Local Incentive Compatibility in Moral Hazard Problems: A Unifying Approach*

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Abstract

I suggest a unifying new approach to moral hazard. Once local incentive compatibility (L-IC) is satisfied, the problem of verifying global incentive compatibility (G-IC) is shown to be isomorphic to the well-understood problem of comparing two classes of distribution functions. In the one-signal case, the sufficient conditions for the validity of the first-order approach (FOA) provided by Rogerson and Jewitt are related to first and second order stochastic dominance, respectively. New conditions relying on other stochastic orders are presented. Conlon's multi-signal justifications can be related to particular multivariate extensions of the usual stochastic orders. However, there are several ways in which these orders can be extended into higher dimensions. New multi-signal conditions that rely on the more tractable orthant orders are thus provided. When the standard FOA is invalid it may be possible to construct a valid "modified" FOA. The modified FOA correctly solves Mirrlees' famous counterexample.

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1 Introduction

The principal-agent model of moral hazard is among the core models of microeconomic theory and central to the economics of information. The problem is conceptually simple; a principal must design a contract to induce the agent to take the desired action. From the agent's point of view the intended action must be made preferable to all other actions. Thus, a multitude of incentive compatibility constraints must be satisfied. Unfortunately, it is generally difficult to determine which constraints bind and to make robust predictions about the structure of optimal contracts.

In response, much of the literature has focused on environments where the only binding constraint is the "local" incentive compatibility constraint (L-IC). In such cases, ensuring the agent has no incentive to deviate marginally from the intended action guarantees global incentive compatibility (G-IC), i.e. larger deviations can be ruled out too. Indeed, the classic first-order approach (FOA) simply uses the agent's first-order condition to summarize G-IC. The optimal contract is then easily derived. The FOA has a long history, dating back to Holmström (1979) and Mirrlees (1976, 1999). Rogerson (1985) and Jewitt (1988) have provided sufficient conditions under which the FOA is valid. However, although there are similarities in the structure of their proofs, the techniques they use are quite different. Moreover, despite criticizing the stringency of his assumptions, most textbooks on the topic prove Rogerson's result, but, as Conlon (2009a) observes, none even state Jewitt's. In short, Jewitt's result may be underappreciated and there is little in the current literature to unify the two results. Similarly, Conlon (2009a) uses two different approaches to obtain his generalizations of Rogerson's and Jewitt's conditions to multi-signal environments.[1,2]

With these observations in mind, the primary objective of this paper is to propose an accessible and unifying approach to the moral hazard problem. From this methodological contribution flows two distinct sets of insights that enable previous results to be extended in several different directions. First, it provides a unified methodology to understand Rogerson's, Jewitt's, and Conlon's classic results on the validity of the

[1] An earlier paper by Sinclair-Desgagné (1994) also extended Rogerson's conditions to the multi-signal model. However, Conlon (2009a) relaxes Sinclair-Desgagné's assumptions. Jewitt (1988) also offered two different multi-signal justifications of the FOA. Conlon further generalized one of these.

[2] Ke (2012a) proposes a fixed-point method for justifying the FOA. Araujo and Moreira (2001) propose a general Lagrangian approach to solve moral hazard problems when the FOA is not valid. See also Ke (2012b).

FOA.[3] There is a common thread to these results. Once this is identified, it becomes a straightforward matter to provide new justifications of the FOA. Secondly, it is also possible to obtain insights into environments where the FOA is not valid.

The approach relies on "translating" the problem of verifying global incentive compatibility into a problem that is familiar to, and well-understood by, any economist. In particular, I will show that checking G-IC (once L-IC is satisfied) is isomorphic to the problem of comparing two classes of risky prospects, or two classes of distribution functions. Given this equivalence, many of the results follow by simply calling upon well-known results from the literature on stochastic dominance. The remainder of this introduction outlines the main results.

Any contract translates into a distribution of wages (where the distribution is determined in part by the agent's action). For brevity, I will refer to a contract as monotonic if the agent's utility is nondecreasing in the outcome or state. A contract is concave if the agent's utility is concave in the state. With this terminology, Rogerson's (1985) and Jewitt's (1988) proofs can be decomposed into two concise parts. In Rogerson's case, the first part is to identify conditions under which any monotonic and L-IC contract is also G-IC. The second part is then to identify additional conditions under which the candidate contract is in fact monotonic. In Jewitt's case, contracts are both monotonic and concave.

The two first columns in the top row of Table 1 summarize the conclusions in step 1 of Rogerson and Jewitt, respectively. For future reference, the third column identifies a natural extension. In comparison, the second row summarizes the notions of first, second, and third order stochastic dominance (FOSD, SOSD, and TOSD, respectively) between two lotteries, G and H.[4] Note that Jewitt weakens Rogerson's assumption on the distribution function, but in exchange has to strengthen the assumptions imposed on the shape of the contract. This trade-off is remarkably similar to the one encountered when FOSD and SOSD are compared. This is of course no

[3] Jewitt's (1988) original proof is made complicated by the fact that it relies on results in an unpublished working paper. The full proof is published in Conlon (2009b). In the existing literature, Conlon (2009a) comes closest to methodologically unifying Rogerson's and Jewitt's results. Specifically, Conlon (2009a, footnote 7) observes that Rogerson's proof relies on integration by parts, and that a second round of integration by part can be used to prove Jewitt's result. He does not ask, for instance, what can be obtained from further rounds of integration by parts. As mentioned, Conlon's (2009a) multi-signal results rely on two different approaches.

[4] See Hadar and Russell (1969), Rothschild and Stiglitz (1970), Whitmore (1970), and Menezes et al (1980). For textbooks on stochastic orders, see Müller and Stoyan (2002) and Shaked and Shantikumar (2007).

coincide, and much can be gained from exploring the relationship between the two rows in the table. As the third column reveals, once the pattern is identified it is easy to develop a third set of conditions to validate the FOA.[5] Indeed, infinitely many extensions to higher order stochastic dominance are possible. Moreover, by appealing to related stochastic orders (the increasing convex orders), it is possible to obtain another infinite sequence of justifications of the FOA in which contracts are convex.

[TABLE 1 ABOUT HERE (SEE THE LAST PAGES)]

Conlon (2009a) generalizes Rogerson's and Jewitt's conditions to the multi-signal model. Both Jewitt and Conlon encounter obstacles in the multi-signal model. For instance, it is not trivial to check Conlon's conditions. These difficulties can be explained by the direction in which Jewitt and Conlon seek to extend the results into higher dimensions. However, it turns out that there are several ways in which FOSD and SOSD can be extended from one dimension to many dimensions. Some are more tractable than others. This simple insight immediately leads to a number of new multivariate justifications of the FOA. Central to these new results are the so-called orthant orders.[6] There are a number of advantages to these new justifications. For instance, they are simpler to check. Moreover, one of the new justifications handles the important special case in which signals are independent particularly well.

Conlon emphasizes that his so-called CISP condition need not hold if signals are independent and each signal separately satisfies Rogerson's convexity assumption. Indeed, CISP must fail if there are sufficiently many i.i.d. signals. Conlon explains this failure with the observation that "with many signals, the principal tends to become very well informed about the agent's action and, even in the one-signal case, [Rogerson's condition] must fail when the signal becomes very accurate". While the logic is compelling, it turns out that one of the new justification of the FOA proposed in this paper in fact applies to any situation with independent signals, as long as each signal separately complies with Rogerson's assumption. Interestingly, the implication is that the FOA may not be valid with a single very accurate signal, but that it may be valid with a multitude of inaccurate independent signals, even though the latter in combination provide very precise information.

[5] To make the second step in the proof work, assumptions on the agent's utility function and on the likelihood ratio are also needed. There is an appealing pattern in those assumptions as well.

[6] Jewitt presents a second multi-signal justification for settings with two independent signals. This justification is in fact based on the lower orthant order, and is thus further generalized here.

It should be stressed that the new multivariate justifications should be seen as complements to Conlon's justifications. There are three natural ways in which FOSD can be extended from one to many signals. CISP essentially boils down to one such extension. As the preceding paragraph may suggest, the other two extensions impose weaker conditions on the distribution function. On the other hand, different assumptions on the likelihood ratio and the utility function must be made in each of the three multivariate extensions to Rogerson's conditions.[7] Similar observations apply to Conlon's generalization of Jewitt's conditions, which is based on one of the three possible multivariate extension of SOSD. This is discussed further in Section 5.

The second contribution of the paper is to examine environments in which the FOA is not necessarily valid. In general, some actions may not even be implementable. For implementable actions, L-IC may or may not be sufficient for G-IC. However, I characterize a set of actions for which L-IC is guaranteed to be sufficient for G-IC.

I then identify a model where L-IC is sufficient for G-IC for *any* implementable action. Here, the FOA can be applied on the "feasible set" of implementable actions, which is easily identified. This method of analysis is valid whenever Grossman and Hart's (1983) spanning condition is satisfied. Although this simple model was proposed three decades ago, no complete analysis has been offered until now.

As a special case, the modified FOA is valid in textbook settings with two states (but a continuum of actions). The method also easily solves Mirrlees' (1999) original counterexample, the purpose of which was to demonstrate how and why the standard FOA may fail. Indeed, a conceptually much simpler counterexample can be constructed using the insights of this part of the paper.

A natural extension of the spanning condition is briefly considered. This generalization encompasses a setting with three states, for example. Here, L-IC may not be sufficient for G-IC. Nevertheless, it turns out to be possible to identify a small set of sufficient "non-local" incentive compatibility constraints. In one special case, any L-IC contract is G-IC if and only if the agent has no incentive to deviate to the largest or the smallest action level. From an analytical point of view, one advantage of the model is that it is rather simple. It may therefore find future use as a tractable testbed for situations in which the FOA is not valid.

[7] John Conlon (private communication) has suggested that CISP and the new conditions presented in this paper be thought of as representing different regions of the frontier of conditions that justify the FOA. This observation leads to the question of whether these conditions can be combined to derive even more justifications for the FOA. This question is left for future research.

2 Model and preliminaries

A risk averse agent takes a costly action that is not verifiable to others. The set of possible actions is some closed and bounded interval, $[\underline{a}, \overline{a}]$. The agent's action determines the joint distribution of $n \geq 1$ verifiable signals, denoted $\mathbf{x} = (x_1, ..., x_n)$. If the action is a, the cumulative distribution function is $F(\mathbf{x}|a)$, where it is assumed that the domain, $\mathcal{X} = \times_{i=1}^n [\underline{x}_i, \overline{x}_i]$, is convex, compact and independent of a. Define $\underline{\mathbf{x}} = (\underline{x}_1, ..., \underline{x}_n)$ and $\overline{\mathbf{x}} = (\overline{x}_1, ..., \overline{x}_n)$.[8] It is assumed that $F(\mathbf{x}|a)$ has no mass points and is continuously differentiable in $\mathbf{x}$ and a to the requisite degree, with $f(\mathbf{x}|a)$ denoting the density for fixed a. Assume that $f(\mathbf{x}|a)$ is strictly positive. Let $\overline{F}(\mathbf{x}|a)$ denote the survival function, i.e. the probability that the vector of signals is greater than $\mathbf{x}$. Generally, $\overline{F}(\mathbf{x}|a) \neq 1 - F(\mathbf{x}|a)$ when there are two or more signals.

The agent faces a contract that, to him, is fixed. He receives wage $w(\mathbf{x})$ if the outcome is $\mathbf{x}$, in which case utility is $v(w(\mathbf{x})) - a$.[9] The agent's expected utility (assuming it exists) given action a is then

$$EU(a) = \int v(w(\mathbf{x}))f(\mathbf{x}|a)d\mathbf{x} - a. \tag{1}$$

Evidently, costs are assumed to be linear in the action. For instance, think of the agent's action, a, as being his choice of what cost of effort to incur. The linearity is convenient since it implies that only the first term in (1) has curvature, which simplifies the search for necessary and sufficient conditions (which is pursued in Section 6). Incidentally, Rogerson (1985) chose this parameterization too, although he only pursued sufficient conditions. Conlon (2009a, footnote 3) also observes that curvature in the cost function can be important, and thus chooses the same parameterization.

The agent's utility function $v(w)$ is strictly increasing and differentiable to the requisite degree. Moreover, the agent is strictly risk averse, or $v''(\cdot) < 0$. The domain of the utility function is some interval which may or may not be the entire real line. Finally, utility is unbounded below and/or above. The latter assumption is invoked only in Sections 6 and 7.

