Doubts and Equilibria[1]

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August 2011

1. We are grateful to Ignacio Palacios-Huerta and Joel Sobel for their helpful suggestions. Cabrales gratefully acknowledges the financial support from the Spanish Ministry of Science and Technology under grants CONSOLIDER INGENIO 2010 (CSD2006-0016), and SEJ2006-11665-C02-00. Uriarte gratefully acknowledges the financial support from the Spanish Ministry of Science and Technology and FEDER, grant number SEJ 2006-05455, and the University of the Basque Country, UPV 00043.321-15836/2010. The usual disclaimer applies.

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Abstract

In real life strategic interactions decision-makers are likely to entertain doubts about the degree of optimality of their play. To capture this feature of real choice-making, we present a model based on the doubts felt by an agent about how well is playing a game. The doubts are coupled with (and mutually reinforced by) imperfect discrimination capacity, which we model by means of similarity relations. These cognitive features, together with an adaptive learning process guiding agents' choice behavior leads to doubt-based selection dynamic systems. We introduce the concept of Mixed Strategy Doubt Equilibrium and study its theoretical relevance.

Keywords: Doubts; Similarity Relations; Adaptive Behaviour;
PsycINFO Classification code: 3040
JEL Classification codes: C72, D70, D78

1. Introduction

The period preceding the moment when a decision is taken is often pervaded with doubts about the consequences associated to the alternatives. Even experienced decision-makers face doubts in their domain of expertise. The literature on decision-making under risk (known probabilities) or uncertainty (unknown probabilities), both theoretical and experimental, has devoted little attention to doubt as a cognitive mechanism influencing choice. Our paper introduces doubts in a theoretical model of choice with doubts in strategic environments.

Doubts are commonly related to uncertainty, but doubts and uncertainty should be distinguished. It can be said that doubts appear mainly as a *consequence* of uncertainty. While uncertainty is a characteristic of the environment for the decision maker, doubts are the manner in which uncertainty is perceived and interpreted by the human mind. Doubts, in the Ramsey-Savage tradition, were taken into account with the purpose of building a model of rational choice under uncertainty: *"if he* (the subject) *were doubtful his choice would not be decided so simply. I propose to lay down axioms and definitions concerning the principles governing choices of this kind" (*Ramsey 1928).

For Ramsey the degree of certainty (or doubt) is measured by the subjective belief about the consequences of a certain action. We, on the other hand, assume that the agents measure their doubts by observing the choices made by their fellow agents. That is, the level of doubts that an agent feels depends on the proportion of people who have adopted her same strategy. We also assume that doubts are strictly decreasing in the proportion of people choosing her strategy (we shall see below that this is not necessarily conformity).

In our view, the essence of a choice behaviour by doubtful agents is that, if given the opportunity, they will switch actions. More precisely, the higher the level of doubts about how good the currently chosen action is, the higher the probability of switching to a new one. Then, to understand and measure doubts, for example in an experimental laboratory, the subjects should have the opportunity to repeat the act of taking a decision from the same choice set. For this reason, we think that neither the normative approach nor the psychological (or descriptive) approach to choice under risk and uncertainty[1] have yet isolated completely the influence of doubts on choices.

Individuals and organizations learn from direct experience and from the

[1] Such as prospect theory, cumulative prospect theory and support theory, Kahneman and Tversky 1979, Tversky and Kahneman 1992, Tversky and Kohler 1994.

experience of others, obtaining sources of information that might be used to reduce uncertainty.[2] This is why economists have understood for a long time that imitation of *common* behavior is a widespread human decision-making strategy.[3] Imitation generates in turn adaptive dynamics which can be modelled by means of several selections dynamics of a different nature (deterministic or stochastic). One way of understanding the consequences of doubts on choices is by studying the long-run outcome of the dynamics generated by the (action) switching behaviour of the doubtful agents. To this end, we relate doubts with strategic decisions in a dynamic model in which doubtful agents interact frequently. Let us think of a game continuously played by two player populations, each of whom chooses from a set of pure strategies. A player population is composed of many (doubtful) agents playing pure strategies. Our purpose is to characterize the long-run outcome of the *doubt-based selection dynamics* for constant-sum 2×2 games with a unique equilibrium in mixed strategies. The rest point of the *dynamics* is called Mixed Strategy Doubt Equilibrium (MSDE).[4]

Some of the results of the paper depend on two limiting cases of doubtful behaviour which could be thought of as simple heuristics[5]: the *doubt-less mode of play* and the *doubt-full mode of play.* More specifically, we study the relationship between the MSDE and the Mixed Strategy Nash Equilibrium. The MSDE is not a Nash equilibrium and has the following feature: the more popular/common a strategy is, the lower its expected payoffs. This is a *general* characteristic of equilibria with decreasing doubt functions. But in the *doubt-full* mode of play the MSDE approaches the Nash equilibrium and it is also asymptotically stable, whereas in the *doubt-less* mode, the MSDE is asymptotically unstable. Thus the Nash equilibrium concept requires *non-*

[2]See Levitt and March (1988) or Henisz and Delios (2003) for empirical work about imitation by organizations and for theoretical work on imitation by individuals see Schlag (1998).

[3]See e.g. Alchian (1950), Smallwood and Conlisk (1979) or Nelson and Winter (1982).

[4]In models of selection dynamics, the treatment of observation with noise has usually led to different versions of the replicator dynamics (see, for example, Weibull, 1995). The main departure of the present paper is the way we model that noise by means of similarity relations (Rubinstein, 1988 and Uriarte, 1999)) with thresholds determined by the level of doubts. As a consequence, when doubts are strictly decreasing, the derived doubt-based selection dynamics are not payoff monotonic. Only when agents display constant doubts, the adjusting behavior would lead us to a doubt-based selection dynamics that is closely related to the replicator dynamics.

[5]As in Gigerenzer and Todd (1999).

conformity behaviour or *highly skeptical agents.*

This paper, by insisting on doubts related with imperfect perception, highlights the need of more evidence from fuzzier, that is, more realistic, experimental environments. A related line of research would be in the field of choice theory; that is the study of choice behaviour under doubts to tackle the problem of choice under uncertainty.

2. A Model of Doubt-based Selection Dynamics

2.1 Notation

Consider a noncooperative finite game G in normal form, with $K = \{1, 2, ..., n\}$ denoting the set of players. For each player $k \in K$, let $S_k = \{1, 2, ..., m_k\}$ be her finite set of pure strategies, for some integer $m_k \geqslant 2$.

Imagine that there exist n large populations, one for each of the n player positions in the game. Members of the n populations chosen at random - one member from each player population- are repeatedly matched to play the game. In what follows, we shall speak of *players* when referring to the game G and we shall speak of *agents* when referring to the members of the populations. Each agent is characterized by a pure strategy. From now on, we shall refer to the agent ki as a member of the player population $k \in K$ who plays pure strategy $i \in S_k$. Let $f_{ki}(t) \in F_{ki} = [0, 1]$ be the relative frequency of ki agents at time t, with $f(t)$ being the vector collecting such probabilities. Time index suppressed, $\pi_{ki}(f)$ will denote agent ki's expected payoff given the population state f. Without loss of generality, we may assume that payoffs are strictly positive and smaller than one; that is, $\pi_{ki}(f) \in \Pi_{ki} = [m, M]$, $m > 0$ and $M < 1$. Finally, $\overline{\pi}_k(f) = \sum_{i=1}^{m_k} f_{ki}(f)\ \pi_{ki}(f)$ is the average payoff in player population $k \in K$. To simplify notation, we shall denote $\pi_{ki}(f)$ as π_{ki}.

2.2 Doubts

Doubtful behaviour. We assume that the game is played by boundedly rational players who have doubts about how well they are playing. More precisely, every agent of each player population is endowed with a (primitive) function that we call the "doubt function". This function, denoted d_{ki}, measures the doubts felt by agent ki about how good is his current strategy $i \in S_k$, available to player population $k \in K = \{1, 2, ..., n\}$, as a response to the strategies that the remaining players are using. Each agent ki relates his doubts to $f_{ki} \in F_{ki}$, the proportion of individuals, in his player population, who are equally using his current strategy $i \in S_k$.

We shall assume that the agents are endowed with a strictly decreasing doubt function. That is, an agent's doubts about how well is playing gradually decrease when he observes (or is informed of) a gradual increase in the number of agents from his player population playing the same strategy as the one he is currently using. The underlying logic of this assumption is a belief on the part of agents about the collective *wisdom of crowds*, combined with the cognitive ease of trusting others relative to thinking through the decision problem.[6]

We may distinguish different degrees of trust on the *wisdom of crowds* to calibrate one's doubts. We shall classify them into two broad groups, each with a type of doubtful behaviour (for a philosophical approach to doubts, see section 5, below).

a) The *Herding doubts agent* (or, in short, the *Herding agent*): a typical agent in this group believes in "the wisdom of crowd" and so his doubts are very sensitive to the level of popularity, $f_{ki} \in (0, 1)$, of his current strategy $i \in S_k$.

b) The *Skeptical agent*: this type of agent is suspicious about "the wisdom of crowd" and has high level of doubts for any $f_{ki} \in (0, 1)$.

We formalize these types of doubtful behaviour as follows.

The doubt functions. Formally, let us consider the following set of strictly decreasing and differentiable doubt functions:

$$D = \left\{ d_{ki} : F_{ki} \to [0,1] : \widehat{f}_{ki} > \widetilde{f}_{ki} \Rightarrow d_{ki}(\widehat{f}_{ki}) < d_{ki}(\widetilde{f}_{ki}) \right\}$$

When an element $d_{ki} \in D$ is interpreted as a doubt function, $d_{ki}(f_{ki})$, for some $f_{ki} \in F_{ki}$ known by the ki agent, measures the doubts (about how well is playing the game) felt by the agent ki when the proportion of agents in player population k playing strategy $i \in S_k$ at time t is $f_{ki} \in F_{ki}$.

Let $m < M$, with both m and M in $(0, 1)$. We will be working with the following types of agents:

- *Herding agents:* they are endowed with doubt functions in the set $D_m \subset D$ such that $d_{ki}(f_{ki}) < m$ for all $f_{ki} \in [0, 1]$.

[6] A similar argument is made in Smallwood and Conlisk (1979).

- *Skeptical agents*: they are endowed with doubt functions in the set $D_M \subset D$, such that $d_{ki}(f_{ki}) > M$, for all $f_{ki} \in [0, 1]$.

