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Network Formation in the Presence of Contagious Risk *

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Abstract

There are a number of domains where agents must collectively form a network in the face of the following trade-off: each agent receives benefits from the direct links it forms to others, but these links expose it to the risk of being hit by a cascading failure that might spread over multi-step paths. Financial contagion, epidemic disease, and the exposure of covert organizations to discovery are all settings in which such issues have been articulated.

Here we formulate the problem in terms of strategic network formation, and provide asymptotically tight bounds on the welfare of both optimal and stable networks. We find that socially optimal networks are, in a precise sense, situated just beyond a phase transition in the behavior of the cascading failures, and that stable graphs lie slightly further beyond this phase transition, at a point where most of the available welfare has been lost. Our analysis enables us to explore such issues as the trade-offs between clustered and anonymous market structures, and it exposes a fundamental sense

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in which very small amounts of "over-linking" in networks with contagious risk can have strong consequences for the welfare of the participants.

1 Introduction

Social networks have particular features that distinguish them from biological and physical networks as a class, and which are important for the propagation of networked agents' behaviors. Two kinds of models have been used to shed light on the structure of social networks. Probabilistic models, such as small-world models [23, 26] and preferential attachment [8], posit a few simple rules describing the probabilistic formation of links. The application of such models to social phenomena presumes that the networks are exogenous from the point-of-view of the phenomena being studied. Strategic models, on the other hand, presume that network formation and agents' behaviors are closely connected. This paper contributes to the study of this second kind of network formation. Recent surveys of endogenous, or strategic, network formation include [25] and the relevant chapters of [19].

A common approach in the strategic network formation literature (e.g. the *connections model* [20]) assumes that links are costly for an agent to form or maintain, and that benefits come from the indirect access to others that the network provides, as measured by distances [13, 15, 20], component sizes [7], or point-to-point connectivity [6]. There are many instances, however, in which this cost-benefit trade-off is inverted. Benefits come from direct links, while the cost is that of exposure to a failure that propagates through the network. In financial markets, benefits come from transacting with others, but counterparty risk, the risk to an agent that its partners cannot complete their side of a transaction, is increased to the extent that the partners are exposed to the failures of their other partners. The possibility exists that a single agent's failure can cause his partners to fail, and so on, leading to a cascade of financial collapse [1, 2, 12, 18]. Even when cascades do not happen, the fear of cascades can lead to market behavior that is costly for all agents, such as happened with the capital markets shutdown in the financial crisis of Fall 2008. Epidemiology provides still other examples of this inverted cost-benefit structure, wherein the pattern of social contacts has significant implications for the spread of disease. This is demonstrated in a model of HIV transmission in a structured population in [21], while [14] demonstrates the importance of network structure for the construction of containment strategies for a smallpox bioterror event; an analogous cost-benefit structure is also present in needle-sharing practices among intravenous drug users [9]. Additionally, [17] observes that clandestine organizations are subject to the risk of being exposed and compromised, and that this risk may be mitigated or magnified by the network structure of agent contacts.

In our model, individuals first construct a social network. In this network, each node fails spontaneously with a small probability q. After this initial phase of spontaneous node failures, each edge transmits the failure with a small probability p. We can think of the set of edges that transmit failure as a random subgraph of the social network, and now the nodes which fail are all those in a component of the random subgraph containing a node which has spontaneously failed. Classical results on Bernoulli random graphs can be viewed as statements about random subgraphs of a complete network, and one of the technical contributions of this paper lies in generalizing this point of view from complete graphs to arbitrary graphs of given minimal degree.

We examine networks that are optimal with respect to a Rawlsian social welfare criterion as well as networks that are stable in a sense different from (but closely related to) the stability concepts in [20] and [16]. In addition to the probabilities p and q, we use two other parameters: a and b, which measure the value of a direct link and the cost of failure. We are interested in a region of the model's parameter space where there is a tension between the desire for more direct links and the fear of failure. We have two kinds of results. Our general results provide welfare upper-bounds for optimal and stable networks, and we see that for small p and q any stable network has small welfare. Specific results for the case where $p = q$ describe the structure of optimal and stable networks, and demonstrate that the upper bounds are approximately achievable by forming cliques of appropriate size. Consequently, for small $p = q$, the welfare-loss from stable networks is large. Further results for the $p = q$ case describe the welfare cost to constructing optimal networks when agents are anonymous; the social planner can choose the degree of an individual node but not the agents at the other end of its edges. We also show that the welfare cost of anonymity is large.

Our formulation of the payoffs is intended to capture the basic trade-off in a simple way, using very few parameters. Links confer benefits that scale linearly in the degree, and failures spread through direct probabilistic contagion across edges. One can imagine more complex models for both of these aspects of the payoff, with more complex notions of the way in which a node's links increase its payoff, and more complex mechanisms for the spread of failures. For example, Amini et al [5] extend the traditional graph contagion framework to better model financial networks, studying contagion in random networks with inhomogeneous degrees and an arbitrary distribution of weights on edges. Extending our analysis of strategic network formation to models with this greater level of complexity is an interesting direction for further analysis. Here we will see that the present model already exhibits rich behavior, and suggests avenues for pursuing such generalizations.

2 The Model

In this paper, we develop a model to capture the underlying trade-off between the benefit of link formation and the problem of contagious risk, using simple definitions for the payoffs arising from these underlying processes. The model is formulated as follows. To begin with, we have a set V of n agents, and agents can choose to form bilateral relationships with one another, resulting in an undirected graph $G = (V, E)$. An agent receives a payoff of $a > 0$ from each relationship in which it takes part. Once the network is formed, a random process creates cascading failures as follows. First nodes fail independently with probability q, and then failed nodes have a probability of p of causing their neighbors to fail as well, with the failure potentially continuing to spread from these newly failed nodes. In more detail:

- First, each agent randomly experiences a failure, independently with probability $q > 0$. We refer to these as the *root failures* in the graph.
- Next, we declare each edge of G to be *live* independently with probability p and *blocked* with probability $1 - p$. We think of the live edges as those that transmit failure, and the blocked edges as those that do not transmit failure. Any node that can reach a root failure using a path consisting entirely of live edges is declared to fail also.

If an agent fails, it loses any benefit from the links it forms, and instead it pays a cost of $b > 0$. We assume that there is an upper bound Δ on the number of links any one node is able to form. Much of the interesting behavior in this model turns out to take place in graphs where the average degree is close to $1/p$. As a result, we want to have Δ larger than $1/p$, but not so large that any single node can dominate the structure of the graph. In particular, we assume that $\Delta = c^*/p$ for a constant $c^* > 1$.

Letting d_i denote the degree of node i in G, and ϕ_i denote the probability that it fails (taken over the random choices of root failures and live edges), we can write i's expected payoff as

$$\pi_i = ad_i(1-\phi_i) - b\phi_i = ad_i - (ad_i + b)\phi_i.$$

We employ a Rawlsian notion of welfare. In particular, we measure the "quality" of a graph via its *minimum welfare* (henceforth abbreviated *min-welfare*), the minimum payoff of any node in the graph. A socially optimal graph is one that maximizes this quantity. This notion of welfare is convenient for our analysis, and well-founded in principles of distributive justice [24]. Min-welfare satisfies criteria of anonymity and the weak Pareto principle.

One could study strategic network formation by defining a non-cooperative game whose outcomes are graphs. However, in such non-cooperative models, small details of the specification of the game will determine the precise structure of equilibrium networks. To capture the notion that it takes two nodes to agree on the formation of a link, but any node can unilaterally withdraw from its links, network theorists, following [16, 20], identified *stable networks* as a class of networks that we could expect to be equilibrium outcomes of any interesting network formation game.

We say that a graph is *stable* if (i) no node can strictly increase its payoff by deleting all its incident links (hence removing itself from the network), and (ii) there is no pair of nodes (i, j) such that (i, j) is not an edge of G, but both i and j would have higher payoffs, with at least one of them strictly higher, if (i, j) were added to G. Our definition of stability similar to the notion of *pairwise Nash stability* [19], which modifies (i) to allow a node to drop any subset of its incident links. Thus, any pairwise Nash stable graph is also stable under our definition, and so our upper bounds on the welfare of all stable graphs also apply to all pairwise Nash stable graphs.

When we consider the structures of socially optimal and stable graphs, much of the interesting behavior emerges in a natural range of the parameters a, b, p, and q motivated by the following considerations. Suppose we had just two nodes i and j, and suppose that i is deciding whether to link to j. If i forms the link, it receives a benefit of a but there is a probability of pq that j will fail and that this failure will spread to i. We want i to be willing to form the link to j under these conditions, and so we assume $a > bqp$. Otherwise no links will form. On the other hand, suppose that i knew that j were going to fail, so that the only thing protecting i from failure is the transmission probability p. Under these conditions we do not want i to form the link to j, so we assume $a < bp$. Otherwise there will be no strategic component to the analysis. Analogously, suppose that i knew that any failure at j would definitely spread to i, so that the only thing protecting i from failure is the chance $1-q$ that j does not fail. Under these conditions we also do not want i to form the link to j, so we assume $a < bq$.

In our analysis, we focus on the range of parameters in which these bounds hold by arbitrarily large constant factors. That is, we consider the case in which p and q are small, the quantity a exceeds bqp by a large amount, and in turn that $\min(bp, bq)$ exceeds a by a large amount. Our assumption is that for a small constant $\delta > 0$, we have

$$\delta^{-1} bqp < a < \delta \min(bp, bq).$$

For ease of future reference, we call this *Assumption* $\mathcal{P}(\delta)$ and refer to δ as the key *separation parameter* in our model. Finally, we consider the case in which the number of nodes n is arbitrarily large compared to these other quantities (and/or their reciprocals).

3 An Upper Bound on the Optimal Min-Welfare

We begin by establishing an upper bound on the min-welfare of any graph. Min-welfare is directly related to degree and we will see that critical graphs are those in which the average degree is close to $1/p$, and thus the average direct benefit from links is close to a/p. Suppose that the min-welfare in a graph G exceeds $\dfrac{(1+\varepsilon)a}{p}$. Then all node degrees must be at least $\dfrac{(1+\varepsilon)}{p}$. If there is a node that can reach many others along live-edge paths with reasonable probability, then this node experiences a large probability of failure, and hence has a sharply reduced payoff, which will ultimately contradict our assumption that G has large min-welfare.

Now, how do we show that some node has a reasonably high chance of reaching many others on live-edge paths? There is a connection to the basic random graph model $\mathcal{G}(n, p)$, in which an edge is inserted between each pair among n nodes independently with probability p. We can think of $\mathcal{G}(n, p)$ equivalently as the model in which one starts with an n-node clique and, declares each edge to be live independently with probability p, and then considers the live-edge subgraph. The challenge in our case is that our graphs G are not necessarily cliques, or even close to being cliques, and relatively little is known about adapting results from $\mathcal{G}(n, p)$ to the case of arbitrary underlying base graphs [3]. Fortunately, however, we are able to prove a result that is strong enough for our purposes, adapting techniques for analyzing connected components in $\mathcal{G}(n, p)$ to the setting of live-edge subgraphs of arbitrary underlying graphs.

We begin with this part of the analysis, as follows.

(3.1) *For all $\varepsilon > 0$ there exist constants $\alpha, \beta > 0$ such that the following holds. Let H be a graph in which each node has degree at least $r \geq \dfrac{1+\varepsilon}{p}$. Construct a random subgraph of H by declaring each edge to be "live" with probability p. Then for every node $i \in V$, the number of nodes reachable from i on live-edge paths is at least αr with probability at least β.*

Proof. Let i be any node in H. We now describe a method for exploring the live edges outward from i, based on Karp's analysis of random subgraphs of the bidirected complete graph [22] and Alon and Spencer's analysis of infinite branching processes [4]. We first take all the nodes (if any) that i can reach via live edges and put them in a queue. We then repeatedly delete a node j from the queue and add to the queue all the nodes (if any) that j can reach via live edges, other than the ones already "discovered" (added to the queue) in previous iterations. Notice, crucially, that the outcome of the random live/blocked decision for each edge (j, j') is only examined once in this process, when one of nodes j or j' first comes to the front of the queue. Thus, we can assume that the live/blocked status of (j, j') is first determined at that moment.

For a small constant $\alpha > 0$, we say that this process *succeeds* if at least αr nodes are added to the queue before the queue ever becomes empty. If the process succeeds with probability at least β, for a constant $\beta > 0$, then our result follows.

Let Q_t be the number of nodes in the queue at the end of iteration t, where we define $Q_0 = 1$ to indicate that i starts in the queue. We have

$$Q_t = Q_{t-1} - 1 + X_t,$$

where the "-1" is because we delete a node j_t from the queue in iteration t (with $j_1 = i$), and X_t is a random variable equal to the number of not-yet-discovered nodes that j_t can reach via live edges. (This is where it is useful to assume that the live/blocked status of edges from j_t to not-yet-discovered nodes is only determined when j_t reaches the front of the queue.) Unrolling this recurrence, we have

$$Q_t = \left(\sum_{u=1}^{t} X_u\right) - t.$$

We are interested in showing that the probability of $Q_t > 0$ for all t from 1 until at least αr nodes have been discovered (added to the queue); in this case, the search for nodes using live-edge paths continues successfully for a sufficient number of steps, as required.

The expectation of X_t, prior to the point at which at least αr nodes have been discovered, can be determined as follows. The node j_t has degree at least r in H, and at most αr nodes have been discovered by the process thus far, so there are at least $(1-\alpha)r$ edges emanating from j_t leading to not-yet-discovered nodes. We choose α small enough that $(1-\alpha)r \geq \dfrac{1+\varepsilon/2}{p}$; since each of these edges is live with probability p, we have $\mathbf{E}[X_t] \geq 1+\varepsilon/2$. Thus, until αr nodes have been discovered, we can think of the queue length as a random walk on the integers with positive drift; as a result, there is a positive probability that the walk never returns to 0, which is the result we want.