[8] The assumption that $\mathcal{X}$ is a hyperrectangle is for simplicity. If it is not a hyperrectangle, then let $\times_{i=1}^n [\underline{x}_i, \overline{x}_i]$ be the smallest hyperrectangle for which $\mathcal{X} \subseteq \times_{i=1}^n [\underline{x}_i, \overline{x}_i]$. In the one-signal case, the support is simply denoted $[\underline{x}, \overline{x}]$.

[9] Additive separability is important. While it is a standard assumption in the literature, there are exceptions. Alvi (1997) and Fagart and Fluet (2012) provide conditions that justify the FOA without additive separability.

2.1 Incentive compatibility

If the principal wishes to induce action $a^* \in [\underline{a}, \overline{a}]$, this action must provide the agent with higher expected utility than any other action, or

$$EU(a^*) \geq EU(a) \text{ for all } a \in [\underline{a}, \overline{a}] \,, \tag{G-IC$_{a^*}$}$$

in which case the contract $w(\mathbf{x})$ is said to be globally incentive compatible. Put differently, given the agent signs the contract in the first place, G-IC$_{a^*}$ ensures that a^* is an optimal action. The participation constraint is ignored, for now. If $a^* \in (\underline{a}, \overline{a})$, a minimum requirement is that $EU(a)$ attains a stationary point at a^*, or

$$\int v(w(\mathbf{x})) f_a(\mathbf{x}|a) d\mathbf{x} - 1 = 0. \tag{L-IC$_{a^*}$}$$

Of course, the stationary point may in principle be a local minimum or a saddle-point. Nevertheless, I will refer to the condition $EU'(a^*) = 0$ as the local incentive compatibility condition.[10] Thus, any contract that satisfies $EU'(a^*) = 0$ will be termed L-IC$_{a^*}$ and any contract that satisfies $EU(a^*) \geq EU(a)$ for all $a \in [\underline{a}, \overline{a}]$ is G-IC$_{a^*}$. The implementation of $\underline{a}$ and $\overline{a}$ is discussed in Section 4.

3 From local to global incentive compatibility

In this section I develop an alternative approach to the moral hazard problem. The intention is to provide a framework that not only conceptually unifies most existing results but which can also be used to guide the search for further generalizations.

3.1 An auxiliary problem

To develop the new approach, an auxiliary problem is introduced. Consider $a^* \in (\underline{a}, \overline{a})$ fixed. Think of this as the action the principal seeks to implement.

[10] Note that I restrict attention to contracts that give the agent bounded utility. In principle, if v is unbounded above, any action could be implemented by specifying a contract that provides unbounded utility to the agent. Hence, contracts are assumed to yield bounded utility and to be integrable. Note that any monotonic contract must be bounded since $\mathcal{X}$ is compact.

Next, fix $\mathbf{x}$ and think of a as a variable. Let

$$f^L(\mathbf{x}|a,a^*) = f(\mathbf{x}|a^*) + (a-a^*)f_a(\mathbf{x}|a^*)$$

and

$$F^L(\mathbf{x}|a,a^*) = F(\mathbf{x}|a^*) + (a-a^*)F_a(\mathbf{x}|a^*)$$

be the tangent lines to $f(\mathbf{x}|a)$ and $F(\mathbf{x}|a)$, respectively, at $a = a^*$.

Now switch the roles of $\mathbf{x}$ and a. Holding a (and a^*) fixed, consider the function $F^L(\mathbf{x}|a,a^*)$. Note that $F^L(\mathbf{x}|a,a^*)$ is not necessarily monotonic in $\mathbf{x}$, nor is it necessarily bounded between 0 and 1. Nevertheless, the following thought experiment is proposed. Think of $f^L(\mathbf{x}|a,a^*)$ and $F^L(\mathbf{x}|a,a^*)$ as (admittedly odd) density and distribution functions, respectively. It is easy to see that F^L can be obtained by integrating f^L over $\mathbf{x}$. Now consider an artificial problem where the agent faces distribution function $F^L(\mathbf{x}|a,a^*)$ rather than $F(\mathbf{x}|a)$.

In defence of these unusual "distributions", note, for now, that F^L does in fact have the key properties that $F^L(\underline{\mathbf{x}}|a,a^*) = 0$ and $F^L(\overline{\mathbf{x}}|a,a^*) = 1$. This claim follows from the fact that $F_a(\underline{\mathbf{x}}|a) = F_a(\overline{\mathbf{x}}|a) = 0$. Now, recall that the standard proof of the equivalence between the two definitions of univariate FOSD in Table 1 relies only on $G(\underline{x}) = H(\underline{x}) = 0$, $G(\overline{x}) = H(\overline{x}) = 1$, and the relative magnitudes of G and H, but not on monotonicity nor on the fact that proper distribution functions are bounded between 0 and 1. See also the discussion following Proposition 1, below.

"Expected utility" in the auxiliary problem is simply

$$EU^L(a|a^*) = \int v(w(\mathbf{x}))f^L(\mathbf{x}|a,a^*)d\mathbf{x} - a, \tag{2}$$

or

$$EU^L(a|a^*) = EU(a^*) + (a-a^*)\left[\int v(w(\mathbf{x}))f_a(\mathbf{x}|a^*)d\mathbf{x} - 1\right]. \tag{3}$$

Evidently, the last term disappears if L-IC$_{a^*}$ is satisfied, in which case $EU^L(a|a^*) = EU(a^*)$ for all a. Stated differently, L-IC$_{a^*}$ *on its own* places a lot of structure on the contract, which can now be utilized. In particular, it follows from (3) that once L-IC$_{a^*}$ is satisfied, G-IC$_{a^*}$ can equivalently be expressed as the requirement that

$$EU^L(a|a^*) \geq EU(a) \text{ for all } a \in [\underline{a}, \overline{a}]$$

or

$$\int v(w(\mathbf{x}))f^L(\mathbf{x}|a,a^*)d\mathbf{x} \geq \int v(w(\mathbf{x}))f(\mathbf{x}|a)d\mathbf{x} \text{ for all } a \in [\underline{a},\overline{a}]\,. \tag{4}$$

In essence, the continuum of incentive compatibility constraints in the original problem has been replaced with a continuum of comparisons of risky prospects. For instance, if $v(w(\mathbf{x}))$ is monotonic, it is fruitful to ask whether F^L first order stochastically dominates F. The point is that such comparisons are commonplace in economics, and that a large literature may now be accessed to inform the analysis. Proposition 1 records this conclusion.

Proposition 1 *Fix $a^* \in (\underline{a},\overline{a})$. Any L-IC$_{a^*}$ contract is G-IC$_{a^*}$ if and only if (4) holds.*[11]

Note that (4) is satisfied if and only if

$$\int v(w(\mathbf{x}))\left[\kappa+\varepsilon f^L(\mathbf{x}|a,a^*)\right]d\mathbf{x} \geq \int v(w(\mathbf{x}))\left[\kappa+\varepsilon f(\mathbf{x}|a)\right]d\mathbf{x} \text{ for all } a \in [\underline{a},\overline{a}] \tag{5}$$

and all $\varepsilon > 0$ and all κ. It is trivial to select κ and $\varepsilon > 0$ in such a manner that both bracketed terms are proper densities, i.e. they are strictly positive and integrate to one.[12] The equivalence of (4) and (5) implies that even though f^L is not a proper density, stochastic dominance results can still be invoked. Thus, I will frequently abuse terminology and say that f^L dominates f in some stochastic order.

3.2 An illustration

Consider the one-signal case. If $F(x|a)$ is convex in a, then its tangent line, $F^L(x|a,a^*)$, lies everywhere below the function itself. Thus, $F^L(\cdot|a,a^*)$ first order stochastically dominates $F(\cdot|a)$ for all a. Consequently, any L-IC$_{a^*}$ contract that is monotonic must necessarily be G-IC$_{a^*}$. Moreover, the argument holds regardless of a^*. Thus, if it can be established that the FOA candidate contract is monotonic then the FOA is itself valid. Figure 1 visualizes the auxiliary problem and the approach suggested here.

[FIGURE 1 ABOUT HERE (SEE THE LAST PAGES)]

[11] Proposition 1 can be generalized to allow for a non-linear cost function, $c(a)$. If $c(a)$ is convex, then L-IC$_{a^*}$ and (4) are sufficient for G-IC$_{a^*}$, for all $a^* \in (\underline{a},\overline{a})$. More generally, (4) is sufficient for G-IC$_{a^*}$ if a^* is on the convex hull of $c(a)$. If a^* is not on the convex hull of $c(a)$ then (4) is necessary.

[12] Since $\int f_a(\mathbf{x}|a)d\mathbf{x} = 0$, both $f(\mathbf{x}|a)$ and $f^L(\mathbf{x}|a,a^*)$ integrate to one.

For convenience, the following easy lemma notes necessary and sufficient conditions for F^L to ith order stochastically dominate F, $i = 1, 2, 3$, regardless of (a, a^*). Obviously, the characterization can be extended to higher stochastic orders.

Lemma 1 (Ordering the real and auxiliary distributions) *Assume there is a single signal. Then, $F^L(\cdot|a, a^*)$ first order stochastically dominates $F(\cdot|a)$ for all $a \in [\underline{a}, \overline{a}]$ and all $a^* \in [\underline{a}, \overline{a}]$ if and only if*

$$F_{aa}(x|a) \geq 0 \textit{ for all } x \in [\underline{x}, \overline{x}] \textit{ and all } a \in [\underline{a}, \overline{a}] .$$

Secondly, $F^L(\cdot|a, a^*)$ second order stochastically dominates $F(\cdot|a)$ for all $a \in [\underline{a}, \overline{a}]$ and all $a^* \in [\underline{a}, \overline{a}]$ if and only if

$$\int_{\underline{x}}^{x} F_{aa}(y|a)dy \geq 0 \textit{ for all } x \in [\underline{x}, \overline{x}] \textit{ and all } a \in [\underline{a}, \overline{a}] .$$

Finally, $F^L(\cdot|a, a^*)$ third order stochastically dominates $F(\cdot|a)$ for all $a \in [\underline{a}, \overline{a}]$ and all $a^* \in [\underline{a}, \overline{a}]$ if and only if

$$\begin{aligned} \int_{\underline{x}}^{x} \int_{\underline{x}}^{z} F(y|a)dydz &\geq 0 \textit{ for all } x \in [\underline{x}, \overline{x}] \textit{ and all } a \in [\underline{a}, \overline{a}] , \\ \textit{and } \int_{\underline{x}}^{\overline{x}} F_{aa}(y|a)dy &\geq 0 \textit{ for all } a \in [\underline{a}, \overline{a}] . \end{aligned}$$

Proof. The first part follows from the fact that a function is convex if and only if it lies everywhere above its tangent line. For the second part, $\int_{\underline{x}}^{x} F(y|a)dy$ is likewise everywhere above its tangent line (as a function of a) if and only if it is convex, or $\int_{\underline{x}}^{x} F_{aa}(y|a)dy \geq 0$. Now, the tangent line to $\int_{\underline{x}}^{x} F(y|a)dy$ at $a = a^*$ is

$$\int_{\underline{x}}^{x} F(y|a^*)dy + (a - a^*) \int_{\underline{x}}^{x} F_a(y|a^*)dy = \int_{\underline{x}}^{x} F^L(y|a, a^*)dy.$$

It follows that $\int_{\underline{x}}^{x} F_{aa}(y|a)dy \geq 0$ for all $x \in [\underline{x}, \overline{x}]$ and all $a \in [\underline{a}, \overline{a}]$ is necessary and sufficient for $\int_{\underline{x}}^{x} F^L(y|a, a^*)dy \leq \int_{\underline{x}}^{x} F(y|a)dy$ for all x, a, and a^*. Of course, for fixed (a, a^*), the latter condition coincides with the definition that $F^L(\cdot|a, a^*)$ second order stochastically dominates $F(\cdot|a)$. The proof for $i = 3$ is analogous. ■

Many results of the type presented in Lemma 1 are utilized in the analysis. Since the proofs are trivial and in any event analogous to the proof of Lemma 1, I will for the most part omit the formal proofs.