Now, let $\delta \in (0,\, m)$ be small enough so that $1 - \delta \in (M, 1)$. Inside these two types of agents, we should note the following:

1. The *Doubt-less agent:* this agent is endowed with a function in the class $D^{\delta} \subset D_m$, such that $d_{ki}(f_{ki}) < \delta$ for all $f_{ki} \in [0, 1]$. When δ is sufficiently small, we say that the agent ki is in the *doubt-less mode* because, whatever the level of popularity of his current strategy, $f_{ki} \in (0, 1)$, his doubts are almost zero. The agent endowed with such a function strongly believes in his current strategy.

2. The *Doubt-full agent* (or the *Cartesian skeptical agent* [7]): this agent is endowed with a doubt function in the class $D^{1-\delta} \subset D_M$, such that $d_{ki}(f_{ki}) > 1 - \delta$ for all $f_{ki} \in [0, 1]$. Whatever the level of popularity of his current strategy, f_{ki}, his doubts are almost one. Thus, this type of agent is very suspicious of "the wisdom of crowd" to trust in his current strategy.

An Index for dissatisfied agents. Doubts accentuate the imperfect discrimination capacity of the human mind; hence, we shall assume that agents observe with some noise the expected payoffs and the popularity attached to the pure strategies available to their player population. We shall model that imperfection by means of *similarity relations* (see Rubistein 1988, Aizpurúa et al, 1993, and Uriarte 1999).

Our adaptive agents are current users of some strategy and, very likely, past and future user of some others. Inside a player population its members are likely to share their experience and information about the game. This naturally leads to imitation processes which give rise to observational learning, herding and other forms of convergent behavior.

The flows of agents among the strategies derive from the level of satisfaction felt with their current strategy. To avoid the use of different parameters determining the level of doubts, we will be working with just one type of doubtful agents: either they are all *Herding agents* or they are all *Skeptical agents.*

[7] The doubts of the *Skeptical agent* are not as those of the *Herding agent,* which are popularity depending. We can think that individuals might have life experience built-in doubts which are systematically used as a method for reasoning or as a procedure for decision-making. Those methodological doubts could also be reinforced by philosophical principles, as advised, for instance, by Hume (2007) and Descartes (2008, and http://plato.stanford.edu/entries/descartes-epistemology/).

Let

$$\alpha_{ki} = \alpha_{ki}(\pi_{ki}, f_{ki}, \pi_{kj}, f_{kj}),\ i \neq j$$

denote the proportion of ki strategists who feel dissatisfied with strategy i at time t. In the Appendix A we justify and microfound the following choice of this function via a model of (correlated) similarities relations[8] :

$$\alpha_{ki} = \frac{\lambda_{ki}}{\Sigma_{i=1}^{m_k} \lambda_{ki}} = \frac{\lambda_{ki}}{\lambda_k}$$

Where, for some $f_{ki} \in (0,1)$, $\lambda_{ki}(\pi_{ki}) = \frac{\pi_{ki}}{\pi_{ki} - d_{ki}(f_{ki})}$ is a continuous and strictly deacreasing function which is used to build similarities on the frequency space F_{ki} (a detailed account of λ_{ki} and α_{ki} is given in Appendix A). This function determines the size of the similarity interval in F_{ki}. And the doubt level, $d_{ki}(f_{ki})$, determines the size of the similarity interval on Π_{ki}. Thus, both $d_{ki}(f_{ki})$ and $\lambda_{ki}(\pi_{ki})$ are the thresholds of their corresponding similarity interval. Note that no matter the type of agents, - *Herding* or *Skeptical* -, the sign of α_{ki} is positive.

In Appendix A, we show how the agent ki builds a procedural preference on $F_{ki} \times \Pi_{ki}$ compatible with a pair of similarity relations (in the same spirit of Rubinstein 1988, Aizpurúa et al. 1993 and Uriarte 1999). Given a vector $(\pi_{ki}, f_{ki}) \in F_{ki} \times \Pi_{ki}$ attached to strategy i, the thickness of its corresponding indifference set is sensitive to both $d_{ki}(f_{ki})$ and π_{ki}. The higher the doubts and/or the smaller the payoffs, the thicker the indifference set will be; hence, the higher is the distance from (π_{ki}, f_{ki}) to its preferred set and so the more dissatisfied the agent ki will feel. It can be seen that the variations of this distance are captured by the properties of λ_{ki} (that is, the variations of λ_{ki} due to changes in π_{ki} and f_{ki}). Hence the λ_{ki} function could be taken as a measure of the degree of dissatisfaction of the agent ki with respect to his current strategy $i \in S_k$; $\Sigma_{i=1}^{m_k} \lambda_{ki} = \lambda_k$ will the be the total dissatisfaction level in population $k \in K$.

The limit case of the *herding agents*, the *doubt-less agent,* would be highly satisfied with his current strategy because his doubts are almost zero and hence the indifference set will be almost a singleton. On the other hand, it can be seen (in Appendix A) that the *skeptical agent* has indifference sets covering the whole choice space $F_{ki} \times \Pi_{ki}$ and thus will feel highly dissatisfied.

[8]For the definition of similarity relation, see Rubinstein (1988). For the definitin of correlated similarity relation, see Aizpurúa et al. (1993).

And the *Cartesian skeptical* agent endowed with hyperbolic and *universal* doubts, as describe above, will be continuously switching and experimenting new strategies.

Doubt-Based Selection Dynamics. We assume that time is divided into discrete periods of length τ. In every period, $1-\tau$ is the probability that the agent does retain his current strategy; thus, τ is the probability that each agent does not retain his current strategy. We make now the following assumption to build a selection dynamic model[9].

When an agent feels dissatisfied with his current strategy, she will choose a new strategy with a probability that is equal to the proportion of agents playing that strategy.

Hence, $\tau\frac{\lambda_{ki}}{\lambda_k}f_{ki}$ will denote the proportion of ki strategists who will choose a new strategy (that is, the *outflow*), and, since a particular strategy is chosen with a probability that is equal to the proportion of agents playing that strategy, then $\sum_{j=1}^{m_k}\frac{\lambda_{kj}}{\lambda_k}f_{kj}f_{ki}=\tau\frac{\overline{\lambda}_k}{\lambda_k}f_{ki}$ is the proportion of agents who will choose strategy i (that is, the *inflow*).

Therefore,

$$f_{ki}(t+\tau)=f_{ki}(t)-\tau\frac{\lambda_{ki}}{\lambda_k}f_{ki}+\tau\frac{\overline{\lambda}_k}{\lambda_k}f_{ki}$$

As $\tau\longrightarrow 0$, in the limit we get the *doubt-based* selection dynamic equation:

$$\overset{\bullet}{f}_{ki}=f_{ki}\left[\frac{\overline{\lambda}_k-\lambda_{ki}}{\lambda_k}\right]\text{..}(1)$$

To gain some intuition, let us now look at equation (1) in a less compact way. Let G be a two-population constant-sum game with $S_I=\{U,D\}$ and $S_{II}=\{L,R\}$ denoting player I and player II's strategy sets, respectively. Let x denote the probability of playing U, y the probability of playing L and $I=[(x^*,1-x^*),(y^*,1-y^*)]$ the Mixed Strategy Nash Equilibrium, with $x^*>0$ and $y^*>0$.

We denote the four doubt functions $d_i\in D$ (where $i=U,D,L,R$). From (1), the *doubt-based* selection dynamics for G is represented by the following system:

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[9]For a justification see, for example, Binmore et al. (1995).

$$\dot{x} = \frac{x(1-x)}{\pi_U(\pi_D - d_D) + \pi_D(\pi_U - d_U)}(\pi_U d_D - \pi_D d_U) \equiv G_1(x,y)F_1(x,y) \qquad (2)$$

$$\dot{y} = \frac{y(1-y)}{\pi_L(\pi_R - d_R) + \pi_R(\pi_L - d_L)}(\pi_L d_R - \pi_R d_L) \equiv G_2(x,y)F_2(x,y) \qquad (3)$$

Clearly, a stationary point for the *doubt-based* system (2)-(3), with $x^* > 0$ and $y^* > 0$, requires $\pi_U d_D = \pi_D d_U$ and $\pi_L d_R = \pi_R d_L$. We call this point the Mixed Strategy Doubt Equilibrium (MSDE).

2.3 Mixed Strategy Nash Equilibrium (MSNE) and Mixed Strategy Doubt Equilibrium (MSDE)

We should distinguish between the Mixed Strategy Nash Equilibrium (MSNE) and the Mixed Strategy Doubt Equilibrium (MSDE) for the doubt-based dynamic system (2)-(3).

1. In a MSNE the requirement is that all strategies in the support of the equilibrium have equal payoffs; that is:

$$\pi_{ki}(f^*) = \pi_{kj}(f^*) \text{ for all } i,j \text{ with } f_i^* > 0 \text{ and } f_j^* > 0 \text{ and all } k$$

2. From (2)-(3) we deduce that for a MSDE the requirement is (recall the assumption $d_{ki} = d \in D$):

$$\frac{\pi_{ki}(f^*)}{d_{ki}(f_i^*)} = \frac{\pi_{kj}(f^*)}{d_{kj}(f_j^*)} \text{ for all } i,j \text{ with } f_i^* > 0 \text{ and } f_j^* > 0 \text{ and all } k$$

Note that in this case, the expected payoffs to the strategies in the support of the equilibrium need not be equal, as it is required in the MSNE. We have the following result:

Proposition 1

Suppose that all the agents are endowed with a doubt function $d_{ki} = d$. Then for all k and all i,j, with $0 < f_{kj}^* < f_{ki}^* < 1$, since the doubt functions are strictly decreasing, $d(f_{ki}^*) < d(f_{kj}^*)$; thus, in order to satisfy the Mixed Strategy Doubt Equilibrium condition, we must have $\pi_{ki}(f^*) < \pi_{kj}(f^*)$.

Proof: Direct from the Mixed Strategy Doubt Equilibrium (MSDE) condition.

In words, the more frequent strategies in a MSDE should have lower expected payoffs. This situation is clearly distinct from a Nash equilibrium and is a general feature of the (decreasing) doubt-based dynamic system.[10]

Relation with the Literature. The major departure of the present paper is how it is modelled the agents' imperfect observation of expected payoffs and popularity attached to his current strategy. Contrary to the standard approach (cf. chapter 5 of Weibull, 1995), modelling the noisy observation by means of similarity relations whose thresholds depend on the level of doubts, we can obtain doubt-based selection dynamics that are not payoff monotonic. We show (see section 6, below) that only when doubts are constant the doubt-based system is just the standard replicator dynamics multiplied by a positive function.

When doubts are very high, we show that:

i) the MSDE converges to the Mixed Strategy Nash Equilibrium (MSNE).

ii) *any interior* MSNE is asymptotically stable for the doubt-based selection dynamics.