We can briefly verify this in more detail for our particular case as follows. Let $S_t = \sum_{u=1}^{t} X_u$; by the Chernoff Bound, we have

$$\Pr[S_t \leq t] < \Pr[S_t \leq (1-\varepsilon/4)\mathbf{E}[S_t]] < e^{-\frac{1}{2}\frac{\varepsilon^2}{16}t}.$$

Now, the sum $\sum_{t=1}^{\infty} e^{-\frac{1}{2}\frac{\varepsilon^2}{16}t}$ converges; we choose t_0 large enough that $\sum_{t=t_0}^{\infty} e^{-\frac{1}{2}\frac{\varepsilon^2}{16}t} < 1$. For p sufficiently small, there is a positive probability that X_1, the number of nodes i can reach directly via live edges, is at least t_0. It then follows that $S_t > t$ for all $t < t_0$. Finally, for all t we have $\Pr[S_t \leq t \mid X_1 \geq t_0] \leq \Pr[S_t \leq t] < e^{-\frac{1}{2}\frac{\varepsilon^2}{16}t}$; summing over t we obtain

$$\sum_{t=0}^{\infty} \Pr[S_t \leq t \mid X_1 \geq t_0] < 1. \qquad \blacksquare$$

Next, we simply want to argue that if a node can reach many other nodes via live-edge paths with reasonably large probability, then it has a large probability of failing and hence a negative payoff. To do this, we first state a simple lemma about the union of many independent events, and then we use this to draw the resulting conclusion for a node's payoff.

(3.2) *Consider a collection of independent events $\mathcal{E}_1, \ldots, \mathcal{E}_n$, each of probability $p > 0$. Then the probability of their union is at least* $\min(\frac{1}{3}, \frac{2}{3}np)$.

Proof. If $p \geq \frac{1}{3}$ then the result follows immediately. Otherwise, if $np \leq \frac{2}{3}$, then we have

$$\begin{aligned}\Pr\left[\bigcup_{j=1}^{n} \mathcal{E}_j\right] &\geq \sum_{j=1}^{n} \Pr\left[\mathcal{E}_j\right] - \sum_{j,j'} \Pr\left[\mathcal{E}_j \cap \mathcal{E}_{j'}\right] \\ &= np - \binom{n}{2} p^2 \\ &\geq np - \frac{1}{2}(np)^2 \\ &= np(1 - \frac{1}{2}np) \\ &\geq \frac{2}{3}np.\end{aligned}$$

Otherwise, we can choose a subset S of $k \leq n$ of the events such that $\frac{2}{3} < kp \leq 1$. We have

$$\begin{aligned}\Pr\left[\bigcup_{j \in S} \mathcal{E}_j\right] &\geq \sum_{j \in S} \Pr\left[\mathcal{E}_j\right] - \sum_{j,j' \in S} \Pr\left[\mathcal{E}_j \cap \mathcal{E}_{j'}\right] \\ &= kp - \binom{k}{2} p^2 \\ &\geq kp - \frac{1}{2}(kp)^2 \\ &= kp(1 - \frac{1}{2}kp) \\ &\geq \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3}.\end{aligned}$$

■

Now, for a node i, let the set of nodes it can reach on live-edge paths in G be called its *live component*, and let $r_i(G)$ be a random variable denoting the size of i's live component.

(3.3) *For all $\gamma_0, \gamma_1 > 0$ there exist $\alpha, \delta > 0$ such that when $p, q \leq \alpha$ and Assumption $\mathcal{P}(\delta)$ holds, we have the following. If G is a graph with a node i for which $r_i(G) \geq \dfrac{\gamma_1}{p}$ with probability at least γ_0, then the payoff of node i satisfies $\pi_i(G) < -bq$. (We note that the right-hand side is the payoff i would receive if it had no links).*

Proof. If i can reach at least $\gamma_1 p^{-1}$ nodes on live-edge paths, then by (3.2), the probability that it fails is at least $\min(\frac{1}{3}, \frac{2}{3}\gamma_1 p^{-1} q)$. Removing the conditioning on this event, the probability it fails is at least $\phi_i \geq \min(\frac{1}{3}\gamma_0, \frac{2}{3}\gamma_0\gamma_1 p^{-1} q)$. We also have $d_i \leq \Delta = c^* p^{-1}$.

If $\phi_i \geq \frac{1}{3}\gamma_0$, then

$$\begin{aligned} \pi_i &\leq ad_i - b\phi_i \leq ac^* p^{-1} - \frac{1}{3}b\gamma_0 \\ &\leq \delta bc^* - \frac{1}{3}b\gamma_0 = b(\delta c^* - \frac{1}{3}\gamma_0) \end{aligned}$$

where the last line is less than $-bq$ for δ sufficiently small and $q < \frac{1}{3}\gamma_0$.

If $\phi_i \geq \frac{2}{3}\gamma_0\gamma_1 p^{-1} q$, then defining $\gamma_2 = \frac{2}{3}\gamma_0\gamma_1$, we have

$$\begin{aligned} \pi_i &\leq ad_i - b\phi_i \\ &\leq ac^* p^{-1} - \gamma_2 bqp^{-1} \\ &= ac^* p^{-1} - (\gamma_2 p^{-1} - 1)bq - bq \\ &< \delta c^* b - (\gamma_2 p^{-1} - 1)bq - bq. \end{aligned}$$

This last line is less than $-bq$ provided that $1 + \delta c^* - \gamma_2 p^{-1} < 0$, which holds provided that p is sufficiently small relative to δ. ■

Finally, combining (3.1) with (3.3), we get an immediate consequence for the payoffs when all nodes have large degrees. The upper bound on min-welfare follows directly from this.

(3.4) *For all $\varepsilon > 0$, there exist $\alpha, \delta > 0$ such that when $p, q \leq \alpha$ and Assumption $\mathcal{P}(\delta)$ holds, we have the following. If each node has degree at least $\frac{(1+\varepsilon)}{p}$, then for each node i we have $\pi_i < -bq$.*

Proof. For each node $i \in V$, (3.1) implies that we have $r_i(G) \geq \frac{\gamma_1}{p}$ with probability at least γ_0, It then follows from (3.3) that $\pi_i < -bq$. ■

(3.5) *For all $\varepsilon > 0$, there exist $\alpha, \delta > 0$ such that when $p, q \leq \alpha$ and Assumption $\mathcal{P}(\delta)$ holds, no graph can have min-welfare greater than $\frac{(1+\varepsilon)a}{p}$.*

Proof. Choose $\alpha, \delta > 0$ as in (3.4), and suppose by way of contradiction that there is a graph with min-welfare greater than $\frac{(1+\varepsilon)a}{p}$. It follows that every node i has degree greater than $\frac{1+\varepsilon}{p}$. But then by (3.4) we have $\pi_i < -bq$, contradicting the assumption that the min-welfare is greater than $\frac{(1+\varepsilon)a}{p}$. ■

4 Super-Critical Payoffs and Anonymous Markets

We now show that the upper bound in Section 3 can essentially be achieved, in an asymptotic sense, and also consider some of the structural implications of this fact.

To begin with, it is instructive to think about the analysis in Section 3 in terms of the random graph $\mathcal{G}(k, r)$.[1] One of the central facts about $\mathcal{G}(k, r)$ is that in a small window around probability $r = 1/k$, the expected size of the largest connected component jumps from a constant value to a constant fraction of k. This is the basic phase transition for $\mathcal{G}(k, r)$, and (3.1) in Section 3 is a reflection of this phase transition for an arbitrary underlying graph.

In order for a graph to achieve *super-critical payoffs* — those of the form $\frac{(1+\varepsilon)a}{p}$ for some $\varepsilon > 0$ — it must lie on the side of the phase transition where the live components are likely to be large, proportional to $1/p$. For this to be possible, it must cross the phase transition by little enough that these large components do not eliminate the payoff of the nodes. We now show how to do this, constructing a family of graphs built from disjoint cliques that achieve min-welfare of the form $\frac{(1+\varepsilon)a}{p}$.

Some Basic Facts about $\mathcal{G}(k, r)$. We begin by carefully stating some quantitative results about the phase transition in $\mathcal{G}(k, r)$ in a form that will be useful for the analysis.

(4.1) *Let C_i denote the component containing node i in $\mathcal{G}(k, r)$. If we fix some*

[1] Since n and p are basic parameters in our model, we adopt the different variable names k and r in discussing $\mathcal{G}(k, r)$. Also, in keeping with standard terminology, we will often refer informally to $\mathcal{G}(k, r)$ as "a random graph," as though it is a single graph rather than a distribution over graphs.

other node j and look at the event $j \in C_i$, then we have

$$\Pr[j \in C_i] = \frac{1}{k} \cdot \mathbf{E}[|C_i|] - \frac{1}{k}.$$

Proof.

$$\begin{aligned}
\Pr[j \in C_i] &= \sum_{s=1}^{k} \Pr[|C_i| = s] \cdot \Pr[j \in C_i \mid |C_i| = s] \\
&= \sum_{s=1}^{k} \frac{s-1}{k} \Pr[|C_i| = s] \\
&= \frac{1}{k} \sum_{s=1}^{k} (s-1) \Pr[|C_i| = s] \\
&= \frac{1}{k} \cdot \mathbf{E}[|C_i|] - \frac{1}{k}
\end{aligned}$$

■

Thus, looking at the probability a node belongs to i's component is equivalent to looking at the expected size of i's component.

The following pair of standard results describe the contrasting behavior of component sizes on opposite sides of $r = 1/k$.

(4.2) *Fix $x < 1$, and consider the component of a given node i in $\mathcal{G}(k, r)$, where $kr = x$. Then for k sufficiently large, we have the following:*

(i) The probability that i's component exceeds size c decreases exponentially in c.

(ii) Consequently, the expected size of i's component is bounded by a constant $c = c(x)$, independent of k, and the maximum size of any component in the graph is thus $O(\log k)$.

(4.3) *There is an increasing function $\theta : [1, \infty] \to [0, 1]$ that is continuously differentiable on $(1, \infty)$ and continuously differentiable from the right at $x = 1$, with $\theta(1) = 0$ and $\theta'(1)$ a positive real number, such that the following holds. Fix $x > 1$ and $\varepsilon > 0$, and consider $\mathcal{G}(k, r)$, where $kr = x$. Then for $k = k(x, \varepsilon)$ sufficiently large, we have the following:*

(i) With probability $1 - \exp(-k)$*, there is a component of size between* $(1 - \varepsilon)\theta(x)k$ *and* $(1 + \varepsilon)\theta(x)k$*.*

(ii) Conditioned on not belonging to the giant component in (i), the probability that a node i *belongs to a component of size greater than* c *decreases exponentially in* c*.*

(iii) Consequently, the expected size of i*'s component is between* $(1-\varepsilon)^2\theta(x)^2k$ *and* $(1+\varepsilon)^2\theta(x)^2k + c$ *for a constant* $c = c(x)$*.*

Point (iii) follows from (i) and (ii) by considering that with probability $(1\pm\varepsilon)\theta(x)$, node i belongs to a component of size $(1 \pm \varepsilon)\theta(x)k$, and with the remaining probability i belongs to a component of expected size at most c.

A Family of Graphs with Super-Critical Payoffs. For parameters k and s, let $F_s(k)$ denote the disjoint union of s cliques of size k. We will show that $F_s\left(\dfrac{1+\gamma}{p}\right)$, for arbitrary $s \geq 1$ and a small constant $\gamma > 0$, achieves super-critical node payoffs.

For our construction, we will focus on the special case $p = q$. A nice feature of this special case is that we can represent the spread of failures in $F_s(k)$ in the following equivalent way. We imagine a single "failure node" i^* associated with each clique, and attached to each real node in the clique, resulting in a clique on $k+1$ nodes. There is a transmission probability p on the edges from i^* to each node in its clique, as there is on all other edges. In this view, a node i fails if it is in the same live-edge component as i^*; in other words, the probability i fails is the probability it belongs to the same component as a given fixed node i^* in $\mathcal{G}(k+1, r)$. By (4.1) we know this is

$$\frac{1}{k+1} \cdot \mathbf{E}\left[|C_i|\right] - \frac{1}{k+1},$$

where C_i denotes the live-edge component of i.

With $p = q$, we define σ to be the ratio $a/bp = a/bq$; by assumption $\mathcal{P}(\delta)$, we have $\sigma < \delta$, and we assume as usual that δ and p are sufficiently small. We let the number of nodes k in each clique be $(1+\gamma)/p$ for a small value $\gamma > 0$ that we determine below.

First, (4.3)(iii) implies that the probability ϕ_i that i fails satisfies

$$(1-\varepsilon_0)\theta(1+\gamma)^2 \leq \phi_i \leq (1+\varepsilon_0)\theta(1+\gamma)^2$$

for a constant ε_0 that goes to 0 with p. Thus, the payoff to a node i is

$$\begin{aligned}
\pi_i &\geq \frac{a(1+\gamma)}{p} - \left(b + \frac{a(1+\gamma)}{p}\right)(1+\varepsilon_0)\theta(1+\gamma)^2 \\
&= \frac{a(1+\gamma)}{p} - b(1+\sigma(1+\gamma))(1+\varepsilon_0)\theta(1+\gamma)^2 \\
&\geq \frac{a(1+\gamma)}{p} - 2b\theta(1+\gamma)^2 \\
&= \frac{\sigma bp(1+\gamma)}{p} - 2b\theta(1+\gamma)^2 \\
&= b(\sigma(1+\gamma) - 2\theta(1+\gamma)^2).
\end{aligned}$$

Now, let

$$h_0(x) = \sigma x - 2\theta(x)^2,$$

so that

$$\pi_i \geq bh_0(1+\gamma).$$

We have

$$h_0'(x) = \sigma - 4\theta(x)\theta'(x).$$

Since $\theta(1) = 0$ and $\theta'(1)$ is a positive real number, we have $h_0'(1) = \sigma$, and hence the function $h_0(x)$ is strictly increasing over the interval $x \in [1, w_0]$ for a constant w_0 depending on σ. Since $h_0(1) = \sigma$, we have $h_0(w_0) = \sigma(1+\sigma_0)$ for a constant $\sigma_0 > 0$ depending on σ.