Of course, Rogerson's (1985) assumption is exactly that $F_{aa}(x|a) \geq 0$. His proof of the validity of the FOA is based on the observation that, in the one-signal case, integration by parts yields

$$EU(a) = v(w(\overline{x})) - \int_{\underline{x}}^{\overline{x}} F(x|a) dv(w(x)) - a, \tag{6}$$

and it follows that $EU(a)$ is concave when the contract is monotonic (i.e., when $dv(w(x)) \geq 0$). The condition $\int_{\underline{x}}^{x} F_{aa}(y|a)dy \geq 0$ is Jewitt's (1988) assumption (2.10a).[13] Conlon (2009a) points out that a second round of integration by parts can be used to prove concavity in Jewitt's model.

In fact, all the new justifications of the FOA that will be presented in Sections 4 and 5 can be shown to imply concavity. However, proving concavity in some cases requires repeated (and remarkably tedious) application of integration by parts. The method of proof I pursue is different and substantially less labor-intensive; the strategy is simply to invoke various stochastic orders. Indeed, the new results were discovered precisely by searching for usable stochastic orders, but it would be possible to rewrite the proofs in a more conventional manner by proving concavity directly.

Incidentally, note that Lemma 1 signifies that not only are Rogerson's and Jewitt's conditions sufficient, they are in fact the weakest conditions that can be imposed to ensure that L-IC implies G-IC for all a when the only characteristics of the contracts that are exploited are monotonicity or monotonicity and concavity. Thus, their results cannot be strengthened without imposing more structure on the contract (Section 6 contains a formal proof). In other words, the one-way implications ($\Downarrow$) in the first row of Table 1 can be converted into two-way implications ($\Updownarrow$), thereby cementing the analogy between the two rows.

Sections 6 and 7 examine environments where L-IC$_{a^*}$ does not imply G-IC$_{a^*}$ for all a^*, or where the agent's expected utility is not necessarily concave in a. In such cases, the FOA may be invalid.

[13] Jewitt also imposes another assumption, (2.10b), but this assumption is redundant; see Conlon (2009a, 2009b). Assumptions (2.11) and (2.12) are used in the other step of his proof (see below).

4 Justifying the first-order approach: One signal

Thus far, focus has been on interior a^*, where L-IC is necessary for utility maximization. However, boundary actions must be considered too, and so this section starts by clearing that technicality.

Thus, consider the corners, $\underline{a}$ and $\overline{a}$. With Rogerson's convexity assumption, $EU'(\overline{a}) = 0$ (or L-IC$_{\overline{a}}$) is sufficient for G-IC$_{\overline{a}}$ among monotonic contracts. Indeed, if $EU'(\overline{a}) \geq 0$, it follows from (3) that $EU^L(\overline{a}|\overline{a}) = EU(\overline{a}) \geq EU^L(a|\overline{a})$ for all $a \in [\underline{a}, \overline{a}]$. Then, G-IC$_{\overline{a}}$ follows if $EU^L(a|\overline{a}) \geq EU(a)$. However, as long as the FOA contract is monotonic, (4) proves this is the case. Hence, at $\overline{a}$, any monotonic contract that satisfies $EU'(\overline{a}) \geq 0$ is G-IC$_{\overline{a}}$. Similarly, any monotonic contract that satisfies $EU'(\underline{a}) \leq 0$ is G-IC$_{\underline{a}}$ (a constant-wage contract is a special case).

Hence, given Rogerson's assumption, it is meaningful to replace the global incentive compatibility constraint with the condition that

$$EU'(a^*) \begin{cases} \leq 0 \text{ if } a^* = \underline{a} \\ = 0 \text{ if } a^* \in (\underline{a}, \overline{a}) \\ \geq 0 \text{ if } a^* = \overline{a} \end{cases} . \tag{7}$$

A more general conclusion can be obtained. Specifically, if enough structure is imposed on F to ensure that L-IC$_{a^*}$ implies G-IC$_{a^*}$ for any interior a^* – among whatever subset of contracts is being considered (e.g. monotonic or monotonic and concave contracts) – then that structure also implies that actions at the corners are easily handled too. One version of the FOA is then to replace G-IC$_{a^*}$ with L-IC$_{a^*}$, solve the principal's problem, and then compare the solution to the optimal implementation of $\underline{a}$ and $\overline{a}$ using $EU'(\underline{a}) \leq 0$ and $EU'(\overline{a}) \geq 0$, respectively. For expositional simplicity, I will assume the second best action is in the interior, but this assumption is evidently innocent and easily checked. Rogerson (1985) makes a similar assumption.

Returning to the main task at hand, justifying the FOA, recall the proof strategy. In the first step, sufficient conditions are given for L-IC$_a$ to imply G-IC$_a$ among a subset of contracts, for any a. In the second step, sufficient conditions are derived to ensure the FOA solution belongs to the relevant subset of contracts. Lemma 1 reveals the conditions required to invoke FOSD, SOSD, and TOSD, respectively. It remains to match these conditions with another set of assumptions that guarantees that the contract takes a form such that these stochastic orders are useful.

To this end, recall the following equivalent definitions of these stochastic orders. Assuming differentiability, the distribution G ith order stochastically dominates distribution H if the former is preferred to the latter for all utility functions $u(x)$ with the property that the first i derivatives of $-u(-x)$ are positive. I will refer to such functions as *i-antitone*. This terminology is inspired by a multivariate concept; see Section 5. Note that the derivatives of $u(x)$ alternates in sign, i.e. $(-1)^{s-1}u^{(s)} \geq 0$ for all $s = 1, 2, ..., i$, where $u^{(s)}$ denotes the sth derivative. Using difference operators, it is also possibly to extend the definition to utility functions that are not necessarily differentiable; see e.g. Müller and Stoyan (2002, Section 1.6). These stochastic orders are sometimes referred to as the *i-increasing concave* (i-icv) orders. That is, 1-icv, 2-icv, and 3-icv are just different names for FOSD, SOSD, and TOSD, respectively. For future reference, a related set of orders, the *i-increasing convex orders* (i-icx), apply to situations in which the first i derivatives of $u(x)$ are all positive. Such functions will be said to be *i-monotone*. The next step is to make sure that the endogenous function $v(w(x))$ is either i-antitone or i-monotone.

As in Jewitt (1988), assume the principal is risk neutral. Let $B(a)$ denote the expected gross benefit to the principal if the agent's action is a. In many applications, $B(a)$ is simply the expected value of x. Apart from incentive compatibility, the only other constraint is a participation constraint. Let $\overline{u}$ denote the agent's reservation utility. It will be assumed the constraint-set is non-empty, i.e. that there exists a contract that satisfies both the participation constraint and L-IC for some a.

The FOA relies on L-IC being sufficient for G-IC. If this is the case, the principal's problem can be written as follows:

$$\max_{w,a} B(a) - \int_{\underline{x}}^{\overline{x}} w(x)f(x|a)dx$$

$$st. \quad \int_{\underline{x}}^{\overline{x}} v(w(x))f(x|a)dx - a \geq \overline{u}$$

$$\int_{\underline{x}}^{\overline{x}} v(w(x))f_a(x|a)dx - 1 = 0.$$

Assume the likelihood-ratio

$$l(x|a) = \frac{f_a(x|a)}{f(x|a)}$$

is bounded below. As in Rogerson and Jewitt, assume that the monotone likelihood ratio property (MLRP) is satisfied, or $l_x(x|a) \geq 0$. This assumption implies $F_a(x|a) \leq 0$, i.e. higher actions make low signals less likely. Finally, assume, in this section and the next, that it is optimal to offer a wage $w(x)$ in state x that is in the interior of the domain of $v(\cdot)$. For a fixed utility function, this assumption is typically satisfied if the agent's reservation utility is high enough.[14] In this case, $w(x)$ is characterized by a first order condition which can be written

$$\frac{1}{v'(w(x))} = \lambda + \mu l(x|a^*), \tag{8}$$

where $\lambda > 0$ is the multiplier of the participation constraint and $\mu \geq 0$ the multiplier of the local incentive compatibility constraint.[15] If $a^* = \underline{a}$, a flat wage is optimal ($\mu = 0$). However, if $a^* > \underline{a}$ then $\mu > 0$, in which case the MLRP implies a monotonic wage schedule. Since $v(w(x))$ is nondecreasing, FOSD can be invoked.

Jewitt (1988) imposes more substantial joint conditions on the utility function and likelihood ratio. To aid the analysis, Jewitt defines the function

$$\omega(z) = v(v'^{-1}(1/z)),\ \ z > 0.^{16}$$

Note that $\omega'(z) > 0$ if and only if $v''(w) < 0$, which has already been assumed. Jewitt adds the assumption that $\omega''(z) \leq 0$ and $l_{xx}(x|a) \leq 0$. From (8),

$$v(w(x)) = \omega\left(\lambda + \mu l(x|a^*)\right).$$

Hence, Jewitt's assumptions imply that whenever (8) is satisfied, $v(w(x))$ is increasing and concave, or 2-antitone. SOSD can now be invoked. As the next lemma shows, it turns out that the pattern can be continued. Conditions are imposed on the inner function $l(x|a)$ and the outer function $\omega(z)$ to guarantee that the composite function $v(w(x)) = \omega\left(\lambda + \mu l(x|a^*)\right)$ has desirable properties

[14] See e.g. Jewitt et al (2008), and in particular Gutiérrez (2012) for a detailed discussion. As can be seen from (8), below, this also explains why $l(x|a)$ must be bounded.

[15] One of the contributions in Rogerson (1985) and Jewitt (1988) is to establish that $\mu \geq 0$. In fact, Jewitt's (1988) paper appears to be cited more often for this result (and its very elegant proof) than for his conditions justifying the FOA. As in Conlon (2009a), I omit the proof here. Rogerson (1985) allows the principal to be risk averse. It is considerable harder to allow a risk averse principal in Jewitt's framework; see Conlon (2009a).

[16] To clarify, $v'^{-1}(\cdot)$ refers to the inverse of $v'(\cdot)$.

Lemma 2 *(i)* $\omega(\lambda + \mu l(x|a))$ *is i-monotone in* x *if* ω *is i-monotone and* $l(x|a)$ *is i-monotone in* x. *(ii)* $\omega(\lambda + \mu l(x|a))$ *is i-antitone in* x *if* ω *is i-antitone and* $l(x|a)$ *is i-antitone in* x.

Proof. Repeated differentiation yields the result. ■

Thus, if $l_{xxx}(x|a) \geq 0$ and $\omega'''(z) \geq 0$ are added to Jewitt's assumptions, then $v(w(x))$ is 3-antitone and TOSD can be invoked.

Table 2 summarizes the main conclusions thus far. The first row identifies sufficient conditions for L-IC$_a$ (or rather (7)) to imply G-IC$_a$ among contracts that are 1-antitone, 2-antitone, and 3-antitone, respectively, for all a. The second row identifies sufficient conditions for the FOA candidate solution in (8) to be such a contract. The validity of the FOA follows by imposing both sets of assumptions.

[TABLE 2 ABOUT HERE (SEE THE LAST PAGES)]

Proposition 2 *Assume the second best action is in* $(\underline{a}, \overline{a})$. *Assume the joint conditions in one of the columns of Table 2 are satisfied. Then, the FOA is valid.*

Obviously, Table 2 and Proposition 2 can be extended to stochastic dominance of higher order (4-icv, 5-icv, etc.). In fact, there is a well-defined limit to the sequence of higher order stochastic dominance, namely the Laplace transform order. See e.g. Müller and Stoyan (2002).

Evidently, the assumptions in the first row of Table 2 become weaker as one moves rightward from one column to the next. As for the second row, consider the following possible utility functions:

$$v_1(w) = 1 - e^{-\alpha w},\ v_2(w) = \ln w,\ v_3(w) = \frac{1}{\beta} w^{\beta},$$

where $\alpha > 0$ and $\beta < 1$, with $\beta \neq 0$. The domain of the first function is $(-\infty, \infty)$ while the domain of the latter two is $(0, \infty)$ (or convex subsets thereof). Of course, the first utility function exhibits constant absolute risk aversion, while the other two exhibits constant relative risk aversion. For these functions, $\omega(z)$ can be shown to be

$$\omega_1(z) = 1 - \frac{1}{\alpha z},\ \omega_2(z) = \ln z, \text{ and } \omega_3(z) = \frac{1}{\beta}(z)^{\frac{\beta}{1-\beta}}$$

respectively. Thus, the first two functions are i-antitone for *any* $i \geq 1$. The third function satisfies $\omega_3'(z) > 0$, $\omega_3''(z) \leq 0$ if and only if $\beta \leq 0.5$, i.e. if the agent is sufficiently risk averse. However, when $\beta \leq 0.5$, ω_3 is i-antitone for *any* $i \geq 1$. Thus, in these examples, the assumptions on $\omega(z)$ in the third column of Table 1 are not any stronger than those in the second column. Hence, the main strengthening from Jewitt's conditions to the new conditions in the third column is in the added requirement that $l_{xxx}(x|a) \geq 0$. Incidentally, all Jewitt's (1988, page 1183) examples have the feature that $l(x|a)$ is i-antitone for any $i \geq 1$.