These two results together provide a doubt-based justification of the MSNE. Thus, only the agents who are skeptical find their way to the unique interior Nash equilibrium (Proposition 3). But those who are not skeptical, such as the herding agents, do not converge to that equilibrium (Proposition 4). These results happen to agree with ordinary ideas, such as skepticism is a good guide for action or too much trust in the wisdom of crowd is not a good strategy and neither is conformity.

It seems harder to find a microeconomic justification, or a natural interpretation, of the asymptotic stability of the Nash equilibrium in the Matching Pennies Game obtained with the parameter c used in the adjusted replicator dynamics (see section 5.2.2 in Weibull, 1995).

3. Doubt-based selection dynamics in constant sum games

In this section we shall explore the relationship between the MSNE and

[10] We believe that this feature of the equilibria of the *doubt-based selection dynamics system* is a relevant and testable implication of our model. We can provide, upon request, some preliminary evidence to support it.

MSDE for different levels of doubts.

3.1 Relationship between a MSNE and a MSDE

Let us recall what game theorists say about a MSNE:

"The point of randomizing is to keep the other player(s) just indifferent between the strategies that the other player is randomizing among. One randomizes to keep one's rivals guessing and not because of any direct benefit to oneself." (Kreps 1990, p 408).

We shall see below that the doubt-based model seems to capture that state of players' mutual guessing that characterizes a MSNE. Assume that we are dealing with 2×2 constant sum games having a unique mixed equilibrium with full support. Consider Player I; how would this player interpret different values of (his own probability) x, say 0.2 and 0.6? A rational Player I knows that Player II is randomizing to keep him indifferent between the strategies he is randomizing among. Therefore, in terms of our model of doubts, $x = 0.2$ and $x = 0.6$ would induce in the Player I's rational mind the same level of doubts as to which is the best probability distribution, because both of them get the same expected payoff. But, for the same reason, Player I's equilibrium strategy in the game will induce the same level of doubts as 0.2 or 0.6. In other words, Player I does not see, in a preference sense, any real difference between different probability distributions in the open unit interval (0,1). As a consequence, he must have (nearly) equal level of doubts at any x in (0,1). The same will happen to Player II.

The above suggests that we should ask first, *which are the level of doubts embedded in the players' mutual guessing that characterizes steady states very close to the MSNE.* This is answered in Proposition 2 below, where we show that, if all agents are playing in the *doubt-full mode,* any interior MSNE coincides with an MSDE ; that is, an MSNE is a Mixed Strategy Doubt-Full Equilibrium (MSDFE).

The second issue to deal with is the following: how is the MSNE reached? or,which is the equilibrating process that may lead to the MSNE? This will be answered in Propositions 4 and 5 below.

Let G be a two-population, two-strategy, constant-sum game with $I = [(x^*, 1 - x^*), (y^*, 1 - y^*)]$, $x^* > 0$, $y^* > 0$, denoting its MSNE.

Proposition 2

1.The (Euclidean) distance between an MSDE and MSNE converges to zero as δ goes to zero if every agent plays with a doubt function in the $D^{1-\delta}$ class; that is, for the *doubt-full* or *Cartesian skeptical* agents. Hence, any MSNE, with $(x^*, y^*) \in (0,1) \times (0,1)$, is an MSDFE.

2.For any interior point of the simplex $A = [(x', 1-x'), (y', 1-y')]$ (i.e. with $0 < x' < 1$ and $0 < y' < 1$) there is a sequence of functions $d^\delta \in D^\delta$ such that the (Euclidean) distance between an MSDE and A converges to zero as δ goes to zero. That is, if every agent plays in a *doubt-less mode*, any interior point of the simplex can be a MSDLE for some kind of doubt-less behavior.

Proof: See appendix B

This means that if $I = [(x^*, 1-x^*), (y^*, 1-y^*)]$ is the MSNE of G, then it is compatible (in the sense of Proposition 1) with agents playing in any of the two modes of play, *doubt-full* or *doubt-less*.

3.2 Learning to Play a Mixed Strategy Nash Equilibrium (MSNE)

We have seen that an MSNE and an MSDE satisfy different equilibrium properties and therefore, in general, they do not coincide. However, from Proposition 2, we know that an MSNE could be converted into an MSDE when all agents are *Cartesian skeptical* or play in the *doubt-full mode*. In other words, an MSNE could be converted into a rest point of the doubt-based dynamic system (2)-(3). Hence, now we are ready to answer the question: how do the boundedly rational player populations learn to coordinate in the MSNE? Proposition 4, below, shows that an introspective element, such as doubts, could be crucial for learning to play optimally.

We know that a fully rational player must avoid being guessed by the opponents and that to achieve this he will behave in such a way so as to create a random sequence of choices. This suggests that a *doubt-less* mode of playing -that implies almost no strategy switching behavior- would be far from being an adjusting process leading to the Nash equilibrium. It seems that, in an equilibrating process, what makes more sense is that players be very skeptical; that is , that they should behave in the *doubt-full* mode. In our deterministic dynamic model, the *Cartesian skeptical* agents will have a tendency to keep trying new strategies and, thus, generating not a truly random sequences of choices, but individual processes of trial-and-error adjustments which could find their way to the MSNE. In Proposition 3 below we show that this is the case: if every agent behaves as if he were constantly with hyperbolic doubts, the agents' adjusting behavior would lead them to the MSNE and endow the equilibrium with a strong stability property. Proposition 4 below shows that the *doubt-less* mode of play has just the opposite consequence.

Proposition 3

Let G be a two-population, two-strategy, constant-sum game with $I^* \equiv [(x^*, 1 - x^*), (y^*, 1 - y^*)]$, any $x^* > 0$ and $y^* > 0$, denoting its MSNE. Then a point close to I^* is asymptotically stable for the *doubt-based* dynamic system (2)-(3) if every agent plays in the *doubt-full mode of play* (that is, if they are all *Cartesian skeptical*).

Proof: See appendix B

Proposition 4

Let G be a game as in Proposition 3. For any interior point of the simplex $A = [(x', 1 - x'), (y', 1 - y')]$ (i.e. with $0 < x' < 1$ and $0 < y' < 1$). If every agent is in the *doubt-less mode* of play and if the initial conditions of the doubt-based dynamic system (2)-(3) are different from A, there is a sequence of functions $d^\delta \in D^\delta$ such that the system diverges to a corner of the simplex. That is, if every agent plays in a *doubt-less mode*, any interior point of the simplex can be a *source* for some kind of doubt-less behavior.

Proof: See appendix B

One may then ask about how to explain the modes of play of Proposition 3 and 4 would arise. Needless to say, doubts are a subjective feeling and hence it is difficult to ascertain the precise reason why they may arise in each particular case. Proposition 3 suggests that the origin of high level of doubts (i.e of being skeptical) lies in the fact that every agent seems to be aware that the proportion with which each available strategy is being played and the sequence that the agents, as a player population, are producing is not random. *Cartesian skeptical* agents have developed *a priori* a theory that make them to be aware and adapted to face this setting. Thus, the *hyperbolic* and *universal* doubts felt by every member of each player population would be what the context demands. If not by a theory, smart agents would develop high doubts from the fear of being guessed and exploited by the opponent. As a consequence, since agents are very unhappy with their current strategies a high proportion of agents will experiment with new strategies in the next period. The fear and the doubts of the agents will continue to be high and, joint with the choices that exploit the variations both in the payoffs and in the strategy proportions, the adjusting behavior would lead the system to the Mixed Strategy Nash Equilibrium. Once in the equilibrium, payoffs are equalized across strategies and the doubt levels continue to be very high and equal across strategies too. Thus, the *doubt-full mode* of play advised by the Cartesian theory of doubts endow the MSNE with strong stability properties.

An interpretation of Proposition 4 is that the extreme sensitivity to the "opinions" of others, leads play to a situation where players imitate, whenever doubtful, the current most fashionable action. This creates a tendency to diverge in population behavior. In addition, the doubt-less agents are quite satisfied with their current strategies and do not feel the need to experiment with new strategies to exploit the differences in payoffs and strategy proportions. Hence, a low level of imitation and strategy adjustment takes place, and the populations diverges very slowly to a situation where initially popular strategies dominate.

4. Example

Without loss of generality, let us consuder the following class of doubt functions: $d_{ki}(f_{ki}) = (1 - f_{ki})^{\alpha}$. Assuming that $\alpha \in (0, \infty)$, we would obtain a large enough subclass of doubt functions in the set D. Note, in particular, that this class contains the two extreme types of doubt functions mentioned above: when α is very small, near zero, the doubt parameter characterizing agent ki, denoted as $H = \frac{1}{\alpha}$, would be very high, for any $f_{ki} \in (0, 1)$. Then the function will have a graph looking like the one of figure 2, and we shall say that the agent is *Cartesian skeptical* or is in the *doubt-full mode of play.* When α is very high, the graph of d_{ki} is close to the axes, as in figure 1, and so the doubt parameter, $H = \frac{1}{\alpha}$, is very small, for any $f_{ki} \in (0, 1)$. This is the agent in the *doubt-less mode of play* .

As in Binmore et al. (1995), we approach equation

$$\overset{\bullet}{f}_{ki} = f_{ki}\left[\frac{\overline{\lambda}_k - \lambda_{ki}}{\lambda_k}\right] \qquad (4)$$

by means of the equation

$$f_{ki}(t+\tau) - f_{ki}(t) = \tau f_{ki}\left[\frac{\overline{\lambda}_k(t) - \lambda_{ki}(t)}{\lambda_k(t)}\right] \qquad (5)$$

where the step size $\tau = 0.01$. We shall consider, like Binmore et al.(1995), that the system has converged on a point when the first 15 decimals are unchanging.

The Penalty Kick Game[11]

Palacios-Huerta (2003) found that the equilibrium theory predictions are observed in the professional players' behavior: (i) their choices follow a random process and (ii) that the probability that a goal will be scored must be the same across each player's strategies and equal to the equilibrium scoring probability (that is, in the Mixed Strategy Nash Equilibrium each player is indifferent among the available strategies). Palacios-Huerta and Volij (2007) extend this result by observing that professional players are capable of transferring their skills from the field to the laboratory, a completely unknown setting for them, and yet behave in a way that is significantly near the Nash equilibrium.

Palacios-Huerta and Volij (2007), from a sample of 2,717 penalty kicks collected from European first division football (soccer) leagues during the period 1995-2004, built the following two player (Player I: the kicker and Player II: goal keeper) two strategy (Left, Right) game.

	(y) L	R
(x)L	0.60, 0.40	0.95 , 0.05
R	0.90, 0.10	0.70, 0.30

where $\pi_I(i,j)$ denotes the kicker's probability of scoring when he chooses i and the goalkeeper chooses j, for $i,j \in \{L,R\}$ .The Mixed Strategy Nash Equilibrium of this game is: $x^* = 0.363\,64, y^* = 0.454\,55$.