Returning to the lower bound on π_i, we choose $\gamma = w_0 - 1$, and so

$$\pi_i \geq bh_0(w_0) = \frac{a}{\sigma p} \cdot \sigma(1+\sigma_0) = \frac{a(1+\sigma_0)}{p}.$$

Consequently, the payoff to each node exceeds $\frac{a}{p}$ by a multiplicative factor greater than 1 that depends on σ.

Comparison to an Anonymous Structure. The construction above achieves super-critical payoffs by allowing nodes to cluster into communities of an appropriate size, and thus to insulate themselves from failures originating in other communities. Drawing on a market motivation, it is interesting to ask whether super-critical payoffs can be achieved through structures that are based instead on *anonymous* interaction, where nodes can specify the number of partners they want to connect to, but have no control over who these partners are — the partners are chosen uniformly at random from the population. As we now show, in fact, anonymous interaction structures are not able to yield super-critical payoffs.

To define these anonymous structures precisely, we use the *configuration model* for random graphs [10, 11, 27]. Each of the n nodes is assigned k "half-edges"; these half-edges are then matched up uniformly at random into pairs, with each matching pair of half-edges forming an edge in the resulting random graph. Note that the pairing may cause two edges to go between the same pair of nodes, or for a node to form an edge that loops to itself; we remove these parallel edges and self-loops to obtain the final graph. Failures then propagate in this graph according to our model, spreading from root failures along live-edge paths.

With high probability, the local neighborhood of a node in this random graph will have a particularly simple structure, as follows. For node i, define $B(i, \ell)$ to be the ball of radius ℓ centered at i, i.e. the induced subgraph of G on the set of all nodes reachable from i in ℓ or fewer hops. For fixed integers k, ℓ and any node i, the probability that $B(i, \ell)$ is a tree of depth ℓ and degree k (i.e. one whose internal nodes all have degree k and whose leaves are all at distance ℓ from the root) tends to 1 as $n \to \infty$.

For our analysis, we will therefore connect the propagation of failures in the configuration model to a related, simpler model based on an infinite k-regular tree. In particular, let $\mathcal{B}(k, r)$ denote the distribution over trees obtained by starting with an infinite k-regular tree and including each edge in the random tree with probability r. We now have a pair of results analogous to (4.2) and (4.3).

(4.4) *Let $x < 1$, and consider a tree generated from $\mathcal{B}(k, r)$ where $kr = x$.*

(i) The probability that the tree's size exceeds size c decreases exponentially in c.

(ii) The expected size of the tree is bounded by a constant $c = c(x)$.

(4.5) *There is an increasing function $\tau : [1, \infty] \to [0, 1]$ that is continuously differentiable on $(1, \infty)$ and continuously differentiable from the right at $x = 1$, with $\tau(1) = 0$ and $\tau'(1)$ a positive real number, such that the following holds. Consider a tree generated from $\mathcal{B}(k, r)$, and let $\psi_r(k)$ be the probability that it has an infinite node set.*

(i) If $kr > 1$, then $\psi_r(k) > \tau(kr)$.

(ii) For all integers $c_0, c_1 > 1$ and $k \geq c_0 c_1 / r$, we have

$$\psi_r(k) \geq 1 - (1 - \psi_r(c_0/r))^{c_1} > 1 - (1 - \tau(c_0))^{c_1}.$$

(iii) Conditioned on not having an infinite node set, the probability that the tree's size exceeds c decreases exponentially in c. Its expected size is thus bounded by a constant $c = c(x)$.

Proof. Part (iii) of the claim is a standard result; parts (i) and (ii) are formulated in ways that are adapted to our present purposes, and we give proofs of them here.

First we prove (i). The probability that the tree is infinite is the unique solution to $z = 1 - (1 - pz)^k$ in the interval $(0, 1)$. Define τ to be the unique solution to $\tau = 1 - e^{-x\tau}$ in the interval $(0, 1)$. Writing $f_0(v) = (1 - pv)^k$ and $f_1(v) = e^{-xv}$, we have

$$f_0(v) = (1 - pv)^k = (1 - pv)^{x/p} < e^{-xv} = f_1(v).$$

Thus, the curve $y = 1 - f_0(v)$ lies above the curve $y = 1 - f_1(v)$ on the interval $(0, 1)$, and so $y = 1 - f_0(v)$ intersects the line $y = v$ to the right of where $y = 1 - f_1(v)$ intersects it. It follows that $z > \tau$, and hence we can take $\tau = \tau(x)$ as our function.

To prove (ii), consider $k' = \frac{c_0 c_1}{r}$ subtrees of the root in the complete k-ary tree (before edges are randomly included), and group them into c_1 blocks of $\frac{c_0}{r}$ subtrees each. For any block, if we consider just the root and the subtrees in a single block, the probability that the resulting random tree is infinite is at least $\psi_r(c_0/r)$ (since the root has this degree in the restricted tree, and the nodes in the subtrees have degree $k \geq c_0/r$). The tree is infinite if it is infinite in any of the blocks, and so the probability it is infinite is at least

$$1 - (1 - \psi_r(c_0/r))^{c_1} > 1 - (1 - \tau(c_0))^{c_1}$$

where the latter inequality follows directly from (i). ■

We now want to show that when each node forms k links in the anonymous structure, for any $k = \frac{1+\beta}{p}$, the node payoffs can be at most a/p as $n \to \infty$. Clearly this is true for $\beta \leq 0$, so we consider the case of an arbitrary $\beta > 0$.

When the random graph G is sampled using the configuration model, for any node i the probability that the ball $B(i, \ell)$ is a tree of degree k and depth ℓ is

$1 - o(1)$ as $n \to \infty$. Applying 4.5(i), the probability that i belongs to a live path of length ℓ is at least $\tau(1+\beta) - o(1)$; for n sufficiently large, this probability is at least $\tau(1+\beta/2)$. In the event that i belongs to a live path of length ℓ, it fails with probability at least $1-(1-q)^\ell$. By taking ℓ large enough, we may assume that $\tau(1+\beta/2)(1-(1-q)^\ell) \geq \tau(1+\beta/3)$ and thus node i fails with probability at least $\tau(1+\beta/3)$.

Thus, if n is sufficiently large we have

$$\begin{aligned}
\pi_i &\leq \left(\frac{a(1+\beta)}{p}\right)\left(1-\tau\left(1+\frac{\beta}{3}\right)\right) - b\tau\left(1+\frac{\beta}{3}\right) \\
&= b\sigma(1+\beta)\left(1-\tau\left(1+\frac{\beta}{3}\right)\right) - b\tau\left(1+\frac{\beta}{3}\right).
\end{aligned}$$

Let

$$h_1(x) = \sigma(1+3x)\left(1-\tau\left(1+x\right)\right) - \tau\left(1+x\right),$$

so that $\pi_i \leq bh_1(\beta/3)$. By (4.5)(ii), we know that for $y \geq 4$, we have

$$\tau(y) \geq 1-(1-\tau(2))^{\lfloor y/2 \rfloor} \geq 1-(1-\tau(2))^{y/4}.$$

We can thus choose $w_1 \geq 4$ such that

$$\tau(y) \geq 1 - \frac{1}{1+y}$$

for all $y \geq w_1$. If $1+x \geq w_1$, we have

$$h_1(x) \leq \sigma(1+3x)\left(\frac{1}{2+x}\right) - \frac{1+x}{2+x} < 0,$$

provided $\sigma < \frac{1}{3}$. Now, if $\sigma \leq \frac{1}{3}\sup_{y\in[1,w_1]}\tau'(y)$, then we have the following for all $x \in [0, w_1 - 1]$:

$$\begin{aligned}
h_1'(x) &= 3\sigma(1-\tau(1+x)) - (\sigma + 3\sigma x + 1)\tau'(1+x) \\
&\leq 3\sigma - \tau'(1+x) \leq 0.
\end{aligned}$$

Thus, for all $x \in [0, w_1-1]$, we have $h_1(x) \leq h_1(0) = \sigma$. Since we also have $h_1(x) < 0$ for $x \geq w_1 - 1$, it follows that $h_1(x) \leq \sigma$ for all $x \geq 0$.

Thus, for any $\beta > 0$, we have $\pi_i \leq b\sigma = \frac{a}{p}$ when each node forms $k = \frac{1+\beta}{p}$ links. Since $\pi_i \leq \frac{a}{p}$ when nodes form at most $k \leq \frac{1}{p}$ links, it follows that for any constant c, if nodes form $\frac{c}{p}$ links then $\pi_i \leq \frac{a}{p}$ provided n is sufficiently large as a function of c.

Clustered vs. Anonymous Markets. It is instructive to consider why a union of disjoint cliques was able to achieve qualitatively higher payoffs than an anonymous interaction pattern. In particular, the nodes in the cliques we constructed are linking at a degree beyond the phase transition point, whereas attempting to do this in the anonymous structure has negative effects on the payoff.

A quantitative way to think about the contrast is to observe that in the union of cliques, the failure probability of a node i was approximately controlled by a conjunction of two events: i belonging to the giant component of the clique, and the "failure node" i^* also belonging to the giant component of the clique. As a result, the failure probability involves a term of the form $\theta(x)^2$, and this has a derivative of 0 at $x = 1$ — hence, it is safe to increase x a bit past 1 without blowing up the failure probability. On the other hand, in the anonymous structure, once i belongs to the giant component, it fails with overwhelming probability; thus, i's failure probability involves a term of the form $\tau(x)$, which has a strictly positive derivative at $x = 1$, and this makes it unprofitable to increase x even arbitrarily little past 1. This is the fundamental difference between the behavior of the two kinds of structures in the region just past the phase transition.

5 Upper Bound on the Min-Welfare of Any Stable Network

We now show that any stable graph must have small min-welfare. (We defer the proof that stable graphs exist to the next section.) To upper-bound the min-welfare, we proceed as follows. Recall that we assumed that Δ is an upper bound on the number of links any one node is able to form. So nodes of degree Δ cannot form further edges. We first show, in (5.1) , that if two nodes i and j are not connected by an edge, and neither is at the maximum degree Δ, then at least one of them must have a large failure probability — this is what dissuades the other from forming the link.

It follows that in a stable network, all low-degree nodes of low failure probability must form a clique, since any unlinked pair of them would have an incentive to connect. If the number of nodes n is sufficiently large, we can then find a node i that is far from this clique. Hence node i, and every node within a large number of steps of i, must have large degrees; we can thus apply an analogue of (3.1) to show i has a large failure probability, and this will conclude the proof.

(5.1) *Suppose Assumption $\mathcal{P}(\delta)$ holds. Let G be a stable graph, and let i and j*

be two nodes of G such that (i, j) is not an edge of G, and the degrees of i and j are each strictly less than Δ. Then we have $\max(\phi_i, \phi_j) \geq \dfrac{(1-\delta)a}{(1+\delta c^*)bp}$.

Proof. Since the degrees of i and j are each strictly less than Δ, at least one of i or j does not have a strictly higher payoff if the edge (i, j) is included; let us assume it is node i. Thus, if G' denotes the graph G with the edge (i, j) included, then we have $\pi_i(G') \leq \pi_i(G)$.

We imagine evaluating failure in G' by first making all random root failure decisions and all random live/blocked decisions in G, then determining which additional nodes fail, and finally deciding whether the edge (i, j) is live or blocked and determining further failures. Let $\Phi_i(G)$ be the event that i fails in G before (i, j) is examined, and let $\mathcal{F}_{ij}(G)$ be the event that (i, j) is live and j fails in G. Then $\Phi_i(G') = \Phi_i(G) \cup \mathcal{F}_{ij}(G)$, so

$$\Pr\left[\Phi(G')\right] \leq \Pr\left[\Phi_i(G)\right] + \Pr\left[\mathcal{F}_{ij}(G)\right].$$

Since $\Pr\left[\mathcal{F}_{ij}(G)\right] = p\phi_j$, we have

$$\phi_i(G') - \phi_i(G) \leq p\phi_j(G).$$

Now,

$$\pi_i(G') = a(d_i + 1) - (ad_i + a + b)\phi_i(G'),$$

so the fact that $\pi_i(G') \leq \pi_i(G)$ implies that

$$a(d_i + 1) - (ad_i + a + b)\phi_i(G') \quad \leq \quad ad_i - (ad_i + b)\phi_i(G)$$

and hence

$$\begin{aligned} a &\leq (ad_i + a + b)(\phi_i(G') - \phi_i(G)) + a\phi_i(G) \\ &\leq (ad_i + a + b)p\phi_j(G) + a\phi_i(G) \\ &\leq (1 + \delta c^*)bp\phi_j(G) + \delta bp\phi_i(G), \end{aligned}$$

where the last line follows from the fact that $a < \delta bp$ and $d_i + 1 \leq c^* p^{-1}$.

Now, if $\phi_i(G) \geq \dfrac{a}{bp}$, we are done. Otherwise, we have

$$a \leq (1 + \delta c^*)bp\phi_j(G) + \delta a,$$

so

$$(1 - \delta)a \leq (1 + \delta c^*)bp\phi_j(G),$$

and hence

$$\phi_j(G) \geq \frac{(1-\delta)a}{(1+\delta c^*)bp}. \quad \blacksquare$$

Following our informal plan above, we note that a stable graph might have some low-degree nodes, so we require the following direct adaptation of (3.1), which applies to nodes that are far from all low-degree nodes.

