr{P}$ ignoring the investor's participation constraint.[8] An optimal incentive compatible contract, denoted by $s^*(p)$, satisfies[9]

$$s^*(p) \in \arg\max_s \mathbb{E}_p\,[x - s] \tag{1a}$$

subject to

$$\mathbb{E}_p[s] - c \geq \mathbb{E}_q[s], \tag{1b}$$

$$\mathbb{E}_p[s] - c \geq 0, \tag{1c}$$

$$\forall i \in \Omega : s_i \geq 0, \tag{1d}$$

$$\forall i \in \{1,\ldots,n\} : s_i \geq s_{i-1}, \tag{1e}$$

$$\forall i \in \{1,\ldots,n\} : x_i - s_i \geq x_{i-1} - s_{i-1}. \tag{1f}$$

As is standard with risk-neutrality and limited liability, the entrepreneur earns an agency rent, and the investor designs the contract to minimize the agency rent. As shown, the incentive constraint (1b) binds; that is,

$$\mathbb{E}_p\,[s^*(p)] - c = \mathbb{E}_q\,[s^*(p)] > 0. \tag{2}$$

Due to limited liability, the entrepreneur has to receive at least zero in all states and more than zero in at least one state to induce effort. Since all states have a positive probability under the baseline technology q, the entrepreneur gets a positive expected utility from shirking. The best the investor can do is to give the entrepreneur the same positive expected utility in equilibrium. In particular, the participation constraint (1c) does not bind.

As a benchmark, consider the contracting problem without the monotonicity constraints (1e) and (1f), such that the only contractual constraint is the entrepreneur's limited liability (1d). As shown, (2) also holds in this limited liability benchmark.

Lemma 1. *Consider the contracting problem (1) without the monotonicity constraints (1e) and*

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[8]The investor's participation constraint is given by $\mathbb{E}_p\,[x - s] \geq I$. I discuss its implications in the following sections.
[9]By a slight abuse of notation, s denotes both the vector and the random variable.

(1f). Consider an entrepreneur $p \in \mathcal{P}$ and a feasible contract s. If the contract s is optimal, then $s_i = 0$ for $i \notin \arg\max_{i \in \Omega} l_i(p)$. In particular, for $j \in \arg\max_{i \in \Omega} l_i(p)$, an optimal contract $s^*(p)$ is given by $s_i^*(p) = \mathbb{1}_{\{i=j\}} \frac{c}{p_j - q_j}$, $i \in \Omega$, where $\mathbb{1}$ denotes the indicator function.

In the limited liability benchmark, the investor minimizes the entrepreneur's agency rent by rewarding him only in states with the maximum likelihood ratio, which has the highest incentive effect per unit of agency rent. In Example 1, the investor would offer the contract $s^*(p') = \left(0, \frac{5}{8}, 0\right)$ to entrepreneur p' and the contract $s^*(p'') = \left(0, 0, \frac{5}{4}\right)$ to entrepreneur p''. Clearly, the contract $s^*(p')$ violates the entrepreneur's monotonicity constraint (1e) and the contract $s^*(p'')$ violates the investor's monotonicity constraint (1f).

Next, I study the full contracting problem (1). It is useful to introduce two additional definitions.

Definition 3. *For each entrepreneur $p \in \mathcal{P}$, the cumulative likelihood ratio $L(p) = (L_i(p))_{i \in \Omega} \in \mathbb{R}^{n+1}$ is defined as follows: $L_i(p) := \frac{\sum_{j=i}^n (p_j - q_j)}{\sum_{j=i}^n q_j}$, $i \in \Omega$. Denote the maximum cumulative likelihood ratio by $L^*(p) := \max_{i \in \Omega} L_i(p)$.*

Lemma 2. *For all $p \in \mathcal{P}$, the cumulative likelihood ratio $L(p)$ is single-peaked.*

As discussed below, the cumulative likelihood ratio determines optimal contract design in the full contracting problem. It can be written as a weighted average of the likelihood ratio:

$$L_i(p) = \frac{\sum_{j=i}^n q_j l_j(p)}{\sum_{j=i}^n q_j}, \ i \in \Omega.$$

Lemma 2 shows that the cumulative likelihood ratio inherits the single-peaked property from the likelihood ratio.

Definition 4. *For each $j \in \{1, \ldots, n\}$, the tranche $s^j \in \mathbb{R}_+^{n+1}$ is defined as $s_i^j := \mathbb{1}_{\{i \geq j\}}(x_j - x_{j-1})$, $i \in \Omega$, where $\mathbb{1}$ denotes the indicator function.*

The tranches s^j, $j \in \{1, \ldots, n\}$, partition the cash flow x_i in each state $i \in \Omega$, that is, $\sum_{j=1}^n s_i^j = x_i$, $i \in \Omega$. They are the building blocks of optimal contracts in the full contracting problem.

Proposition 1. *Consider the contracting problem (1) and an entrepreneur $p \in \mathscr{P}$. Consider a permutation $\{i_1, \ldots, i_n\}$ of the set $\{1, \ldots, n\}$ such that $L_{i_1}(p) \geq L_{i_2}(p) \geq \cdots \geq L_{i_n}(p)$. For $j \in \{1, \ldots, n\}$, define*

$$\bar{c}_j := \sum_{k=1}^{j} \left(x_{i_k} - x_{i_k - 1} \right) L_{i_k}(p) \sum_{i=i_k}^{n} q_i.$$

We have $0 =: \bar{c}_0 \leq \bar{c}_1 \leq \cdots \leq \bar{c}_n = \pi$. Let $m \in \{1, \ldots, n\}$ such that $\bar{c}_{m-1} < c \leq \bar{c}_m$. Then an optimal contract $s^*(p)$ is given by $s^*(p) = \sum_{k=1}^{m} \lambda_{i_k} s^{i_k}$, where $\lambda_{i_k} = 1$ for $k \in \{1, \ldots, m-1\}$, and

$$\lambda_{i_m} = \frac{c - \bar{c}_{m-1}}{(x_{i_m} - x_{i_m - 1}) L_{i_m}(p) \sum_{i=i_m}^{n} q_i} \in (0, 1].$$

In particular, there exist two thresholds $\phi_1(p), \phi_2(p) \in [0, x_n]$ such that an optimal contract $s^*(p)$ is given by

$$s_i^*(p) = \min \left\{ \max \left\{ 0, x_i - \phi_1(p) \right\}, \phi_2(p) \right\}, \ i \in \Omega. \tag{3}$$

Proposition 1 gives optimal contracts in closed form.[10,11] In the full contracting problem, the entrepreneur's contractual payoff has to be nondecreasing. Thus, if the investor decides to reward the entrepreneur in a state $i < n$, she has to reward the entrepreneur in all higher states $j > i$ as well. Since the cumulative likelihood ratio $L_i(p)$ is a weighted average of the likelihood ratios for these states $j \geq i$, the investor first uses a fraction λ_{i_1} of the contract tranche s^{i_1} with the maximum cumulative likelihood ratio $L_{i_1}(p)$, which has the highest incentive effect per unit of agency rent. Since the investor's contractual payoff has to be nondecreasing, we require $\lambda_{i_1} \leq 1$. If the contract tranche s^{i_1} is not enough to generate sufficient incentives, the investor adds a fraction λ_{i_2} of the contract tranche s^{i_2} with the second highest cumulative likelihood ratio $L_{i_2}(p)$, and so on. Since $L(p)$ is single-peaked, a tranche that is added to the contract can be chosen to be adjacent to a tranche that has been added before, which yields the specific shape of the contract in (3).

Example 2. *Consider an entrepreneur with a project generating cash flows $x_i = i$, $i \in \{0, 1, 2, 3\}$, and the baseline technology $q = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4} \right)$. Entrepreneur $p = \left(\frac{1}{20}, \frac{1}{20}, \frac{3}{5}, \frac{3}{10} \right)$ implies the cumulative likelihood ratio $L(p) = \left(0, \frac{4}{15}, \frac{4}{5}, \frac{1}{5} \right)$, with the expected value of effort $\pi = \frac{13}{20}$.*

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[10]Note that the thresholds $\phi_1(p)$ and $\phi_2(p)$ are derived in closed form in the proof of Proposition 1.

[11]The contracting problems considered in Lemma 1 and Proposition 1 do not always have a unique solution. This is not relevant for the purpose of this paper, since I focus on an entrepreneur's agency rent, and different optimal contracts imply the same agency rent.

Figure 1a plots the optimal contracts for Example 2, for the three thresholds $\bar{c}_1$, $\bar{c}_2$, and $\bar{c}_3 = \pi$, defined in Proposition 1. If $c = \bar{c}_1$, the investor uses the contract tranche s^2 with the maximum cumulative likelihood ratio $L_2(p)$ (dots). If $c = \bar{c}_2$, she adds the contract tranche s^1 with the second highest cumulative likelihood ratio $L_1(p)$ (triangles). If $c = \pi$, she adds the contract tranche s^3 with the lowest positive cumulative likelihood ratio $L_3(p)$ (squares).

I refer to the entrepreneur's contractual payoff $s^*(p)$ specified in (3) as a capped bonus. Using (3), the investor's contractual payoff can be written as

$$x_i - s_i^*(p) = \min\{x_i, \phi_1(p)\} + \max\{0, x_i - \phi_1(p) - \phi_2(p)\}, \ i \in \Omega. \tag{4}$$

Following the conventions of the literature, I refer to a contractual payoff of the form $\min\{x, \phi\}$ for some $\phi \in [0, x_n]$ as debt, and a contractual payoff of the form $\max\{0, x - \phi\}$ for some $\phi \in [0, x_n]$ as equity. Thus, we can interpret the investor's contractual payoff as a combination of debt and equity. The relative importance of debt and equity depends on the thresholds $\phi_1(p)$ and $\phi_2(p)$, which in turn depend on the entrepreneur's type $p \in \mathscr{P}$.

Relative to the existing literature (see, e.g., Innes, 1990; Poblete and Spulber, 2012), this section shows that extending the assumption of a monotone likelihood ratio to a single-peaked likelihood ratio generates a relevant class of incentive and financial contracts.

The contracting problem determines both the entrepreneur's incentive contract $s^*(p)$ and the investor's financial contract $x - s^*(p)$. In particular, the results have implications for the optimal compensation of firm insiders such as managers. For example, Innes (1990) provides a stylized theory of equity compensation (see, e.g., Edmans and Gabaix, 2016). My theory rationalizes a much larger class of incentive contracts: capped bonuses.

Corollary 1. *Consider an entrepreneur $p \in \mathscr{P}$. If $L^*(p) = L_1(p)$, there exists a threshold $\phi(p) \in [0, x_n]$ such that an optimal contract $s^*(p)$ is given by $s_i^*(p) = \min\{x_i, \phi(p)\}$, $i \in \Omega$. If $L^*(p) = L_n(p)$, there exists a threshold $\phi(p) \in [0, x_n]$ such that an optimal contract $s^*(p)$ is given by $s_i^*(p) = \max\{0, x_i - \phi(p)\}$, $i \in \Omega$.*

If the cumulative likelihood ratio peaks in state $i = 1$, an optimal incentive contract for the entrepreneur is debt: $s^*(p) = \min\{x, \phi(p)\}$. The investor's contractual payoff is equity:

$x - s^*(p) = \max\{0, x - \phi(p)\}$. In contrast, if the cumulative likelihood ratio peaks in state $i = n$, an optimal incentive contract for the entrepreneur is equity: $s^*(p) = \max\{0, x - \phi(p)\}$. The investor's contractual payoff is debt: $x - s^*(p) = \min\{x, \phi(p)\}$, which corresponds to the result in Innes (1990). In particular, the model rationalizes straight debt financing and straight equity financing for different types of entrepreneurs.

In Example 1, the investor has to reward entrepreneur p' in state $i = 2$ in addition to state $i = 1$, due to the entrepreneur's monotonicity constraint, and the optimal incentive contract for entrepreneur p' is debt: $s^*(p') = \min\{x, \frac{5}{8}\}$. The investor's contractual payoff is equity: $x - s^*(p') = \max\{0, x - \frac{5}{8}\}$. The investor has to reward entrepreneur p'' in state $i = 1$ in addition to state $i = 2$, due to the investor's monotonicity constraint, and the optimal incentive contract for entrepreneur p'' is equity: $s^*(p'') = \max\{0, x - \frac{3}{4}\}$. The investor's contractual payoff is debt: $x - s^*(p'') = \min\{x, \frac{3}{4}\}$.

4 Agency Rents and Pledgeable Income

In this section, I provide a complete characterization of entrepreneurs' agency rents, which is the key departure from the existing literature.[12] Consider again an arbitrary entrepreneur $p \in \mathscr{P}$. Using Assumption 1 and the entrepreneur's binding incentive constraint (2), we can write the investor's expected utility (1a) under an optimal contract $s^*(p)$ as

$$\mathbb{E}_p[x - s^*(p)] = \mathbb{E}_q[x] + \pi - c - \mathbb{E}_q[s^*(p)] =: \bar{I}(p). \tag{5}$$

Since the investor makes the contract offer in my model, (5) is equal to the entrepreneur's (expected) pledgeable income, defined as the maximum expected income that can be pledged to the investor without jeopardizing the entrepreneur's incentives and without violating any of the contractual constraints, which is a standard measure of financial constraints (see, e.g., Holmstrom and Tirole, 1997; Tirole, 2006). I therefore refer to an entrepreneur with a higher agency rent and a

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[12]The existing literature determines the shape of optimal contracts but does not characterize agency rents (see, e.g., Innes, 1990; Poblete and Spulber, 2012), which is necessary to study the investor's choice between entrepreneurs.

lower pledgeable income as more financially constrained.[13] In particular, the investor's participation constraint is given by $\bar{I}(p) \geq I$.