However, extensions in other directions beckon. Except for Rogerson's conditions, the conditions mentioned above assume that the composite function $\omega(\lambda + \mu l(x|a))$ is increasing and concave. Now consider the possibility that it is convex. Note that the outer function ω may be convex even if v is concave. The i-icx orders, defined above, are relevant for such cases. Note that the utility function $v_3(w)$ mentioned above leads to an i-monotone $\omega(z)$ function if and only if $\beta \in [\frac{i-1}{i}, 1)$.

For distribution functions G and H, an equivalent definition of 1-icx is that $\overline{G}(x) \geq \overline{H}(x)$ for all $x \in [\underline{x}, \overline{x}]$. An equivalent definition for 2-icx is that

$$\int_x^{\overline{x}} \overline{G}(z)dz \geq \int_x^{\overline{x}} \overline{H}(z)dz \text{ for all } x \in [\underline{x}, \overline{x}],$$

and so on for higher increasing-convex orders. The orders 1-icx, 2-icx, and 3-icx are the counterparts to FOSD, SOSD, and TOSD, respectively, for risk loving agents.[17] Note that 1-icx in fact coincides with FOSD (or 1-icv), meaning that Rogerson's conditions can also be seen as the starting point to the sequence of conditions developed next. The following proposition, and its proof, is analogous to Proposition 2. It can of course also be extended to higher icx orders.[18]

Proposition 3 *Assume the second best action is in $(\underline{a}, \overline{a})$. Assume the joint conditions in one of the columns of Table 3 are satisfied. Then, the FOA is valid.*

[TABLE 3 ABOUT HERE (SEE THE LAST PAGES)]

[17] Note that the random variable X dominates the random variable Y in the s-icx order if and only if $-Y$ dominates $-X$ in the s-icv order.

[18] Jewitt offers a supremely convincing argument for his concavity assumption on $\int_{\underline{x}}^{x} F(z|a)dz$ in the special case where the signal x is production. Proposition 3 thus covers other cases.

5 Multi-signal justifications of the FOA

Jewitt's (1988) Theorem 2 and Theorem 3 were the first attempts at providing multi-signal justifications for the FOA. These results assume there are exactly two signals, and that they are independent. Moreover, ω is assumed to be concave. Sinclair-Desgagné (1994) generalized Rogerson's conditions to the case where there are multiple (not necessarily independent) signals. Finally, Conlon (2009a) further generalized Rogerson's conditions and offered an extension to Jewitt's Theorem 3, which he refers to as "Jewitt's (1988) main set of multisignal conditions".[19]

Here, I will verify that Conlon's results can be understood as appealing to multi-signal versions of FOSD and SOSD, respectively. Indeed, once the isomorphism in Section 3 has been established, it invites the search for other useful multivariate stochastic orders. Thus, Jewitt's Theorem 2 can be resurrected and extended once the proper stochastic order, which turns out to be the lower orthant order, has been identified. Another related order, the upper orthant order, leads to complementary results. By appealing to higher orthant orders, it turns out to be possible to offer generalizations that are close in spirit to Jewitt's Theorem 3 as well.

5.1 Multivariate FOSD and related stochastic orders

Müller and Stoyan (2002) make the following very useful observation about extending the common stochastic orders from a univariate setting to a multivariate environment. Specifically, comparing two distribution functions, G and H, there are three equivalent definitions of FOSD in the univariate setting, namely: (i) G is preferred to H for all non-decreasing utility function, (ii) $G(x) \leq H(x)$ for all x, and (iii) $\overline{G}(x) \geq \overline{H}(x)$ for all x. The point is that none of these definitions are equivalent when there are multiple signals. Consequently, there are three plausible ways of extending FOSD, which leads to the following definitions:

1. *G first order stochastically dominates H* if G is preferred to H for all non-decreasing utility functions.

2. *G dominates H in the lower orthant order* if $G(\mathbf{x}) \leq H(\mathbf{x})$ for all $\mathbf{x}$.

[19] The results in Jewitt (1988), Sinclair-Desgagné (1994), and Conlon (2009) all rely on proving that the agent's expected payoff is concave in his action. Ke (2012a) takes an alternative approach to justifying the FOA. See Section 6.

3. *G dominates H in the upper orthant order* if $\overline{G}(\mathbf{x}) \geq \overline{H}(\mathbf{x})$ for all $\mathbf{x}$.

Using Conlon's (2009a) notation and terminology, let $\mathbf{E}$ be an increasing set. A set is increasing if $\mathbf{x} \in \mathbf{E}$ and $\mathbf{y} \geq \mathbf{x}$ implies $\mathbf{y} \in \mathbf{E}$. It is well-known that an equivalent definition of FOSD is that G has more probability mass in all increasing sets than H does; see Müller and Stoyan (2002, Theorem 3.3.4). Thus, FOSD is stronger than the orthant orders. However, all three orders can be used to derive separate multi-signal justifications of the FOA.

Returning to the principal-agent model at hand, let

$$P(\mathbf{x} \in \mathbf{E}|a) = \int_{\mathbf{x} \in \mathbf{E}} f(\mathbf{y}|a)d\mathbf{y}$$

denote the probability that the vector of signals is in the increasing set $\mathbf{E}$, given action a. Let

$$P^L(\mathbf{x} \in \mathbf{E}|a, a^*) = P(\mathbf{x} \in \mathbf{E}|a^*) + (a - a^*)P_a(\mathbf{x} \in \mathbf{E}|a^*)$$

denote the counterpart in the auxiliary problem. Now, Conlon (2009a) proposes a *concave increasing-set probability (CISP) condition*, specifically that $P_{aa}(\mathbf{x} \in \mathbf{E}|a) \leq 0$ for all increasing sets and all $a \in [\underline{a}, \overline{a}]$. Evidently, the CISP condition implies that $P^L(\mathbf{x} \in \mathbf{E}|a, a^*) \geq P(\mathbf{x} \in \mathbf{E}|a)$ for all all $a \in [\underline{a}, \overline{a}]$. In other words, $F^L(\mathbf{x}|a, a^*)$ first order stochastically dominates $F(\mathbf{x}|a)$. Hence, expected payoff in the auxiliary problem is greater than in the original problem as long as the FOA contract is monotonic, as continues to be the case as long as the (multivariate) MLRP holds. This explains Conlon's (2009a, Proposition 4) extension of Rogerson's conditions.

Conlon (2009a) devotes considerable effort to examining CISP and deriving sufficient conditions for its applicability. However, CISP can be weakened, even without moving to conditions that can be used to invoke SOSD. In particular, recall that the orthant orders are weaker than FOSD. They also have the desirable property that equivalent statements of these orders can be given in term of the class of utility functions for which one distribution is preferred to another. Specifically, it can be shown that G dominates H in the upper orthant order if and only if G is preferred to H for all Δ-*monotone* utility functions (Müller and Stoyan (2002, Theorem 3.3.15)). If the utility function $u(\mathbf{x})$ is n times differentiable, then it is Δ-monotone if and only if

$$\frac{\partial^{k_1+\ldots+k_n} u(\mathbf{x})}{\partial x_1^{k_1} \ldots \partial x_n^{k_n}} \geq 0$$

for all $k_i \in \{0,1\}$, $i = 1, ..., n$, with $k_1 + ... + k_n \geq 1$. In words, all the mixed partial derivatives must be non-negative. See Müller and Stoyan (2002) for a formal definition, in term of difference operators, that allows $u(\mathbf{x})$ to be non-differentiable. Similarly, G dominates H in the lower orthant order if and only if G is preferred to H for all utility functions with the property that $u(-\mathbf{x})$ is Δ-*antitone*, i.e. $-u(-\mathbf{x})$ is Δ-monotone, or

$$\frac{\partial^{k_1+...+k_n}\left[-u(-\mathbf{x})\right]}{\partial x_1^{k_1}...\partial x_n^{k_n}} \geq 0$$

for all $k_i \in \{0,1\}$, $i = 1, ..., n$, with $k_1 + ... + k_n \geq 1$. Thus, the cross-partial derivatives alternate in sign as more cross-partials are added. Note that the relationship between the lower orthant order and the upper orthant order is similar to the relationship between the i-icv order and the i-icx order in the univariate case.

As in the one-signal case, the FOA implies that $v(w(\mathbf{x})) = \omega(\lambda + \mu l(\mathbf{x}|a))$. Jewitt (1988) and Conlon (2009a) observe that the multipliers remain positive in the multi-signal model. The next Lemma summarizes some pertinent observations about the composite function. The proof is straightforward and is thus omitted.

Lemma 3 *(i) $\omega(\lambda + \mu l(\mathbf{x}|a))$ is Δ-monotone in $\mathbf{x}$ if $l(\mathbf{x}|a)$ is Δ-monotone in $\mathbf{x}$ and ω is n-monotone. (ii) $\omega(\lambda + \mu l(\mathbf{x}|a))$ is Δ-antitone in $\mathbf{x}$ if $l(\mathbf{x}|a)$ is Δ-antitone in $\mathbf{x}$ and ω is n-antitone.*

Lemma 3 provides conditions under which the FOA contract belongs to one of the classes of functions that are useful when one of the orthant orders apply. However, it remains to impose conditions on the distribution function such that the orthant orders can indeed be invoked. To this end, note that if $F_{aa}(\mathbf{x}|a) \geq 0$ then $F^L(\mathbf{x}|a, a^*)$ dominates $F(\mathbf{x}|a)$ in the lower orthant order. Likewise, if $\overline{F}_{aa}(\mathbf{x}|a) \leq 0$ then $F^L(\mathbf{x}|a, a^*)$ dominates $F(\mathbf{x}|a)$ in the upper orthant order. These conditions coincide in the one-signal case, where they collapse to Rogerson's condition. Finally, Conlon's (2009a) CISP condition implies both $\overline{F}_{aa}(\mathbf{x}|a) \leq 0$ and $F_{aa}(\mathbf{x}|a) \geq 0$. New justifications of the FOA are now possible.

Proposition 4 *Assume the second best action is in $(\underline{a}, \overline{a})$. Then, the FOA is valid if either:*

1. *$\overline{F}_{aa}(\mathbf{x}|a) \leq 0$ for all $\mathbf{x}$ and all a, $l(\mathbf{x}|a)$ is Δ-monotone in $\mathbf{x}$ for all a, and ω is n-monotone, or*

2. *$F_{aa}(\mathbf{x}|a) \geq 0$ for all $\mathbf{x}$ and all a, $l(\mathbf{x}|a)$ is Δ-antitone in $\mathbf{x}$ for all a, and ω is n-antitone.*

Proof. For the first part, $\overline{F}_{aa}(\mathbf{x}|a) \leq 0$ implies that $F^L(\mathbf{x}|a, a^*)$ dominates $F(\mathbf{x}|a)$ in the upper orthant order. Hence, expected payoff in the auxiliary problem is higher than in the original problem as long as utility is Δ-monotone. The remaining conditions ensure this is the case, since they allow Lemma 3 to be invoked. The proof of the second part of the proposition is analogous. ■

Note the rather pleasing similarities between the conditions on ω and $l(\mathbf{x}|a)$, and their pattern, in the univariate case (Propositions 2 and 3) and the multivariate case (Proposition 4). Specifically, the conditions that must be added as another signal becomes available are similar to the conditions that must be added in the univariate case when the stochastic order is weakened by one degree (see also Corollary 1, below).

Conlon (2009a) makes the point that if the n signals are independent and each satisfies Rogerson's conditions, then the joint distribution function may nevertheless fail the CISP condition. In this sense, the CISP condition is a strong assumption. In contrast, the lower orthant order is more amenable to such extensions.