Football matches are continuously played and players' game is based on the study of the opponents in the field and watching their play on TV and videotapes, so that their behavior in the penalty kicks is collected and analyzed. Thus, there is a history of play of each player and, hence, an interactive learning process. Thus, a natural issue is to investigate the type of dynamic process that may lead to the result found by Palacios-Huerta (2003). The *doubt-based* model seems to be a suitable model for this task.

The *doubt-based* selection dynamic system (2)-(3) corresponding to this game is the following:

[11] We avoid on purpose an example based on the ubiquitous Matching Pennies Game. Proposition 3 is valid for any interior MSNE of a 2×2 constant-sum game.

$$
\begin{aligned}
\dot{x} &= \frac{x(1-x)((0.95-0.35y)x^{\alpha}-(0.2y+0.7)(1-x)^{\alpha})}{2(0.95-0.35y)(0.2y+0.7)-(0.95-0.35y)x^{\alpha}-(0.2y+0.7)(1-x)^{\alpha}} \\
\dot{y} &= \frac{y(1-y)((0.1+0.3x)y^{\alpha}-(0.3-0.25x)(1-y)^{\alpha})}{2(0.1+0.3x)(0.3-0.25x)-(0.1+0.3x)y^{\alpha}-(0.3-0.25x)(1-y)^{\alpha}}
\end{aligned}
$$

The vector field defining (2)-(3) is

$$
\begin{aligned}
F(x,y) &= \Big(\frac{x(1-x)((0.95-0.35y)x^{\alpha}-(0.2y+0.7)(1-x)^{\alpha})}{2(0.95-0.35y)(0.2y+0.7)-(0.95-0.35y)x^{\alpha}-(0.2y+0.7)(1-x)^{\alpha}}, \\
&\quad \frac{y(1-y)((0.1+0.3x)y^{\alpha}-(0.3-0.25x)(1-y)^{\alpha})}{2(0.1+0.3x)(0.3-0.25x)-(0.1+0.3x)y^{\alpha}-(0.3-0.25x)(1-y)^{\alpha}}\Big)
\end{aligned}
$$

We compute first the derivative $DF(x,y)$ and then evaluate $DF(x,y)$ at $(0.363\,64, 0.454\,55)$ to get the following Jacobian matrix:

$$
DF(0.363\,64, 0.454\,55) = \begin{bmatrix} \frac{\alpha}{1.\,581\,8-2\times 0.363\,64^{\alpha}} & \frac{0.146\,29(-0.2\times 0.636\,36^{\alpha}-0.35\times 0.363\,64^{\alpha})}{0.790\,91-0.363\,64^{\alpha}} \\ 0.592\,88\frac{0.25\times 0.545\,45^{\alpha}+0.3\times 0.454\,55^{\alpha}}{0.209\,09-0.454\,55^{\alpha}} & \frac{\alpha}{0.418\,18-2\times 0.454\,55^{\alpha}} \end{bmatrix}
$$

It is easy to see that for values of $\alpha \in (0, 0.231\,88)$, all the eigenvalues of $DF(0.363\,64, 0.454\,55)$ have negative real parts and the associated determinants are all positive. Thus, the equilibrium $(0.363\,64, 0.454\,55)$ is a *spiral sink*, for those values of α, and, therefore, it is asymptotically stable.

5. Constant doubt-based selection dynamics

The individual choice model that we are going to use in this section is derived from a choice procedure introduced by Aizpurúa, Ichiishi, Nieto and Uriarte (1993) in the space of simple lotteries. We consider now the case when the level of doubts felt is constant, for any value of $f_{ki} \in F_{ki}$. This means that society has no influence upon the doubt level of the agents. Formally,

Assumption 3

$$
d_{ki}(f_{ki}) = \epsilon_k \in (0,1)
$$

We assume that the constant level of doubts ϵ_k felt by agent ki induces *threshold levels* in both expected payoffs and strategy frequencies and that these threshold levels are described by means of similarity relations.

As in the previous case, it is by means of Assumption 3 about the doubt function that we may define a similarity relation on $\Pi_{ki} = (0,1]$ and correlated similarity relations on $F_{ki} = [0,1]$. Suppose that (π_{ki}, f_{ki}) is the vector of expected payoff-strategy proportion attached to strategy i at time t.

The similarity relation on Π_{ki}, denoted $S\Pi_{ki}$, is assumed to be of the difference type and it is defined as follows

$$\pi_{ki} S\Pi_{ki} \pi'_{ki} \Leftrightarrow |\pi_{ki} - \pi'_{ki}| \leq \epsilon_k$$

On F_{ki}, we define now the correlated similarity relations as follows. First, for all $\pi_{ki} > \varepsilon_k > 0$ we build the function $\phi_{ki} : \Pi_{ki} \to (1, \infty]$ as follows,

$$\phi_{ki}(\pi_{ki}) = \frac{\pi_{ki}}{\pi_{ki} - \epsilon_k} > 1$$

Then, we can establish the following similarity relation (of the ratio-type) between f_{ki} and other frequencies in F_{ki}, such as f'_{ki}, given π_{ki}.

$$f_{ki} SF_{ki}(\pi_{ki}) f'_{ki} \Leftrightarrow \frac{1}{\phi_{ki}(\pi_{ki})} \leqq \frac{f_{ki}}{f'_{ki}} \leqq \phi_{ki}(\pi_{ki})$$

We call $SF_{ki}(\pi_{ki})$ a **correlated** similarity relation because the similarity on F_{ki} depends on the level of expected payoff π_{ki} at period t. For values of $\pi_{ki} \leqq \epsilon_k$ the function ϕ_{ki} is not defined and we assume that in that case that $SF_{ki}(\pi_{ki})$ is the degenerate similarity relation (see Rubinstein (1988)).

Remark:The threshold level in the frequency space is inversely related to expected payoffs: $\frac{\partial \phi_{ki}(\pi_{ki})}{\partial \pi_{ki}} < 0$. This means that as the expected payoffs at stake increases, the discrimination on the frequency space F_{ki} increases.

Assumption

We proceed as in the previous case (for simplicity we shall write ϕ_{ki} instead of $\phi_{ki}(\pi_{ki})$). Let the ratio

$$\frac{\phi_{ki}}{\Sigma_{i=1}^{m_k} \phi_{ki}} = \frac{\phi_{ki}}{\phi_k}$$

denote the proportion of ki strategists who feel dissatisfied with strategy i. Note that, everything equal, this function increases with ϕ_{ki}. Hence, an increase in ϕ_{ki}, due to a decrease in the expected payoffs π_{ki}, will increase the proportion of dissatisfied ki strategists.

As before, $\tau \frac{(\phi_{ki} - 1)}{\phi_k} f_{ki}$ denotes the proportion of ki strategists who will choose a new strategy at time t (the *outflow*). Since a particular strategy is chosen with a probability that is equal to the proportion of agents playing that strategy, then $\tau \sum_{j=1}^{m_k} \frac{\phi_{kj}}{\phi_k} f_{kj} f_{ki} = \tau \frac{\overline{\phi}_k}{\phi_k} f_{ki}$ denotes the proportion of agents who choose strategy i ; i.e. the *inflow* (where $\overline{\phi}_k = \sum_{j=1}^{m_k} \phi_{kj} f_{kj}$ is the average perception in player population k at time t).

Therefore

$$f_{ki}(t+\tau) = f_{ki}(t) - \tau \frac{\phi_{ki}}{\phi_k} f_{ki} + \tau \frac{\overline{\phi}_k}{\phi_k} f_{ki} \text{................}(6)$$

Proposition 5

As $\tau \to 0$, equation (6) becomes

$$\overset{\bullet}{f}_{ki} = f_{ki} \left[\frac{\overline{\phi}_k - \phi_{ki}}{\phi_k} \right] \text{.................................}(7)$$

1. If for all player position $k \in K = \{1, 2, ..., n\}$, the strategy set S_k consists of two elements, i.e. if $m_k = 2$ then, equation (7) is just the standard Replicator Dynamics (RD) multiplied by a positive function (i.e. is aggregate monotonic).

2. If $m_k > 2$, then we obtain a selection dynamics that approximates the RD, but preserves only the positive sign of the RD (i.e. is weakly payoff positive).

Proof: See appendix B

6. Concluding Remarks

In 2×2 games with Mixed Strategy Nash Equilibria, the introduction of agents with doubts coupled with (and mutually reinforced by) imperfect discrimination capacity, permits a departure from the long-run behavior of traditional selection dynamic systems. For instance, if we assume that the feeling of doubts is sensitive to the popularity of a pure strategy, then we obtain doubt-based selection dynamics that are not payoff monotonic. The main feature of the doubt-based system is that its equilibrium does not require expected payoffs to be equalized across strategies. Nevertheless, the curvature of the decreasing doubt functions has strong implications on the long run behavior of the system. If agents do not beleive in the wisdom of crowd, are very *skeptical* and thus play in the *doubt-full mode*,- i.e. agents are endowed with an extremely concave doubt function-, a Mixed Strategy Nash Equilibrium is a Mixed Strategy Doubt-Full Equilibrium and it is shown to be asymptotically stable. But stability is lost when agents have *herding* *doubts;* that is, doubts that are influenced by the relative popularity of each of the pure strategies available to his player role. We have shown this result when agents are in the *doubt-less mode*, i.e. when they are endowed with an extremely convex doubt function. Herding doubts lead to the *Mixed Strategy Doubt Equilibrium* in which the most popular strategies receive lower expected payoffs.

The present behaviorally-based theoretical work could be applicable to experimental research. Our view is that experiments should be designed to capture the decision schemes that are actually used by subjects. Two features of bounded rationality are, in our view, embedded in those schemes: doubts and imperfect perception. It is known that similarity judgments are part of observed decision procedures (see Tversky (1977), Rubinstein (1988) and Arieli et al. (2009)). We think that feedback on the popularity of different strategies would be important to consider, as well as less sharply defined payoffs. However, subjects in experiments usually do not have information about the proportion of people using each strategy. For example, the only experiment from those surveyed in chapter 3 of Camerer (2003) in which agents are given that information is the one carried out by Tang (2001). We suspect, though, that the highly precise (and, we would argue, unnatural) form of the feedback given to subjects eliminates the "doubt" considerations that are important in the build-up of our model. It would be unrealistic to assume that the agents get the correct numbers. We believe that more evidence, and hopefully, from "fuzzier" (more realistic) environments would be useful to confront some predictions made in this work. Hence, a translation of our theoretical model into an experimental design should be our next task.