Using Lemma 1, it is straightforward to calculate the agency rent in the limited liability benchmark, which is given by $\mathbb{E}_q[s^*(p)] = \frac{c}{l^*(p)}$, where $s^*(p)$ is an optimal contract from Lemma 1. The agency rent is linearly increasing in the disutility of effort c and decreasing in the maximum likelihood ratio $l^*(p)$. Intuitively, an entrepreneur with a high maximum likelihood ratio has a high informativeness of cash flows, which implies a low agency rent and a high pledgeable income. In Example 1, both entrepreneurs have the same maximum likelihood ratio, which therefore implies the same agency rent and pledgeable income in the limited liability benchmark. In particular, we have $\bar{I}(p') = \bar{I}(p'') = \frac{7}{16} > \frac{1}{5} = I$ and the investor is indifferent between providing capital to entrepreneur p' and entrepreneur p''.

The contribution of this section is to determine entrepreneurs' agency rents in the full contracting problem. For an entrepreneur $p \in \mathscr{P}$, I call the mapping $[0, \pi] \ni c \mapsto \mathbb{E}_q[s^*(p)] \in \mathbb{R}_+$ the agency rent function, where $s^*(p)$ is an optimal contract from Proposition 1.

Proposition 2. *Consider an entrepreneur $p \in \mathscr{P}$ and a permutation $\{i_1, \ldots, i_n\}$ of the set $\{1, \ldots, n\}$ such that $L_{i_1}(p) \geq L_{i_2}(p) \geq \cdots \geq L_{i_n}(p)$. Let $(\bar{c}_j)_{j \in \{0, \ldots, n\}}$ be the partition of the interval $[0, \pi]$ from Proposition 1. Let $m \in \{1, \ldots, n\}$ such that $\bar{c}_{m-1} < c \leq \bar{c}_m$, then*

$$\mathbb{E}_q[s^*(p)] = \frac{c}{L_{i_m}(p)} + \sum_{k=1}^{m-1} (x_{i_k} - x_{i_k-1}) \left(1 - \frac{L_{i_k}(p)}{L_{i_m}(p)}\right) \sum_{i=i_k}^{n} q_i.$$

In particular, for $c \in (\bar{c}_{m-1}, \bar{c}_m)$, we have

$$\frac{\partial \mathbb{E}_q[s^*(p)]}{\partial c} = \frac{1}{L_{i_m}(p)} = \frac{\sum_{i=i_m}^{n} q_i}{\sum_{i=i_m}^{n} (p_i - q_i)} > 0.$$

The agency rent function is continuous, increasing, piecewise linear, (weakly) convex, and equal

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[13]The investor can provide capital to a single entrepreneur. Thus, only an entrepreneur with the minimum agency rent and maximum pledgeable income can attract financing provided that $\bar{I}(p) \geq I$. The interpretation of the pledgeable income as a measure of financial constraints is much broader. For example, consider a setting in which an entrepreneur $p \in \mathscr{P}$ faces competitive investors but requires positive net worth to finance the project cost I (see, e.g., Holmstrom and Tirole, 1997). In this case, $\bar{I}(p) < I$ such that the entrepreneur requires net worth $A \geq \bar{A}(p) := I - \bar{I}(p) > 0$ to be able to invest. This implies that an entrepreneur with a lower pledgeable income $\bar{I}(p)$ requires a higher minimum net worth $\bar{A}(p)$ to be able to invest and is therefore more financially constrained.

to zero at $c = 0$.

Proposition 2 gives agency rents in closed form. The shape of the agency rent function mirrors the design of the optimal contract discussed in Section 3. If the investor uses a fraction λ_{i_m} of the contract tranche s^{i_m} with cumulative likelihood ratio $L_{i_m}(p)$ to incentivize the entrepreneur, the marginal agency rent is given by the inverse of the cumulative likelihood ratio, $\frac{1}{L_{i_m}(p)}$. The investor first uses the tranche with the maximum cumulative likelihood ratio and the lowest marginal agency rent, and adds tranches with lower cumulative likelihood ratios and higher marginal agency rents.
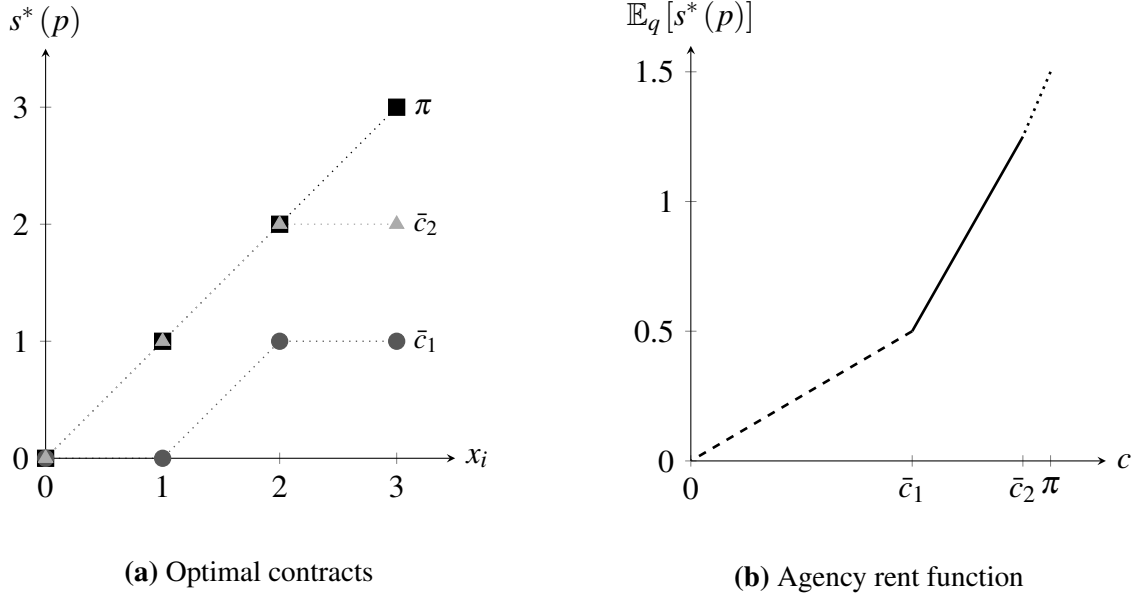
Figure 1b plots the agency rent function for Example 2. If $0 < c < \bar{c}_1$, the investor uses a fraction λ_2 of the contract tranche s^2 with the maximum cumulative likelihood ratio $L_2(p)$ and marginal agency rent $\frac{1}{L_2(p)}$ (dashed line). If $\bar{c}_1 < c < \bar{c}_2$, the investor adds a fraction λ_1 of the contract tranche s^1 with the second highest cumulative likelihood ratio $L_1(p)$ and marginal agency rent $\frac{1}{L_1(p)}$ (solid line). If $\bar{c}_2 < c < \pi$, the investor adds a fraction λ_3 of the contract tranche s^3 with the lowest positive cumulative likelihood ratio $L_3(p)$ and marginal agency rent $\frac{1}{L_3(p)}$ (dotted line).

In contrast to the limited liability benchmark, the agency rent function is (weakly) convex due to the monotonicity constraints. This implies that small variations in the disutility of effort can lead to large changes in an entrepreneur's agency rent and pledgeable income. In other words, small changes in the economic environment can lead to large changes in entrepreneurs' financial constraints. Moreover, the size of this effect depends on an entrepreneur's type such that the same change in the economic environment can significantly change the financial constraints of some types of entrepreneurs, while barely affecting other types of entrepreneurs.

The result implies that the pledgeable income $\bar{I}(p)$ is strictly decreasing in the disutility of effort c. A higher disutility of effort reduces the expected surplus of effort $\pi - c$ and increases the agency rent. Since $\mathbb{E}_q[x] \leq I$, there exists a threshold $\bar{c}(p) \in [0, \pi)$ such that $\bar{I}(p) \geq I$ if and only if $c \leq \bar{c}(p)$. In particular, since the agency rent function depends on the entrepreneur's type $p \in \mathscr{P}$, entrepreneurs can have different thresholds $\bar{c}(p)$.

In Appendix B, I show that it is possible to derive an entrepreneur's agency rent as a function of the expected value of effort $\pi \geq c$, for a given disutility of effort $c > 0$. The key insight is that an entrepreneur's agency rent depends only on ratio $\frac{\pi}{c}$, which I refer to as the productivity of effort.

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Figure 1: Optimal Contracts and Agency Rents

(a) Optimal contracts

(b) Agency rent function

This figure considers Example 2. Figure 1a plots the optimal contracts $s^*(p)$ from Proposition 1 for the three thresholds $\bar{c}_1$, $\bar{c}_2$, and $\bar{c}_3 = \pi$, defined in Proposition 1, where $\bar{c}_1 = \frac{2}{5}$, $\bar{c}_2 = \frac{3}{5}$, and $\bar{c}_3 = \pi = \frac{13}{20}$. Figure 1b plots the agency rent function from Proposition 2.

In other words, a high agency rent can be the result of a high disutility of effort or a low expected value of effort, which we can simply refer to as a low productivity of effort.

5   Optimal Agents

In this section, I characterize the investor's decision between different types of entrepreneurs $p \in \mathscr{P}$. In the absence of frictions, the investor would be indifferent between all entrepreneurs $p \in \mathscr{P}$ since all entrepreneurs generate the same nonnegative net present value $\mathbb{E}_q[x] + \pi - c - I$ under effort. In the presence of frictions, the investor's choice between entrepreneurs amounts to choosing the entrepreneur with the lowest agency rent, or equivalently, the entrepreneur with the highest pledgeable income, provided that the entrepreneur's pledgeable income exceeds the project cost. In Section 5.1, I study the two classes of entrepreneurs from Corollary 1, who are either financed by debt or by equity. I extend the analysis to general entrepreneurs who are financed by a combination of debt and equity in Section 5.2. Section 5.3 extends the analysis to entrepreneurs with different expected values of effort.

5.1 Debt and Equity Entrepreneurs

In this section, I solve the investor's investment decision for the two subsets of entrepreneurs from Corollary 1. First, consider the subset $\mathscr{P}^1 := \{p \in \mathscr{P} | L^*(p) = L_1(p)\}$. Corollary 1 shows that for each entrepreneur $p \in \mathscr{P}^1$, an optimal incentive contract for the entrepreneur is debt and the investor's contractual payoff is equity. Since these types of entrepreneurs are financed by equity, I refer to them as equity entrepreneurs.

Proposition 3. *Let $\pi \leq x_1 q_0$.*[14] *There exists a unique entrepreneur $p^1 \in \mathscr{P}^1$ such that for all $c \in (0, \pi]$, the agency rent of entrepreneur p^1 is lower than the agency rent of all other entrepreneurs in $\mathscr{P}^1$.*

The optimal equity entrepreneur p^1 has the lowest agency rent, the highest pledgeable income, and is thus least financially constrained within the set of equity entrepreneurs $\mathscr{P}^1$.

Next, consider the subset $\mathscr{P}^n := \{p \in \mathscr{P} | L^*(p) = L_n(p)\}$. Corollary 1 shows that for each entrepreneur $p \in \mathscr{P}^n$, an optimal incentive contract for the entrepreneur is equity and the investor's contractual payoff is debt. Since these types of entrepreneurs are financed by debt, I refer to them as debt entrepreneurs.

Proposition 4. *There exists a unique entrepreneur $p^n \in \mathscr{P}^n$ such that for all $c \in (0, \pi)$, the agency rent of entrepreneur p^n is lower than the agency rent of all other entrepreneurs in $\mathscr{P}^n$.*

The optimal debt entrepreneur p^n has the lowest agency rent, the highest pledgeable income, and is thus least financially constrained within the set of debt entrepreneurs $\mathscr{P}^n$. Importantly, $\mathscr{P}^n$ is equal to the set of entrepreneurs satisfying the monotone likelihood ratio property (MLRP).[15] Thus, the optimal debt entrepreneur p^n has the lowest agency rent among all MLRP entrepreneurs.

I next determine whether the optimal equity entrepreneur $p^1 \in \mathscr{P}^1$ or the optimal debt entrepreneur $p^n \in \mathscr{P}^n$ has a lower agency rent, which provides a complete characterization of the

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[14]The restriction $\pi \leq x_1 q_0$ ensures that we consider the largest possible set of entrepreneurs $\mathscr{P}^1$. In general, the set of entrepreneurs $\mathscr{P}$ is defined as a set $\mathscr{P} \subset \mathbb{R}^{n+1}$ such that all $p \in \mathscr{P}$ satisfy $\forall j \in \Omega : \sum_{i=0}^{j}(p_i - q_i) \leq 0$, $\sum_{i=0}^{n}(p_i - q_i) = 0$, $l(p)$ is single-peaked, $\mathbb{E}_p[x] - \mathbb{E}_q[x] = \pi$, and $\forall i \in \Omega : 0 \leq p_i \leq 1$. A sufficiently low π ensures that the last set of constraints does not bind. See Appendix B for a detailed discussion of how $\mathscr{P}$ depends on π.