Corollary 1 *Assume there are $n \geq 2$ independent signals, with distribution functions $F^i(x_i|a)$ and likelihood ratio $l^i(x_i|a)$, $i = 1, 2, ..., n$. Assume the second best action is in $(\underline{a}, \overline{a})$. Then, the FOA is valid if*

1. *Each signal satisfies Rogerson's condition; $F^i_{aa}(x_i|a) \geq 0$ and $l^i_x(x_i|a) \geq 0$ for all $i = 1, 2, ..., n$, and*

2. *ω is n-antitone.*

Proof. The MLRP implies that F^i is decreasing in a. Since F^i is also convex, it follows that the product $F(\mathbf{x}|a) = \Pi F^i(x_i|a)$ is also convex in a. When signals are independent, $l(\mathbf{x}|a) = \Sigma l^i(x_i|a)$. Hence, $l(\mathbf{x}|a)$ is Δ-antitone. The second part of Proposition 4 can now be invoked. ■

Jewitt (1988, Theorem 2) reports a special case of this corollary, with $n = 2$. In this case, the second condition requires ω to be increasing and concave, which is of course precisely Jewitt's one-signal condition. At first sight, Jewitt's result may seem peculiar because it combines Rogerson's and Jewitt's one-signal conditions. Indeed, Conlon (2009a) does not devote much attention to this result. However, he does supply the following generalization (with a proof in Conlon (2009b)), while attributing it to Jewitt.

Assume there are two signals, and that the likelihood ratio is increasing and submodular in the two signals. Then, Conlon (2009b) proves the FOA is valid if $F_{aa}(\mathbf{x}|a) \geq 0$ (which he calls the lower quadrant convexity condition (LQCC)). Submodularity means that the cross-partial derivative is non-positive. Thus, with $n = 2$, $l(\mathbf{x}|a)$ is Δ-antitone. In other words, this result is a special case of Proposition 4. However, Conlon (2009a) concludes that "it is not clear how to extend this beyond the two-signal case." The resolution to the conundrum comes from the observation that the submodular order and the lower orthant order coincide in the bivariate case. As Proposition 4 demonstrates, the latter is well suited for extensions to many signals.

Before proceeding to Jewitt's and Conlon's other results, it is worthwhile to comment on one aspect of the previous results. All the stochastic orders invoked in this paper are so-called *integral stochastic orders*, meaning that they can be expressed as follows: G dominates H if G is preferred to H for all utility functions in some class $\mathcal{U}$. For an introduction to integral stochastic orders, see Müller and Stoyan (2002). The set $\mathcal{U}$ is referred to as a generator of the stochastic order. For example, one generator for FOSD is the set of all increasing functions. Importantly, the integral stochastic orders invoked until now have well-defined "small" generators. In the case of univariate FOSD, this is the set of nondecreasing step-functions, which can be thought of as being at the "corner" of the set of increasing functions because any increasing function can be approximated by a combination of step-functions. The existence of a small generator is crucial in being able to obtain equivalent characterizations of an integral stochastic order. For instance, step-functions are used to prove the equivalence between the two definitions of univariate FOSD in Table 1. Unfortunately, not all integral stochastic orders have small generators. In particular, this problem arises when multi-variate SOSD is considered.

5.2 Multivariate SOSD and related stochastic orders

Among the ingredients in Jewitt's (1988, Theorem 3) second set of conditions and Conlon's (2009a, Proposition 2) extension thereof, are the assumptions that $l(\mathbf{x}|a)$ is increasing and concave in $\mathbf{x}$ and that ω is increasing and concave. These assumptions imply that $v(w(\mathbf{x}))$ is increasing and concave in $\mathbf{x}$. Naturally, this points in the direction of SOSD.

However, to close the proof, Jewitt and Conlon add conditions that on the surface appear different in nature from those in all previous results. In particular, they utilize the state-space formulation of the principal-agent model and assume that for each realization of the state, ϑ, each signal $x_i(a, \vartheta)$, is concave in a. The joint assumptions then ensure that the agent's problem is concave in a.

Conlon (2009a, p. 258) observes that "it is not immediately obvious how to express the condition, that $\mathbf{x}(a, \vartheta)$ is concave in a, using the Mirrlees notation [where everything is expressed in terms of $F(\mathbf{x}|a)$]." It is in fact impossible to do so. As Müller and Stoyan (2002, p. 98) succinctly put it, "there is no hope of finding a 'small' generator" for SOSD, and thus it is not possible to express SOSD with a set of conditions directly on $F(\mathbf{x}|a)$.[20] Conlon (2009b) explains the difficulties in the context of the principal-agent model.

There are, however, other stochastic orders that not only have a familiar flavor but that are also better suited for the Mirrlees formulation. Consider the following orders, defined in Shaked and Shantikumar (2007):

1. *G dominates H in the lower orthant-concave order* if
$$\int_{\underline{x}_1}^{x_1} \cdots \int_{\underline{x}_n}^{x_n} G(y_1, ..., y_n) dy_n \cdots dy_1 \leq \int_{\underline{x}_1}^{x_1} \cdots \int_{\underline{x}_n}^{x_n} H(y_1, ..., y_n) dy_n \cdots dy_1 \text{ for all } \mathbf{x}.$$

2. *G dominates H in the upper orthant-convex order* if
$$\int_{x_1}^{\overline{x}_1} \cdots \int_{x_n}^{\overline{x}_n} \overline{G}(y_1, ..., y_n) dy_n \cdots dy_1 \geq \int_{x_1}^{\overline{x}_1} \cdots \int_{x_n}^{\overline{x}_n} \overline{H}(y_1, ..., y_n) dy_n \cdots dy_1 \text{ for all } \mathbf{x}.$$

Denuit and Mesfioui (2010) examine these and related stochastic orders. It can be shown that if G dominates H in the upper orthant-convex order then G is preferred to H for any utility function for which
$$\frac{\partial^{k_1+...+k_n} u(\mathbf{x})}{\partial x_1^{k_1} ... \partial x_n^{k_n}} \geq 0$$

[20] Another way to finish Jewitt's and Conlon's proofs would be to replace their assumption on $\mathbf{x}(a, \vartheta)$ with the (somewhat facetious) assumption that (4) holds true for all increasing and concave functions $v(w(\cdot))$ and all pairs (a, a^*), but that is hardly satisfying either.

for all $k_i \in \{0,1,2\}$, $i = 1, ..., n$, with $k_1 + ... + k_n \geq 1$. Similarly, if G dominates H in the lower orthant-concave order then G is preferred to H for any utility function for which $-u(-\mathbf{x})$ has the above property. In the univariate case, these orders obviously reduce to 2-icx and SOSD, respectively. The following proposition then follows from the usual logic.

Proposition 5 *Assume the second best action is in $(\underline{a}, \overline{a})$. Then, the FOA is valid if either:*

1. $\int_{\mathbf{y}\geq\mathbf{x}} \overline{F}_{aa}(\mathbf{y}|a)d\mathbf{y} \leq 0$ *for all* $\mathbf{x}$ *and all* a*,* ω *is* $2n$*-monotone, and*

$$\frac{\partial^{k_1+...+k_n} l(\mathbf{x}|a)}{\partial x_1^{k_1} ... \partial x_n^{k_n}} \geq 0$$

for all a *and for all* $k_i \in \{0,1,2\}$*,* $i = 1, ..., n$*, with* $k_1 + ... + k_n \geq 1$*, or*

2. $\int_{\mathbf{y}\leq\mathbf{x}} F_{aa}(\mathbf{y}|a)d\mathbf{y} \geq 0$ *for all* $\mathbf{x}$ *and all* a*,* ω *is* $2n$*-antitone, and*

$$\frac{\partial^{k_1+...+k_n} \left(-l(-\mathbf{x}|a)\right)}{\partial x_1^{k_1} ... \partial x_n^{k_n}} \geq 0$$

for all a *and for all* $k_i \in \{0,1,2\}$*,* $i = 1, ..., n$*, with* $k_1 + ... + k_n \geq 1$*.*

Jewitt's one-signal conditions imply the univariate function $v(w(x))$ has a negative second derivative. There are several ways in which this property can be extended into higher dimensions; requiring multivariate concavity is but one of them. Conlon's aim was precisely to include concavity in the sufficient conditions, but the Mirrlees formulation of the model was not up to the task. Thus, if the goal is sufficient conditions in Mirrlees notation then the most fruitful concept of "curvature" in the multi-signal model is not concavity. "Small" generators aside, to understand this result note that among the stochastic orders invoked in this paper, all but multivariate SOSD can be defined in terms only of the *sign* of certain derivatives. For multivariate SOSD, however, conditions must also be imposed upon the relative magnitude of various second derivatives; multivariate concavity is a messier concept. This is a significant difference, which on its own explains the difference in tractability.

Proposition 5 thus illustrates the price of escaping Conlon's conundrum. To recover sufficient conditions in the Mirrless notation, Conlon's implicit assumption about the relative magnitude of second derivatives must be replaced by conditions on the sign of higher-order derivatives.

A counterpart to Corollary 1 is also possible for the lower orthant-concave order. The proof is analogous to the proof of Corollary 1 and is thus omitted.

Corollary 2 *Assume there are $n \geq 2$ independent signals, with distribution functions $F^i(x_i|a)$ and likelihood ratio $l^i(x_i|a)$, $i = 1, 2, ..., n$. Assume the second best action is in $(\underline{a}, \overline{a})$. Then, the FOA is valid if*

1. *Each signal satisfies Jewitt's one-signal condition; $\int_{\underline{x}}^{x} F(y|a)dy \geq 0$, $l_x^i(x_i|a) \geq 0$, and $l_{xx}^i(x_i|a) \leq 0$ for all $i = 1, 2, ..., n$, and*

2. *ω is $2n$-antitone.*

Together, Corollary 1 and Corollary 2 offer an argument in favor of multi-signal conditions based on the orthant orders, like Propositions 4 and 5 in the current paper, over conditions based on the more demanding multivariate notions of FOSD and SOSD, like Jewitt's Theorem 3 or Conlon's (2009a) propositions. In practice, the orthant orders may also be easier to check. The second part of the corollaries captures the other side of the trade-off, namely that more conditions must be imposed on the underlying utility functions. However, the discussion following Proposition 2 reveals that this may be a small price to pay for a multi-signal extension.

On the other hand, Conlon (2009a) advocates for the state-space formulation, in large part because of the stringency of Rogerson's conditions in the Mirrlees formulation. As explained above, any justification that seeks to build on concavity of the agent's utility in the vector of signals (i.e., multivariate SOSD) must necessarily be phrased in the state-space model. The results in this section suggest that both formulations are indispensable and clarifies the way in which they are complementary.

Though it is not pursued here, there seems to be no conceptual obstacle to extending the result to higher multivariate stochastic orders, e.g. to various versions of multivariate TOSD.

6 Sufficient local conditions

As Mirrlees (1999) pointed out early on, the FOA is not always valid. The remainder of the paper studies such environments. For brevity, I focus on the one-signal case.

In general, the FOA may identify a contract and a target action for which L-IC is not sufficient for G-IC. However, in the current section I identify a subset of actions for which L-IC is in fact sufficient for G-IC, even when e.g. Rogerson's or Jewitt's global conditions on the primitives are violated. In the next section, I present a more specific model in which the FOA can be modified to identify the optimal contract.

As in the beginning of Section 3, fix x vary a. Let $F^C(x|a)$ denote the convex hull of $F(x|a)$, when a is thought of as the variable.[21] Note that Rogerson's assumption that $F(x|a)$ is convex in a for all x is in fact *equivalent* to assuming that $F(x|a) = F^C(x|a)$ for all x and all a, i.e. F always coincides with its convex hull.

Next, fix some $a^* \in (\underline{a}, \overline{a})$ that the principal would like to induce. Assume that $F(x|a)$ coincides with its convex hull (again as a function of a) at a^* for all x, or $F(x|a^*) = F^C(x|a^*)$ for all x. This *local* condition is evidently weaker than Rogerson's *global* condition. In this case, $F^L(\cdot|a, a^*)$ first order stochastically dominates $F(\cdot|a)$ for all a; this is a "local" version of Lemma 1, with a^* held fixed.[22] Figure 2 illustrates. It now follows that any monotonic and L-IC$_{a^*}$ contract is G-IC$_{a^*}$. Thus, a local counterpart to Rogerson's condition has been identified; when attempting to induce a^* with a L-IC$_{a^*}$ contract, what matters is whether $F(x|a)$ coincides with its convex hull at a^*. Consequently, if the MLRP is satisfied and $F(x|a^*) = F^C(x|a^*)$ for all x, then the optimal contract that implements a^* is described by (8).