(5.2) *For all $\varepsilon > 0$ there exist constants $\alpha, \beta > 0$ such that the following holds. Let H be a graph, and let A be the set of nodes of degree less than $\dfrac{1+\varepsilon}{p}$. Let i be a node of distance greater than $\dfrac{1}{p}$ from A. Construct a random subgraph of H by declaring each edge to be "live" with probability p. Then the number of nodes reachable from i on live-edge paths is at least αp^{-1} with probability at least β.*

Proof. Consider the node-discovery process described in the proof of (3.1), starting from the node i, and recall that we declare it to *succeed* if it adds at least $\dfrac{\alpha}{p}$ nodes to the queue before it ever becomes empty, for the small constant $\alpha < 1$ used there. The event that the process succeeds depends only on the live/blocked decisions for nodes within distance $\dfrac{\alpha}{p}$ of i, and all such nodes have degrees at least $\dfrac{1+\varepsilon}{p}$; hence, for this whole time we can apply the argument used in (3.1). $\blacksquare$

Finally, we conclude the proof strategy outlined at the beginning of the section, resulting in our upper bound.

(5.3) *Let $n > \Delta^{\Delta+2}$. For all $\varepsilon > 0$ there exist $\alpha, \delta > 0$ such that when $p, q \leq \alpha$ and Assumption $\mathcal{P}(\delta)$ holds, no stable graph can have min-welfare greater than $\dfrac{\varepsilon a}{p}$.*

Proof. Suppose by way of contradiction that $G = (V, E)$ is a stable graph in which $\pi_i \geq \dfrac{\varepsilon a}{p}$ for all $i \in V$.

Let $A \subseteq V$ denote the set of all nodes i of G for which $d_i < \Delta$ and $\phi_i < \dfrac{(1-\delta)a}{(1+\delta c^*)bp}$. Since any node in A is able to form an additional edge, (5.1) implies that there must be an edge between each pair of nodes in A — in other words, A induces a clique in G.

Let $B \subseteq V$ denote the set of all nodes in G of degree equal to Δ. For any $i \in V - (A \cup B)$, we have $\phi_i \geq \frac{(1-\delta)a}{(1+\delta c^*)bp}$. Since $\pi_i \geq \frac{\varepsilon a}{p}$ by assumption, we have

$$\frac{\varepsilon a}{p} \leq ad_i - b\phi_i \leq ad_i - \frac{(1-\delta)a}{(1+\delta c^*)p}$$

and hence

$$d_i \geq \frac{\varepsilon}{p} + \frac{1-\delta}{(1+\delta c^*)p}.$$

For δ sufficiently small, the right-hand side of this inequality is at least $\frac{1+\varepsilon_1}{p}$ for a constant $\varepsilon_1 > 0$ Choosing $\varepsilon_2 = \min(\varepsilon_1, c^* - 1)$, it follows that all nodes $i \in V - A$ have degree at least $\frac{1+\varepsilon_2}{p}$.

Now, for any $j \in A$, there are at most $1 + \Delta + \Delta^2 + \cdots + \Delta^{\Delta+1} < \Delta^{\Delta+2} < n$ nodes within distance $\Delta + 1$ of j, and hence within distance Δ of some node in A. Hence there is some node $i \in V$ at distance greater than $\Delta > p^{-1}$ from A. For this node i, (5.2) implies that $r_i \geq \frac{\gamma_1}{p}$ with probability at least γ_0, for constants $\gamma_0, \gamma_1 > 0$. By (3.3), it follows that $\pi_i < -bq$, contradicting the assumption that the min-welfare of G is greater than $\frac{\varepsilon a}{p}$. ■

6 Existence of Stable Networks

Finally, we show that there exist arbitrarily large stable networks. As with our constructions in Section 4, we will consider graphs that consist of disjoint cliques — graphs $F_s(k+1)$ with $k = \frac{1+\gamma}{p}$ for an appropriately chosen $\gamma > 0$.

The challenge is to find a k where the union of cliques is stable, and this requires some care for the following reason. Stability requires that no unlinked pair of nodes wants to form an edge — this can be achieved by making k sufficiently large that creating a link between two cliques brings about too large an increase in failure probability to the nodes forming the link. Unfortunately, making k large also raises the failure probability of each node i based simply on its current set of edges — so we must not raise k so high that a node i wants to drop all its existing links. The crux of the problem is thus the following: is there a k that is large enough to discourage the formation of cross-clique links, but not so large that nodes will drop their current links? The main part of our analysis will be to show that such a k exists.

As in Section 4, we consider the case in which $p = q$; defining σ to be the ratio $a/bp = a/bq$, we have $\sigma < \delta$, and we assume δ and p are sufficiently small.

(6.1) *Given a, b, p, q as above, there exists $\gamma > 0$ such that with $k = \dfrac{1+\gamma}{p}$, the union of cliques $F_s(k+1)$ is stable.*

Proof. For the analysis of the construction, we will work with the function $\theta(x)$ defined in (4.3), as well as the related function $\lambda(x) = x(1-\theta(x)^2)$. Observe that $\lambda(1) = 1$, since $\theta(1) = 0$. Taking derivatives, we have

$$\lambda'(x) = (1-\theta(x)^2) - 2x\theta(x)\theta'(x),$$

and hence $\lambda'(1) = 1$. Thus we have

(6.2) *For some constant $w > 1$, the function $\lambda(x)$ is strictly increasing on the closed interval $[1, w]$.*

As in Section 4, we analyze the failure process by attaching a single "failure node" i^* to each clique. The probability ϕ_i that node i fails is the probability that i belongs to the same live-edge component as i^* in the $(k+2)$-node clique where i^* is added to i's clique. The payoff to node i is

$$\pi_i = ak - (ak+b)\phi_i.$$

If i drops all its edges, it receives a payoff of $-bq < 0$. If i forms an edge to a node j in another clique, it receives an added benefit of a, and incurs an increased expected loss of at least

$$(ak+b)p\phi_i(1-\phi_i).$$

There are four terms here; the second and third represent the chance that j's failure (which is $\phi_j = \phi_i$ by symmetry) spreads to i, and the fourth term represents the fact that this only matters if i had not already failed in its own clique. In more detail: the payoff to node i before the addition of this edge is $ak-(ak+b)\phi_i$, and afterward it is

$$a(k+1) - (ak+a+b)(\phi_i + p\phi_i - p\phi_i^2),$$

so the change in payoff is less than $a - (ak+b)p\phi_i(1-\phi_i)$.

Now, what is ϕ_i? By (4.1) and (4.3), we have

$$(1-\varepsilon_1)^2\theta(p(k+2))^2 \leq \phi_i \leq (1+\varepsilon_1)^2\theta(p(k+2))^2 + c_1 p$$

for a constant ε_1 that goes to zero as p does. By choosing a slightly larger ε_2, and using the fact that $\theta(\cdot)$ has a bounded first derivative, we have

$$(1-\varepsilon_2)\theta(1+\gamma)^2 \leq \phi_i \leq (1+\varepsilon_2)\theta(1+\gamma)^2,$$

with ε_2 going to zero as p does.

In the expression $\phi_i(1-\phi_i)$, provided the upper bound $(1+\varepsilon_2)\theta(1+\gamma)^2 \leq \frac{1}{2}$, we have

$$\begin{aligned} \phi_i(1-\phi_i) &\geq (1-\varepsilon_2)\theta(1+\gamma)^2(1-(1-\varepsilon_2)\theta(1+\gamma)^2) \\ &\geq (1-\varepsilon_2)\theta(1+\gamma)^2(1-\theta(1+\gamma)^2). \end{aligned}$$

Since $a = \sigma bp$ and $k = (1+\gamma)/p$, if we write $\sigma_1 = \sigma(1+\gamma)$, then we have $ak = \sigma(1+\gamma)b = \sigma_1 b$. Now we have

$$\begin{aligned} &(ak+b)pk\phi_i(1-\phi_i) \\ &\quad = b(1+\sigma_1)(1+\gamma)\phi_i(1-\phi_i) \\ &\quad \geq b(1+\sigma_1)(1+\gamma)(1-\varepsilon_2)\theta(1+\gamma)^2(1-\theta(1+\gamma)^2) \\ &\quad = b(1+\sigma_1)(1-\varepsilon_2)\lambda(1+\gamma)\theta(1+\gamma)^2 \\ &\quad \triangleq f_1(\gamma), \end{aligned}$$

where the last line is taken as the definition of $f_1(\gamma)$.

Observe that $f_1(0) = 0$, and by (6.2), there is an $x_1 < 1$ such that the function $f_1(x)$ is strictly increasing for x in a closed interval $[0, x_1]$.

We also have

$$\begin{aligned} (ak+b)\phi_i &\leq b(1+\sigma_1)(1+\varepsilon_2)\theta(1+\gamma)^2 \\ &\triangleq f_0(\gamma), \end{aligned}$$

where once again the last line is taken as the definition of $f_0(\gamma)$. We see that $f_0(x)$ is also strictly increasing in $[0, x_1]$ (and beyond this interval as well).

Now, since $\lambda(1) = 1$ and $\lambda(\cdot)$ is monotone increasing on $[0, x_1]$, for any small enough $\varepsilon_2 > 0$, there is a unique $x_0 < x_1$ such that

$$\lambda(1+x_0) = \frac{1+\varepsilon_2}{1-\varepsilon_2}.$$

Moreover, $f_1(x) > f_0(x)$ for all $x \in (x_0, x_1]$, and the value of x_0 goes to 0 as ε_2 goes to 0. Also, we observe that for $\gamma \in (x_0, x_1]$, we have

$$f_1(\gamma) > f_0(\gamma) \geq b\theta(1+\gamma)^2.$$

Now, we choose σ small enough that $2\sigma < \theta(1+x_1)^2$. We then choose ε_2 small enough (by choosing p small enough) so that $f_1(x_0) = f_0(x_0) < b\sigma$. Finally, let $g(\gamma) = ak = (1+\gamma)b\sigma$. Since $b\sigma < g(\gamma) < 2b\sigma$ for all $\gamma \in (x_0, x_1]$, it follows that $f_0(x_0) < g(x_0)$ but $g(x_1) < f_1(x_1)$. Therefore, since $f_0(\cdot)$ and $f_1(\cdot)$ are continuous functions, there exist $\gamma^*, \gamma^{**} \in (x_0, x_1]$ for which $g(\gamma^*) = f_1(\gamma^*)$ and $g(\gamma^{**}) \geq f_0(\gamma^{**})$, with $\gamma^* < \gamma^{**}$.

We choose any $\gamma \in [\gamma^*, \gamma^{**}]$ as the value of γ we use to define k. With this value of k, the payoff i receives from keeping all its edges is

$$\pi_i = ak - (ak+b)\phi_i \geq g(\gamma) - f_0(\gamma) \geq 0,$$

and hence i prefers to keep its edges rather than deleting all of them. The change in payoff i receives from linking to a node j in a different clique is less than

$$\begin{aligned} &k^{-1}\left(ak - (ak+b)pk\phi_i(1-\phi_i)\right) \\ &\quad \leq k^{-1}\left(ak - b(1+\sigma_1)(1-\varepsilon_2)\lambda(1+\gamma)\theta(1+\gamma)^2\right) \\ &\quad = k^{-1}\left(g(\gamma) - f_1(\gamma)\right) \\ &\quad \leq 0, \end{aligned}$$

and hence i will not form this link. Thus, the graph $F_s(k+1)$ is stable. ∎

A Stable Graph with Unequal Clique Sizes. We observe that starting with a set of disjoint cliques $F_s(k+1)$, we can create a different stable graph by adding one additional clique Γ of size $\ell < k+1$ on a disjoint set of nodes. The size ℓ can be chosen in any way such that the payoffs of nodes in the clique Γ each exceed $-bq$. In this way, nodes in Γ will not want to drop their incident edges. Moreover, there is still no edge that can form so as to improve the payoffs of both its endpoints, since any edge involving a node i in Γ must have its other end at a node j in one of the cliques of size $k+1$, in which case the argument for (6.1) shows that i would not want to form the link.

In particular, this means that we can take ℓ to be a clique yielding the maximum possible node payoff over all clique sizes, as in Section 4; this shows how certain nodes in a stable graph can have higher payoffs than others.

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Network Formation in the Presence of Contagious Risk *

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Abstract

There are a number of domains where agents must collectively form a network in the face of the following trade-off: each agent receives benefits from the direct links it forms to others, but these links expose it to the risk of being hit by a cascading failure that might spread over multi-step paths. Financial contagion, epidemic disease, the exposure of covert organizations to discovery, and electrical power networks are all settings in which such issues have been articulated.

Here we formulate the problem in terms of strategic network formation, and provide asymptotically tight bounds on the welfare of both optimal and stable networks. We find that socially optimal networks are, in a precise sense, situated just beyond a phase transition in the behavior of the cascading failures, and that stable graphs lie slightly further beyond this phase transition, at a point where most of the available welfare has been lost. Our analysis enables us to explore such issues as the trade-offs between clustered and anonymous market structures, and it exposes a fundamental sense

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in which very small amounts of "over-linking" in networks with contagious risk can have strong consequences for the welfare of the participants.

1 Introduction

Social networks have particular features that distinguish them from biological and physical networks as a class, and which are important for the propagation of networked agents' behaviors. Two kinds of models have been used to shed light on the structure of social networks. Probabilistic models, such as small-world models [27, 30] and preferential attachment [10], posit a few simple rules describing the probabilistic formation of links. The application of such models to social phenomena presumes that the networks are exogenous from the point-of-view of the phenomena being studied. Strategic models, on the other hand, presume that network formation and agents' behaviors are closely connected. This paper contributes to the study of this second kind of network formation. Recent surveys of endogenous, or strategic, network formation include [29] and the relevant chapters of [22].