[15]A monotone likelihood ratio always implies a monotone cumulative likelihood ratio. While the converse is not true for general probability distributions, it is true for the set of probability distributions with a single-peaked likelihood ratio.

investor's decision for debt and equity entrepreneurs.

Proposition 5. *Let $\pi \le x_1 q_0$. Consider the optimal equity entrepreneur $p^1 \in \mathscr{P}^1$ and the optimal debt entrepreneur $p^n \in \mathscr{P}^n$ from Propositions 3 and 4, respectively. There are two cases.*

(i) *If $l^*(p^n) > \frac{q_1}{q_1 + \cdots + q_n} l^*(p^1)$, then there exists a threshold $\bar{c} \in (0, \pi)$ such that for all $c \in (0, \bar{c})$, the optimal debt entrepreneur p^n has a lower agency rent than the optimal equity entrepreneur p^1, and for all $c \in (\bar{c}, \pi]$, the optimal equity entrepreneur p^1 has a lower agency rent than the optimal debt entrepreneur p^n.*

(ii) *If $l^*(p^n) < \frac{q_1}{q_1 + \cdots + q_n} l^*(p^1)$, then for all $c \in (0, \pi]$, the optimal equity entrepreneur p^1 has a lower agency rent than the optimal debt entrepreneur p^n.*

If the optimal equity and debt entrepreneurs p^1 and p^n have the same maximum likelihood ratio $l^*(p)$ and would therefore require identical agency rents in the limited liability benchmark, we obtain the first case of Proposition 5. In this case, if the disutility of effort is low, the investor chooses the optimal debt entrepreneur p^n, provided that the investor's participation constraint is satisfied, that is, provided that $\bar{I}(p^n) \ge I$. If the disutility of effort high, the investor chooses the optimal equity entrepreneur p^1, provided that the investor's participation constraint is satisfied, that is, provided that $\bar{I}(p^1) \ge I$.

Since the pledgeable income of each entrepreneur is decreasing in the disutility of effort, there are cases in which both $\bar{I}(p^1) < I$ and $\bar{I}(p^n) < I$. In these cases, the investor is unwilling to provide capital to any debt or equity entrepreneur independent of their types.

Using the results from Appendix B (discussed in Section 4), I can rephrase the result in terms of the expected value of effort π. If the expected value of effort is high, the investor prefers the optimal debt entrepreneur p^n.[16] If the expected value of effort is low, the investor prefers the optimal equity entrepreneur p^1.

The reason for the change in the investment decision is the changing exposure to the contractual constraints. If the productivity of effort is high (low disutility of effort or high expected value of effort) and the investor gives the optimal equity entrepreneur p^1 a small fraction of the cash flow

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[16]Notice that there is an upper bound for the choice of π as discussed in Appendix B. However, there always exists a c such that both regions exist.

in state $i = 1$, she is forced to give the entrepreneur the same level in higher states due to the entrepreneur's monotonicity constraint (Assumption 6). In contrast, the investor gives the optimal debt entrepreneur p^n a small fraction of the cash flow in state $i = n$ and no monotonicity constraint binds. As a result, the optimal debt entrepreneur p^n requires a lower agency rent. If the productivity of effort is low, the investor gives the optimal debt entrepreneur p^n a high fraction of the cash flow in state $i = n$ and is therefore forced to give the entrepreneur a high fraction of the cash flows in lower states due to the investor's monotonicity constraint (Assumption 7). In contrast, giving the optimal equity entrepreneur p^1 a high fraction of the cash flow in state $i = 1$, forces the investor to give the entrepreneur the same level but lower fractions in higher states. As a result, the optimal equity entrepreneur p^1 requires a lower agency rent if the productivity of effort is sufficiently low.
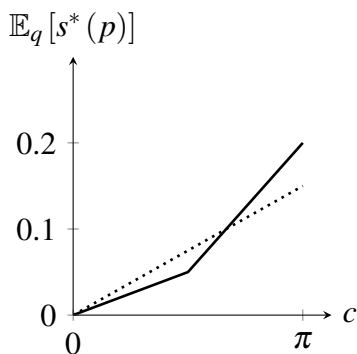
Figure 2 plots the agency rent functions for the equity entrepreneur p' (dotted line) and the debt entrepreneur p'' (solid line) from Example 1. Both entrepreneurs have the same maximum likelihood ratio and would therefore require identical agency rents in the limited liability benchmark. If the productivity of effort is high, the investor is forced to reward the equity entrepreneur p' in state $i = 2$ in addition to state $i = 1$, whereas no monotonicity constraint binds for the debt entrepreneur p''. As a result, the agency rent of the equity entrepreneur p' is higher. If the productivity of effort is low, the investor is forced reward the debt entrepreneur p'' in state $i = 1$ in addition to state $i = 2$. A sufficiently low productivity of effort implies that the investor is forced to give the debt entrepreneur p'' a high fraction of the cash flow in both states. In contrast, she is forced to give the equity entrepreneur p' only the same level but a lower fraction in state $i = 2$. As a result, the agency rent of the debt entrepreneur p'' is higher if the productivity of effort is sufficiently low.

The differences in entrepreneurs' types $p \in \mathscr{P}$ can be interpreted as differences in entrepreneurs' production technologies. For example, they capture differences in entrepreneur characteristics such as human capital or differences in project characteristics such as assets. The result relates differences in entrepreneurs' production technologies to differences in their financial constraints. Specifically, if the productivity of effort is high, entrepreneurs with a production technology p^n are less financially constrained compared to entrepreneurs with a production technology p^1. In contrast, if the productivity of effort is low, entrepreneurs with a production technology p^1 are less financially constrained compared to entrepreneurs with a production technology p^n.

In addition, the result relates financial constraints to financial contracts. Specifically, if the productivity of effort is high, debt financing is associated with lower financial constraints compared to equity financing. In contrast, if the productivity of effort is low, equity financing is associated with lower financial constraints compared to debt financing.

Since $\mathscr{P}^n$ is equal to the set of MLRP entrepreneurs, the optimal debt entrepreneur p^n is also the optimal MLRP entrepreneur. The result implies that the investor would not choose any MLRP entrepreneur if the productivity of effort is sufficiently low. Importantly, these are the environments in which agency rents are high and agency problems are thus likely to matter most for firms and investors. My theory suggests that it is exactly in these environments that MLRP might not be a plausible assumption.

Figure 2: Agency Rent Functions

The figure plots the agency rent functions for entrepreneurs p' (dotted line) and p'' (solid line) from Example 1.

The second case of Proposition 5 shows that there are environments in which the investor always chooses the optimal equity entrepreneur p^1, provided that the investor's participation constraint is satisfied, that is, provided that $\bar{I}(p^1) \geq I$. This is the case when the optimal debt entrepreneur p^n has a significantly lower maximum likelihood ratio than the optimal equity entrepreneur p^1 and would therefore have a considerably higher agency rent, even in the limited liability benchmark.

5.2 General Entrepreneurs

In this section, I study general entrepreneurs $p \in \mathscr{P}$, who are financed by a combination of debt and equity. To begin, I study the existence of an optimal entrepreneur.

Lemma 3. *The set of entrepreneurs $\mathscr{P}$ is compact.*

Since the set $\mathscr{P}$ is compact, and since the mapping $\mathscr{P} \ni p \mapsto \mathbb{E}_q[s^*(p)] \in \mathbb{R}_+$ is continuous, the extreme value theorem applies and an optimal entrepreneur $p^* \in \mathscr{P}$ exists.

Corollary 2. *There exists an optimal entrepreneur $p^* \in \mathscr{P}$; that is, $\min_{p \in \mathscr{P}} \mathbb{E}_q[s^*(p)]$ exists, and we have $\min_{p \in \mathscr{P}} \mathbb{E}_q[s^*(p)] = \mathbb{E}_q[s^*(p^*)]$.*

An optimal entrepreneur p^* has the lowest agency rent, the highest pledgeable income, and is thus least financially constrained within the full set of entrepreneurs $\mathscr{P}$. The investor is willing to provide capital to an optimal entrepreneur p^* if $\bar{I}(p^*) \geq I$.

Lemma 4. *Consider an entrepreneur $p \in \mathscr{P}$. Let $p \neq q$. There exist states $j, m \in \{1, \ldots, n\}$, $1 \leq m \leq j \leq n$, such that $p_0 < q_0$; for all $i \in \{1, \ldots, m-1\}$, $p_i \leq q_i$; for all $i \in \{m, \ldots, j\}$, $p_i > q_i$; and for all $i \in \{j+1, \ldots, n\}$, $p_i = q_i$.*

Lemma 4 shows that an entrepreneur $p \in \mathscr{P}$ reduces the probability of a region of low cash flow states $\{0, \ldots, m-1\}$ and increases the probability of a region of high cash flow states $\{m, \ldots, j\}$. Put differently, by exerting effort, an entrepreneur shifts probability mass from low to high states.

Lemma 5. *Let $\pi > 0$. Consider an entrepreneur $p \in \mathscr{P}$. Let $j, m \in \{1, \ldots, n\}$, $1 \leq m \leq j \leq n$, be defined as in Lemma 4. For all $c \in [0, \pi]$ and all $i \in \Omega$, we have $s_i^*(p) \leq \min\{x_i, x_j\}$, which holds with equality for $c = \pi$.*

Lemma 5 determines an upper bound for an optimal contract. If an entrepreneur $p \in \mathscr{P}$ affects the probability of cash flow states only up to a state $j \in \{m, \ldots, n\}$, then the bound is given by a debt contract with face value x_j. The investor's claim therefore satisfies

$$x_i - s_i^*(p) \geq x_i - \min\{x_i, x_j\} = \max\{0, x_i - x_j\}, \ i \in \Omega.$$

A lower x_j therefore implies a larger equity component in the entrepreneur's financing. I therefore refer to an entrepreneur with a lower x_j as more equity-like, consistent with the notion of debt and equity entrepreneurs from Section 5.1. Intuitively, an entrepreneur who improves a lower region of the cash flow distribution requires more debt-like incentives and therefore more equity-like financing. An entrepreneur who improves a higher region of the cash flow distribution requires more equity-like incentives and more debt-like financing.

Proposition 6. *Let $\pi > 0$ and $0 < m_1 < m_2 < n$. Consider two entrepreneurs $p', p'' \in \mathscr{P}$. Let $j \leq n - m_2$ such that the entrepreneurs positively affect the states in the regions $\{m_1, \ldots, m_1 + j\}$ and $\{m_2, \ldots, m_2 + j\}$, respectively, that is, $p'_i > q_i \Leftrightarrow i \in \{m_1, \ldots, m_1 + j\}$ and $p''_i > q_i \Leftrightarrow i \in \{m_2, \ldots, m_2 + j\}$. Then there exists a $\tilde{c} \in [0, \pi)$ such that for all $c \in (\tilde{c}, \pi]$, entrepreneur p' has a lower agency rent than entrepreneur p''.*

Similar to the intuition for Proposition 5, if the productivity of effort is high (low disutility of effort or high expected value of effort) and the investor gives the more equity-like entrepreneur p' a small fraction of the cash flows in low states, she is forced to give the entrepreneur the same level in higher states due to the entrepreneur's monotonicity constraint (Assumption 6). In contrast, the investor gives the more debt-like entrepreneur p'' a small fraction of the cash flows in higher states such that the entrepreneur's monotonicity constraint binds less. If the productivity of effort is low, the investor gives the more debt-like entrepreneur p'' a high fraction of the cash flows in high states and is therefore forced to give the entrepreneur a high fraction of the cash flows in lower states due to the investor's monotonicity constraint (Assumption 7). In contrast, if the investor gives the more equity-like entrepreneur p' a high fraction of the cash flows in low states, she is forced to give the entrepreneur the same level but lower fractions in higher states. As a result, the more equity-like entrepreneur p' requires a lower agency rent if the productivity of effort is sufficiently low.

5.3 Entrepreneurs with Different Net Present Values

This section extends the model to entrepreneurs with different expected values of effort π and therefore different net present values.[17] In this case, the investor takes into account both agency

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[17]Given that agency rents depend only on the productivity of effort (see Appendix B), I could also consider the case where entrepreneurs differ in their disutilities of effort or differ in both their expected values and disutilities of effort.

rents and the expected value of effort, which both affect an entrepreneur's pledgeable income.

Denote by $\mathscr{P}_\pi$ the set of entrepreneurs with expected value of effort π. Further denote the set of optimal entrepreneurs for a given disutility of effort $c \in [0, \pi]$ by $\mathscr{P}_\pi^*$; that is,

$$\mathscr{P}_\pi^* := \arg\min_{p \in \mathscr{P}_\pi} \mathbb{E}_q\left[s^*(p)\right].$$

The set $\mathscr{P}_\pi^*$ contains all entrepreneurs with the lowest agency rent, potentially including the optimal debt entrepreneur or the optimal equity entrepreneur from Section 5.1.