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Appendices

A. Satisficing Procedural Preferences based on Similarity Judgements

It is safe to say that doubts are closely related to imperfect discrimination capacity. Thus, we will assume that agents observe the expected payoff and the popularity attached to their current pure strategy with some noise. We model this imperfection by means of an extension of Rubinstein type of similarity relations (see Rubistein 1988), which we call correlated similarity relations (see Aizpurúa et al, 1993 and Uriarte 1999).[12]. Then the agents would build a procedural preference relation compatible with those similarity relations, as in Rubinstein (1988).

We proceed as follows, **first** we shall build the correlated similarity relations and **second**, we show how the agent may proceed to build his prefence and decide about the pure strategies.

To be more specific, let $(\pi_{ki},\ f_{ki})$ be the vector of expected payoff-proportion of agents of player population k attached to strategy $i \in S_k$ at time t with $f_{ki} \in (0,1)$.

I. Correlated Similaritites on Π_{ki} and F_{ki}

The doubt function serves to build correlated similarity relations on both Π_{ki} **and** F_{ki}. Let (π_{ki}, f_{ki}) and $(\overline{\pi}_{ki}, \overline{f}_{ki})$ be two vectors in $\Pi_{ki} \times F_{ki}$, with $\overline{f}_{ki}$, $f_{ki} \in (0,1)$.

(a) On the space of expected payoffs, Π_{ki}:

The doubt function d_{ki} defines correlated similarities of the difference-type as follows: given f_{ki} we say that $\overline{\pi}_{ki}$ is similar to π_{ki}, (formally written as $\overline{\pi}_{ki} S\Pi[f_{ki}]\pi_{ki}$), if and only if $|\overline{\pi}_{ki} - \pi_{ki}| \leqq d_{ki}(f_{ki})$, where $|.|$ stands for absolute value. Thus, there is one similarity relation on Π_{ki}, for each $\overline{f}_{ki} \in (0,1)$

Then the similarity interval of π_{ki}, given f_{ki} is:

$$[\pi_{ki} - d_{ki}(f_{ki}), \pi_{ki} + d_{ki}(f_{ki})]$$

Note that $d_{ki}(f_{ki})$, the doubt level felt by $\sum$ agent ki given the proportion f_{ki}, becomes the threshold level in the definition of this type of similarity relation. If f_{ki} increases, the threshold, $d_{ki}(f_{ki})$, decreases and so the similarity intervals of π_{ki} shrink (giving rise to the vertical cone-shaped form in figure 2).This means that when f_{ki} increases, the discrimination capacity on the space of expected payoffs to strategy i increases (probably because the accumulated experience with strategy i has increased due to the increased number of agents from population k currently playing strategy i). When f_{ki} is such that $\pi_{ki} - d_{ki}(f_{ki}) \leq m$ and $\pi_{ki} + d_{ki}(f_{ki}) \geq M$, the whole set Π_{ki} $= [m, N]$ is similar to π_{ki} and when $f_{ki} = 1$ only π_{ki} is similar to itself. This

[12] Rather than being constant, correlated similarities depend on the value of some relevant parameter.

variations in perception induces a vertical wedge type form, as it can be seen in figure 2.

Notice that for a *Cartesian skeptical agent*, the similarity interval is

$$[\pi_{ki} - d_{ki}(f_{ki}), \pi_{ki} + d_{ki}(f_{ki})] = [m, M]$$

That is, since in this case $d_{ki} \in D_M$, then $d_{ki}(f_{ki}) > M$ for all $f_{ki} \in (0, 1)$, thus m is similar to M, and hence the similarity on Π_{ki} is degenerate (see Rubistein 1988).

(b) On the strategy frequency space, F_{ki}:

The doubt function d_{ki} defines correlated similarity relations of the ratio-type by means of the continuous function $\lambda_{ki} : \Pi_{ki} \to R$, which is defined as follows: for a given $f_{ki} \in (0, 1)$ and d_{ki},

$$\lambda_{ki}(\pi_{ki}) = \frac{\pi_{ki}}{\pi_{ki} - d_{ki}(f_{ki})}$$

The properties of λ_{ki} are the following:

(i) given f_{ki} and d_{ki}, if π_{ki} increases, $\lambda_{ki}(\pi_{ki})$ decreases continuously and thus, the similarity interval shrinks. This means that when the expected payoffs at stake increase, the discrimination efforts on the frequency space, F_{ki}, increases (generating a kind of horizontal wedge type form, as it is shown in figure 1)

(ii) keeping the function d_{ki}, and π_{ki} constant, if the frequency f_{ki} increases, then $\lambda_{ki}(\pi_{ki})$ decreases and so the similarity intervals of the higher frequency shrink.

Note that:

1. For the *herding agents*: $d_{ki} \in D_m$ and the function $\lambda_{ki} > 1$ is then used to define on F_{ki} correlated similarity relations of the ratio-type whose similarity interval for that f_{ki} is:

$$[f_{ki}/\lambda_{ki}(.), \;\; f_{ki}.\lambda_{ki}(.)]$$

2. For the *skeptical agents*: $d_{ki} \in D_M$, then $\lambda_{ki} < 0$ will define a *degenerate* similarity relation (see Rubistein 1988). Thus, when doubts are of a skeptical nature, the similarity relations on both Π_{ki} and F_{ki} are *degenerate*; that is, 0 is similar to 1. Formally, on F_{ki}, given a $f_{ki} \in (0, 1)$, the correlated similarity relation $SF_{ki}[\pi_{ki}, \, f_{ki}]$ will induce the following similarity intervals for f_{ki}:

$$[f_{ki}/\lambda_{ki}(.),\ \ f_{ki}/\lambda_{ki}(.)] = [0,1]$$

The size of this degenerate similarity interval does not change with π_{ki}; it remains constant for any value of π_{ki}.

II. Procedural Preference on $\Pi_{ki} \times F_{ki}$:

Based on a model developed in Uriarte (1999), we show now how the above two correlated similarity relations build a (non-complete and non-transitive) preference-indifference relation defined on the space of expected payoffs and frequencies, $\Pi_{ki} \times F_{ki}$, attached to pure strategy $i \in S_k$. Let us assume that each agent ki compares pairs of alternatives in $\Pi_{ki} \times F_{ki}$ with the aid of a pair of correlated similarity relations to decide which of the two is preferred. The agent may define a procedural preference $\succsim_{ki}$on $\Pi_{ki} \times F_{ki}$ by means of the pair of correlated similarities and know his aspiration set U at each t (which we identify with the upper contour set of the vector (π_{ki}, f_{ki}) at t , $U = U_\alpha \cup U_\beta \cup U_\delta$; see figure 1). That is, given a pair of vectors $(\overline{\pi}_{ki}, \overline{f}_{ki})$ and (π_{ki}, f_{ki}) in $\Pi_{ki} \times F_{ki}$, the vector $(\overline{\pi}_{ki}, \overline{f}_{ki})$ will be declared to be preferred to (π_{ki}, f_{ki}), i.e. $(\overline{\pi}_{ki}, \overline{f}_{ki}) \succ_{ki} (\pi_{ki}, f_{ki})$, whenever the agent ki perceives that one of the following three conditions is met. (Note that since $(\overline{\pi}_{ki}, \overline{f}_{ki})$ is to be preferred, the conditional similarity relation $S\Pi$ on Π_{ki} given $\overline{f}_{ki}$ and the conditional similarity relation SF on F_{ki} given $\overline{\pi}_{ki}$ and $\overline{f}_{ki}$ are to be used):

Condition α : $\overline{\pi}_{ki} > \pi_{ki}$, and *no* $\overline{\pi}_{ki} S\Pi[\overline{f}_{ki}]\pi_{ki}$; while $\overline{f}_{ki} SF[\overline{\pi}_{ki}, \overline{f}_{ki}] f_{ki}$.

In words, $\overline{\pi}_{ki}$ is bigger than π_{ki} and, given $\overline{f}_{ki}$, $\overline{\pi}_{ki}$ is perceived to be *not similar* to π_{ki} ; while , $\overline{f}_{ki}$ is perceived to be *similar* to f_{ki}. U_α in figure 1 is the area implied by this condition.

Condition β : $\overline{f}_{ki} > f_{ki}$ and *no* $\overline{f}_{ki} SF[\overline{\pi}_{ki}, \overline{f}_{ki}] f_{ki}$;while $\overline{\pi}_{ki} S\Pi[\overline{f}_{ki}]\pi_{ki}$.

In words, $\overline{f}_{ki}$ is bigger than f_{ki} and, given $\overline{\pi}_{ki}$ and $\overline{f}_{ki}$, $\overline{f}_{ki}$ is perceived to be *not similar* to f_{ki}; while, given $\overline{f}_{ki}$, $\overline{\pi}_{ki}$ is perceived to be *similar* to π_{ki}.U_β in Figure 1 is the area implied by this condition.

Condition δ : $\overline{\pi}_{ki} > \pi_{ki}$ and *no* $\overline{\pi}_{ki} S\Pi_{ki}[\overline{f}_{ki}]\pi_{ki}$; $\overline{f}_{ki} > f_{ki}$ and *no* $\overline{f}_{ki} SF[\overline{\pi}_{ki}, \overline{f}_{ki}] f_{ki}$.

That is, vector $(\overline{\pi}_{ki}, \overline{f}_{ki})$ is strictly bigger than (π_{ki}, f_{ki}) and no similarity is perceived in both instances. U_δ in figure 1 is the area implied by this condition.

Indifference: Whenever both expected payoffs and strategy proportions are perceived to be *similar*, then the two vectors will be declared *indifferent*; i.e. when $\overline{\pi}_{ki}S\Pi[\overline{f}_{ki}]\pi_{ki}$, $\pi_{ki}S\Pi[f_{ki}]\overline{\pi}_{ki}$, $\overline{f}_{ki}SF[\overline{\pi}_{ki},\overline{f}_{ki}]f_{ki}$ and $f_{ki}SF[\pi_{ki},f_{ki}]\overline{f}_{ki}$, then $(\overline{\pi}_{ki},\overline{f}_{ki})\sim_{ki}(\pi_{ki},f_{ki})$.

When none of these four situations takes place, then the two vectors would be non-comparable (see figure 1).

The distance to the aspiration set U depends on how thick the indifference set of $(\pi_{ki},\ f_{ki})$ is. We assume here that agents are preference-satisficers; that is, they choose a strategy to reduce the distance from $(\pi_{ki},\ f_{ki})$ to U. The *Herding agent* can achieve this by reducing doubts by means of playing more popular strategies and/or increasing expected payoffs. The smaller (greater) that distance the more satisfied (dissatisfied) the *ki*-agent will be with his current strategy. It can be seen that the properties (i) and (ii) of λ_{ki} capture the changes in the thickness of the indifference sets. Hence, the λ_{ki} function can be thought of as an index of how dissatisfied the *ki*-agent is with his current strategy. Notice that a *doubt-less agent*'s indifference classes will consists of almost singletons: $\sim[(\pi_{ki},\ f_{ki})]\cong(\pi_{ki},\ f_{ki})$ conveying the idea that with almost no doubts about the goodness of the current strategy, the *doubt-less agent* feels very satisfied and, very likely, will not switch to a different strategy.