A common approach in the strategic network formation literature (e.g. the *connections model* [23]) assumes that links are costly for an agent to form or maintain, and that benefits come from the indirect access to others that the network provides, as measured by distances [15, 18, 23], component sizes [9], or point-to-point connectivity [6]. There are many instances, however, in which this cost-benefit trade-off is inverted. Benefits come from direct links, while the cost is that of exposure to a failure that propagates through the network. In financial markets, benefits come from transacting with others, but counterparty risk, the risk to an agent that its partners cannot complete their side of a transaction, is increased to the extent that the partners are exposed to the failures of their other partners. The possibility exists that a single agent's failure can cause his partners to fail, and so on, leading to a cascade of financial collapse [1, 2, 14, 21]. Even when cascades do not happen, the fear of cascades can lead to market behavior that is costly for all agents, such as happened with the capital markets shutdown in the financial crisis of Fall 2008. Epidemiology provides still other examples of this inverted cost-benefit structure, wherein the pattern of social contacts has significant implications for the spread of disease. This is demonstrated in a model of HIV transmission in a structured population in [24], while [17] demonstrates the importance of network structure for the construction of containment strategies for a smallpox bioterror event; an analogous cost-benefit structure is also present in needle-sharing practices among intravenous drug users [11]. Additionally, [20] observes that clandestine organizations are subject to the risk of being exposed and compromised, and that this risk may be mitigated or magnified by the network structure of agent contacts. Asavathiratham et al [7, 8] use a similar model for analyzing cascading failures in power grids.

In our model, individuals first construct a social network. In this network, each node fails spontaneously with a small probability q. After this initial phase of spontaneous node failures, each edge transmits the failure with a small probability p. We can think of the set of edges that transmit failure as a random subgraph of the social network, and now the nodes which fail are all those in a component of the random subgraph containing a node which has spontaneously failed. Classical results on Bernoulli random graphs can be viewed as statements about random subgraphs of a complete network, and one of the technical contributions of this paper lies in generalizing this point of view from complete graphs to arbitrary graphs.

We examine networks that are optimal with respect to a Rawlsian social welfare criterion as well as networks that are stable in a sense different from (but closely related to) the stability concepts in [23] and [19]. In addition to the probabilities p and q, we use two other parameters: a and b, which measure the value of a direct link and the cost of failure. We are interested in a region of the model's parameter space where there is a tension between the desire for more direct links and the fear of failure. We have two kinds of results. Our general results provide welfare upper bounds for optimal and stable networks, and we see that for small p and q any stable network has small welfare. Specific results for the case where $p = q$ describe the structure of optimal and stable networks, and demonstrate that the upper bounds are approximately achievable by forming cliques of appropriate size. Consequently, for small $p = q$, the welfare-loss from stable networks is large: stable networks have significantly smaller welfare than the maximum possible. Further results for the $p = q$ case describe the welfare cost to constructing optimal networks when agents are anonymous; the social planner can choose the degree of an individual node but not the agents at the other end of its edges. We also show that the welfare cost of anonymity is large.

Our formulation of the payoffs is intended to capture the basic trade-off in a simple way, using very few parameters. Links confer benefits that scale linearly in the degree, and failures spread through direct probabilistic contagion across edges. One can imagine more complex models for both of these aspects of the payoff, with more complex notions of the way in which a node's links increase its payoff, and more complex mechanisms for the spread of failures. For example, Amini et al [5] extend the traditional graph contagion framework to better model financial networks, studying contagion in random networks with inhomogeneous degrees and an arbitrary distribution of weights on edges. Extending our analysis of strategic network formation to models with this greater level of complexity is an interesting direction for further analysis. Here we will see that the present model already exhibits rich behavior, and suggests avenues for pursuing such generalizations.

2 The Model

In this paper, we develop a model to capture the underlying trade-off between the benefit of link formation and the problem of contagious risk, using simple definitions for the payoffs arising from these underlying processes. The model is formulated as follows. To begin with, we have a set V of n agents, and agents can choose to form bilateral relationships with one another, resulting in an undirected graph $G = (V, E)$. An agent receives a payoff of $a > 0$ from each relationship in which it takes part. Once the network is formed, a random process creates cascading failures as follows. First, nodes fail independently with probability q, and then failed nodes have a probability of p of causing their neighbors to fail as well, with the failure potentially continuing to spread from these newly failed nodes. In more detail:

- First, each agent randomly experiences a failure, independently with probability $q > 0$. We refer to these as the *root failures* in the graph.

- Next, we declare each edge of G to be *active* independently with probability p and *blocked* with probability $1 - p$. We think of the active edges as those that transmit failure, and the blocked edges as those that do not transmit failure. Any node that can reach a root failure using a path consisting entirely of active edges is declared to fail also.

If an agent fails, it loses any benefit from the links it forms, and instead it pays a cost of $b > 0$. We assume that there is an upper bound Δ on the number of links a node is able to form. Much of the interesting behavior in this model turns out to take place in graphs where the average degree is close to $1/p$. As a result, we want to have Δ larger than $1/p$, but not so large that any single node can dominate the structure of the graph. In particular, we assume that $\Delta = c^*/p$ for a constant $c^* > 1$.

Letting d_i denote the degree of node i in G, and ϕ_i denote the probability that it fails (taken over the random choices of root failures and active edges), we can write i's expected payoff as

$$\pi_i = ad_i(1 - \phi_i) - b\phi_i = ad_i - (ad_i + b)\phi_i.$$

We employ a Rawlsian notion of welfare. In particular, we measure the "quality" of a graph via its *minimum welfare* (henceforth abbreviated *min-welfare*), the minimum payoff of any node in the graph. A socially optimal graph is one that maximizes this quantity. This notion of welfare is convenient for our analysis, and well-founded in principles of fairness (knows as distributive justice [28]). Min-welfare satisfies criteria of anonymity and the weak Pareto principle.

One could study strategic network formation by defining a non-cooperative game whose outcomes are graphs. However, in such non-cooperative models, small details of the specification of the game will determine the precise structure of equilibrium networks. To avoid this problem and capture the notion that it takes two nodes to agree on the formation of a link, but any node can unilaterally withdraw from its links, network theorists, following [19, 23], identified *stable networks* as a class of networks that we could expect to be equilibrium outcomes of any interesting network formation game.

We say that a graph is *stable* if (i) no node can strictly increase its payoff by deleting all its incident links (hence removing itself from the network), and (ii) there is no pair of nodes (i, j) such that (i, j) is not an edge of G, but both i and j would have higher payoffs, with at least one of them strictly higher, if (i, j) were added to G.

Our definition of stability is similar to the notion of *pairwise Nash stability* [22], which modifies (i) to allow a node to drop any subset of its incident links. In the settings that motivate our model, we can view our definition of stability as capturing strategic settings in which nodes have the ability to withdraw from the system, but not to selectively break bilateral agreements with certain nodes while keeping others operational. In the context of the paper's technical content, we can think about the relation between our notion of stability and the pairwise Nash notion as follows. Any pairwise Nash stable graph is also stable under our definition, and because of this all the results in the paper carry over to the case of pairwise Nash stability except for the result of Section 6 — proving the existence of a stable network. In particular, the large gap between the achievable min-welfare in optimal and stable networks continues to hold as is if we use pairwise Nash stability as our underlying notion. The networks we construct in Section 6, on the other hand, are only shown to be stable under our definition, and it is an open question to show that pairwise Nash stable networks exist.

When we consider the structures of socially optimal and stable graphs, much of the interesting behavior emerges in a natural range of the parameters a, b, p, and q motivated by the following considerations. Suppose we have just two nodes i and j, and suppose that i is deciding whether to link to j. If i forms the link, it receives a benefit of a but there is a probability of pq that j will fail and that this failure will spread to i. We want i to be willing to form the link to j under these conditions, and so we assume $a > bqp$. Otherwise no links will form. On the other hand, suppose that i knew that j were going to fail, so that the only thing protecting i from failure is the transmission probability p. Under these conditions we do not want i to form the link to j, so we assume $a < bp$. Otherwise nodes will want to form as many links as possible. Analogously, suppose that i knew that any failure at j would definitely spread to i, so that the only thing protecting i from failure is the chance $1 - q$ that j does not fail. Under these conditions we also do not want i to form the link to j, so we assume $a < bq$.

In our analysis, we focus on the range of parameters in which these bounds hold by arbitrarily large constant factors. That is, we consider the case in which p and q are small, the quantity a exceeds bqp by a large amount, and in turn that $\min(bp, bq)$ exceeds a by a large amount. Our assumption is that for a small constant $\delta > 0$, we have

$$\delta^{-1}bqp < a < \delta \min(bp, bq).$$

For ease of future reference, we call this *Assumption* $\mathcal{P}(\delta)$ and refer to δ as the key *separation parameter* in our model. Finally, we consider the case in which the number of nodes n is arbitrarily large compared to these other quantities (and/or their reciprocals).

In what follows, Sections 3 and 5 give general upper bounds on the min-welfare for optimal and stable graphs respectively. Sections 4 and 6 give constructions of networks establishing the existence of certain properties: large min-welfare for optimal graphs in the former section, and the existence of stable graphs in the latter section. For these constructions in Sections 4 and 6, we employ the additional assumption that $p = q$ (whereas for Sections 3 and 5 we allow arbitrary p and q). One consequence of combining the construction in Section 4 with the upper bound in Section 3 is that when $p = q$, the ratio between the welfare achievable in optimal and stable graphs diverges as δ goes to zero.

3 An Upper Bound on the Optimal Min-Welfare

We begin by establishing an upper bound on the min-welfare of any graph. Min-welfare is directly related to degree and we will see that critical graphs are those in which the average degree is close to $1/p$, and thus the average direct benefit from links is close to a/p. Suppose that the min-welfare in a graph G exceeds $\frac{(1+\varepsilon)a}{p}$. Then all node degrees must be at least $\frac{(1+\varepsilon)}{p}$. If there is a node that can reach many others along active-edge paths with reasonable probability, then this node experiences a large probability of failure, and hence has a sharply reduced payoff, which will ultimately contradict our assumption that G has large min-welfare.

Now, how do we show that some node has a reasonably high chance of reaching many others on active-edge paths? There is a connection to the basic random graph model $\mathcal{G}(n,p)$, in which an edge is inserted between each pair among n nodes independently with probability p. We can think of $\mathcal{G}(n,p)$ equivalently as the model in which one starts with an n-node clique and declares each edge to be active independently with probability p, and then considers the active-edge subgraph. The challenge in our case is that our graphs G are not necessarily cliques, or even close to being cliques, and relatively little is known about adapting results from $\mathcal{G}(n,p)$ to the case of arbitrary underlying base graphs [3]. Fortunately, however, we are able to prove a result that is strong enough for our purposes, adapting techniques for analyzing connected components in $\mathcal{G}(n,p)$ to the setting of active-edge subgraphs of arbitrary underlying graphs.

We begin with adapting random graph techniques to apply to a random subgraph of an arbitrary graph.

(3.1) *For all $\varepsilon > 0$ there exist constants $\alpha, \beta > 0$ such that the following holds. Let H be a graph in which each node has degree at least $r \geq \frac{1+\varepsilon}{p}$. Construct a random subgraph of H by declaring each edge to be "active" with probability p. Then for every node $i \in V$, the number of nodes reachable from i on active-edge paths is at least αr with probability at least β.*

Proof. Let i be any node in H. We analyze the process of breadth-first search over the active edges, based on Karp's analysis of random subgraphs of the bidirected complete graph [26] and Alon and Spencer's analysis of infinite branching processes [4]. We first take all the nodes (if any) that i can reach via active edges and put them in a queue. We then repeatedly delete a node j from the queue and add to the queue all the nodes (if any) that j can reach via active edges, other than the ones already "discovered" (added to the queue) in previous iterations. Notice, crucially, that the outcome of the random active/blocked decision for each edge (j, j') is only examined once in this process, when one of nodes j or j' first comes to the front of the queue. Thus, we can assume that the active/blocked status of (j, j') is first determined at that moment.

For a small constant $\alpha > 0$, we say that this process *succeeds* if at least αr nodes are added to the queue before the queue ever becomes empty. If the process succeeds with probability at least β, for a constant $\beta > 0$, then our result follows.

Let Q_t be the number of nodes in the queue at the end of iteration t, where we define $Q_0 = 1$ to indicate that i starts in the queue. We have

$$Q_t = Q_{t-1} - 1 + X_t,$$

where the "-1" is because we delete a node j_t from the queue in iteration t (with $j_1 = i$), and X_t is a random variable equal to the number of not-yet-discovered nodes that j_t can reach via active edges. (This is where it is useful to assume that the active/blocked status of edges from j_t to not-yet-discovered nodes is only determined when j_t reaches the front of the queue.) Unrolling this recurrence, we have

$$Q_t = \left(\sum_{u=1}^{t} X_u \right) - t.$$

We are interested in showing that with probability at least β, we have $Q_t > 0$ for all t from 1 until at least αr. If this is true, the search for nodes using active-edge paths continues successfully until at least αr nodes have been discovered (added to the queue), as required.

The expectation of X_t, prior to the point at which at least αr nodes have been discovered, can be determined as follows. The node j_t has degree at least r in H, and at most αr nodes have been discovered by the process thus far, so there are at least $(1 - \alpha)r$ edges emanating from j_t leading to not-yet-discovered nodes. We choose α small enough that $(1 - \alpha)r \geq \dfrac{1 + \varepsilon/2}{p}$; since each of these edges is active with probability p, we have $\mathbf{E}[X_t] \geq 1 + \varepsilon/2$. Thus, until αr nodes have been discovered, we can think of the queue length as a random walk on the integers with positive drift; as a result, there is a positive probability that the walk never returns to 0, which is the result we want.