Proposition 7. *Consider a level of the expected value of effort $\pi_1 > c > 0$. Consider an entrepreneur $p' \in \mathscr{P}_{\pi_1}$ with $p' \notin \mathscr{P}_{\pi_1}^*$ and $\bar{I}(p') \geq I$. There exists a lower level of the expected value of effort $\pi_2 < \pi_1$ and an entrepreneur $p'' \in \mathscr{P}_{\pi_2}$ such that the investor prefers entrepreneur p'' over entrepreneur p'.*

Proposition 7 shows that a less productive entrepreneur can require a lower agency rent, which offsets the loss in expected surplus. An entrepreneur with an expected value of effort π_1 generates the net present value $\mathbb{E}_q[x] + \pi_1 - c - I$. If the investor chooses a less productive entrepreneur with an expected value of effort $\pi_2 < \pi_1$, the net present value reduces by $\pi_1 - \pi_2 > 0$, implying a welfare loss for the economy. As a result, the investor may prefer an inefficient entrepreneur from a set of potential entrepreneurs if this entrepreneur requires a sufficiently lower agency rent relative to the otherwise efficient choice. Thus, the investor's capital allocation decision can be distorted due to the presence of agency rents.

6 Discussion

In this section, I discuss several implications of the model.

6.1 Entrepreneurial Finance

While the model proposed in this paper has implications for firms' capital structure and financial constraints in general, it specifically applies to entrepreneurial finance and in particular to venture

capitalists (VCs). As the investor in the model, VCs choose the entrepreneurs they invest in from a number of investment opportunities and design financial contracts, in particular to mitigate agency problems (see, e.g., Kaplan and Strömberg, 2001, 2004).

Kaplan and Strömberg (2000) study how VCs choose the entrepreneurs they invest in. VCs prepare detailed investment analyses, which discuss a large number of deal characteristics (e.g., the entrepreneur's product) and risk factors (e.g., demand uncertainty). In my model, I capture deal characteristics which are subject to the entrepreneur's moral hazard by $p \in \mathscr{P}$, and the general uncertainty which entrepreneurs cannot control by q. My theory suggests that these deal characteristics affect an entrepreneur's ability to obtain financing, even if they do not affect the entrepreneur's net present value. The bias of the investor depends on the productivity of effort as discussed in detail in Section 5. For example, the productivity of effort can differ across industries. A more competitive industry has a lower expected value of effort and a lower productivity of effort compared to a less competitive industry. My theory suggests that the differences in the productivity of effort lead to systematic differences in the structure of the supply of capital to entrepreneurs.

Kaplan and Strömberg (2003) show that the VC's contractual payoff corresponds in most cases to a combination of debt and equity, in line with the financial contracts in my model. Moreover, VCs tailor financial contracts to the characteristics of individual entrepreneurs and, in line with the mechanism I propose, they in turn take the contract terms into account when making their investment decision.

6.2 Firms' Financial Constraints

The corporate finance literature has identified a number of explanations for the existence of firms' financial constraints (see, e.g., Tirole, 2006). These theories imply that a number of firm characteristics are likely to affect firms' financial constraints and several empirical papers attempt to measure firms' financial constraints (see, e.g., Fazzari et al., 1988; Kaplan and Zingales, 1997; Graham and Harvey, 2001; Whited and Wu, 2006; Hadlock and Pierce, 2010).

My theory suggests a potentially important determinant of firms' financial constraints: firms' production technologies. I show that there is a systematic relationship between firms' financial

constraints (i.e., firms' pledgeable income) and firms' production technologies $p \in \mathscr{P}$. In addition, my theory maps production technologies $p \in \mathscr{P}$ to financial contracts. As shown, an entrepreneur who has the strongest impact on a higher region of the output distribution receives more debt-like financing and an entrepreneur who has the strongest impact on a lower region of the output distribution receives more equity-like financing. The results thus relate financial constraints to financial contracts. In particular, if the productivity of effort is high, debt financing is associated with lower financial constraints compared to equity financing. In contrast, if the productivity of effort is low, equity financing is associated with lower financial constraints compared to debt financing. In line with this view of financial constraints, Hoberg and Maksimovic (2014) differentiate between debt and equity financing and find significant differences between firms who desire equity and firms who desire debt financing. My paper provides a unified theory of debt and equity financing that can rationalize these differences.

6.3 Debt and Equity Financing

In the model, entrepreneurs are financed by a combination of debt and equity. Indeed, in reality, small firms rely on both debt and equity financing (see, e.g., Berger and Udell, 1998). An important question is how the supply of debt and equity financing for firms depends on fluctuations in the macroeconomic environment.[18] While my theory does not feature macroeconomic shocks, some of the insights can be used to derive implications for the supply of debt and equity financing. Specifically, we can think of the expected value of effort π as being a function of the macroeconomic environment such as the business cycle. For example, π is lower in a downturn compared to an upturn. An increase in π leads to an increase in the productivity of effort, which reduces agency rents and increases the net present value of entrepreneurs' projects, implying an increase in entrepreneurs' pledgeable incomes. Simply put, an increase in the productivity of effort relaxes financial constraints for all entrepreneurs. Importantly, the change in entrepreneurs' pledgeable incomes differs across entrepreneurs. Specifically, the model implies that firms who require debt financing are associated with lower financial constraints compared to firms who require equity fi-

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[18]For example, see Covas and Den Haan (2011) for an analysis of the cyclical behavior of debt and equity financing by publicly listed firms.

nancing if the productivity of effort is high (e.g., in an upturn) and vice versa if the productivity of effort is low (e.g., in a downturn). Simply put, an entrepreneur who finds it easier to raise financing relative to similarly profitable entrepreneurs in an upturn, might find it relatively harder to raise financing in a downturn. The broader interpretation of the result is that the structure of the supply of capital for entrepreneurs with different technologies who require different types of financial contracts changes systematically in response to changes in the macroeconomic environment.

6.4 Innovation and Technological Change

Since my model features differences in characteristics of entrepreneurs which can be interpreted as different types of ideas and innovations, it has implications for innovation and technological change (see, e.g., Tinbergen, 1974, 1975).[19] For illustration, consider the optimal debt entrepreneur p^n and the optimal equity entrepreneur p^1 from Section 5.1. For example, the two entrepreneurs have different ideas for an innovation that improves an existing drug as discussed in the introduction. As shown, whether entrepreneur p^1 or entrepreneur p^n obtains financing depends on the productivity of effort. If the productivity of effort is low, entrepreneur p^1 obtains financing rather than entrepreneur p^n and it is his innovation that is realized.

Consider a productivity shock, which leads to a reduction in the disutility of effort or an increase in the expected value of effort, implying an increase in the productivity of effort. If the productivity of effort increases sufficiently, then entrepreneur p^n obtains financing rather than entrepreneur p^1 and it is his innovation that is realized. Thus, an increase in the productivity of effort leads to changes in the composition of entrepreneurs and the types of projects and ideas, which are realized. This interpretation of the model as a theory of technological change does not require differences in the productivity of different entrepreneurs to drive the adoption of new technologies. Instead, technological change is driven only by the contracting problem between entrepreneurs and investors.

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[19]See Acemoglu and Autor (2011) for a survey.

7   Concluding Remarks

In this paper, I consider a theory of financial constraints of entrepreneurs with different production technologies. The theory is based on a standard risk-neutral principal-agent model with contractual constraints: limited liability and monotonicity. Different entrepreneurs generate different probability distributions of output under unverifiable effort. The paper provides a complete characterization of optimal incentive and financial contracts, agency rents, and pledgeable incomes, and relates entrepreneurs' production technologies to financial contracts and financial constraints.

The central idea in the paper is that an entrepreneur's type is a probability distribution $p \in \mathscr{P}$, which determines the entrepreneur's output distribution under effort. The entrepreneur's type $p \in \mathscr{P}$ captures entrepreneur specific characteristics such as human capital and project specific characteristics such as the characteristics of the assets required for the project. The view taken here is that these characteristics affect not only the value of a project but also the nature of the contracting problem between the entrepreneur and an investor. As shown, in the presence of contractual constraints, these characteristics affect entrepreneurs' financial constraints.

The theory has implications for optimal incentive contracts for firm insiders and optimal financial contracts. The results therefore also apply to questions of optimal compensation (see, e.g., Edmans and Gabaix, 2016). In this context, the principal is a firm hiring an agent such as a manager. The results then imply a joint theory of incentives and hiring, treated largely separately (and hence separably) in the literature (see, e.g., Oyer and Schaefer, 2011).

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A Proofs

A.1 Proof of Lemma 1

From Assumption 1 and the incentive constraint (1b), it follows that

$$\mathbb{E}_p\left[x-s\right] = \mathbb{E}_q\left[x\right] + \pi - \mathbb{E}_p\left[s\right] = \mathbb{E}_q\left[x\right] + \pi - c - \left(\mathbb{E}_p\left[s\right] - c\right) \le \mathbb{E}_q\left[x\right] + \pi - c - \mathbb{E}_q\left[s\right].$$

I will show that this quantity is less than or equal to $\mathbb{E}_q\left[x\right] + \pi - c - \frac{c}{l^*(p)}$, by showing that we have $\mathbb{E}_q[s] \ge \frac{c}{l^*(p)}$. Using (1b), it follows that

$$l^*(p)\mathbb{E}_q[s] = l^*(p)\sum_{i=0}^{n} q_i s_i \ge \sum_{i=0}^{n} q_i s_i \left(\frac{p_i}{q_i} - 1\right) = \mathbb{E}_p[s] - \mathbb{E}_q[s] \ge c, \tag{6}$$

and hence $\mathbb{E}_q[s] \ge \frac{c}{l^*(p)}$. Thus, we have

$$\mathbb{E}_p\left[x-s\right] \le \mathbb{E}_q\left[x\right] + \pi - c - \frac{c}{l^*(p)}. \tag{7}$$

Notice that there exist feasible contracts such that (7) holds with equality, for example, the contracts in the statement of the lemma. In particular, s is optimal if and only if $l^*(p)\mathbb{E}_q[s] = c$.

If s is optimal, then all inequalities in (6) turn to equalities, in particular,

$$l^*(p)\sum_{i=0}^{n} q_i s_i = \sum_{i=0}^{n} q_i s_i \left(\frac{p_i}{q_i} - 1\right),$$

which can be rewritten as

$$\sum_{i=0}^{n} q_i s_i \left(l^*(p) - \left(\frac{p_i}{q_i} - 1\right)\right) = \sum_{i=0}^{n} q_i s_i \left(l^*(p) - l_i(p)\right) = 0.$$

All summands are nonnegative, and all summands must therefore be equal to zero. As $q_i > 0$, $i \in \Omega$, it follows that $s_i = 0$ if $l_i(p) < l^*(p)$. Hence, if s is optimal, then $s_i = 0$ if $i \notin \arg\max_{i \in \Omega} l_i(p)$. ∎

A.2 Lemma A.1

Lemma A.1. *Each technology $p \in \mathscr{P}$ has the following properties:*

(i) $\forall j \in \Omega : \sum_{i=j}^n (p_i - q_i) \geq 0.$

(ii) $p_0 \leq q_0$ *and* $p_n \geq q_n.$

(iii) $p_0 < q_0$ *if and only if* $p \neq q.$

Proof. First-order stochastic dominance of p over q is defined as $\forall j \in \Omega : \sum_{i=0}^j (p_i - q_i) \leq 0$. Given the restriction $\sum_{i=0}^n (p_i - q_i) = 0$, we have for every $j \in \{0, \ldots, n-1\}$,

$$\sum_{i=0}^j (p_i - q_i) \leq 0 \Leftrightarrow 0 \leq \sum_{i=j+1}^n (p_i - q_i).$$

In particular, we have $p_0 \leq q_0$ and $p_n \geq q_n$.

Assumption 4 implies that the minimum value of $l(p)$ is achieved at the ends of the domain, that is,

$$\min_{i \in \Omega} l_i(p) = \min\{l_0(p), l_n(p)\}.$$

The inequalities $p_0 \leq q_0$ and $p_n \geq q_n$ imply $l_0(p) \leq 0$ and $l_n(p) \geq 0$. If $p_0 = q_0$, then

$$\min_{i \in \Omega} l_i(p) = 0,$$

and for all $i \in \Omega$, we have $p_i \geq q_i$. As $\sum_{i=0}^n p_i = \sum_{i=0}^n q_i$, it follows that $p = q$. Hence, if $p \neq q$, then necessarily, $p_0 < q_0$. $\blacksquare$

A.3 Proof of Lemma 2

For $i \in \{0, \ldots, n-1\}$, we have $L_i(p) \geq L_{i+1}(p)$ if and only if

$$\frac{(p_i - q_i) + (p_{i+1} - q_{i+1}) + \cdots + (p_n - q_n)}{q_i + q_{i+1} + \cdots + q_n} \geq \frac{(p_{i+1} - q_{i+1}) + \cdots + (p_n - q_n)}{q_{i+1} + \cdots + q_n},$$

which is equivalent to

$$\frac{p_i - q_i}{q_i} \geq \frac{(p_{i+1} - q_{i+1}) + \cdots + (p_n - q_n)}{q_{i+1} + \cdots + q_n},$$

and also equivalent to

$$\frac{p_i - q_i}{q_i} \geq \frac{(p_i - q_i) + \cdots + (p_n - q_n)}{q_i + \cdots + q_n}.$$

Hence, we have the following equivalence:

$$L_i(p) \geq L_{i+1}(p) \Leftrightarrow l_i(p) \geq L_{i+1}(p) \Leftrightarrow l_i(p) \geq L_i(p). \tag{8}$$

Consider the following representation of the cumulative likelihood ratio:

$$L_i(p) = \frac{\sum_{j=i}^{n} q_j l_j(p)}{\sum_{j=i}^{n} q_j}. \tag{9}$$

As the case $p = q$ is obvious, we consider the case $p \neq q$, which implies $p_0 < q_0$ by Lemma A.1. From Assumption 4, there is an $m \in \{1, \ldots, n\}$ such that $l_m(p) \geq l_{m+1}(p) \geq \cdots \geq l_n(p)$ and $l_0(p) \leq l_1(p) \leq \cdots \leq l_m(p)$.