The S*keptical agent* will have indifference sets that will cover the entire choice space because the similarity intervals on both the Π_{ki} and F_{ki} spaces are degenerate. Thus, we will say that the *Skeptical agent*'s preference relation is *degenerate*. Thus, for tis type of agent any pair of expected payoffs will be similar, as well as any pair of strategy frequencies. Hence, in terms of preferences, the agent will not perceive real differences between any two different vectors in $\Pi_{ki}\times F_{ki}$ and he will declare to be indifferent among them. Thus, having the thickest indifference sets that are possible, the upper contour sets (i.e agents's aspiration set) will appear to be unreachable and the *Skeptical* agent will feel highly dissatisfied.

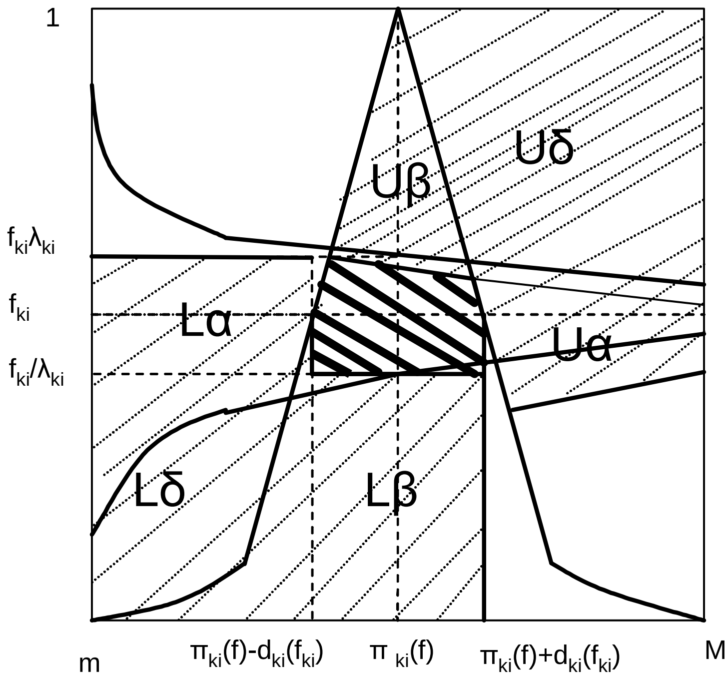

Figure 1: Preference-indifference relation compatible with correlated similarities. Relative to $(\pi_{ki},\, f_{ki})$, $U = U_\alpha \cup U_\beta \cup U_\delta$ denotes the upper-contour (or aspiration) set, $L = L_\alpha \cup L_\beta \cup L_\delta$ the lower contour set and the darker area is the indifference set.

Place here Figure 1

B. The Index of Dissatisfied Agents

Given the expected payoffs and the frequencies attached to each of the pure strategies of population k, we propose the index of dissatisfied agents with pure strategy i to be represented by the agent ki's dissatisfaction level relative to the total dissatisfaction level of population k (or, the agent kís threshold divided by player population ḱs threshold on F_{ki}):

$$\alpha_{ki} = \alpha_{ki}(\pi_{ki}, f_{ki}, \pi_{kj}, f_{kj}) = \frac{\lambda_{ki}}{\sum_{i=1}^{m_k} \lambda_{ki}} = \frac{\lambda_{ki}}{\lambda_k},\ i \neq j$$

To avoid the use of different doubt parameters, we will only assume that either they are all *herding agents* or *skeptical* ones; no mixed populations of doubtful agents are allowed. Further, we assume that the *herding doubts agents* do perceive the changes in payoffs and frequencies. But we cannot assume the same for the *skeptical agents*. This is so because, as said above, the *skeptical agent* has similarity intervals that are degenerate and, as a consequence, in terms of preferences, he does not distinguish between any two different vectors in $\Pi_{ki} \times F_{ki}$. Therefore, the stimulus intensity received by this agent from any vector $(\pi_{ki},\ f_{ki})$ would be the same and hence the response probability is the same for each strategy. Furthermore, since a *Cartesian skeptical* agent is endowed with *universal doubts* (see section 5), he will always be dissatisfied and continuously experimenting with every available strategy, no matter the level of payoffs and popularity attached to each strategy. For this reason, we may say that this type of agents react in a "non-standard" way to the changes in the expected payoffs and strategy frequencies.

The properties of α_{ki} follow naturally from those of λ_{ki}; hence, the properties of α_{ki} for the *herding agent ki* are the following:

1. The proportion of dissatisfied agents with their current pure strategy $i \in S_k$ will decrease if expected payoffs to strategy $i \in S_k$, π_{ki}, increase.

$$\frac{\partial \alpha_{ki}}{\partial \pi_{ki}} = \frac{\frac{\partial \lambda_{ki}}{\partial \pi_{ki}} \lambda_k - \frac{\partial \lambda_{ki}}{\partial \pi_{ki}} \lambda_{ki}}{\lambda_k^2} = \frac{\frac{-d_{ki}(f_{ki})}{(\pi_{ki} - d_{ki}(f_{ki}))^2} (\lambda_k - \lambda_{ki})}{\lambda_k^2} < 0$$

2. The proportion of dissatisfied agents with their current pure strategy $i \in S_k$ should increase if expected payoffs to strategy $j \in S_k$, π_{kj}, increase.

$$\frac{\partial \alpha_{ki}}{\partial \pi_{kj}} = \frac{-\frac{\partial \lambda_{kj}}{\partial \pi_{kj}} \lambda_{ki}}{\lambda_k^2} = \frac{\frac{-d_{kj}(f_{kj})}{\left(\pi_{kj} - d_{kj}(f_{kj})\right)^2} (-\lambda_{ki})}{\lambda_k^2} > 0$$

3. If agents ki's doubts decrease, because the popularity of strategy i,

f_{ki}, has increased, the proportion of dissatisfied should decrease too.

$$\frac{\partial \alpha_{ki}}{\partial f_{ki}} = \frac{\frac{\partial \lambda_{ki}}{\partial f_{ki}}\lambda_k - \frac{\partial \lambda_{ki}}{\partial f_{ki}}\lambda_{ki}}{\lambda_k^2} = \frac{\frac{\pi_{ki}\frac{\partial d_{ki}(f_{ki})}{\partial f_{ki}}}{(\pi_{ki}-d_{ki}(f_{ki}))^2}(\lambda_k - \lambda_{ki})}{\lambda_k^2} < 0$$

4. If the popularity of strategy $j \in S_k$, f_{kj}, increases, the proportion of dissatisfied agents with their current pure strategy $i \in S_k$ should increase.

$$\frac{\partial \alpha_{ki}}{\partial f_{kj}} = \frac{-\frac{\partial \lambda_{kj}}{\partial f_{kj}}\lambda_{ki}}{\lambda_k^2} = \frac{\frac{\pi_{kj}\frac{\partial d_{kj}(f_{kj})}{\partial f_{kj}}}{\left(\pi_{kj}-d_{kj}(f_{kj})\right)^2}(-\lambda_{ki})}{\lambda_k^2} > 0$$

C. Proofs of propositions

Let

	(y) L	R
(x)U	a_{11}, b_{11}	a_{12} , b_{12}
D	a_{21}, b_{21}	a_{22}, b_{22}

denote the 2×2 constant-sum game G, and $I^* \equiv [(x^*, 1-x^*), (y^*, 1-y^*)]$,with $x^* > 0$ and $y^* > 0$, the Mixed strategy Nash Equilibrium of G. To get this equilibrium, we may assume, without loss of generality, that $a_{11} > a_{21}$, $b_{11} < b_{12}$, $a_{12} < a_{22}$, and $b_{22} < b_{21}$. Recall that payoffs are normalized so that they take values on $[m, M]$. To avoid the use of four different doubt parameters, we shall assume that the four doubt functions are the same: $d_D = d_U = d_R = d_L = d$. The doubt-based selection dynamics (for definition (a) of λ_{ki}) are represented by the following system:

$$\begin{aligned}
\dot{x} &= \frac{x(1-x)}{\pi_U(\pi_D - d_D) + \pi_D(\pi_U - d_U)}(\pi_U d_D - \pi_D d_U) \text{...} C.1 \\
&= \frac{x(1-x)}{\pi_U(\pi_D - d_D) + \pi_D(\pi_U - d_U)}((a_{11}y + a_{12}(1-y))d_D(1-x) - (a_{21}y + a_{22}(1-y))d_U(x)) \\
&\equiv G_1(x,y)F_1(x,y)
\end{aligned}$$

$$\begin{aligned}\dot{x} &= \frac{x(1-x)}{\pi_U(\pi_D - d_D) + \pi_D(\pi_U - d_U)}(\pi_U d_D - \pi_D d_U) \text{..} C.2 \\ &= \frac{x(1-x)}{\pi_U(\pi_D - d_D) + \pi_D(\pi_U - d_U)}((a_{11}y + a_{12}(1-y))d_D(1-x) - (a_{21}y + a_{22}(1-y))d_U(x)) \\ &\equiv G_2(x,y)F_2(x,y)\end{aligned}$$

Proof of Proposition 2:

1. We must first show that a Mixed Strategy Nash Equilibrium (MSNE) converges to a Mixed Strategy Doubt-Full Equilibrium (MSDFE) as δ converges to zero in the class of doubt functions $D^{1-\delta} \subset D_M$.

An interior rest point of C.1-C.2, (i.e. a MSDE), satisfies:

$$\begin{aligned}(a_{11}y + a_{12}(1-y))\, d_D(1-x) - (a_{21}y + a_{22}(1-y))\, d_U(x) &= 0 \\ (b_{11}x + b_{21}(1-x))\, d_R(1-y) - (b_{12}x + b_{22}(1-x))\, d_L(y) &= 0\end{aligned}$$

Then, if $d_i \in D^{1-\delta}$ for $i \in \{U, D, L, R\}$,

$$\lim_{\delta \to 0} \frac{d_U(x)}{d_D(1-x)} = \lim_{\delta \to 0} \frac{d_L(y)}{d_R(1-y)} = 1, \text{ for all } (x,y) \in (0,1) \times (0,1)$$

Now suppose that we are in the MSNE, $(x^*, y^*) \in (0,1) \times (0,1)$, of G and that $d_i \in D^{1-\delta}$. Then, the strategies available to each player get the same expected payoff; that is $a_{11}y^* + a_{12}(1-y^*) = a_{21}y^* + a_{22}(1-y^*)$ and $b_{11}x^* + b_{21}(1-x^*) = b_{12}x^* + b_{22}(1-x^*)$. Thus,

$$\lim_{\delta \to 0} \frac{(a_{11}y^* + a_{12}(1-y^*))\, d_D(1-x^*)}{(a_{21}y^* + a_{22}(1-y^*))\, d_U(x^*)} = \lim_{\delta \to 0} \frac{(b_{11}x^* + b_{21}(1-x^*))\, d_R(1-y^*)}{(b_{12}x^* + b_{22}(1-x^*))\, d_L(y^*)} = 1$$

This, plus continuity, establishes the result.