We can briefly verify this in more detail for our particular case as follows. Let $S_t = \sum_{u=1}^{t} X_u$; by the Chernoff Bound, we have

$$\Pr[S_t \leq t] < \Pr[S_t \leq (1-\varepsilon/4)\mathbf{E}[S_t]] < e^{-\frac{1}{2}\frac{\varepsilon^2}{16}t}.$$

Now, the sum $\sum_{t=1}^{\infty} e^{-\frac{1}{2}\frac{\varepsilon^2}{16}t}$ converges; let us choose t_0 large enough that

$$\sum_{t=t_0}^{\infty} e^{-\frac{1}{2}\frac{\varepsilon^2}{16}t} < 1.$$

For p sufficiently small, there is a positive probability that X_1, the number of nodes i can reach directly via active edges, is at least t_0. It then follows that $S_t > t$ for all $t < t_0$. Finally, for all t we have $\Pr[S_t \leq t \mid X_1 \geq t_0] \leq \Pr[S_t \leq t] < e^{-\frac{1}{2}\frac{\varepsilon^2}{16}t}$ (as $X_1 \geq t_0$ is negatively correlated with $S_t \leq t$); summing over t we obtain

$$\sum_{t=0}^{\infty} \Pr[S_t \leq t \mid X_1 \geq t_0] < 1. \qquad \blacksquare$$

Next, we simply want to argue that if a node can reach many other nodes via active-edge paths with reasonably large probability, then it has a large probability of failing and hence a negative payoff. To do this, we first state a simple lemma about the union of many independent events, and then we use this to draw the resulting conclusion for a node's payoff.

(3.2) *Consider a collection of independent events $\mathcal{E}_1, \ldots, \mathcal{E}_n$, each of probability $p > 0$. Then the probability of their union is at least* $\min(\frac{1}{3}, \frac{2}{3}np)$.

Proof. If $p \geq \frac{1}{3}$ then the result follows immediately. If $p < \frac{1}{3}$ and $np \leq \frac{2}{3}$, then we have

$$\begin{aligned}
\Pr\left[\bigcup_{j=1}^{n} \mathcal{E}_j\right] &\geq \sum_{j=1}^{n} \Pr[\mathcal{E}_j] - \sum_{j,j'} \Pr[\mathcal{E}_j \cap \mathcal{E}_{j'}] \\
&= np - \binom{n}{2} p^2 \\
&\geq np - \frac{1}{2}(np)^2 \\
&= np(1 - \frac{1}{2}np) \\
&\geq \frac{2}{3}np.
\end{aligned}$$

Finally, if $p < \frac{1}{3}$ and $np < \frac{2}{3}$, we can choose a subset S of $k \leq n$ of the events such that $\frac{2}{3} < kp \leq 1$. We have

$$\begin{aligned} \Pr\left[\bigcup_{j\in S}\mathcal{E}_j\right] &\geq \sum_{j\in S}\Pr\left[\mathcal{E}_j\right] - \sum_{j,j'\in S}\Pr\left[\mathcal{E}_j \cap \mathcal{E}_{j'}\right] \\ &= kp - \binom{k}{2}p^2 \\ &\geq kp - \frac{1}{2}(kp)^2 \\ &= kp(1-\frac{1}{2}kp) \\ &\geq \frac{2}{3}\cdot\frac{1}{2} = \frac{1}{3}. \end{aligned}$$

■

Now, for a node i, let the set of nodes it can reach on active-edge paths in G be called its *active component*, and let $r_i(G)$ be a random variable denoting the size of i's active component. We are ready to prove that if a node's active component $r_i(G)$ is large with sufficiently high probability, than the node must have negative payoff.

(3.3) *For all $\gamma_0, \gamma_1 > 0$ there exist $\alpha, \delta > 0$ such that when $p, q \leq \alpha$ and Assumption $\mathcal{P}(\delta)$ holds, we have the following. If G is a graph with a node i for which $r_i(G) \geq \dfrac{\gamma_1}{p}$ with probability at least γ_0, then the payoff of node i satisfies $\pi_i(G) < -bq$. (We note that the right-hand side is the payoff i would receive if it had no links).*

Proof. If i can reach at least $\gamma_1 p^{-1}$ nodes on active-edge paths, then by (3.2), the probability that it fails is at least $\min(\frac{1}{3}, \frac{2}{3}\gamma_1 p^{-1} q)$. Removing the conditioning on this event, the probability it fails is at least $\phi_i \geq \min(\frac{1}{3}\gamma_0, \frac{2}{3}\gamma_0\gamma_1 p^{-1} q)$. We also have $d_i \leq \Delta = c^* p^{-1}$.

If $\phi_i \geq \frac{1}{3}\gamma_0$, then

$$\begin{aligned} \pi_i &\leq ad_i - b\phi_i \leq ac^* p^{-1} - \frac{1}{3}b\gamma_0 \\ &\leq \delta bc^* - \frac{1}{3}b\gamma_0 = b(\delta c^* - \frac{1}{3}\gamma_0) \end{aligned}$$

where the last line is less than $-bq$ for δ sufficiently small and $q < \frac{1}{3}\gamma_0$.

If $\phi_i \geq \frac{2}{3}\gamma_0\gamma_1 p^{-1}q$, then defining $\gamma_2 = \frac{2}{3}\gamma_0\gamma_1$, we have

$$\begin{aligned}\pi_i &\leq ad_i - b\phi_i \\ &\leq ac^* p^{-1} - \gamma_2 bqp^{-1} \\ &= p^{-1}c^*(a - \gamma_2 c^{*-1}bq). \\ &< p^{-1}c^*(\delta bq - \gamma_2 c^{*-1}bq) \\ &= p^{-1}c^*(\delta - \gamma_2 c^{*-1})bq.\end{aligned}$$

Now if $\delta < \frac{1}{2}\gamma_2 c^{*-1}$, then $\delta - \gamma_2 c^{*-1} < -\frac{1}{2}\gamma_2 c^{*-1}$, and so we have

$$\begin{aligned}\pi_i &< p^{-1}c^*(\delta - \gamma_2 c^{*-1})bq \\ &< p^{-1}c^*(-\frac{1}{2}\gamma_2 c^{*-1})bq \\ &= -\frac{1}{2}\gamma_2 p^{-1}bq.\end{aligned}$$

Finally, if $p < \frac{1}{2}\gamma_2$, then we have

$$\pi_i < -\frac{1}{2}\gamma_2 p^{-1}bq < -bq$$

as required. ■

Finally, combining (3.1) with (3.3), we get an immediate consequence for the payoffs when all nodes have large degrees.

(3.4) *For all $\varepsilon > 0$, there exist $\alpha, \delta > 0$ such that when $p, q \leq \alpha$ and Assumption $\mathcal{P}(\delta)$ holds, we have the following. If each node has degree at least $\frac{(1+\varepsilon)}{p}$, then for each node i we have $\pi_i < -bq$.*

Proof. For each node $i \in V$, (3.1) implies that we have $r_i(G) \geq \frac{\gamma_1}{p}$ with probability at least γ_0, It then follows from (3.3) that $\pi_i < -bq$. ■

The upper bound on min-welfare follows directly from the above claims.

(3.5) *For all $\varepsilon > 0$, there exist $\alpha, \delta > 0$ such that when $p, q \leq \alpha$ and Assumption $\mathcal{P}(\delta)$ holds, no graph can have min-welfare greater than $\frac{(1+\varepsilon)a}{p}$.*

Proof. Choose $\alpha, \delta > 0$ as in (3.4), and suppose by way of contradiction that there is a graph with min-welfare greater than $\frac{(1+\varepsilon)a}{p}$. It follows that every node i has degree greater than $\frac{1+\varepsilon}{p}$. But then by (3.4) we have $\pi_i < -bq$, contradicting the assumption that the min-welfare is greater than $\frac{(1+\varepsilon)a}{p}$. ■

4 Super-Critical Payoffs and Anonymous Market Structures

We consider networks formed under two rules: Non-anonymous market formation, wherein individuals are assigned links to particular other individuals, and anonymous market formation, wherein the market designer chooses the degree of each market participant, but the requisite number of links are formed at random. In this section, we will consider how the optimal networks differ under the two regimes.

We now show that the upper bound in Section 3 can essentially be achieved, in an asymptotic sense, and also consider some of the structural implications of this fact.

To begin with, it is instructive to think about the analysis in Section 3 in terms of the random graph $\mathcal{G}(k, r)$.[1] One of the central facts about $\mathcal{G}(k, r)$ is that in a small window around probability $r = 1/k$, the expected size of the largest connected component jumps from a constant value to a constant fraction of k. This is the basic phase transition for $\mathcal{G}(k, r)$, and (3.1) in Section 3 is a reflection of this phase transition for an arbitrary underlying graph.

In order for a graph to achieve payoffs of the form $\frac{(1+\varepsilon)a}{p}$ for some $\varepsilon > 0$, it must lie on the side of the phase transition where the active components are likely to be large, proportional to $1/p$. For this to be possible, it must cross the phase transition by little enough that these large components do not eliminate the payoff of the nodes. We say that a graph achieves *super-critical payoffs* if the payoff is at least $\frac{(1+\varepsilon)a}{p}$ for some $\varepsilon > 0$, and construct a family of graphs built from disjoint

[1] Since n and p are basic parameters in our model, we adopt the different variable names k and r in discussing $\mathcal{G}(k, r)$. Also, in keeping with standard terminology, we will often refer informally to $\mathcal{G}(k, r)$ as "a random graph," as though it is a single graph rather than a distribution over graphs.

cliques that achieves super-critical payoffs.

Some Basic Facts about $\mathcal{G}(k,r)$. We begin by carefully stating some quantitative results about the phase transition in $\mathcal{G}(k,r)$ in a form that will be useful for the analysis.

(4.1) *Let C_i denote the component containing node i in $\mathcal{G}(k,r)$. If we fix some other node j and look at the event $j \in C_i$, then we have*

$$\Pr\left[j \in C_i\right] = \frac{1}{k} \cdot \mathbf{E}\left[|C_i|\right] - \frac{1}{k}.$$

Proof.

$$\begin{aligned}
\Pr\left[j \in C_i\right] &= \sum_{s=1}^{k} \Pr\left[|C_i| = s\right] \cdot \Pr\left[j \in C_i \mid |C_i| = s\right] \\
&= \sum_{s=1}^{k} \frac{s-1}{k} \Pr\left[|C_i| = s\right] \\
&= \frac{1}{k} \sum_{s=1}^{k} (s-1) \Pr\left[|C_i| = s\right] \\
&= \frac{1}{k} \left(\sum_{s=1}^{k} s \Pr\left[|C_i| = s\right] - \sum_{s=1}^{k} \Pr\left[|C_i| = s\right]\right) \\
&= \frac{1}{k} \cdot \mathbf{E}\left[|C_i|\right] - \frac{1}{k}
\end{aligned}$$

■

Thus, the probability a node belongs to i's component is essentially proportional to the expected size of i's component.

The following pair of standard results describe the well-known contrasting behavior of component sizes on opposite sides of $r = 1/k$ (see Chapter 5 of [25]): a random graph with expected degree less than 2 consists of many small components, while above expected degree 2 a large fraction of the nodes all belong to a single giant component.

(4.2) *Fix $x < 1$, and consider the component of a given node i in $\mathcal{G}(k, r)$, where $kr = x$. Then for k sufficiently large, we have the following:*

(i) The probability that i's component exceeds size c decreases exponentially in c.

(ii) Consequently, the expected size of i's component is bounded by a constant $c = c(x)$, independent of k, and the maximum size of any component in the graph is $O(\log k)$ with high probability.

(4.3) *There is an increasing function $\theta : [1, \infty] \to [0, 1]$ that is continuously differentiable on $(1, \infty)$ and continuously differentiable from the right at $x = 1$, with $\theta(1) = 0$ and $\theta'(1)$ a positive real number, such that the following holds. Fix $x > 1$ and $\varepsilon > 0$, and consider $\mathcal{G}(k, r)$, where $kr = x$. Then for $k = k(x, \varepsilon)$ sufficiently large, we have the following:*

(i) With probability $1 - \exp(-k)$, there is a component of size between $(1 - \varepsilon)\theta(x)k$ and $(1 + \varepsilon)\theta(x)k$.

(ii) Conditioned on not belonging to the giant component in (i), the probability that a node i belongs to a component of size greater than c decreases exponentially in c.

(iii) Consequently, the expected size of i's component is between $(1-\varepsilon)^2(\theta(x))^2k$ and $(1 + \varepsilon)^2(\theta(x))^2k + c$ for a constant $c = c(x)$.

Point (iii) follows from (i) and (ii) by considering that with probability $(1\pm\varepsilon)\theta(x)$, node i belongs to a component of size $(1 \pm \varepsilon)\theta(x)k$, and with the remaining probability i belongs to a component of expected size at most c.

A Family of Graphs with Super-Critical Payoffs. For parameters k and s, let $F_s(k)$ denote the disjoint union of s cliques of size k.

(4.4) *When $p = q$, the graph $F_s\left(\frac{1+\gamma}{p}\right)$, for arbitrary $s \geq 1$ and a small constant $\gamma > 0$, achieves super-critical node payoffs.*

Proof. A useful feature of the special case $p = q$ is that we can represent the spread of failures in $F_s(k)$ in the following equivalent way. We imagine a single "failure node" i^* associated with each clique, and attached to each real node in the clique, resulting in a clique on $k+1$ nodes. There is a transmission probability p on the edges from i^* to each node in its clique, as there is on all other edges. In this view, a node i fails if it is in the same active-edge component as i^*; in other words, the probability i fails is the probability it belongs to the same component as a given fixed node i^* in $\mathcal{G}(k+1, r)$. By (4.1) we know this probability is

$$\frac{1}{k+1} \cdot \mathbf{E}\left[|C_i|\right] - \frac{1}{k+1},$$

where C_i denotes the active-edge component of i.

With $p = q$, we define σ to be the ratio $a/bp = a/bq$; by assumption $\mathcal{P}(\delta)$, we have $\sigma < \delta$, and we assume as usual that δ and p are sufficiently small. We let the number of nodes k in each clique be $(1+\gamma)/p$ for a small value $\gamma > 0$ that we determine below.