If it exists, consider $i \in \{m, \ldots, n-1\}$. For all $j \geq i$, we have $l_i(p) \geq l_j(p)$. From representation (9), we have $l_i(p) \geq L_i(p)$, and by (8), we have $L_i(p) \geq L_{i+1}(p)$; that is, $L(p)$ is nonincreasing on $\{m, \ldots, n\}$. It remains to prove that $L(p)$ is single-peaked on $\{0, \ldots, m\}$. As $l_0(p) < 0 = L_0(p)$, it follows by (8) that $L_0(p) < L_1(p)$. Consider the first index $j \in \{1, \ldots, m\}$ such that $L_{j-1}(p) > L_j(p)$. If such an index does not exist, then $L(p)$ is nondecreasing and hence single-peaked on $\{0, \ldots, m\}$. If $j = m$, then $L(p)$ is single-peaked on $\{0, \ldots, m\}$. Let $1 \leq j \leq m-1$. I will show that for any $i \in \{j, \ldots, m-1\}$, the inequality $l_i(p) \geq L_i(p)$ holds, which implies $L_i(p) \geq L_{i+1}(p)$ by (8). For $i = j$, it is true, since from $L_{j-1}(p) > L_j(p)$ and (8), it follows that $l_j(p) \geq l_{j-1}(p) > L_j(p)$. By induction, if $l_i(p) \geq L_i(p)$ for some $i \in \{j, \ldots, m-2\}$, then by (8), $L_i(p) \geq L_{i+1}(p)$, and

$$l_{i+1}(p) \geq l_i(p) \geq L_i(p) \geq L_{i+1}(p).$$

Hence, for all $i \in \{j, \ldots, m-1\}$, we have $l_i(p) \geq L_i(p)$, and $L(p)$ is nonincreasing on $\{j, \ldots, m\}$. By construction of j, $L(p)$ is nondecreasing on $\{0, \ldots, j\}$. As a result, $L(p)$ is single-peaked on $\{0, \ldots, m\}$. ∎

A.4 Proof of Proposition 1

I first show that an optimal contract $s^*(p)$ satisfies $s_0^*(p) = 0$. We observe that the contract $\tilde{s}_i :=$ $s_i^*(p) - s_0^*(p)$, $i \in \Omega$, is feasible and satisfies

$$\mathbb{E}_p[x - \tilde{s}] = \mathbb{E}_p[x - s^*(p)] + s_0^*(p) > \mathbb{E}_p[x - s^*(p)],$$

unless $s_0^*(p) = 0$.

Next, I show that an optimal contract $s^*(p)$ satisfies $\mathbb{E}_p[s^*(p)] - c = \mathbb{E}_q[s^*(p)]$. Assume that, on the contrary, $\mathbb{E}_p[s^*(p)] - c > \mathbb{E}_q[s^*(p)]$. Rewrite this relation in the following form:

$$c < \mathbb{E}_p[s^*(p)] - \mathbb{E}_q[s^*(p)] = \sum_{i=1}^{n}(p_i - q_i)s_i^*(p) = \sum_{i=1}^{n}(p_i - q_i)\sum_{k=1}^{i}\left(s_k^*(p) - s_{k-1}^*(p)\right)$$
$$= \sum_{k=1}^{n}\left(s_k^*(p) - s_{k-1}^*(p)\right)\sum_{i=k}^{n}(p_i - q_i).$$

In the latter sum, all summands are non-negative. Hence, there exists a $j \in \{1, \ldots, n\}$ such that

$$\left(s_j^*(p) - s_{j-1}^*(p)\right)\sum_{i=j}^{n}(p_i - q_i) > 0.$$

Consider a new contract $\tilde{s}_i := s_i^*(p) - \delta\mathbb{1}_{\{i \geq j\}}$, where $\mathbb{1}$ denotes the indicator function and $\delta > 0$ is chosen in such a way that

$$\delta < \min\left\{s_j^*(p) - s_{j-1}^*(p), \frac{\mathbb{E}_p[s^*(p)] - E_q[s^*(p)] - c}{\sum_{i=j}^{n}(p_i - q_i)}\right\}.$$

The contract $\tilde{s}$ is feasible, since $\tilde{s}_i - \tilde{s}_{i-1} = s_i^*(p) - s_{i-1}^*(p) - \delta\mathbb{1}_{\{i=j\}}$, and

$$\mathbb{E}_p[\tilde{s}] - \mathbb{E}_q[\tilde{s}] = \sum_{k=1}^{n}\left(\tilde{s}_k - \tilde{s}_{k-1}(p)\right)\sum_{i=k}^{n}(p_i - q_i) = \mathbb{E}_p[s^*(p)] - E_q[s^*(p)] - \delta\sum_{i=j}^{n}(p_i - q_i) > c.$$

Finally,

$$\mathbb{E}_p[x - \tilde{s}] = \mathbb{E}_p[x - s^*(p)] + \delta\sum_{i=j}^{n}p_i > \mathbb{E}_p[x - s^*(p)],$$

a contradiction.

The previous two statements imply that the set of feasible contracts can be reduced to the set

$$\mathscr{S} = \left\{ s \in \mathbb{R}^{n+1} \,\middle|\, s_0 = 0, \forall i \in \{1,\ldots,n\} : s_i - s_{i-1} \in [0, x_i - x_{i-1}], \sum_{k=1}^{n} (s_k - s_{k-1}) \sum_{i=k}^{n} (p_i - q_i) = c \right\}.$$

I next show that each $s \in \mathscr{S}$ can be written as a linear combination of the n tranches from Definition 4. For $j \in \{1,\ldots,n\}$, set

$$\lambda_j = \frac{s_j - s_{j-1}}{x_j - x_{j-1}}.$$

Due to the first monotonicity constraint (Assumption 6), we have $s_j \geq s_{j-1}$, which implies $\lambda_j \geq 0$. Due to the second monotonicity constraint (Assumption 7), we have $s_j - s_{j-1} \leq x_j - x_{j-1}$, which implies $\lambda_j \leq 1$. In particular, we have $s = \lambda_1 s^1 + \cdots + \lambda_n s^n$. As a result, we can write the set $\mathscr{S}$ as

$$\mathscr{S} = \left\{ \lambda_1 s^1 + \cdots + \lambda_n s^n \,\middle|\, \lambda \in [0, 1]^n, \sum_{k=1}^{n} \lambda_k (x_k - x_{k-1}) \sum_{i=k}^{n} (p_i - q_i) = c \right\}.$$

We observe that the equality constraint can be written as

$$\sum_{k=1}^{n} \lambda_k (x_k - x_{k-1}) L_k(p) \sum_{i=k}^{n} q_i = c.$$

The set $\mathscr{S}$ is compact and nonempty, as $0 \leq c \leq \pi$ and the values 0 and π are attained, respectively, with $\lambda_1 = \cdots = \lambda_n = 0$ and $\lambda_1 = \cdots = \lambda_n = 1$. In particular, an optimal contract exists.

Using the relation $\mathbb{E}_p[x - s] = \mathbb{E}_q[x] + \pi - c - \mathbb{E}_q[s]$, we can rewrite the contracting problem as

$$s^*(p) \in \arg\min_{s \in \mathscr{S}} \mathbb{E}_q[s].$$

We will make use of the equality

$$\mathbb{E}_q[s] = \sum_{k=1}^{n} (s_k - s_{k-1}) \sum_{i=k}^{n} q_i = \sum_{k=1}^{n} \lambda_k (x_k - x_{k-1}) \sum_{i=k}^{n} q_i.$$

Hence, we get the following unified form of the problem:

$$\min_{\lambda \in [0,1]^n} \sum_{k=1}^{n} \lambda_k (x_k - x_{k-1}) \sum_{i=k}^{n} q_i$$

subject to

$$\sum_{k=1}^{n} \lambda_k (x_k - x_{k-1}) L_k(p) \sum_{i=k}^{n} q_i = c.$$

Consider a permutation $\{i_1, \ldots, i_n\}$ of the set $\{1, \ldots, n\}$ such that $L_{i_1}(p) \geq L_{i_2}(p) \geq \cdots \geq L_{i_n}(p)$. For $j \in \{1, \ldots, n\}$, let

$$\bar{c}_j = \sum_{k=1}^{j} (x_{i_k} - x_{i_k-1}) L_{i_k}(p) \sum_{i=i_k}^{n} q_i.$$

Then $0 =: \bar{c}_0 \leq \bar{c}_1 \leq \cdots \leq \bar{c}_n = \pi$. Let $m \in \{1, \ldots, n\}$ such that $\bar{c}_{m-1} < c \leq \bar{c}_m$. Equivalently,

$$\bar{c}_{m-1} < c \leq \bar{c}_{m-1} + (x_{i_m} - x_{i_m-1}) L_{i_m}(p) \sum_{i=i_m}^{n} q_i.$$

Set $\lambda_{i_k} = 1$ for $k \in \{1, \ldots, m-1\}$, $\lambda_{i_k} = 0$ for $k \in \{m+1, \ldots, n\}$, and

$$\lambda_{i_m} = \frac{c - \bar{c}_{m-1}}{(x_{i_m} - x_{i_m-1}) L_{i_m}(p) \sum_{i=i_m}^{n} q_i}.$$

Then, by construction

$$\begin{aligned}
\sum_{k=1}^{n} \lambda_k (x_k - x_{k-1}) L_k(p) \sum_{i=k}^{n} q_i &= \sum_{k=1}^{m} \lambda_{i_k} (x_{i_k} - x_{i_k-1}) L_{i_k}(p) \sum_{i=i_k}^{n} q_i \\
&= \sum_{k=1}^{m-1} (x_{i_k} - x_{i_k-1}) L_{i_k}(p) \sum_{i=i_k}^{n} q_i + \lambda_{i_m} (x_{i_m} - x_{i_m-1}) L_{i_m}(p) \sum_{i=i_m}^{n} q_i \\
&= \bar{c}_{m-1} + \lambda_{i_m} (x_{i_m} - x_{i_m-1}) L_{i_m}(p) \sum_{i=i_m}^{n} q_i = c,
\end{aligned}$$

and the contract $\hat{s} = \sum_{k=1}^{m} \lambda_{i_k} s^{i_k}$ is feasible. I will prove that it is optimal. Consider another contract $\tilde{s} = \sum_{k=1}^{n} \mu_k s^k \in \mathscr{S}$. In particular,

$$c = \sum_{k=1}^{n} \mu_k (x_k - x_{k-1}) L_k(p) \sum_{i=k}^{n} q_i.$$

Consequently,

$$\mu_{i_m}\left(x_{i_m}-x_{i_m-1}\right)\sum_{i=i_m}^{n}q_i = \frac{c-\sum_{k\neq i_m}\mu_k\left(x_k-x_{k-1}\right)L_k(p)\sum_{i=k}^{n}q_i}{L_{i_m}(p)},$$

and we can write the objective function in the form

$$
\begin{aligned}
\mathbb{E}_q\left[\tilde{s}\right] &= \sum_{k=1}^{n}\mu_k\left(x_k-x_{k-1}\right)\sum_{i=k}^{n}q_i\\
&= \frac{c-\sum_{k\neq i_m}\mu_k\left(x_k-x_{k-1}\right)L_k(p)\sum_{i=k}^{n}q_i}{L_{i_m}(p)} + \sum_{k\neq i_m}\mu_k\left(x_k-x_{k-1}\right)\sum_{i=k}^{n}q_i\\
&= \frac{c}{L_{i_m}(p)} + \sum_{k\neq i_m}\mu_k\left(x_k-x_{k-1}\right)\left(1-\frac{L_k(p)}{L_{i_m}(p)}\right)\sum_{i=k}^{n}q_i\\
&\geq \frac{c}{L_{i_m}(p)} + \sum_{k=1}^{m-1}\left(x_{i_k}-x_{i_k-1}\right)\left(1-\frac{L_{i_k}(p)}{L_{i_m}(p)}\right)\sum_{i=i_k}^{n}q_i,
\end{aligned}
$$

where in the last inequality, I use

$$\mu_k\left(x_k-x_{k-1}\right)\left(1-\frac{L_k(p)}{L_{i_m}(p)}\right)\sum_{i=k}^{n}q_i \geq 0$$

when $k=i_j$, $j>m$, since in this case $\frac{L_k(p)}{L_{i_m}(p)}\leq 1$, and

$$\mu_k\left(x_k-x_{k-1}\right)\left(1-\frac{L_k(p)}{L_{i_m}(p)}\right)\sum_{i=k}^{n}q_i \geq \left(x_k-x_{k-1}\right)\left(1-\frac{L_k(p)}{L_{i_m}(p)}\right)\sum_{i=k}^{n}q_i$$

when $k=i_j$, $j<m$, since in this case $\frac{L_k(p)}{L_{i_m}(p)}\geq 1$. It remains to observe that

$$\frac{c}{L_{i_m}(p)} + \sum_{k=1}^{m-1}\left(x_{i_k}-x_{i_k-1}\right)\left(1-\frac{L_{i_k}(p)}{L_{i_m}(p)}\right)\sum_{i=i_k}^{n}q_i = \sum_{k=1}^{m-1}\left(x_{i_k}-x_{i_k-1}\right)\sum_{i=i_k}^{n}q_i+\lambda_{i_m}\left(x_{i_m}-x_{i_m-1}\right)\sum_{i=i_m}^{n}q_i = \mathbb{E}_q\left[\hat{s}\right].$$

Hence, for all $s\in\mathscr{S}$, we have $\mathbb{E}_q\left[s\right]\geq\mathbb{E}_q\left[\hat{s}\right]$.