[FIGURE 2 ABOUT HERE (SEE THE LAST PAGES)]

For completeness, the following lemma states a stronger version of the above observation. Under the additional mild assumption that $F_a(x|a) < 0$ for all $x \in (\underline{x}, \overline{x})$, *all* monotonic and L-IC$_{a^*}$ contracts are G-IC$_{a^*}$ *if and only if* $F(x|a)$ coincides with its convex hull at a^* for all x. The lemma is illustrated in Figure 2. The significance of this result is that Rogerson's global convexity assumption is in fact the weakest assumption which ensures that monotonicity and L-IC is sufficient for G-IC for *all* a.

Lemma 4 *Assume there is a single signal and that $F_a(x|a) < 0$ for all $x \in (\underline{x}, \overline{x})$ and all a. Fix $a^* \in (\underline{a}, \overline{a})$. Then, all monotonic and L-IC$_{a^*}$ contracts are G-IC$_{a^*}$ if and only if $F(x|a)$ coincides with its convex hull at a^* for all x.*

[21] Recall that the convex hull of a function $g(a)$ is the highest convex function that is always below $g(a)$; see Rockafellar (1970).

[22] Holding x fixed, $F^L(x|a, a^*)$ is the tangent line to $F(x|a)$ through a^*. Since by assumption $F(x|a^*) = F^C(x|a^*)$ and $a^* \in (\underline{a}, \overline{a})$, $F^L(x|a, a^*)$ is also the tangent line to $F^C(x|a)$ through a^*. Since $F^C(x|a)$ is convex, it follows that $F^L(x|a, a^*) \le F^C(x|a)$. Finally, by definition, $F^C(x|a) \le F(x|a)$. It follows that $F^L(x|a, a^*) \le F(x|a)$ for all x and all a, as claimed.

Proof. The "if" part was proven in the text. For the other direction, assume there is some x such that $F(x|a)$ does not coincide with its convex hull at a^*. Note that such an x must necessarily be in $(\underline{x}, \overline{x})$. It suffices to find some monotonic and L-IC$_{a^*}$ contract that is not G-IC$_{a^*}$. Consider a step contract that delivers utility v_0 if the outcome is worse than x, and utility v_1 otherwise. The agent's expected utility is $EU(a) = v_1 + (v_0 - v_1) F(x|a) - a$ with $EU'(a^*) = (v_0 - v_1) F_a(x|a^*) - 1$. Since $F_a(x|a^*) < 0$ and utility is assumed to be continuous and unbounded above and/or below, there exists a pair (v_0, v_1) that satisfies L-IC$_{a^*}$ and monotonicity ($v_1 > v_0$). However, because $F(x|a)$ does not coincide with its convex hull at a^* there is an alternative action that yields higher payoff for the agent. ■

It is of course possible to obtain similar local versions of the other results in this paper that can be characterized using the Mirrlees formulation. For example, a local version of Jewitt's condition requires that the antiderivative of $F(x|a)$ coincides with its convex hull at a^*.

In an ambitious recent paper, Ke (2012a) notes that even when L-IC is not sufficient for G-IC for all actions, the FOA may nevertheless still identify the optimal contract. For instance, this occurs if the solution a^* happens to have the property that $F(x|a)$ coincides with its convex hull at a^*. It may also occur if L-IC$_{a^*}$ is generally not sufficient for G-IC$_{a^*}$, but just happens to be sufficient with the specific contract identified by the FOA. Thus, Ke (2012a) proposes a clever fixed-point method designed to identify conditions under which the FOA produces the correct solution. Think of the principal as "targeting" some action, a^*, by constructing the appropriate FOA contract as described in (8). The agent responds to this by taking a utility-maximizing action, a', which may be different from the targeted action. However, if a' and a^* coincide then a^* is implementable with a FOA contract. The first question is then whether there exists a target action a^* – a fixed-point – where a' coincides with a^*. The second question is whether the optimal action according to the FOA is among these fixed-points. Note that it is immaterial whether actions the principal would not want to induce can or cannot be implemented with a FOA contract.

To illustrate Ke's (2012a) results, consider the case where there are n independent signals, as in Corollaries 1 and 2 of the current paper. It is a corollary of Ke's Proposition 2 that the FOA is valid if $\omega''(\cdot) \leq 0$, MLRP holds, and

$$\frac{\partial l^i(x_i|a)}{\partial a} \leq 0 \text{ and } \frac{\partial l^i_x(x_i|a)}{\partial a} \geq 0.$$

Compared to Corollaries 1 and 2, Ke can thus dispense with Rogerson's and Jewitt's assumptions on the distribution function. In exchange, he has to impose conditions on how the likelihood ratio depends on a. Of course, he also requires $\omega(\cdot)$ to be only 2-antitone, but as argued earlier there does not seem to be too much loss in assuming in addition that $\omega(\cdot)$ is n-antitone or $2n$-antitone, as in Corollaries 1 and 2.

Ke's (2012a) approach does not necessarily imply that the agent's utility is concave in his action. However, it does rely on the specific form taken by a FOA contract, described in (8). This, in turn, assumes that the only constraints are the participation constraint and the incentive compatibility constraints. In contrast, Rogerson's convexity condition validates the FOA whenever the contract is monotonic. As long as this is satisfied, any L-IC contract is G-IC, even if more types of constraints are in play, such as a binding minimum wage, a non-bankruptcy condition, monotonicity of the principal's rewards, and so on. Likewise, Lemma 4, above, requires monotonicity. As highlighted by Innes (1990), this can in many cases be justified for exogenous reasons. For instance, if the agent could sabotage the outcome before it is observed by the principal, then any contract in effect becomes monotonic. Given this monotonicity, however, Lemma 4 identifies a "robust" set of actions where the incentive compatibility problem is fundamentally the same (and described by L-IC only) regardless of what kind of additional constraints are added to the problem. Moreover, the conditions in Lemma 4 does not rule out that the agent's payoff is non-concave.[23] Note also that the isomorphism in Section 3 in no way requires the contract to take a specific form such as e.g. (8).

7 A modified FOA

In this section, I consider a more specialized environment in which $F(x|a)$ is described by

$$F(x|a) = \sum_{i=1}^{k} p_i(a) G_i(x) + \left(1 - \sum_{i=1}^{k} p_i(a)\right) H(x), \tag{9}$$

[23] Building on the example in Figure 2, imagine that for all $x \in (\underline{x}, \overline{x})$, a' is the smallest interior action for which $F(x|a) = F^C(x|a)$. Then, any monotonic and L-IC$_{a'}$ contract must make the agent indifferent between $\underline{a}$ and a', while any other action is strictly worse. A related result is presented in Section 7.2.

or

$$F(x|a) = \sum_{i=1}^{k} p_i(a)\left(G_i(x) - H(x)\right) + H(x),$$

where $G_i(\cdot)$ and $H(\cdot)$ are non-identical distribution functions with support $[\underline{x}, \overline{x}]$ and where $p_i(a) \geq 0$ for all i and all a, with $\sum_{i=1}^{k} p_i(a) \leq 1$ for all a. Assume that p_i is continuously differentiable for all i and all a. In the special case where G_i and H are degenerate, the model describes a situation with a finite number $(k+1)$ of outcomes.

The FOA is not generally valid in this model. However, the structure in (9) can in some cases be exploited to construct a modified and valid FOA.

The special case in which $k = 1$ has a long history (see the next subsection). Nevertheless, the first complete analysis of this environment is presented here. The main insight is that the FOA is easily modified. There are two steps. First, the set of implementable actions is characterized, which happens to be straightforward. Second, it turns out that L-IC is in fact sufficient for G-IC on the set of implementable actions. The resulting modified FOA simplifies some classic examples in the literature. These include a counterexample due to Mirrlees in which he demonstrates how the FOA may fail. I develop a simpler and much more easily interpretable counterexample.

To the best of my knowledge, cases with $k > 1$ have not been analyzed before.[24] When $k = 2$, it is possible to characterize a small set of non-local incentive compatibility constraints that may come into play, representing those actions that are the most tempting deviations. Once this set has been characterized, modifying the FOA becomes a simple matter. Finally, when only monotonic contracts are feasible, I identify conditions for arbitrary k under which the only relevant non-local incentive compatibility constraint is that associated with the very lowest action, $\underline{a}$.

As a preliminary step, note that if $a^* \in (\underline{a}, \overline{a})$ and L-IC$_{a^*}$ is satisfied, then

$$EU^L(a|a^*) - EU(a) = \sum_{i=1}^{k} \left[p_i(a^*) + (a - a^*)p_i'(a^*) - p_i(a)\right] C_i, \tag{10}$$

where

$$C_i = \int v(w(x))d\left(G_i(x) - H(x)\right). \tag{11}$$

Obviously, C_i is endogenously determined. Note that the ith bracketed terms in (10) is the difference between the tangent line to $p_i(a)$ through a^* and $p_i(a)$ itself. This is

[24] I would like to thank John Conlon for suggesting that I examine the $k > 1$ case too.

positive for all a if and only if a^* is on the concave hull of p_i.[25]

7.1 The spanning condition ($k = 1$)

Assume first that $k = 1$. For notational simplicity, the subscript on G_1, p_1, and C_1 will be dropped, such that $F(x|a)$ is simply written as

$$F(x|a) = p(a)G(x) + (1 - p(a))\, H(x), \tag{12}$$

where $p(a) \in [0, 1]$ for all $a \in [\underline{a}, \overline{a}]$ and G and H are non-identical distribution functions with support $[\underline{x}, \overline{x}]$. While this model is certainly too specialized to capture all principal-agent relationships, it should be stressed that it does have a compelling interpretation. For instance, $p(a)$ could be the proportion of time the parts-supplier (the agent) spends using the new and advanced technology G rather than the less reliable but more user-friendly old technology, H. Given such interpretations of the model, the most meaningful economic assumption is that $p(a)$ is monotonic. Thus, as is common in the literature, assume that $p'(a) > 0$ for all $a \in (\underline{a}, \overline{a}]$. The case where $p(a)$ is non-monotonic is not that much more difficult. It is discussed briefly later. The assumption that there is a single signal is for notational simplicity, but the analysis does not rely on this assumption.

Distributions of this form have been studied extensively. Grossman and Hart (1983) say that the *spanning condition* is satisfied if $F(x|a)$ can be written as in (12). Since (12) is linear in p, Hart and Holmström (1987) refer to (12) as the Linear Distribution Function Condition (LDFC). The significance of the model and its place in the literature is discussed in detail after the formal analysis.

Typically, additional assumptions are imposed on the curvature of $p(a)$ as well as on the relationship between G and H. For instance, Sinclair-Desgagné (1994, 2009) points out that the FOA is valid if $p(a)$ is concave and $\frac{g(x)}{h(x)}$ is nondecreasing, where g and h are the densities of G and H, respectively. The latter assumption implies the MLRP, while the former ensures concavity of the agent's objective function when he faces a monotonic contract. The second assumption also implies that G first order stochastically dominates H. Without assumptions on $p(a)$, Grossman and Hart (1983) prove that if $\frac{g(x)}{h(x)}$ is nondecreasing then any optimal contract must feature

[25] The concave hull is the lowest concave function that is always above $p_i(a)$.

monotonic wages.[26] Ke (2012a, Proposition 7) shows that the FOA is valid if $p(a)$ is concave, even without the MLRP.

Here, I impose no such conditions on (12). For instance, $p(a)$ may be concave only locally, or not at all, and G and H may cross, as would be the case if H is a mean-preserving spread over G. Indeed, to determine whether L-IC$_{a^*}$ is sufficient for G-IC$_{a^*}$, G and H need not be continuous, though this is of course required for the optimal contract to subsequently take the form in (8). No restrictions are placed on the shape of the contract either (apart from bounded utility).

As in the first part of the paper, the crucial step is to explore the link between L-IC and G-IC. Using the notation in (11), L-IC$_{a^*}$ is

$$p'(a^*)C - 1 = 0 \tag{13}$$

whenever $a^* \in (\underline{a}, \overline{a})$. Since $p'(a^*) > 0$, C must take the strictly positive value $\frac{1}{p'(a^*)}$ in order to satisfy (13). Thus, (10) becomes

$$EU^L(a|a^*) - EU(a) = \frac{p(a^*) + (a - a^*)\, p'(a^*) - p(a)}{p'(a^*)}. \tag{14}$$

Let A_p^C denote the set of actions in $(\underline{a}, \overline{a})$ for which $p(a)$ coincides with its concave hull.[27] By definition, $a^* \in A_p^C$ if and only if (14) is non-negative for any a.