2. We show that for all $(x', y') \in (0,1) \times (0,1)$, there exists a sequence of functions $d^\delta \in D^\delta$ and a δ' low enough that the rest point of C.1-C.2 cannot be any $C \neq [(x', 1-x'), (y', 1-y')]$ for any $\delta \leq \delta'$ and then the result follows.

An interior rest point of C.1-C.2 must satisfy:

$$\begin{aligned}(a_{11}y + a_{12}(1-y))\, d_D(1-x) - (a_{21}y + a_{22}(1-y))\, d_U(x) &= 0 \\ (b_{11}x + b_{21}(1-x))\, d_R(1-y) - (b_{12}x + b_{22}(1-x))\, d_L(y) &= 0\end{aligned}$$

which implies that

$$\begin{aligned}\left(a_{11}y+a_{12}\left(1-y\right)\right)\frac{d_D\left(1-x\right)}{d_U\left(x\right)}-\left(a_{21}y+a_{22}\left(1-y\right)\right)&=0\\ \left(b_{11}x+b_{21}\left(1-x\right)\right)\frac{d_R\left(1-y\right)}{d_L\left(y\right)}-\left(b_{12}x+b_{22}\left(1-x\right)\right)&=0\end{aligned}$$

Let first $x'\leq 1/2$. We construct the doubt functions $d_U(x)$ and $d_D(1-x)$ in D_m as follows:

$$d^{x'}_{ki}(f_{ki})=\begin{cases} m\delta\left(1-f_{ki}\right) & \text{if } f_{ki}\leq x' \\ m\delta\left(1-f_{ki}\right)\frac{(1-f_{ki})^{1/\delta}}{(1-x')^{1/\delta}} & \text{if } f_{ki}>x'\end{cases}$$

where $ki\in\{U,D\}$ and $\delta>0$. Note that as δ approaches 0, the graph of $d^{x'}_{ki}$ function approaches the horizontal axis and the agent is said to be in a *doubt-less* mode.

Now, for $x>x'$

$$\frac{d_D\left(1-x\right)}{d_U\left(x\right)}=\frac{mx}{m\left(1-x\right)\frac{(1-x)^{1/\delta}}{(1-x')^{1/\delta}}}=\frac{x}{1-x}\left(\frac{1-x'}{1-x}\right)^{1/\delta}$$

Since $1-x'>1-x$ we can make $\left(\frac{1-x'}{1-x}\right)^{1/\delta}$ as big as we want by choosing a sufficiently small δ. Then

$$\frac{x}{1-x}\left(\frac{1-x'}{1-x}\right)^{1/\delta}>\frac{\left(a_{21}y+a_{22}\left(1-y\right)\right)}{\left(a_{11}y+a_{12}\left(1-y\right)\right)}$$

Hence,

$$\left(a_{11}y+a_{12}\left(1-y\right)\right)\frac{x}{1-x}\left(\frac{1-x'}{1-x}\right)^{1/\delta}-\left(a_{21}y+a_{22}\left(1-y\right)\right)>0$$

Now, for $x<x'$

$$\frac{d_D\left(1-x\right)}{d_U\left(x\right)}=\frac{mx\left(\frac{x}{1-x'}\right)^{1/\delta}}{m\left(1-x\right)}=\frac{x}{1-x}\left(\frac{x}{1-x'}\right)^{1/\delta}$$

since $x<x'\leq 1/2$, we have that $1-x'>x$ so we can make $\left(\frac{x}{1-x'}\right)^{1/\delta}$ as small as we want by choosing a sufficiently small δ. Then

$$\frac{x}{1-x}\left(\frac{x}{1-x'}\right)^{1/\delta}<\frac{\left(a_{21}y+a_{22}\left(1-y\right)\right)}{\left(a_{11}y+a_{12}\left(1-y\right)\right)}$$

Hence

$$(a_{11}y + a_{12}(1-y))\frac{x}{1-x}\left(\frac{x}{1-x'}\right)^{1/\delta} - (a_{21}y + a_{22}(1-y)) < 0$$

When $x' > 1/2$ let $d_U(x)$ and $d_D(1-x)$ in D as follows:

$$d_{ki}^{x'}(f_{ki}) = \begin{cases} m\delta(1-f_{ki}) & \text{if } f_{ki} \le x' \\ m\delta(1-f_{ki})\frac{(1-f_{ki})^{1/\delta}}{x'^{1/\delta}} & \text{if } f_{ki} > x' \end{cases}$$

where $ki \in \{U, D\}$ and $\delta > 0$. Now, for $x > x'$

$$\frac{d_D(1-x)}{d_U(x)} = \frac{mx}{m(1-x)\frac{(1-x)^{1/\delta}}{x'^{1/\delta}}} = \frac{x}{1-x}\left(\frac{x'}{1-x}\right)^{1/\delta}$$

Since $x > x' > 1/2$, $1-x < 1/2$ we can make$\left(\frac{x'}{1-x}\right)^{1/\delta}$ as big as we want by choosing a sufficiently small δ. Then

$$\frac{x}{1-x}\left(\frac{x'}{1-x}\right)^{1/\delta} > \frac{(a_{21}y + a_{22}(1-y))}{(a_{11}y + a_{12}(1-y))}$$

Hence,

$$(a_{11}y + a_{12}(1-y))\frac{x}{1-x}\left(\frac{x'}{1-x}\right)^{1/\delta} - (a_{21}y + a_{22}(1-y)) > 0$$

For $x < x'$

$$\frac{d_D(1-x)}{d_U(x)} = \frac{mx\left(\frac{x}{x'}\right)^{1/\delta}}{m(1-x)} = \frac{x}{1-x}\left(\frac{x}{x'}\right)^{1/\delta}$$

Since $x < x'$, we can make$\left(\frac{x}{x'}\right)^{1/\delta}$ as small as we want by choosing a sufficiently small δ. Then

$$\frac{x}{1-x}\left(\frac{x}{x'}\right)^{1/\delta} < \frac{(a_{21}y + a_{22}(1-y))}{(a_{11}y + a_{12}(1-y))}$$

Hence

$$(a_{11}y + a_{12}(1-y))\frac{x}{1-x}\left(\frac{x}{x'}\right)^{1/\delta} - (a_{21}y + a_{22}(1-y)) < 0$$

The argument for y is analogous. ■

Proof of Proposition 3

Let $I^* \equiv [(x^*, 1-x^*), (y^*, 1-y^*)] \in (0,1) \times (0,1)$ be an interior Mixed Strategy Nash Equilibrium (MSNE) of G. In this equilibrium, expected payoffs are equalized across strategies; that is, $\pi_U = \pi_D$ and $\pi_L = \pi_R$. From Proposition 2, we also know that an MSNE is a Mixed Strategy Doubt-Full equilibrium (MSDFE); that is, $\pi_U d_D(1-x^*) = \pi_D d_U(x^*)$ and $\pi_L d_R(1-y^*) = \pi_R d_L(y^*)$. Hence, an interior MSNE is a stationary state of the system C.1-C.2 if all agents are *Cartesian Skeptical*.

Thus, $F_1(x^*, y^*) = 0$ and $F_2(x^*, y^*) = 0$, where

$$\begin{aligned} F_1(x,y) &= \pi_U d_D(1-x) - \pi_D d_U(x) \\ &= (a_{11}y + a_{12}(1-y))d_D(1-x) - (a_{21}y + a_{22}(1-y))d_U(x) \\ F_2(x,y) &= \pi_L d_R(1-y) - \pi_R d_L(y) \\ &= (b_{11}x + b_{21}(1-x))d_R(1-y) - (b_{12}x + b_{22}(1-x))d_L(y) \end{aligned}$$

and

$$\begin{aligned} \frac{\partial F_1(x,y)}{\partial x} &= \pi_U \frac{\partial d_D(1-x)}{\partial x} - \pi_D \frac{\partial d_U(x)}{\partial x} \\ \frac{\partial F_1(x,y)}{\partial y} &= (a_{11} - a_{12})\, d_D(1-x) + (a_{22} - a_{21})\, d_U(x) \\ \frac{\partial F_2(x,y)}{\partial x} &= (b_{11} - b_{21})\, d_R(1-y) + (b_{22} - b_{12})\, d_L(y) \\ \frac{\partial F_2(x,y)}{\partial y} &= \pi_L \frac{\partial d_R(1-y)}{\partial y} - \pi_R \frac{\partial d_L(y)}{\partial y} \end{aligned}$$

On the other hand, the Jacobian $J(x,y)$ of the dynamic system C.1-C.2 evaluated at the steady state (x^*, y^*) is:

$$J(x^*, y^*) = \begin{bmatrix} G_1(x^*, y^*) \left.\frac{\partial F_1(x,y)}{\partial x}\right|_{I^*} & G_1(x^*, y^*) \left.\frac{\partial F_1(x,y)}{\partial y}\right|_{I^*} \\ G_2(x^*, y^*) \left.\frac{\partial F_2(x,y)}{\partial x}\right|_{I^*} & G_2(x^*, y^*) \left.\frac{\partial F_2(x,y)}{\partial y}\right|_{I^*} \end{bmatrix}$$

In an MSNE, $\pi_U = \pi_D$, $\pi_L = \pi_R$. If, on the other hand, agents are playing in a *doubt-full mode*, (that is, $d_i \in D^{1-\delta}$ for $i \in \{U, D, L, R\}$ with $\lim_{\delta \to \delta^*} d_U(x) = \lim_{\delta \to \delta^*} d_D(1-x) = \lim_{\delta \to \delta^*} d_L(y) = \lim_{\delta \to \delta^*} d_R(1-y)$ and being nearly 1, for all $(x, y) \in (0,1) \times (0,1)$; $\delta^* > 0$ but nearly zero, as in Proposition 2). Then, writing $d_i(.) = 1$, we would also have $\pi_U d_D = \pi_D d_U$ and $\pi_L d_R = \pi_R d_L$.