In particular, if $\bar{c}_{m-1}<c\leq\bar{c}_m$, then an optimal contract is given by

$$s^*(p) = \sum_{k=1}^{m-1}s^{i_k} + \frac{c-\bar{c}_{m-1}}{\left(x_{i_m}-x_{i_m-1}\right)L_{i_m}(p)\sum_{i=i_m}^{n}q_i}s^{i_m},$$

with the corresponding agency rent

$$\mathbb{E}_q[s^*(p)] = \frac{c}{L_{i_m}(p)} + \sum_{k=1}^{m-1} (x_{i_k} - x_{i_k-1}) \left(1 - \frac{L_{i_k}(p)}{L_{i_m}(p)}\right) \sum_{i=i_k}^{n} q_i.$$

The optimal contract $s^*(p)$ was constructed using the ordering $L_{i_1}(p) \geq L_{i_2}(p) \geq \ldots \geq L_{i_n}(p)$. From the single-peaked property of $L(p)$ (Lemma 2), it follows that one can choose an ordering in such a way that each set $\{i_1, \ldots, i_k\}$ is in fact a segment of the form $\{f, f+1, \ldots, g\}$ with $1 \leq f \leq g \leq n$. In the following, I show that this implies a capped bonus contract, and I derive the values of the thresholds. There are two possibilities for the optimal contract $s^*(p)$.

Case 1: $\{i_1, \ldots, i_{m-1}\} = \{f, \ldots, g\}$, $i_m = f - 1$; that is, the m-th largest value of $L(p)$ is added to the left of already ordered values. In this case, we have $s^*(p) = \lambda_{f-1}s^{f-1} + \sum_{k=f}^{g} s^k$, $\phi_1(p) = \lambda_{f-1}x_{f-2} + (1 - \lambda_{f-1})x_{f-1}$, and $\phi_2(p) = x_g - \phi_1(p)$.

Case 2: $\{i_1, \ldots, i_{m-1}\} = \{f, \ldots, g\}$, $i_m = g + 1$; that is, the m-th largest value of $L(p)$ is added to the right of already ordered values. In this case, we have $s^*(p) = \sum_{k=f}^{g} s^k + \lambda_{g+1}s^{g+1}$, $\phi_1(p) = x_{f-1}$, and $\phi_2(p) = \lambda_{g+1}x_{g+1} + (1 - \lambda_{f+1})x_g - \phi_1(p)$, which completes the proof. ∎

A.5 Proof of Corollary 1

The results follow directly from the construction of optimal contracts in the proof of Proposition 1. In the first case, we have $L_1(p) \geq \cdots \geq L_n(p)$ and we are in Case 2, described at the end of the proof of Proposition 1, with $f = 1$. Thus, $\phi_1(p) = x_0 = 0$ and $s_i^*(p) = \min\{x_i, \phi_2(p)\}$, $i \in \Omega$. In the second case, we have the opposite ordering $L_n(p) \geq \cdots \geq L_1(p)$, which arises in the case of a monotone likelihood ratio $l(p)$, and we are in Case 1, described at the end of the proof of Proposition 1, with $g = n$. Thus, $\phi_2(p) = x_n - \phi_1(p)$ and $s_i^*(p) = \max\{0, x_i - \phi_1(p)\}$, $i \in \Omega$. ∎

A.6 Proof of Proposition 2

The explicit formula for the agency rent was derived as part of the proof of Proposition 1. From this representation, it is clear that the agency rent function is continuous on $(\bar{c}_{m-1}, \bar{c}_m]$, as the function

is linear on $(\bar{c}_{m-1}, \bar{c}_m]$. In particular, it is differentiable on $(\bar{c}_{m-1}, \bar{c}_m)$ with slope $\frac{1}{L_{i_m}(p)}$. To show continuity on $[0, \pi]$, it remains to verify that

$$\lim_{c \searrow \bar{c}_{m-1}} \mathbb{E}_q \left[s^*(p) \right] = \mathbb{E}_q \left[s^*(p) \right] \big|_{c = \bar{c}_{m-1}}.$$

The limit on the left is given by

$$\frac{\bar{c}_{m-1}}{L_{i_m}(p)} + \sum_{k=1}^{m-1} \left(x_{i_k} - x_{i_k - 1} \right) \left(1 - \frac{L_{i_k}(p)}{L_{i_m}(p)} \right) \sum_{i=i_k}^{n} q_i.$$

Using the explicit formula for $\bar{c}_{m-1}$ from Proposition 1, we therefore have

$$\lim_{c \searrow \bar{c}_{m-1}} \mathbb{E}_q \left[s^*(p) \right] = \frac{1}{L_{i_m}(p)} \sum_{k=1}^{m-1} (x_{i_k} - x_{i_k - 1}) L_{i_k}(p) \sum_{i=i_k}^{n} q_i + \sum_{k=1}^{m-1} \left(x_{i_k} - x_{i_k - 1} \right) \left(1 - \frac{L_{i_k}(p)}{L_{i_m}(p)} \right) \sum_{i=i_k}^{n} q_i$$

$$= \sum_{k=1}^{m-1} \left(x_{i_k} - x_{i_k - 1} \right) \sum_{i=i_k}^{n} q_i = \mathbb{E}_q \left[s^*(p) \right] \big|_{c = \bar{c}_{m-1}}.$$

As a result, the agency rent function is continuous and piecewise linear. Since the slopes are nondecreasing, the agency rent function is (weakly) convex. ∎

A.7   Proof of Proposition 3

The proof proceeds by guessing a technology in $\mathscr{P}^1$ and verifying that it is the unique minimum agency rent technology in $\mathscr{P}^1$. Define the technology $p^1 \in \mathscr{P}^1$ as follows: $p_0^1 = q_0 - \frac{\pi}{x_1}$, $p_1^1 = q_1 + \frac{\pi}{x_1}$, and for $i \in \{2, \ldots, n\}$, $p_i^1 = q_i$. By Proposition 2, the agency rent function for entrepreneur p^1 is linear:

$$\mathbb{E}_q \left[s^* \left(p^1 \right) \right] = \frac{(1 - q_0) x_1}{\pi} c.$$

In particular, the agency rent function is linear with slope $\frac{(1-q_0)x_1}{\pi}$. For all entrepreneurs $p \in \mathscr{P}^1$, the agency rent function is (weakly) convex with slope $\frac{1}{L_1(p)}$ at $c = 0$ (i.e., the right derivative). Hence, in order to show that for all $p \in \mathscr{P}^1 \setminus \{p^1\}$, we have $\mathbb{E}_q[s^*(p)] > E_q \left[s^* \left(p^1 \right) \right]$ for all

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$c \in (0, \pi]$, it is sufficient to show that for all $p \in \mathscr{P}^1 \setminus \{p^1\}$, we have

$$\frac{1}{L_1(p)} > \frac{(1-q_0)x_1}{\pi}.$$

Since for all $p \in \mathscr{P}^1$, we have $L_1(p) \geq \cdots \geq L_n(p) \geq 0$, and since $L_1\left(p^1\right) > L_2\left(p^1\right) = \cdots = L_n\left(p^1\right) = 0$, for each $p \in \mathscr{P}^1 \setminus \{p^1\}$, there exists a $j \in \{2, \ldots, n\}$ such that $L_j(p) > 0$. In particular,

$$\pi = \sum_{k=1}^n (x_k - x_{k-1}) L_k(p) \sum_{i=k}^n q_i > x_1 L_1(p) \sum_{i=1}^n q_i = x_1 L_1(p)(1-q_0),$$

which completes the proof. ∎

A.8 Proof of Proposition 4

I study the agency rent function specified in Proposition 2 and use the following definitions: $D_j(p) := (x_j - x_{j-1}) \sum_{i=j}^n (p_i - q_i)$ and $Q_j := (x_j - x_{j-1}) \sum_{i=j}^n q_i$. Consider again the permutation $\{i_1, \ldots, i_n\}$ of the set $\{1, \ldots, n\}$ from Proposition 1. The proof of Proposition 1 implies that the agency rent function has the following coordinates:

$$(0,0), (D_{i_1}(p), Q_{i_1}), (D_{i_1}(p) + D_{i_2}(p), Q_{i_1} + Q_{i_2}), \ldots, (D_{i_1}(p) + \cdots + D_{i_n}(p), Q_{i_1} + \cdots + Q_{i_n}).$$

Since for all $p \in \mathscr{P}^n$, the cumulative likelihood ratio $L(p)$ is nondecreasing, we get the following coordinates of the agency rent function:

$$(0,0), (D_n(p), Q_n), (D_n(p) + D_{n-1}(p), Q_n + Q_{n-1}), \ldots, (D_n(p) + \cdots + D_1(p), Q_n + \cdots + Q_1).$$

In particular, each technology $p \in \mathscr{P}^n$ takes on the same values at the cutoffs $\bar{c}_j = \sum_{k=1}^j D_{i_k}(p)$, $j \in \{1, \ldots, n\}$.

Note that $\mathscr{P}^n$ is equal to the set of technologies with a monotone likelihood ratio (see Footnote 15). The proof of Lemma 3 shows that this set is compact. Since the mapping $\mathscr{P}^n \ni p \mapsto \mathbb{E}_q[s^*(p)] \in \mathbb{R}_+$ is continuous, the extreme value theorem applies and an optimal entrepreneur $p^* \in \mathscr{P}^n$ exists. The proof proceeds by guessing a technology $p^n \in \mathscr{P}^n$ and verifying that it is the

unique minimum agency rent technology in $\mathscr{P}^n$. To show this, I consider an entrepreneur $p \in \mathscr{P}^n$, $p \neq p^n$, and a $c \in (0, \pi)$. I then show that there exists a technology in $\mathscr{P}^n$ with a lower agency rent, which implies that $p^* = p^n$.

The optimal debt technology $p^n \in \mathscr{P}^n$ is the technology with a constant likelihood ratio for all $i < n$; that is, it satisfies for all $i, j \in \{0, \ldots, n-1\}$, $l_i(p^n) = l_j(p^n) < 0$, and $l_n(p^n) > 0$. Consider an entrepreneur $p \in \mathscr{P}^n$, where $p \neq p^n$. Since $p \neq p^n$, there exists a smallest $k \in \{0, \ldots, n-2\}$ such that $l_k(p) < l_{k+1}(p)$. Consider two variations $v^k \in \mathbb{R}$ and $\delta^k \in \mathbb{R}$ that leave the ordering of the cumulative likelihood ratio unchanged.

1. Variation v^k: Reduce p_{k+1} by a small $v^k > 0$ and increase p_k by v^k. This variation reduces the value of the technology by $(x_{k+1} - x_k) v^k$.

2. Variation δ^k: Reduce p_{k+1} by a small $\delta^k > 0$ and increase p_n by δ^k. I choose δ^k such that both variations together leave the value of the technology unchanged, that is,

$$(x_{k+1} - x_k) v^k = (x_n - x_{k+1}) \delta^k \Leftrightarrow v^k = \frac{x_n - x_{k+1}}{x_{k+1} - x_k} \delta^k. \tag{10}$$

Denote the technology with the two variations applied by p^k. We can now calculate and compare the thresholds $\bar{c}_j$ for technology p and $\bar{c}_j^k$ for technology p^k, $j \in \Omega$. Consider two cases. First, let $k = 0$.

1. For $j = 1$, we have

$$\bar{c}_1^k = \left(p_n^k - q_n\right)(x_n - x_{n-1}) = \left((p_n - q_n) + \delta^k\right)(x_n - x_{n-1}) > (p_n - q_n)(x_n - x_{n-1}) = \bar{c}_1.$$

2. We can proceed by iteration. Let $j \in \{2, \ldots, n-1\}$. Assuming $\bar{c}_{j-1}^k > \bar{c}_{j-1}$, we get

$$\bar{c}_j^k = \bar{c}_{j-1}^k + (x_{n-j+1} - x_{n-j}) \left(\sum_{i=n-j+1}^{n} (p_i - q_i) + \delta^k\right)$$
$$> \bar{c}_{j-1} + (x_{n-j+1} - x_{n-j}) \sum_{i=n-j+1}^{n} (p_i - q_i) = \bar{c}_j.$$

Hence, we have $\bar{c}_j^k > \bar{c}_j$ for all $j \in \{1, \ldots, n-1\}$, and therefore the variation has lower agency rents for all disutilities of effort $c \in (0, \pi)$, since the agency rent functions take on the same values at the cutoffs.

Next, consider the case where $k > 0$. Following the same argument as above, I can show that for all $j < n - k$, $\bar{c}_j^k > \bar{c}_j$, and for all $j \geq n - k$, $\bar{c}_j^k = \bar{c}_j$. We can then proceed by iteration. We can apply further variations v^{k-1} and δ^{k-1} to p^k in addition to those applied before (note that $l_{k-1}(p^k) < l_k(p^k)$, due to the increase in p_k).