First, (4.3)(iii) implies that the probability ϕ_i that i fails satisfies

$$(1-\varepsilon_0)(\theta(1+\gamma))^2 \le \phi_i \le (1+\varepsilon_0)(\theta(1+\gamma))^2$$

for a constant ε_0 that goes to 0 with p. Thus, the payoff to a node i is

$$\begin{aligned}
\pi_i &\ge \frac{a(1+\gamma)}{p} - \left(b + \frac{a(1+\gamma)}{p}\right)(1+\varepsilon_0)(\theta(1+\gamma))^2 \\
&= \frac{a(1+\gamma)}{p} - b(1+\sigma(1+\gamma))(1+\varepsilon_0)(\theta(1+\gamma))^2 \\
&\ge \frac{a(1+\gamma)}{p} - 2b(\theta(1+\gamma))^2 \\
&= \frac{\sigma bp(1+\gamma)}{p} - 2b(\theta(1+\gamma))^2 \\
&= b\left[\sigma(1+\gamma) - 2(\theta(1+\gamma))^2\right].
\end{aligned}$$

Now, let

$$h_0(x) = \sigma x - 2(\theta(x))^2,$$

so that

$$\pi_i \ge b h_0(1+\gamma).$$

We have

$$h_0'(x) = \sigma - 4\theta(x)\theta'(x).$$

Since $\theta(1) = 0$ and $\theta'(1)$ is a positive real number, we have $h_0'(1) = \sigma$, and hence the function $h_0(x)$ is strictly increasing over the interval $x \in [1, w_0]$ for a constant w_0 depending on σ. Since $h_0(1) = \sigma$, we have $h_0(w_0) = \sigma(1 + \sigma_0)$ for a constant $\sigma_0 > 0$ depending on σ.

Returning to the lower bound on π_i, we choose $\gamma = w_0 - 1$, and so

$$\pi_i \geq bh_0(w_0) = \frac{a}{\sigma p} \cdot \sigma(1 + \sigma_0) = \frac{a(1 + \sigma_0)}{p}.$$

Consequently, the payoff to each node exceeds $\frac{a}{p}$ by a multiplicative factor greater than 1 that depends on σ. ■

Comparison to an Anonymous Structure. The construction above achieves super-critical payoffs by allowing nodes to cluster into communities of an appropriate size, and thus to insulate themselves from failures originating in other communities. Drawing on a market motivation, it is interesting to ask whether super-critical payoffs can be achieved through structures that are based instead on *anonymous* interaction, where nodes can specify the number of partners they want to connect to, but have no control over who these partners are — the partners are chosen uniformly at random from the population. As we now show, in fact, anonymous interaction structures are not able to yield super-critical payoffs.

To define these anonymous structures precisely, we use the *configuration model* for random graphs [12, 13, 31]. Each of the n nodes is assigned k "half-edges"; these half-edges are then matched up uniformly at random into pairs, with each matching pair of half-edges forming an edge in the resulting random graph. Note that the pairing may cause two edges to go between the same pair of nodes, or for a node to form an edge that loops to itself; we remove these parallel edges and self-loops to obtain the final graph. Failures then propagate in this graph according to our model, spreading from root failures along active-edge paths.

With high probability, the local neighborhood of a node in this random graph will have a particularly simple structure, as follows. For node i, define $B(i, \ell)$ to be the ball of radius ℓ centered at i, i.e. the induced subgraph of G on the set of all nodes reachable from i in ℓ or fewer hops. For fixed integers k, ℓ and any node i, the probability that $B(i, \ell)$ is a tree of depth ℓ and degree k (i.e. one whose internal nodes all have degree k and whose leaves are all at distance ℓ from the root) tends to 1 as $n \to \infty$.

For our analysis, we will therefore connect the propagation of failures in the configuration model to a related, simpler model based on an infinite k-regular tree.

In particular, let $\mathcal{B}(k,r)$ denote the distribution over trees obtained by starting with an infinite k-regular tree and including each edge in the random tree with probability r. We can think of the resulting tree as the outcome of a branching process with branching factor rk, and use this to get a pair of results analogous to (4.2) and (4.3) (see Chapter 2 of [16]).

(4.5) *Let $x < 1$, and consider a tree generated from $\mathcal{B}(k,r)$ where $kr = x$.*

(i) The probability that the tree's size exceeds size c decreases exponentially in c.

(ii) The expected size of the tree is bounded by a constant $c = c(x)$.

(4.6) *There is an increasing function $\tau : [1,\infty] \to [0,1]$ that is continuously differentiable on $(1,\infty)$ and continuously differentiable from the right at $x = 1$, with $\tau(1) = 0$ and $\tau'(1)$ a positive real number, such that the following holds. Consider a tree generated from $\mathcal{B}(k,r)$, and let $\psi_r(k)$ be the probability that it has an infinite node set.*

(i) If $kr > 1$, then $\psi_r(k) > \tau(kr)$.

(ii) For all integers $c_0, c_1 > 1$ and $k \geq c_0c_1/r$, we have

$$\psi_r(k) \geq 1 - (1 - \psi_r\,(c_0/r))^{c_1} > 1 - (1 - \tau(c_0))^{c_1}.$$

(iii) Conditioned on not having an infinite node set, the probability that the tree's size exceeds c decreases exponentially in c. Its conditional expected size is thus bounded by a constant $c = c(x)$.

Proof. Part (iii) of the claim is a standard result; parts (i) and (ii) are formulated in ways that are adapted to our present purposes, and we give proofs of them here.

First we prove (i). The probability that the tree is infinite is the unique solution to $z = 1 - (1 - pz)^k$ in the interval $(0,1)$. Define τ to be the unique solution to $\tau = 1 - e^{-x\tau}$ in the interval $(0,1)$. Writing $f_0(v) = (1 - pv)^k$ and $f_1(v) = e^{-xv}$, we have

$$f_0(v) = (1 - pv)^k = (1 - pv)^{x/p} < e^{-xv} = f_1(v).$$

Thus, the curve $y = 1 - f_0(v)$ lies above the curve $y = 1 - f_1(v)$ on the interval $(0,1)$, and so $y = 1 - f_0(v)$ intersects the line $y = v$ to the right of where

$y = 1 - f_1(v)$ intersects it. It follows that $z > \tau$, and hence we can take $\tau = \tau(x)$ as our function.

To prove (ii), consider $k' = \frac{c_0 c_1}{r}$ subtrees of the root in the complete k-ary tree (before edges are randomly included), and group them into c_1 blocks of $\frac{c_0}{r}$ subtrees each. For any block, if we consider just the root and the subtrees in a single block, the probability that the resulting random tree is infinite is at least $\psi_r(c_0/r)$ (since the root has this degree in the restricted tree, and the nodes in the subtrees have degree $k \geq c_0/r$). The tree is infinite if it is infinite in any of the blocks, and so the probability it is infinite is at least

$$1 - (1 - \psi_r(c_0/r))^{c_1} > 1 - (1 - \tau(c_0))^{c_1}$$

where the latter inequality follows directly from (i). ■

We now want to show an upper bound on the node payoffs in the anonymous structure that contrasts with the payoffs achievable in (4.4).

(4.7) *When each node forms k links in the anonymous structure, for any $k = \frac{1+\beta}{p}$, the node payoffs can be at most a/p as $n \to \infty$.*

Proof. Clearly this is true for $\beta \leq 0$, so we consider the case of an arbitrary $\beta > 0$.

When the random graph G is sampled using the configuration model, for any node i the probability that the ball $B(i, \ell)$ is a tree of degree k and depth ℓ is $1 - o(1)$ as $n \to \infty$. Applying 4.6(i), the probability that i belongs to a active path of length ℓ is at least $\tau(1+\beta) - o(1)$; for n sufficiently large, this probability is at least $\tau(1+\beta/2)$. In the event that i belongs to a active path of length ℓ, it fails with probability at least $1 - (1-q)^\ell$. By taking ℓ large enough, we may assume that $\tau(1+\beta/2)(1-(1-q)^\ell) \geq \tau(1+\beta/3)$ and thus node i fails with probability at least $\tau(1+\beta/3)$.

Thus, if n is sufficiently large we have

$$\begin{aligned}
\pi_i &\leq \left(\frac{a(1+\beta)}{p}\right)\left(1 - \tau\left(1 + \frac{\beta}{3}\right)\right) - b\tau\left(1 + \frac{\beta}{3}\right) \\
&= b\sigma(1+\beta)\left(1 - \tau\left(1 + \frac{\beta}{3}\right)\right) - b\tau\left(1 + \frac{\beta}{3}\right).
\end{aligned}$$

Let

$$h_1(x) = \sigma(1+3x)\left(1 - \tau\left(1+x\right)\right) - \tau\left(1+x\right),$$

so that $\pi_i \leq bh_1(\beta/3)$. By (4.6)(ii), we know that for $y \geq 4$, we have

$$\tau(y) \geq 1 - (1 - \tau(2))^{\lfloor y/2 \rfloor} \geq 1 - (1 - \tau(2))^{y/4}.$$

We can thus choose $w_1 \geq 4$ such that

$$\tau(y) \geq 1 - \frac{1}{1+y}$$

for all $y \geq w_1$. If $1 + x \geq w_1$, we have

$$h_1(x) \leq \sigma(1+3x)\left(\frac{1}{2+x}\right) - \frac{1+x}{2+x} < 0,$$

provided $\sigma < \frac{1}{3}$. Now, if $\sigma \leq \frac{1}{3}\sup_{y \in [1,w_1]} \tau'(y)$, then we have the following for all $x \in [0, w_1 - 1]$:

$$\begin{aligned} h_1'(x) &= 3\sigma(1 - \tau(1+x)) - (\sigma + 3\sigma x + 1)\tau'(1+x) \\ &\leq 3\sigma - \tau'(1+x) \leq 0. \end{aligned}$$

Thus, for all $x \in [0, w_1 - 1]$, we have $h_1(x) \leq h_1(0) = \sigma$. Since we also have $h_1(x) < 0$ for $x \geq w_1 - 1$, it follows that $h_1(x) \leq \sigma$ for all $x \geq 0$.

Thus, for any $\beta > 0$, we have $\pi_i \leq b\sigma = \frac{a}{p}$ when each node forms $k = \frac{1+\beta}{p}$ links. Since $\pi_i \leq \frac{a}{p}$ when nodes form at most $k \leq \frac{1}{p}$ links, it follows that for any constant c, if nodes form $\frac{c}{p}$ links then $\pi_i \leq \frac{a}{p}$ provided n is sufficiently large as a function of c. ■

Clustered vs. Anonymous Markets. It is instructive to consider why a union of disjoint cliques was able to achieve qualitatively higher payoffs than an anonymous interaction pattern. In particular, the nodes in the cliques we constructed are linking at a degree beyond the phase transition point, whereas attempting to do this in the anonymous structure has negative effects on the payoff.

A quantitative way to think about the contrast is to observe that in the union of cliques, the failure probability of a node i was approximately controlled by a conjunction of two events: i belonging to the giant component of the clique, and the "failure node" i^* also belonging to the giant component of the clique. As a result, the failure probability involves a term of the form $(\theta(x))^2$, and this has a derivative of 0 at $x = 1$ — hence, it is safe to increase x a bit past 1 without blowing up the failure probability. On the other hand, in the anonymous structure, once i belongs to the giant component, it fails with overwhelming probability; thus, i's failure probability involves a term of the form $\tau(x)$, which has a strictly positive derivative at $x = 1$, and this makes it unprofitable to increase x even arbitrarily little past 1. This is the fundamental difference between the behavior of the two kinds of structures in the region just past the phase transition.

5 Upper Bound on the Min-Welfare of Any Stable Network

We now show that any stable graph must have small min-welfare. (We defer the proof that stable graphs exist to the next section.) To prove an upper bound on the min-welfare, we proceed as follows. Recall that we assumed that Δ is an upper bound on the number of links any one node is able to form. So nodes of degree Δ cannot form further edges. We first show, in (5.1), that if two nodes i and j both have degree less then the maximum degree Δ and are not connected by an edge, then at least one of them must have a large failure probability — this is what dissuades the other from forming the link.

It follows that in a stable network, all low-degree nodes of low failure probability must form a clique, since any unlinked pair of them would have an incentive to connect. If the number of nodes n is sufficiently large, we can then find a node i that is far from this clique. Hence node i, and every node within a large number of steps of i, must have a large degree; we can thus apply an analogue of (3.1) to show i has a large failure probability, and this will conclude the proof.

(5.1) *Suppose Assumption $\mathcal{P}(\delta)$ holds. Let G be a stable graph, and let i and j be two nodes of G such that (i, j) is not an edge of G, and the degrees of i and j are each strictly less than Δ. Then we have* $\max(\phi_i, \phi_j) \geq \dfrac{(1-\delta)a}{(1+\delta c^*)bp}$.

Proof. Since the degrees of i and j are each strictly less than Δ, at least one of i or j does not have a strictly higher payoff if the edge (i, j) is included; let us assume it is node i. Thus, if G' denotes the graph G with the edge (i, j) included, then we have $\pi_i(G') \leq \pi_i(G)$.