Hence, in an MSNE as an MSDFE :

$$\begin{aligned} G_1(x^*, y^*) &= \frac{x^*(1-x^*)}{2\pi_U\pi_D - \pi_U d_D(1-x^*) - \pi_D d_U(x^*)} \\ &= \frac{x^*(1-x^*)}{\pi_U(2\pi_U - d_D(1-x^*) - d_U(x^*))} \\ &= \frac{x^*(1-x^*)}{2\pi_U(\pi_U - 1)} \\ G_2(x^*, y^*) &= \frac{y^*(1-y^*)}{2\pi_L(\pi_L - 1)} \end{aligned}$$

Thus, the elements of the Jacobian matrix are the following:

$$\begin{aligned} j_{11} &= G_1(x^*, y^*) \left. \frac{\partial F_1(x, y)}{\partial x} \right|_{I^*} \\ &= \frac{x^*(1-x^*)}{2\pi_U(\pi_U - 1)} \left(\pi_U \frac{\partial d_D(1-x)}{\partial x} - \pi_U \frac{\partial d_U(x)}{\partial x} \right)_{I^*} \\ &= \frac{x^*(1-x^*)}{2(\pi_U - 1)} \left(\frac{\partial d_D(1-x)}{\partial x} - \frac{\partial d_U(x)}{\partial x} \right)_{I^*} \end{aligned}$$

$$\begin{aligned} j_{12} &= G_1(x^*, y^*) \left. \frac{\partial F_1(x, y)}{\partial y} \right|_{I^*} \\ &= \frac{x^*(1-x^*)}{2\pi_U(\pi_U - 1)} ((a_{11} - a_{12})\, d_D(1-x^*) + (a_{22} - a_{21})\, d_U(x^*)) \end{aligned}$$

$$\begin{aligned} j_{21} &= G_2(x^*, y^*) \left. \frac{\partial F_2(x, y)}{\partial y} \right|_{I^*} \\ &= \frac{y^*(1-y^*)}{2\pi_L(\pi_L - 1)} ((b_{11} - b_{21})\, d_R(1-y^*) + (b_{22} - b_{12})\, d_L(y^*)) \end{aligned}$$

$$\begin{aligned} j_{22} &= G_2(x^*,y^*)\left.\frac{\partial F_2(x,y)}{\partial y}\right|_{I^*} \\ &= \frac{y^*(1-y^*)}{2(\pi_L-1)}\left(\frac{\partial d_R(1-y)}{\partial y}-\frac{\partial d_L(y)}{\partial y}\right)_{I^*} \end{aligned}$$

Recall that the real part of the eigenvalues of $J(x^*,y^*)$ only depends on the sum of the diagonal terms (the trace of the matrix):

$$\begin{aligned} \text{Trace of } J(x^*,y^*) &= G_1(x^*,y^*)\left.\frac{\partial F_1(x,y)}{\partial x}\right|_{I^*}+G_2(x^*,y^*)\left.\frac{\partial F_2(x,y)}{\partial y}\right|_{I^*} \\ &= \frac{x^*(1-x^*)}{2(\pi_U-1)}\left(\frac{\partial d_D(1-x)}{\partial x}-\frac{\partial d_U(x)}{\partial x}\right)_{I^*} \\ &\quad+\frac{y^*(1-y^*)}{2(\pi_L-1)}\left(\frac{\partial d_R(1-y)}{\partial y}-\frac{\partial d_L(y)}{\partial y}\right)_{I^*} \end{aligned}$$

Since the expected values $\pi_U = a_{11}y^*+a_{12}(1-y^*)$ and $\pi_L = b_{11}x^*+b_{21}(1-x^*)$ are smaller than 1, both $\frac{x^*(1-x^*)}{2(\pi_U-1)}$ and $\frac{y^*(1-y^*)}{2(\pi_L-1)}$ are negative. The sign of $\left(\frac{\partial d_D(1-x)}{\partial x}-\frac{\partial d_U(x)}{\partial x}\right)_{I^*}$ and $\left(\frac{\partial d_R(1-y)}{\partial y}-\frac{\partial d_L(y)}{\partial y}\right)_{I^*}$ is clearly positive (that is, the signs of the derivatives of $d_D(1-x)$ and $d_R(1-y)$ with respect to x and y, respectively, are positive and those of $d_U(x)$ and $d_L(y)$ are negative). Thus, $j_{11}<0$ and $j_{22}<0$ and so the sign of the trace is negative

$$sign\ \left[G_1(x^*,y^*)\left.\frac{\partial F_1(x,y)}{\partial x}\right|_{I^*}+G_2(x^*,y^*)\left.\frac{\partial F_2(x,y)}{\partial y}\right|_{I^*}\right]<0$$

Without loss of generality, we may assume, for an interior equilibrium, that $a_{11}>a_{21}$, $b_{11}<b_{12}$, $a_{12}<a_{22}$, and $b_{22}<b_{21}$. Then it can be seen that the sign of $j_{21}\times j_{12}$ is negative, when the agents are playing in the *absent* or *doubt-full mode:*

$$j_{21}\times j_{12}=\left(\frac{y^*(1-y^*)}{2\pi_L(\pi_L-1)}((b_{11}-b_{12})+(b_{22}-b_{21}))\right)\times\left(\frac{x^*(1-x^*)}{2\pi_U(\pi_U-1)}((a_{11}-a_{21})+(a_{22}-a_{12}))\right)<0$$

Thus, the determinant associated to $J(x^*,y^*)$, Det $J(x^*,y^*)=j_{11}\times j_{22}-j_{21}\times j_{12}$, has a positive sign. Therefore, when every agent is *Cartesian* *skeptical*, the MSNE, $I^* \equiv [(x^*, 1-x^*), (y^*, 1-y^*)]$, is a *sink* and therefore is an asymptotically stable equilibrium.■

Proof of Proposition 4:

Using the same procedure as in Proposition 3, we can easily prove that, under the *doubt-less* mode, the MSDLE $[(1/2, 1/2), (1/2, 1/2)]$ is a *source*. Now, to see the trajectory of initial points different from $[(1/2, 1/2), (1/2, 1/2)]$, we might use the doubt function constructed for the proof of part 2 of Proposition 2.

Note that the denominators of C.1-C.2 are positive in the *doubt-less mode* of play. Hence the sign of $\dot{x}$ and $\dot{y}$ depend on the sign of $(\pi_U d_D - \pi_D d_U)$ and $(\pi_L d_R - \pi_R d_L)$, respectively. Now we can proceed as in the proof of part 2 of Proposition 2.

Let first $x' \leq 1/2$. We construct the doubt functions $d_U(x)$ and $d_D(1-x)$ in D_m as follows:

$$d_{ki}^{x'}(f_{ki}) = \begin{cases} m\delta(1-f_{ki}) & \text{if } f_{ki} \leq x' \\ m\delta(1-f_{ki})\frac{(1-f_{ki})^{1/\delta}}{(1-x')^{1/\delta}} & \text{if } f_{ki} > x' \end{cases}$$

This means that if $x > x'$

$$sign\left[\dot{x}\right] = sign\left[\left(\pi_U - \pi_D\frac{(1-x)^{1/\delta}}{(1-x')^{1/\delta}}\right)\right]$$

Then there is a δ' low enough such that for all $0 < \delta \leq \delta'$, $(1-x)^{1/\delta} / (1-x')^{1/\delta}$ is sufficiently small so that $sign\left[\dot{x}\right] > 0$ and hence if $x(0) > x'$, then $\lim_{t\to\infty} x(t) = 1$.

If on the other hand $x < x'$

$$sign\left[\dot{x}\right] = sign\left[\left(\pi_U\frac{x^{1/\delta}}{(1-x')^{1/\delta}} - \pi_D\right)\right]$$

Since $x < x' \leq 1/2$, we have that $1 - x' > x$ so there is a δ' low enough such that for all $0 < \delta \leq \delta'$, $x^{1/\delta} / (1-x')^{1/\delta}$ is sufficiently small so that $sign\left[\dot{x}\right] < 0$ and hence if $x(0) < x'$, then $\lim_{t\to\infty} x(t) = 0$.

When $x' > 1/2$, we let $d_U(x)$ and $d_D(1-x)$ in D as follows:

$$d_{ki}^{x'}(f_{ki}) = \begin{cases} m\delta(1-f_{ki}) & \text{if } f_{ki} \leq x' \\ m\delta(1-f_{ki})\frac{(1-f_{ki})^{1/\delta}}{x'^{1/\delta}} & \text{if } f_{ki} > x' \end{cases}$$

This means that if $x > x'$

$$sign\left[\dot{x}\right] = sign\left[\left(\pi_U - \pi_D \frac{(1-x)^{1/\delta}}{x'^{1/\delta}}\right)\right]$$

Since $x > x' > 1/2$, $1 - x < 1/2$, there is a $\delta^{'}$ low enough such that for all $0 < \delta \leq \delta^{'}$, we can make$(x')^{1/\delta} / (1-x)^{1/\delta}$ is sufficiently big so that $sign\left[\dot{x}\right] > 0$ and hence if $x(0) > x'$, then $\lim_{t\to\infty} x(t) = 1$.

If on the other hand $x < x'$

$$sign\left[\dot{x}\right] = sign\left[\left(\pi_U \frac{x^{1/\delta}}{x'^{1/\delta}} - \pi_D\right)\right]$$

Since $x < x'$, there is a $\delta^{'}$ low enough such that for all $0 < \delta \leq \delta^{'}$, $x^{1/\delta}/x'^{1/\delta}$ is sufficiently small so that $sign\left[\dot{x}\right] < 0$ and hence if $x(0) < x'$, then $\lim_{t\to\infty} x(t) = 0$. The argument for y is analogous.■

Proof Proposition 5:

(a) Let $S_k = \{1, 2\}$ be player population k's strategy set. Without loss of generality, let us refer to the dynamics of strategy 1. Then, by equation (7), we have

$$\begin{aligned} \dot{f}_{k1} &= f_{k1}\overline{\phi_k} - \phi_{k1}\phi_k \\ &= \frac{\epsilon_k}{D(f)} f_{ki}[\pi_{ki} - \overline{\pi}_k] \end{aligned}$$

where $D(f) \equiv \pi_{k1}(\pi_{k2} - \epsilon_k) + \pi_{k2}(\pi_{k1} - \epsilon_k) > 0$.

Hence, the growth rates $\frac{\dot{f}_{ki}}{f_{ki}}$ equal payoff differences $[\pi_{ki} - \overline{\pi}_k]$ multiplied by a (Lipschitz) continuous, positive function $\frac{\epsilon_k}{D(f)}$. This concludes the proof. (Note that, given ϵ_k, a payoff difference $[\pi_{ki} - \overline{\pi}_k]$ will have stronger dynamic effect if $D(f)$ is low than if it is high; if ϵ_k decreases, the dynamic effect of $[\pi_{ki} - \overline{\pi}_k]$ decreases).

(b) Easy.■