1. Variation v^{k-1}: Reduce p_k^k by a small $v^{k-1} > 0$ and increase p_{k-1}^k by v^{k-1}. This variation reduces the expected value of the technology by $v^{k-1}(x_k - x_{k-1})$.

2. Variation δ^{k-1}: Reduce p_k^k by a small $\delta^{k-1} > 0$ and increase p_n^k by δ^{k-1}. Choose δ^{k-1} such that

$$(x_k - x_{k-1})v^{k-1} = (x_n - x_k)\delta^{k-1} \Leftrightarrow v^{k-1} = \frac{x_n - x_k}{x_k - x_{k-1}}\delta^{k-1}.$$

This variation weakly decreases the agency rent function (strictly for some c). If $k = 1$, we are finished. If $k > 1$, we can then apply the same step again, applying variations v^{k-2} and δ^{k-2}, and continue to get to the last variation of p_0 and p_1. This reduces the agency rent function, as shown above. Hence, we must have $p^n = \arg\min_{p \in \mathscr{P}^n} \mathbb{E}_q[s^*(p)]$.

We can explicitly construct the technology p^n. Note that it satisfies for $i \in \{0, \ldots, n-1\}$, $p_i^n - q_i = \frac{q_i}{q_0}(p_0^n - q_0)$ (i.e., for all $i, j \in \{0, \ldots, n-1\}$, $l_i(p^n) = l_j(p^n)$), and we then have $p_n^n = 1 - \sum_{i=0}^{n-1} p_i^n$, such that only p_0^n needs to be determined by $\mathbb{E}_{p^n}[x] - \mathbb{E}_q[x] = \pi$. We have

$$\mathbb{E}_{p^n}[x] - \mathbb{E}_q[x] = \pi \Leftrightarrow \sum_{i=0}^{n}(p_i^n - q_i)x_i = \pi \Leftrightarrow \sum_{i=0}^{n-1}\frac{q_i}{q_0}(p_0^n - q_0)x_i - x_n\sum_{i=0}^{n-1}\frac{q_i}{q_0}(p_0^n - q_0) = \pi$$

$$\Leftrightarrow \frac{p_0^n - q_0}{q_0}(\mathbb{E}_q[x] - x_n) = \pi \Leftrightarrow p_0^n - q_0 = -\frac{\pi q_0}{x_n - \mathbb{E}_q[x]} < 0.$$

As a result, we have $p_0^n - q_0 = -\frac{\pi q_0}{x_n - \mathbb{E}_q[x]}$, for all $i \in \{1, \ldots, n-1\}$, $p_i^n - q_i = -\frac{\pi q_i}{x_n - \mathbb{E}_q[x]}$, and

$$p_n^n - q_n = -\sum_{i=0}^{n-1}(p_i^n - q_i) = \sum_{i=0}^{n-1}\frac{\pi q_i}{x_n - \mathbb{E}_q[x]} = \frac{\pi \sum_{i=0}^{n-1} q_i}{x_n - \mathbb{E}_q[x]} = \frac{\pi(1 - q_n)}{x_n - \mathbb{E}_q[x]},$$

which completes the proof. $\blacksquare$

A.9 Proof of Proposition 5

From the proof of Proposition 3, we know that the agency rent function of entrepreneur p^1 is linear. Given the (weak) convexity of the agency rent functions, the investor can only prefer p^n to p^1 for some region of disutilities of effort c if the initial slope of the agency rent function of the entrepreneur p^n is lower than the slope of the agency rent function of entrepreneur p^1. The initial slope of the agency rent function for p^n is given by $\frac{1}{L^*(p^n)} = \frac{1}{L_n(p^n)} = \frac{1}{l^*(p^n)}$. The slope of the agency rent function for p^1 is given by $\frac{1}{L^*(p^1)} = \frac{1}{l^*(p^1)} \frac{q_1 + \cdots + q_n}{q_1}$ (see Proposition 2 and use p_1^1). Hence, the slope at $c = 0$ (i.e., the right derivative) of the agency rent function of the optimal equity entrepreneur p^1 is higher than that of the optimal debt entrepreneur p^n if

$$\frac{1}{l^*(p^1)} \frac{q_1 + \cdots + q_n}{q_1} > \frac{1}{l^*(p^n)} \Leftrightarrow l^*(p^n) > \frac{q_1}{q_1 + \cdots + q_n} l^*(p^1). \tag{11}$$

Further, we know that

$$\mathbb{E}_q\left[s^*\left(p^1\right)\right]\big|_{c=\pi} = \sum_{i=1}^n q_i x_1 < \mathbb{E}_q[x] = \mathbb{E}_q\left[s^*\left(p^n\right)\right]\big|_{c=\pi}.$$

Due to the linearity of $\mathbb{E}_q\left[s^*\left(p^1\right)\right]$ in c and the (weak) convexity of $\mathbb{E}_q\left[s^*\left(p^n\right)\right]$ in c, there exists a unique crossing point if (11) holds and $\mathbb{E}_q\left[s^*\left(p^n\right)\right] > \mathbb{E}_q\left[s^*\left(p^1\right)\right]$ for all $c \in (0, \pi]$ if (11) does not hold with a strict inequality. ∎

A.10 Proof of Lemma 3

For $m \in \Omega$, consider the set $\mathscr{P}_m \subset [0,1]^{n+1}$ such that all $p \in \mathscr{P}_m$ satisfy $\forall j \in \Omega : \sum_{i=0}^j (p_i - q_i) \leq 0$, $\sum_{i=0}^n (p_i - q_i) = 0$, $\mathbb{E}_p[x] - \mathbb{E}_q[x] = \pi$, and $l_0(p) \leq \cdots \leq l_m(p) \geq l_{m+1}(p) \geq \cdots \geq l_n(p)$. In particular, the set $\mathscr{P}_m$ is defined by a set of weak inequality constraints. Thus $\mathscr{P}_m$ is closed and $\mathscr{P} = \bigcup_{m \in \Omega} \mathscr{P}_m$ is closed as a finite union of closed sets. The set $\mathscr{P}$ is clearly bounded. Hence, $\mathscr{P}$ is compact. ∎

A.11 Proof of Lemma 4

Lemma A.1 in Appendix A.2 shows that $p_0 < q_0$ if $p \neq q$, and $p_n \geq q_n$. As $p_0 < q_0$, there must exist an index $m \in \{1, \ldots, n\}$ such that $p_m > q_m$, and we take m to be the lowest index with this property. Then, $p_i \leq q_i$ for all $i \in \{1, \ldots, m-1\}$. In particular, $l_i(p) \leq 0$ for all $i \in \{1, \ldots, m-1\}$, $l_m(p) > 0$, and $l_n(p) \geq 0$. From the single-peaked property, we have

$$\min_{m \leq i \leq n} l_i(p) = \min\{l_m(p), l_n(p)\} \geq 0,$$

and hence $p_i \geq q_i$ for all $i \in \{m, \ldots, n\}$. Finally, if $p_h = q_h$ for some $h \in \{m+1, \ldots, n\}$, then the peak of $l(p)$ is between m and h and $l(p)$ is nonincreasing on $\{h, h+1, \ldots, n\}$. Since $l_h(p) = 0$ and $l_n(p) \geq 0$, it follows that $l_i(p) = 0$ for all $i \in \{h, h+1, \ldots, n\}$, and hence $p_i = q_i$ for all $i \in \{h, h+1, \ldots, n\}$. ∎

A.12 Proof of Lemma 5

This result follows directly from the construction of contracts in Proposition 1 and the assumption about p in the lemma. If $j \in \{m, \ldots, n\}$ such that for all $i \in \{m, \ldots, j\}$, $p_i > q_i$, and for all $i \in \{j+1, \ldots, n\}$, $p_i = q_i$, then we have for $i \in \{1, \ldots, j\}$, $L_i(p) > 0$, and for all $i > j$, $L_i(p) = 0$. The investor therefore never includes contract tranches corresponding to states exceeding state j, since these states generate no incentives for the entrepreneur. The investor might use tranches in all states $i \leq j$. If $c = \pi$, an optimal contract exhausts the payoff space with positive cumulative likelihood ratios, that is, for all $i \in \Omega$, $s_i^*(p) = \min\{x_i, x_j\}$. ∎

A.13 Proof of Proposition 6

Lemma 5 directly implies that for entrepreneur p', we have for all $i \in \Omega$ and for all $c \in [0, \pi]$, $s_i^*(p') \leq \min\{x_i, x_{m_1+j}\}$, and $s_i^*(p') = \min\{x_i, x_{m_1+j}\} \Leftrightarrow c = \pi$. For entrepreneur p'', we have for all $i \in \Omega$ and for all $c \in [0, \pi]$, $s_i^*(p'') \leq \min\{x_i, x_{m_2+j}\}$, and $s_i^*(p'') = \min\{x_i, x_{m_2+j}\} \Leftrightarrow c = \pi$.

The agency rent functions therefore satisfy

$$\mathbb{E}_q\left[s^*\left(p'\right)\right]\big|_{c=\pi} = \mathbb{E}_q\left[\min\left\{x, x_{m_1+j}\right\}\right] < \mathbb{E}_q\left[\min\left\{x, x_{m_2+j}\right\}\right] = \mathbb{E}_q\left[s^*\left(p''\right)\right]\big|_{c=\pi},$$

since $m_1 + j < m_2 + j \Leftrightarrow m_1 < m_2$. Since the agency rent functions are continuous, there exists a $\tilde{c} \in [0, \pi)$ such that for all $c > \tilde{c}$, $\mathbb{E}_q\left[s^*\left(p'\right)\right] < \mathbb{E}_q\left[s^*\left(p''\right)\right]$. ∎

A.14 Proof of Proposition 7

Consider an entrepreneur $p' \in \mathscr{P}_{\pi_1}$ and $p' \notin \mathscr{P}_{\pi_1}^*$. Hence, there exists an entrepreneur $p^* \in \mathscr{P}_{\pi_1}^*$, such that $\mathbb{E}_q\left[s^*\left(p^*\right)\right] < \mathbb{E}_q\left[s^*\left(p'\right)\right]$.

We can then scale the technology p^* by $\lambda \in \left(\frac{c}{\pi_1}, 1\right]$, that is, define a new technology $\tilde{p}(\lambda)$ by $\tilde{p}(\lambda) := q + \lambda\left(p^* - q\right)$, such that

$$\mathbb{E}_{\tilde{p}(\lambda)}[x] - \mathbb{E}_q[x] = \lambda\left(\mathbb{E}_{p^*}[x] - \mathbb{E}_q[x]\right) = \lambda\pi_1 \leq \pi_1.$$

Consider the investor's expected payoff with entrepreneur $\tilde{p}(\lambda)$ net of I, given by

$$P(\lambda) := \mathbb{E}_q[x] + \lambda\pi_1 - c - \mathbb{E}_q\left[s^*\left(\tilde{p}(\lambda)\right)\right] - I.$$

We know that

$$P(1) = \mathbb{E}_q[x] + \pi_1 - c - \mathbb{E}_q\left[s^*\left(p^*\right)\right] - I > \mathbb{E}_q[x] + \pi_1 - c - \mathbb{E}_q\left[s^*\left(p'\right)\right] - I \geq 0.$$

Since $\left(\frac{c}{\pi_1}, 1\right] \ni \lambda \mapsto P(\lambda)$ is continuous, there exists a $\hat{\lambda} < 1$ such that

$$P\left(\hat{\lambda}\right) = \mathbb{E}_q[x] + \hat{\lambda}\pi_1 - c - \mathbb{E}_q\left[s^*\left(\tilde{p}\left(\hat{\lambda}\right)\right)\right] - I > \mathbb{E}_q[x] + \pi_1 - c - \mathbb{E}_q\left[s^*\left(p'\right)\right] - I.$$

Set $p'' := \tilde{p}\left(\hat{\lambda}\right)$, which has an expected value of effort of $\pi_2 := \hat{\lambda}\pi_1 < \pi_1$, and the investor prefers entrepreneur p'' to entrepreneur p'. ∎

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B Productivity of Effort and Equivalent Models

I first construct technologies to be a direct function of the expected value of effort. Consider the set of entrepreneurs $\mathscr{P}$ satisfying Assumptions 1, 3, and 4, with $\pi > 0$. The expected value of effort π imposes a constraint on the mean of an entrepreneur's technology $p \in \mathscr{P}$; that is, I require $\mathbb{E}_p[x] - \mathbb{E}_q[x] = \pi$. In particular, a technology $p \in \mathscr{P}$ is not an explicit function of the parameter π. I therefore construct entrepreneurs' technologies to be a direct function of π. For each technology p, define the basic technology $\hat{p} := q + \frac{p-q}{\pi}$, which satisfies $\mathbb{E}_{\hat{p}}[x] - \mathbb{E}_q[x] = 1$ by construction.[20] In other words, a basic technology preserves the shape and is scaled to a unit expected value of effort. Define the set of basic technologies as follows:

$$\hat{\mathscr{P}} := \left\{ q + \frac{p-q}{\pi} \,\middle|\, p \in \mathscr{P} \right\}.$$

I can write the original set $\mathscr{P}$ by rescaling the basic technologies as follows:

$$\mathscr{P} = \left\{ q + \pi\,(\hat{p} - q) \,\middle|\, \hat{p} \in \hat{\mathscr{P}} \right\}.$$

Every technology $p \in \mathscr{P}$ can therefore be written as $p = q + \pi\,(\hat{p} - q)$, where $\hat{p} \in \hat{\mathscr{P}}$ is a basic technology. The basic technology determines the shape of the technology, and the parameter π determines the expected value of effort.