We imagine evaluating failure in G' by first making all random root failure decisions and all random active/blocked decisions in G, then determining which additional nodes fail, and finally deciding whether the edge (i, j) is active or blocked and determining further failures. Let $\Phi_i(G)$ be the event that i fails in G before (i, j) is examined, and let $\mathcal{F}_{ij}(G)$ be the event that (i, j) is active and j fails in G. Then $\Phi_i(G') = \Phi_i(G) \cup \mathcal{F}_{ij}(G)$, so

$$\Pr\left[\Phi(G')\right] \leq \Pr\left[\Phi_i(G)\right] + \Pr\left[\mathcal{F}_{ij}(G)\right].$$

Since $\Pr\left[\mathcal{F}_{ij}(G)\right] = p\phi_j$, we have

$$\phi_i(G') - \phi_i(G) \leq p\phi_j(G).$$

Now,

$$\pi_i(G') = a(d_i + 1) - (ad_i + a + b)\phi_i(G'),$$

so the fact that $\pi_i(G') \leq \pi_i(G)$ implies that

$$a(d_i + 1) - (ad_i + a + b)\phi_i(G') \quad \leq \quad ad_i - (ad_i + b)\phi_i(G)$$

and hence

$$\begin{aligned} a &\leq (ad_i + a + b)(\phi_i(G') - \phi_i(G)) + a\phi_i(G) \\ &\leq (ad_i + a + b)p\phi_j(G) + a\phi_i(G) \\ &\leq (1 + \delta c^*)bp\phi_j(G) + \delta bp\phi_i(G), \end{aligned}$$

where the last line follows from the fact that $a < \delta bp$ and $d_i + 1 \leq c^* p^{-1}$.

Now, if $\phi_i(G) \geq \dfrac{a}{bp}$, we are done. Otherwise, we have

$$a \leq (1 + \delta c^*)bp\phi_j(G) + \delta a,$$

so

$$(1 - \delta)a \leq (1 + \delta c^*)bp\phi_j(G),$$

and hence

$$\phi_j(G) \geq \frac{(1 - \delta)a}{(1 + \delta c^*)bp}. \qquad \blacksquare$$

Following our informal plan above, we note that a stable graph might have some low-degree nodes, so we require the following direct adaptation of (3.1), which applies to nodes that are far from all low-degree nodes.

(5.2) *For all $\varepsilon > 0$ there exist constants $\alpha, \beta > 0$ such that the following holds. Let H be a graph, and let A be the set of nodes of degree less than $\frac{1+\varepsilon}{p}$. Let i be a node of distance greater than $\frac{1}{p}$ from A. Construct a random subgraph of H by declaring each edge to be "active" with probability p. Then the number of nodes reachable from i on active-edge paths is at least αp^{-1} with probability at least β.*

Proof. Consider the node-discovery process described in the proof of (3.1), starting from the node i, and recall that we declare it to *succeed* if it adds at least $\frac{\alpha}{p}$ nodes to the queue before it ever becomes empty, for the small constant $\alpha < 1$ used there. The event that the process succeeds depends only on the active/blocked decisions for nodes within distance $\frac{\alpha}{p}$ of i, and all such nodes have degrees at least $\frac{1+\varepsilon}{p}$; hence, for this whole time we can apply the argument used in (3.1). ■

Finally, we conclude the proof strategy outlined at the beginning of the section, resulting in our upper bound.

(5.3) *Let $n > \Delta^{\Delta+2}$. For all $\varepsilon > 0$ there exist $\alpha, \delta > 0$ such that when $p, q \leq \alpha$ and Assumption $\mathcal{P}(\delta)$ holds, no stable graph can have min-welfare greater than $\frac{\varepsilon a}{p}$.*

Proof. Suppose by way of contradiction that $G = (V, E)$ is a stable graph in which $\pi_i \geq \frac{\varepsilon a}{p}$ for all $i \in V$.

Let $A \subseteq V$ denote the set of all nodes i of G for which $d_i < \Delta$ and $\phi_i < \frac{(1-\delta)a}{(1+\delta c^*)bp}$. Since any node in A is able to form an additional edge, (5.1) implies that there must be an edge between each pair of nodes in A — in other words, A induces a clique in G.

Let $B \subseteq V$ denote the set of all nodes in G of degree equal to Δ. For any $i \in V - (A \cup B)$, we have $\phi_i \geq \frac{(1-\delta)a}{(1+\delta c^*)bp}$. Since $\pi_i \geq \frac{\varepsilon a}{p}$ by assumption, we have

$$\frac{\varepsilon a}{p} \leq ad_i - b\phi_i \leq ad_i - \frac{(1-\delta)a}{(1+\delta c^*)p}$$

and hence

$$d_i \geq \frac{\varepsilon}{p} + \frac{1-\delta}{(1+\delta c^*)p}.$$

For δ sufficiently small, the right-hand side of this inequality is at least $\frac{1+\varepsilon_1}{p}$ for a constant $\varepsilon_1 > 0$. Choosing $\varepsilon_2 = \min(\varepsilon_1, c^* - 1)$, it follows that all nodes $i \in V - A$ have degree at least $\frac{1+\varepsilon_2}{p}$.

Now, for any $j \in A$, there are at most $1 + \Delta + \Delta^2 + \cdots + \Delta^{\Delta+1} < \Delta^{\Delta+2} < n$ nodes within distance $\Delta + 1$ of j, and hence within distance Δ of some node in A. Hence there is some node $i \in V$ at distance greater than $\Delta > p^{-1}$ from A. For this node i, (5.2) implies that $r_i \geq \frac{\gamma_1}{p}$ with probability at least γ_0, for constants $\gamma_0, \gamma_1 > 0$. By (3.3), it follows that $\pi_i < -bq$, contradicting the assumption that the min-welfare of G is greater than $\frac{\varepsilon a}{p}$. ■

6 Existence of Stable Networks

Finally, we show that there exist arbitrarily large stable networks. As with our constructions in Section 4, we will consider graphs that consist of disjoint cliques — graphs $F_s\,(k+1)$ with $k = \frac{1+\gamma}{p}$ for an appropriately chosen $\gamma > 0$.

The challenge is to find a k where the union of cliques is stable, and this requires some care for the following reason. Stability requires that no unlinked pair of nodes wants to form an edge — this can be achieved by making k sufficiently large that creating a link between two cliques brings about too large an increase in failure probability to the nodes forming the link. Unfortunately, making k large also raises the failure probability of each node i based simply on its current set of edges — so we must not raise k so high that a node i wants to drop all its existing links. The crux of the problem is thus the following: is there a k that is large enough to discourage the formation of cross-clique links, but not so large that nodes will drop their current links? The main part of our analysis will be to show that such a k exists.

As in Section 4, we consider the case in which $p = q$; defining σ to be the ratio $a/bp = a/bq$, we have $\sigma < \delta$, and we assume δ and p are sufficiently small.

(6.1) *Given a, b, p, q as above, there exists $\gamma > 0$ such that with $k = \frac{1+\gamma}{p}$, the union of cliques $F_s\,(k+1)$ is stable.*

Proof. For the analysis of the construction, we will work with the function $\theta(x)$ defined in (4.3), as well as the related function $\lambda(x) = x\,[1 - (\theta(x))^2]$. Observe that $\lambda(1) = 1$, since $\theta(1) = 0$. Taking derivatives, we have

$$\lambda'(x) = \left[1 - (\theta(x))^2\right] - 2x\theta(x)\theta'(x),$$

and hence $\lambda'(1) = 1$. Thus we have

(6.2) *For some constant $w > 1$, the function $\lambda(x)$ is strictly increasing on the closed interval* $[1, w]$.

As in Section 4, we analyze the failure process by attaching a single "failure node" i^* to each clique. The probability ϕ_i that node i fails is the probability that i belongs to the same active-edge component as i^* in the $(k+2)$-node clique where i^* is added to i's clique. The payoff to node i is

$$\pi_i = ak - (ak + b)\phi_i.$$

If i drops all its edges, it receives a payoff of $-bq < 0$. If i forms an edge to a node j in another clique, it receives an added benefit of a, and incurs an increased expected loss of at least

$$(ak + b)p\phi_i(1 - \phi_i).$$

There are four terms here; the second and third represent the chance that j's failure (which is $\phi_j = \phi_i$ by symmetry) spreads to i, and the fourth term represents the fact that this only matters if i had not already failed in its own clique. In more detail: the payoff to node i before the addition of this edge is $ak - (ak + b)\phi_i$, and afterward it is

$$a(k + 1) - (ak + a + b)(\phi_i + p\phi_i - p\phi_i^2),$$

so the increase in payoff is less than $a - (ak + b)p\phi_i(1 - \phi_i)$.

Now, what is ϕ_i? By (4.1) and (4.3), we have

$$(1 - \varepsilon_1)^2(\theta(p(k + 2)))^2 \leq \phi_i \leq (1 + \varepsilon_1)^2(\theta(p(k + 2)))^2 + c_1 p$$

for a constant ε_1 that goes to zero as p does. By choosing a slightly larger ε_2, and using the fact that $\theta(\cdot)$ has a bounded first derivative, we have

$$(1 - \varepsilon_2)(\theta(1 + \gamma))^2 \leq \phi_i \leq (1 + \varepsilon_2)(\theta(1 + \gamma))^2,$$

with ε_2 going to zero as p does.

In the expression $\phi_i(1-\phi_i)$, provided the upper bound $(1+\varepsilon_2)\theta(1+\gamma)^2 \leq \frac{1}{2}$, we have

$$\begin{aligned}\phi_i(1-\phi_i) &\geq (1-\varepsilon_2)(\theta(1+\gamma))^2\left[1-(1-\varepsilon_2)(\theta(1+\gamma)^2)\right]\\ &\geq (1-\varepsilon_2)(\theta(1+\gamma))^2\left[1-(\theta(1+\gamma))^2\right].\end{aligned}$$

Since $a = \sigma bp$ and $k = (1+\gamma)/p$, if we write $\sigma_1 = \sigma(1+\gamma)$, then we have $ak = \sigma(1+\gamma)b = \sigma_1 b$. Now we have

$$\begin{aligned}&(ak+b)pk\phi_i(1-\phi_i)\\ &\quad= b(1+\sigma_1)(1+\gamma)\phi_i(1-\phi_i)\\ &\quad\geq b(1+\sigma_1)(1+\gamma)(1-\varepsilon_2)(\theta(1+\gamma))^2\left[1-(\theta(1+\gamma))^2\right]\\ &\quad= b(1+\sigma_1)(1-\varepsilon_2)\lambda(1+\gamma)(\theta(1+\gamma))^2\\ &\quad\triangleq f_1(\gamma),\end{aligned}$$

where the last line is taken as the definition of $f_1(\gamma)$.

Observe that $f_1(0) = 0$, and by (6.2), there is an $x_1 < 1$ such that the function $f_1(x)$ is strictly increasing for x in a closed interval $[0, x_1]$.

We also have

$$\begin{aligned}(ak+b)\phi_i &\leq b(1+\sigma_1)(1+\varepsilon_2)(\theta(1+\gamma))^2\\ &\triangleq f_0(\gamma),\end{aligned}$$

where once again the last line is taken as the definition of $f_0(\gamma)$. We see that $f_0(x)$ is also strictly increasing in $[0, x_1]$ (and beyond this interval as well).

Now, since $\lambda(1) = 1$ and $\lambda(\cdot)$ is monotone increasing on $[0, x_1]$, for any small enough $\varepsilon_2 > 0$, there is a unique $x_0 < x_1$ such that

$$\lambda(1+x_0) = \frac{1+\varepsilon_2}{1-\varepsilon_2}.$$

Moreover, $f_1(x) > f_0(x)$ for all $x \in (x_0, x_1]$, and the value of x_0 goes to 0 as ε_2 goes to 0. Also, we observe that for $\gamma \in (x_0, x_1]$, we have

$$f_1(\gamma) > f_0(\gamma) \geq b(\theta(1+\gamma))^2.$$

Now, we choose σ small enough that $2\sigma < (\theta(1+x_1))^2$. We then choose ε_2 small enough (by choosing p small enough) so that $f_1(x_0) = f_0(x_0) < b\sigma$. Finally, let $g(\gamma) = ak = (1+\gamma)b\sigma$. Since $b\sigma < g(\gamma) < 2b\sigma$ for all $\gamma \in (x_0, x_1]$, it follows that $f_0(x_0) < g(x_0)$ but $g(x_1) < f_1(x_1)$. Therefore, since $f_0(\cdot)$ and $f_1(\cdot)$ are continuous functions, there exist $\gamma^*, \gamma^{**} \in (x_0, x_1]$ for which $g(\gamma^*) = f_1(\gamma^*)$ and $g(\gamma^{**}) \geq f_0(\gamma^{**})$, with $\gamma^* < \gamma^{**}$.

We choose any $\gamma \in [\gamma^*, \gamma^{**}]$ as the value of γ we use to define k. With this value of k, the payoff i receives from keeping all its edges is

$$\pi_i = ak - (ak + b)\phi_i \geq g(\gamma) - f_0(\gamma) \geq 0,$$

and hence i prefers to keep its edges rather than deleting all of them. The change in payoff i receives from linking to a node j in a different clique is less than

$$\begin{aligned} &k^{-1}\left(ak - (ak + b)pk\phi_i(1 - \phi_i)\right) \\ &\leq k^{-1}\left[ak - b(1 + \sigma_1)(1 - \varepsilon_2)\lambda(1 + \gamma)(\theta(1 + \gamma))^2\right] \\ &= k^{-1}\left(g(\gamma) - f_1(\gamma)\right) \\ &\leq 0, \end{aligned}$$

and hence i will not form this link. Thus, the graph $F_s\,(k + 1)$ is stable. ■

A Stable Graph with Unequal Clique Sizes. We observe that starting with a set of disjoint cliques $F_s\,(k + 1)$, we can create a different stable graph by adding one additional clique Γ of size $\ell < k + 1$ on a disjoint set of nodes. The size ℓ can be chosen in any way such that the payoffs of nodes in the clique Γ each exceed $-bq$. In this way, nodes in Γ will not want to drop their incident edges. Moreover, there is still no edge that can form so as to improve the payoffs of both its endpoints, since any edge involving a node i in Γ must have its other end at a node j in one of the cliques of size $k + 1$, in which case the argument for (6.1) shows that i would not want to form the link.

In particular, this means that we can take ℓ to be a clique yielding the maximum possible node payoff over all clique sizes, as in Section 4; this shows how certain nodes in a stable graph can have higher payoffs than others.

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