Proposition 6 *Assume that $p'(a) > 0$ for all $a \in (\underline{a}, \overline{a}]$. Then, there exists a G-IC$_{a^*}$ contract (that yields bounded utility) if and only if $a^* \in A_p^C \cup \{\underline{a}, \overline{a}\}$.*

Proof. Assume $a^* \in (\underline{a}, \overline{a})$ and $a^* \notin A_p^C$. If there is a G-IC$_{a^*}$ contract, then that contract must necessarily be L-IC$_{a^*}$, and so (14) applies. However, since $a^* \notin A_p^C$, there is some $a \in (\underline{a}, \overline{a})$ for which (14) is strictly negative, which contradicts G-IC$_{a^*}$.

For the other direction, assume $a^* \in A_p^C$. Since G and H are distinct, there is some $x \in (\underline{x}, \overline{x})$ for which $G(x) \neq H(x)$, or $F_a(x|a^*) \neq 0$. Now, as in the proof

[26] In their discrete model, Grossman and Hart (1983) allow multiple incentive compatibility constraints to bind. I will show, in the continuous model, that if a can be implemented then all but the local incentive compatibility constraint are redundant.

[27] To compare the model in this section with the more general model in Section 6, note that in the present model $F_a(x|a) = p'(a)(G(x) - H(x))$. Thus, in order to have $F_a(x|a) < 0$ for all $x \in (\underline{x}, \overline{x})$ (as required by MLRP) would necessitate that G first order stochastically dominates H. In this case, however, $p'(a)$ has the opposite sign of F_a. This explains why the concave hull of $p(a)$ is used here, while the convex hull of $F(x|a)$ is used in Section 6.

of Lemma 4, construct a step contract that yields utility v_0 if the outcome is worse than x, and utility v_1 otherwise. As before, v_0 and v_1 can be chosen to satisfy L-IC$_{a^*}$ (contrary to Lemma 4, however, it is possible that $v_0 > v_1$). Since $a^* \in A_p^C$, (14) is everywhere non-negative. Hence, the contract is G-IC$_{a^*}$.

Now assume $a^* \in \{\underline{a}, \overline{a}\}$. By modifying the steps that led to (14), it is easy to see that a step contract that makes $EU'(\underline{a})$ sufficiently small or $EU'(\overline{a})$ sufficiently large is G-IC$_{\underline{a}}$ or G-IC$_{\overline{a}}$, respectively. ■

Thus, the spanning condition allows a succinct formulation of the "feasible set" of implementable actions.[28] Moreover, it should be clear from the proof of Proposition 6 that L-IC$_{a^*}$ is in fact *necessary and sufficient* for G-IC$_{a^*}$, for any $a^* \in A_p^C$.

Proposition 7 *Assume that $p'(a) > 0$ for all $a \in (\underline{a}, \overline{a}]$. If $a^* \in A_p^C$ then any L-IC$_{a^*}$ contract is G-IC$_{a^*}$.*

Proof. Given $p'(a) > 0$, (14) is everywhere positive if $a^* \in A_p^C$. ■

As a consequence of Propositions 6 and 7, a modified FOA suggests itself. In the first step, the feasible set is identified, $A_p^C \cup \{\underline{a}, \overline{a}\}$. The feasible set is closed (but not necessarily convex). In the second step, the FOA is applied to this set (i.e. with the constraint that $a \in A_p^C \cup \{\underline{a}, \overline{a}\}$). In a third step, the solution is compared to the payoff from optimally implementing $\underline{a}$ and $\overline{a}$. The superior contract is then chosen.

To find the optimal contract that implements $\underline{a}$ or $\overline{a}$, it turns out that the continuum of incentive compatibility constraints can again be summarized by one lone condition. For instance, consider implementing $\underline{a}$. Let $\underline{a}^c = \inf A_p^C$ if A_p^C is non-empty and let $\underline{a}^c = \overline{a}$ otherwise. First, $EU'(\underline{a}) \leq 0$ is necessary for G-IC$_{\underline{a}}$. However, if $\underline{a} = \underline{a}^c$, then $EU'(\underline{a}) \leq 0$ is also sufficient for G-IC$_{\underline{a}}$, as proven below.

Consider next the possibility that $\underline{a} < \underline{a}^c$. Then, G-IC$_{\underline{a}}$ obviously necessitates that $EU(\underline{a}) \geq EU(\underline{a}^c)$, such that there is no incentive to pick $\underline{a}^c$ over $\underline{a}$. However, it turns out that $EU(\underline{a}) \geq EU(\underline{a}^c)$ is in fact sufficient for G-IC$_{\underline{a}}$. In particular, $EU(\underline{a}) \geq EU(\underline{a}^c)$ implies $EU'(\underline{a}) \leq 0$ when $\underline{a} < \underline{a}^c$. To implement $\overline{a}$, the relevant counterpart to $\underline{a}^c$ is $\overline{a}^c = \sup A_p^C$ when A_p^C is non-empty and $\overline{a}^c = \underline{a}$ otherwise.

Proposition 8 *Assume that $p'(a) > 0$ for all $a \in (\underline{a}, \overline{a}]$. Then, it is possible to implement the boundary actions, as follows:*

[28] Hermalin and Katz (1991) use tools from convex analysis to characterize the set of implementable actions in a model with a finite set of actions and a finite set of outcomes. Note that their analysis does not reveal when L-IC is sufficient for G-IC.

1. *If* $\underline{a}^c = \underline{a}$ *then* $EU'(\underline{a}) \leq 0$ *is necessary and sufficient for G-IC*$_{\underline{a}}$. *If* $\underline{a}^c > \underline{a}$ *then* $EU(\underline{a}) \geq EU(\underline{a}^c)$ *is necessary and sufficient for G-IC*$_{\underline{a}}$.

2. *If* $\overline{a}^c = \overline{a}$ *then* $EU'(\overline{a}) \geq 0$ *is necessary and sufficient for G-IC*$_{\overline{a}}$. *If* $\overline{a}^c < \overline{a}$ *then* $EU(\overline{a}) \geq EU(\overline{a}^c)$ *is necessary and sufficient for G-IC*$_{\underline{a}}$.

Proof. Necessity is obvious. For sufficiency in the first part of the proposition, consider first the "no-gap" case, $\underline{a}^c = \underline{a}$. Here, the slope of $p(a)$ coincides with the slope of its concave hull at $\underline{a}$. As in the proof of Proposition 6, a modification of (14) then establishes that $EU'(\underline{a}) \leq 0$ is sufficient for G-IC$_{\underline{a}}$. However, this is not necessarily true in the "gap" case, where $\underline{a}^c > \underline{a}$. Note that

$$EU(\underline{a}) - EU(a) = (a - \underline{a}) \left[1 - \frac{p(a) - p(\underline{a})}{a - \underline{a}} C \right],$$

where C is defined in (11). Hence, $EU(\underline{a}) \geq EU(\underline{a}^c)$ implies that the term in brackets must be non-negative when $a = \underline{a}^c$. If C is negative, then the term in brackets is positive for all a, or $EU(\underline{a}) \geq EU(a)$ for all a. That is, the contract is G-IC$_{\underline{a}}$. If C is positive, then the term in brackets is *minimized* at $a = \underline{a}^c$. This follows by definition of the concave hull, since the line from $(\underline{a}, p(\underline{a}))$ to $(\underline{a}^c, p(\underline{a}^c))$ is steeper than the line from $(\underline{a}, p(\underline{a}))$ to any other point on $p(\cdot)$. Hence, if $EU(\underline{a}) \geq EU(\underline{a}^c)$ then $EU(\underline{a}) \geq EU(a)$ for all $a \in [\underline{a}, \overline{a}]$, thus implying G-IC$_{\underline{a}}$. The proof of the second part of the proposition is analogous. ■

The assumption that $p(a)$ is monotonic seems justified on economic grounds. However, it is possible to allow $p(a)$ to be non-monotonic. First, note that the argument following (14) remains valid if $p'(a^*) > 0$ even if $p'(a^{**}) < 0$ for some $a^{**} \neq a^*$. That is, a^* can be implemented, and L-IC$_{a^*}$ is sufficient, if and only if a^* is on the concave hull of $p(a^*)$. By similar reasoning, $a^{**} \in (\underline{a}, \overline{a})$ can be implemented, and L-IC$_{a^{**}}$ is sufficient, if and only if a^{**} is on the convex hull of $p(a^{**})$.[29] Thus, the set of implementable interior actions can be obtained by piecing together the sets of implementable actions with $p'(\cdot) > 0$ and $p'(\cdot) < 0$, respectively. Of course, if $a^* \in (\underline{a}, \overline{a})$ and $p'(a^*) = 0$ then no L-IC$_{a^*}$ contract exist (with bounded utility), as can be seen from (13). Similarly, if $p(a^*) = p(a')$, then a^* cannot be implemented if $a' < a^*$ because it would be cheaper for the agent to pick a' rather than a^*.[30] Note

[29] This is easily seen by multiplying both numerator and denominator in (14) by -1.

[30] Consequently, once $p(a)$ is allowed to be non-monotonic, it is no longer necessarily the case that $\overline{a}$ can be implemented.

that such actions can be on neither the concave hull nor the convex hull of $p(a)$ when $p'(a^*) \neq 0$.

The rest of this subsection is devoted to demonstrating the significance of the spanning condition as well as illustrating some uses of the preceding characterization.

First, it is useful to recognize that the textbook case in which there are two outcomes (but a continuum of actions) is in fact a special case of (12). Specifically, this model corresponds to assuming that G and H are degenerate distributions, with all mass concentrated at opposite ends of the support. Hence, Propositions 6 – 8 make it possible to reexamine some important examples in the literature.

Example 1 (Araujo and Moreira (2001)): Araujo and Moreira (2001) propose a general Lagrangian approach to the moral hazard problem that applies when the FOA is not valid. Their leading example is the following. There are two states, where state 1 is the bad state and state 2 is the good state. The agent picks an effort level, e, from $[\underline{e}, \overline{e}] \subseteq [0,1]$. With effort e, the probability of the good state is $q(e) = e^3$. The cost of effort is $c(e) = e^2$. To reparameterize the model, let $a \equiv c(e) = e^2$ and $p(a) = q(c^{-1}(a)) = a^{\frac{3}{2}}$, $a \in [\underline{a}, \overline{a}] = [\underline{e}^2, \overline{e}^2]$. Note that $p(a)$ is increasing and convex. Thus, A_p^C is empty. In other words, no interior action can be implemented. Moreover, the boundary actions can be implemented, and the only relevant incentive compatibility constraint is that the desired action be preferable to the action on the opposite end of the support. Consequently, this example essentially reduces to the textbook example with two outcomes and two actions, $\underline{a}$ and $\overline{a}$, and is therefore trivial to solve once a participation constraint is added. In contrast, to use their general approach to solve the example, Araujo and Moreira (2001) (having added assumptions on $v(w)$ and on the principal's payoff) construct an algorithm in Mathematica and use this to solve 20 non-linear systems of equations. As expected, they find the optimal action is at a corner. While their method is obviously powerful, using it on their leading example is overkill (not to mention labor intensive) and obscures the intuition. Ke (2012b) proposes another method to solve this problem. Though his method is simpler than that used by Araujo and Moreira (2001), it remains more complicated than the method suggested above. ▲

Mirrlees (1999) offers a famous example to illustrate how the FOA may fail. In their textbook, Bolton and Dewatripont (2005, p. 148) remark that: "This example is admittedly abstract, but this is the only one to our knowledge that addresses the technical issue." Next, I will show that Mirrlees' (1999) example can be analyzed using the techniques presented earlier in this section. In particular, the modified FOA correctly solves the problem. Thereafter, in the hope it will have some pedagogical value, I will provide a more straightforward example of how the FOA may fail. Again, the modified FOA allows the correct solution to be obtained.