Using this parameterization of technologies, I next study how changes in the disutility of effort and the expected value of effort jointly affect agency rents.

Proposition 8. *Consider an entrepreneur $p \in \mathscr{P}$. Consider two sets of parameters (c_1, π_1) and (c_2, π_2), where $c_1, c_2 > 0$. Then*

$$\frac{\pi_1}{c_1} = \frac{\pi_2}{c_2} \Leftrightarrow \mathbb{E}_q\left[s^*(p)\right]\big|_{(c,\pi)=(c_1,\pi_1)} = \mathbb{E}_q\left[s^*(p)\right]\big|_{(c,\pi)=(c_2,\pi_2)}.$$

Proof. I study the agency rent functions specified in Proposition 2 and use the following definitions: $D_j(p) := (x_j - x_{j-1})\sum_{i=j}^n (p_i - q_i)$ and $Q_j := (x_j - x_{j-1})\sum_{i=j}^n q_i$. Consider the permutation $\{i_1, \ldots, i_n\}$ of the set $\{1, \ldots, n\}$ from Proposition 2. The proof of Proposition 1 implies that the

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[20]A basic technology might not be a probability distribution, since for some $i \in \Omega$, we might have $\hat{p}_i < 0$ or $\hat{p}_i > 1$.

agency rent function has the following coordinates:

$$(0,0), (D_{i_1}(p), Q_{i_1}), (D_{i_1}(p) + D_{i_2}(p), Q_{i_1} + Q_{i_2}), \ldots, (D_{i_1}(p) + \cdots + D_{i_n}(p), Q_{i_1} + \cdots + Q_{i_n}).$$

We know that we can write $p = q + \pi(\hat{p} - q)$, where $\hat{p} \in \hat{\mathscr{P}}$ is a basic technology. We have $c \in [\bar{c}_{j-1}, \bar{c}_j]$ is equivalent to

$$\sum_{l=1}^{j-1} D_{i_l}(p) \leq c \leq \sum_{l=1}^{j} D_{i_l}(p) \Leftrightarrow \sum_{l=1}^{j-1} D_{i_l}(q + \pi(\hat{p} - q))) \leq c \leq \sum_{l=1}^{j} D_{i_l}(q + \pi(\hat{p} - q))$$

$$\Leftrightarrow \pi \sum_{l=1}^{j-1} D_{i_l}(\hat{p}) \leq c \leq \pi \sum_{l=1}^{j} D_{i_l}(\hat{p}) \Leftrightarrow \sum_{l=1}^{j-1} D_{i_l}(\hat{p}) \leq \frac{c}{\pi} \leq \sum_{l=1}^{j} D_{i_l}(\hat{p}).$$

The agency rent for $c \in [\bar{c}_{j-1}, \bar{c}_j]$, denoted by $A(p, \pi, c)$, is given by

$$\begin{aligned} A(p, \pi, c) &= Q_{i_1} + \cdots + Q_{i_{j-1}} + \frac{Q_{i_j}}{D_{i_j}(p)} \left(c - \sum_{l=1}^{j-1} D_{i_l}(p) \right) \\ &= Q_{i_1} + \cdots + Q_{i_{j-1}} + \frac{Q_{i_j}}{D_{i_j}(q + \pi(\hat{p} - q))} \left(c - \sum_{l=1}^{j-1} D_{i_l}(q + \pi(\hat{p} - q)) \right) \\ &= Q_{i_1} + \cdots + Q_{i_{j-1}} + \frac{Q_{i_j}}{\pi D_{i_j}(\hat{p})} \left(c - \pi \sum_{l=1}^{j-1} D_{i_l}(\hat{p}) \right) \\ &= Q_{i_1} + \cdots + Q_{i_{j-1}} + \frac{Q_{i_j}}{D_{i_j}(\hat{p})} \left(\frac{c}{\pi} - \sum_{l=1}^{j-1} D_{i_l}(\hat{p}) \right) = A\left(\hat{p}, 1, \frac{c}{\pi}\right). \end{aligned}$$

Hence, the agency rent function can be written as a function of $\frac{c}{\pi}$. The agency rent function of the basic technology is increasing, which implies the equivalence. ∎

Proposition 8 shows that the agency rent function depends only on the productivity of effort. In particular, I can interpret the comparative statics with respect to the expected value of effort or the disutility of effort as changes in entrepreneurs' productivity of effort.

Corollary 3. *Consider two entrepreneurs $p', p'' \in \mathscr{P}$. Consider two sets of parameters (c_1, π_1) and (c_2, π_2), where $c_1, c_2 > 0$ and $\frac{\pi_1}{c_1} = \frac{\pi_2}{c_2}$. Then the following two inequalities are equivalent:*

(i) $\mathbb{E}_q[s^*(p')]\big|_{(c,\pi)=(c_1,\pi_1)} > \mathbb{E}_q[s^*(p'')]\big|_{(c,\pi)=(c_1,\pi_1)}$,

(ii) $\mathbb{E}_q\left[s^*\left(p'\right)\right]\big|_{(c,\pi)=(c_2,\pi_2)} > \mathbb{E}_q\left[s^*\left(p''\right)\right]\big|_{(c,\pi)=(c_2,\pi_2)}.$

Corollary 3 shows that all models with different disutilities and expected values of effort but the same productivity of effort generate the same ranking of entrepreneurs in terms of agency rents.

C   Risk-Averse Entrepreneurs

In this section, I discuss the case of risk-averse entrepreneurs. I assume that entrepreneurs' utility from a contractual payoff s_i is measured by a utility function $u : \mathbb{R} \mapsto \mathbb{R}$, where u is increasing, (strictly) concave, and differentiable, with $u(0) = 0$, $u'(0) \in (0,\infty)$, and $\lim_{y \to \infty} u'(y) = 0$. In particular, if the investor offers a contract s to an entrepreneur $p \in \mathscr{P}$, and the entrepreneur exerts effort, the entrepreneur's expected utility is given by $\mathbb{E}_p[u(s)] - c$.

Consider an arbitrary entrepreneur $p \in \mathscr{P}$. An optimal incentive compatible contract that satisfies the entrepreneur's limited liability, denoted by $s^*(p)$, satisfies

$$s^*(p) \in \arg\max_s \mathbb{E}_p\left[x - s\right] \text{ s.t. } \mathbb{E}_p[u(s)] - c \geq \mathbb{E}_q[u(s)], \; \mathbb{E}_p[u(s)] - c \geq 0, \; \forall i \in \Omega : s_i \geq 0.$$

As in the limited liability benchmark discussed in Section 3, the incentive constraint implies the participation constraint, and the incentive constraint binds. Since the investor has to give the entrepreneur at least zero in all states and a positive amount in some states to satisfy the incentive constraint, we have

$$\mathbb{E}_p\left[u\left(s^*(p)\right)\right] - c = \mathbb{E}_q\left[u\left(s^*(p)\right)\right] > 0.$$

In particular, the entrepreneur earns an agency rent equal to $\mathbb{E}_q\left[u\left(s^*(p)\right)\right] > 0$. I rewrite the incentive constraint as

$$\mathbb{E}_p[u(s)] - c \geq \mathbb{E}_q[u(s)] \Leftrightarrow \sum_{i=0}^n p_i u\left(s_i\right) - c \geq \sum_{i=0}^n q_i u\left(s_i\right) \Leftrightarrow \sum_{i=0}^n (p_i - q_i) u\left(s_i\right) \geq c.$$

The investor's optimization problem can then be written as

$$\max_s -\sum_{i=0}^n p_i s_i \text{ s.t. } \sum_{i=0}^n (p_i - q_i) u\left(s_i\right) \geq c, \; \forall i \in \Omega : s_i \geq 0.$$

The necessary and sufficient conditions for an optimal contract $s^*(p)$ are as follows (see Léonard and Van Long, 1992):

1. $\sum_{i=0}^{n}(p_i - q_i)u\left(s_i^*(p)\right) - c \geq 0$, $\mu \geq 0$, and $\mu\left(\sum_{i=0}^{n}(p_i - q_i)u\left(s_i^*(p)\right) - c\right) = 0$.

2. For all $i \in \Omega$: $-p_i + \mu(p_i - q_i)u'\left(s_i^*(p)\right) \leq 0$ and $s_i^*(p) \geq 0$.

3. For all $i \in \Omega$: $s_i^*(p)\left(-p_i + \mu(p_i - q_i)u'\left(s_i^*(p)\right)\right) = 0$.

I first show that $\mu > 0$ and $\sum_{i=0}^{n}(p_i - q_i)u\left(s_i^*(p)\right) = c$. Assume that this is not the case, then $\mu = 0$, and we have for all $i \in \Omega$, $s_i^*(p)p_i = 0$. In particular, for all $i \in \Omega$ with $p_i \geq q_i$, we have $p_i > 0$, which implies $s_i^*(p) = 0$, a contradiction, since the contract would not satisfy the incentive constraint otherwise.

I next show that $p_i \leq q_i$ implies $s_i^*(p) = 0$. Assume that this is not the case, then there exists an $i \in \Omega$ with $p_i \leq q_i$ and $s_i^*(p) > 0$. In particular, this implies $-p_i + \mu(p_i - q_i)u'\left(s_i^*(p)\right) = 0 \Leftrightarrow p_i = \mu(p_i - q_i)u'\left(s_i^*(p)\right)$. If $p_i < q_i$, then this implies $p_i < 0$, since $u' > 0$, a contradiction. If $p_i = q_i$, this implies that $p_i = q_i = 0$, a contradiction.

Hence, consider states $i \in \Omega$ with $p_i > q_i$. We get the following result:

Lemma C.1. *Let $i \in \Omega$ with $p_i > q_i$. If $u'(0) \leq \frac{p_i}{\mu(p_i - q_i)}$, then $s_i^*(p) = 0$. If $u'(0) > \frac{p_i}{\mu(p_i - q_i)}$, then $s_i^*(p) = (u')^{-1}\left(\frac{p_i}{\mu(p_i - q_i)}\right) > 0$.*

Proof. To show this, let $s_i^*(p) > 0$. We then have

$$-p_i + \mu(p_i - q_i)u'\left(s_i^*(p)\right) = 0 \Leftrightarrow u'\left(s_i^*(p)\right) = \frac{p_i}{\mu(p_i - q_i)}.$$

If $\frac{p_i}{\mu(p_i - q_i)} < u'(0)$, then we have

$$s_i^*(p) = (u')^{-1}\left(\frac{p_i}{\mu(p_i - q_i)}\right) > 0.$$

If $\frac{p_i}{\mu(p_i - q_i)} \geq u'(0)$, we must therefore have $s_i^*(p) = 0$.

It remains to show that if $\frac{p_i}{\mu(p_i - q_i)} < u'(0)$, we have $s_i^*(p) > 0$. Assume that this is not the case,

and there exists an $i \in \Omega$ with $\frac{p_i}{\mu(p_i - q_i)} < u'(0)$ and $s_i^*(p) = 0$. We must then have

$$-p_i + \mu(p_i - q_i)u'(0) \leq 0 \Leftrightarrow u'(0) \leq \frac{p_i}{\mu(p_i - q_i)},$$

a contradiction. $\blacksquare$

We then get the following characterization of the optimal contract:

Lemma C.2. *Let $s_i^*(p) > 0$ and $s_j^*(p) > 0$. We then have $s_i^*(p) > s_j^*(p) \Leftrightarrow \frac{p_i - q_i}{q_i} > \frac{p_j - q_j}{q_j}$.*

Proof. We have

$$s_i^*(p) > s_j^*(p) \Leftrightarrow (u')^{-1}\left(\frac{p_i}{\mu(p_i - q_i)}\right) > (u')^{-1}\left(\frac{p_j}{\mu(p_j - q_j)}\right) \Leftrightarrow \frac{p_i}{\mu(p_i - q_i)} < \frac{p_j}{\mu(p_j - q_j)}$$

$$\Leftrightarrow \frac{q_i + (p_i - q_i)}{p_i - q_i} < \frac{q_j + (p_j - q_j)}{p_j - q_j} \Leftrightarrow \frac{p_i - q_i}{q_i} > \frac{p_j - q_j}{q_j},$$

since we must have $p_i > q_i$ and $p_j > q_j$. $\blacksquare$

In particular, the investor rewards the entrepreneur only in states with a positive likelihood ratio. Further, the investor rewards the entrepreneur more in a state with a higher likelihood ratio. In the risk-neutral limited liability benchmark discussed in Section 3, the investor rewards the entrepreneur only in the state with the maximum likelihood ratio.

This result shows that, if contracts also have to satisfy the monotonicity constraints from Assumptions 6 and 7, the general insight from Section 5 also applies in the case of risk aversion. If an entrepreneur has a high impact in low states of the world, the investor rewards the entrepreneur most in these states in the limited liability benchmark. Introducing monotonicity constraints forces the investor to reward the entrepreneur at least the same amount in higher states as well. If an entrepreneur has a high impact on the project in high states of the world, the investor rewards the entrepreneur most in these states in the limited liability benchmark. Introducing monotonicity constraints might force the investor to reward the entrepreneur in lower states as well.