Example 2 (Mirrlees (1999)): Consider an agent with payoff function

$$U(w,z) = we^{-(z+1)^2} - \left(-e^{-(z-1)^2}\right).$$

It may be helpful to think of this example as a special environment with two outcomes, where, for some reason, the wage in one state is exogenously fixed at 0. The principal controls the "bonus" w (which may be positive or negative) if the other state materializes. The agent's action is $z \in \mathbb{R}$. Think of $e^{-(z+1)^2}$ roughly as the probability of the state in which a bonus is paid out, and think of $-e^{-(z-1)^2}$ as the cost function. This example fits rather well with the model in (12). In particular, with the spanning condition and only two states, the agent's expected utility is separable in the action and the difference between utility in the two states (the bonus). Next, let $a = -e^{-(z-1)^2}$, and note that $a \in [-1, 0)$. Think of the agent as having a two-stage problem. First, he has to decide which cost level, a, to incur, and, second, whether to incur this cost with a z that is above or below 1 (since $z_- = 1 - \sqrt{-\ln(-a)}$ and $z_+ = 1 + \sqrt{-\ln(-a)}$ both yield the same a). Depending on whether $z < 1$ or $z > 1$, expected utility can be written as $V^-(w,a) = wp^-(a) - a$ or $V^+(w,a) = wp^+(a) - a$, respectively, where

$$p^-(a) = e^{-\left(2-\sqrt{-\ln(-a)}\right)^2}, \text{ and } p^+(a) = e^{-\left(2+\sqrt{-\ln(-a)}\right)^2}, \ a \in [-1, 0).$$

Clearly, $p^-(a) > p^+(a)$. Hence, $V^-(w,a) > V^+(w,a)$ if and only if w is strictly positive. It is now possible to split the problem into two entirely conventional problems. In one, the principal is constrained to $w \leq 0$ and the agent's payoff function is effectively $V^+(w,a)$. In the other, the constraint is $w \geq 0$ and the agent's payoff function is $V^-(w,a)$.

For the first problem, it can be shown that $p^+(a)$ is decreasing. Hence, negative wages are indeed necessary for L-IC. Moreover, $p^+(a)$ is convex and so coincides with its convex hull. It follows from the discussion following Proposition 8 that any interior action can be implemented and that the FOA is valid.

The second problem is more interesting. Here, $p^{-}(a)$ is increasing on $[-1, -e^{-4})$, and decreasing on $(-e^{-4}, 0)$. Since only non-negative wages can be used, there is no permissible contract that satisfies L-IC for any $a \geq -e^{-4} \approx -0.0183$. On the remaining support, $p^{-}(a)$ coincides with its concave hull if and only if $a \in [-1, -0.9982] \cup [-0.0217, -e^{-4})$. By Propositions 6 and 7, the modified FOA is valid on this set (and actions in $(-0.9982, -0.0217)$ cannot be implemented).

Next, Mirrlees specifies an objective function for the principal. There is no participation constraint. The principal seeks to maximize $-(z-1)^2 - (w-2)^2$ or, equivalently, $\ln(-a) - (w-2)^2$. The agent's first order condition yields $w = 1/p^{-\prime}(a)$ and $w = 1/p^{+\prime}(a)$, respectively. Substituting this into the principal's objective function and plotting the resulting functions reveals that positive bonuses are superior to negative bonuses and that the solution is at a corner of the feasible set, specifically at $w = 1$ and $a = -0.9982$ (or $z_{-} = 0.957$). This of course coincides with the solution Mirrlees found, but not with the solution one would obtain from the standard FOA (which yields $a = -0.988\,97$ or $z_{-} = 0.895$, as demonstrated by Mirrlees). ▲

Example 3 (Simplified counterexample): There are two outcomes. Let v_1 be the agent's utility (from wages) if the outcome is bad and v_2 be his utility if the outcome is good. The outcomes are worth x_1 and x_2 to the principal, respectively. The probability of the good outcome is $p(a)$, with $p'(a) > 0$. The participation constraint and L-IC constraint yield the system

$$\begin{aligned} v_1 + p(a)(v_2 - v_1) - a &= \overline{u} \\ p'(a)(v_2 - v_1) - 1 &= 0 \end{aligned}$$

with solution

$$v_1 = \overline{u} + a - \frac{p(a)}{p'(a)},\ v_2 = \overline{u} + a + \frac{1 - p(a)}{p'(a)}.$$

If L-IC is sufficient, the risk-neutral principal's expected payoff is

$$\pi(a) = (1 - p(a))(x_1 - v^{-1}(v_1)) - p(a)(x_2 - v^{-1}(v_2)).$$

Assume $p(a) = a + \frac{1}{2}(a^2 - a^3)$, $a \in [0, \overline{a}]$, $\overline{a} \in (\frac{1}{2}, 1]$. Note that if $a = 0$ then $v_1 = v_2 = \overline{u} + a$, so in this special case, with $p(0) = 0$, $\pi(0)$ also describes the optimal way of implementing the lowest action.[31] Here, $p(a)$ is convex when $a < \frac{1}{3}$ and concave when $a > \frac{1}{3}$. However, the relevant set is A_p^C, which is $A_p^C = [\frac{1}{2}, \overline{a})$. Thus, the set of implementable actions is $\{0\} \cup [\frac{1}{2}, \overline{a}]$, with $\underline{a}^c = \frac{1}{2} > \underline{a}$ and $\overline{a}^c = \overline{a}$. Assume $\overline{u} = 2$, $v(w) = \sqrt{w}$ and $x_1 = 5$, $x_2 = 9.4$. Figure 3 plots $\pi(a)$ when $\overline{a} = \frac{2}{3}$.

There are two stationary points. The first, at $a^* = 0.114$, minimizes the principal's payoff and is not even implementable because the agent's payoff is convex whenever $a < \frac{1}{3}$. The second stationary point, at $a^{**} = 0.464$, is the global maximum of $\pi(a)$. However, a^{**} is not implementable either. Though the agent's payoff is locally concave at a^{**}, it is profitable for the agent to deviate to $\underline{a} = 0$. One way to see this is that $v_2 > v_1 > 2 = \overline{u}$ whenever a (futile) attempt is made at implementing an action in $\left(0, \frac{1}{2}\right)$; the agent can then guarantee himself payoff $v_1 > \overline{u}$ by selecting $\underline{a}$ instead. In other words, while there is a L-IC$_{a^{**}}$ contract there exists no G-IC$_{a^{**}}$ contract. The nearest action for which there is a G-IC contract is $a = \frac{1}{2}$. Indeed, recalling that the feasible set is $\{0\} \cup [\frac{1}{2}, \overline{a}]$, it is clear from Figure 3 that the optimal action to induce is $a = \frac{1}{2}$ (with $v_1 = 2$, $v_2 = \frac{26}{9}$). ▲

[FIGURE 3 ABOUT HERE (SEE THE LAST PAGES)]

The spanning condition has often been implicitly imposed in papers with a continuum of actions. Perhaps the most significant example of this is in LiCalzi and Spaeter (2003) who provide two classes of distributions for which Rogerson's conditions are satisfied. Thus, this paper is customarily cited in papers that rely on the FOA. The first family of distributions is

$$F(x|a) = x + \beta(x)\gamma(a), \;\; x \in [0, 1].$$

Obviously, conditions must be imposed on $\beta(\cdot)$ and $\gamma(\cdot)$ to ensure that $F(x|a)$ is a proper distribution function. LiCalzi and Spaeter (2003) identify additional assumptions on both $\beta(\cdot)$ and $\gamma(\cdot)$ which ensure $F_{aa}(x|a) \geq 0$ and the MLRP. Note, however, that these distribution functions are separable in x and a. Thus, although it seems to not have been observed before, it should be clear that $F(x|a)$ could be stated as

[31] In general, the cost of implementing a given action is discontinuous at $\underline{a}$. The highest action, $a = \overline{a}$, can be implemented with any contract for which $EU'(\overline{a}) \geq 0$. However, it is easy to see that any contract with $EU'(\overline{a}) > 0$ cannot be optimal. The reason is that such a contract unnecessarily imposes more risk on the agent ($v_2 - v_1$ is larger).

in (12). Hence, the modified FOA is always valid in this family of distributions, even without LiCalzi and Spaeter's (2003) additional assumptions.[32]

Example 1 in Jewitt et al (2008) uses the Farlie-Gumbel-Morgenstern copula,

$$f(x|a) = 1 + \frac{1}{2}(1-2x)(1-2a),$$

where $x, a \in [0,1]$. This distribution is also separable in x and a and thus can be written as in (12). Finally, example 1 in Kadan and Swinkels (2012) can be written as

$$F(x|a) = p(a)x + (1-p(a))\left(x + x^2 - x^3\right),$$

where $p(a) = 2a^2 - a^3$, $a \in [\frac{2}{3}, 1]$, and $x \in [0,1]$.

7.2 Beyond the spanning condition ($k \geq 2$)

The spanning condition leads to the striking conclusion that the optimal way to induce any implementable action is by offering a contract that takes the form in (8). In other words, the shape of the contract is determined solely by L-IC; the role of G-IC is only to determine whether the action is implementable in the first place. In general, however, G-IC may also change the shape of the optimal contract. The model in (9) can be used to illustrate this property when $k > 1$.

Fix some action a^* that the principal seeks to implement. Let IC_{a,a^*} denote the non-local incentive compatibility constraint that action a^* be no worse than action a for the agent, or

$$EU(a^*) \geq EU(a) \qquad (\text{IC}_{a,a^*})$$

for some $a \neq a^*$. Of course, G-IC$_{a^*}$ is satisfied if and only if IC_{a,a^*} is satisfied for all $a \neq a^*$. When the FOA is valid, these constraints can be ignored since they are implied by L-IC$_{a^*}$. When the FOA is not valid, however, the optimal contract will often (the case with $k = 1$ notwithstanding) be shaped by whichever non-local incentive compatibility constraints are binding.

For the case where $k = 2$, I will in the following characterize a small subset of non-local incentive compatibility constraints that, together with L-IC$_{a^*}$, are sufficient for G-IC$_{a^*}$. Put differently, the remaining non-local incentive compatibility constraints

[32] Ke (2011a, Proposition 9) prove that LiCalzi and Spaeter's (2003) additional assumptions on $\beta(x)$ are not necessary for the validity of the FOA.

are redundant. Intuitively, the relevant constraints represent those actions that are the most tempting deviations for the agent. Once these constraints have been identified it is trivial to derive the optimal contract that induces a^*.

Thus, assume $k = 2$. In this subsection, I will also assume that p_1 and p_2 are twice continuously differentiable. Fix some $a^* \in (\underline{a}, \overline{a})$. For brevity, define $p_i^L(a|a^*) = p_i(a^*) + (a - a^*)p_i'(a^*)$, $i = 1, 2$. Then, (10) can be written

$$EU^L(a|a^*) - EU(a) = \left[p_1^L(a|a^*) - p_1(a)\right] C_1 + \left[p_2^L(a|a^*) - p_2(a)\right] C_2. \tag{15}$$

For now, assume further that $p_1''(a) \neq 0$ for all $a \in [\underline{a}, \overline{a}]$, i.e. p_1 is either strictly concave or strictly convex. Given that the previous literature on the spanning condition ($k = 1$) typically assumes $p(a)$ is (weakly) concave, this seems like a reasonable starting point. Thus, $p_1^L(a|a^*) - p_1(a) \neq 0$ for all $a \neq a^*$ and so

$$EU^L(a|a^*) - EU(a) = \left[p_1^L(a|a^*) - p_1(a)\right] \left(C_1 + \frac{p_2^L(a|a^*) - p_2(a)}{p_1^L(a|a^*) - p_1(a)} C_2\right) \tag{16}$$

whenever $a \neq a^*$. The second term is continuous; by L'Hôpital's rule, the ratio takes on the value $\frac{p_2''(a^*)}{p_1''(a^*)}$ at $a = a^*$. Let

$$m(a^*) \in \arg\min_a \frac{p_2^L(a|a^*) - p_2(a)}{p_1^L(a|a^*) - p_1(a)}, \tag{17}$$

$$M(a^*) \in \arg\max_a \frac{p_2^L(a|a^*) - p_2(a)}{p_1^L(a|a^*) - p_1(a)}. \tag{18}$$

By assumption, the first term in (16) has constant sign. Thus, (16) is positive at all $a \neq a^*$ if and only if it is positive at $a = m(a^*)$ and at $a = M(a^*)$. All the other non-local incentive compatibility constraints are redundant. Note that no conditions on the contract, such as monotonicity, has been imposed.

Proposition 9 *Assume $k = 2$ and $p_1''(a) \neq 0$ for all $a \in [\underline{a}, \overline{a}]$. Then, L-IC$_{a^*}$, IC$_{m(a^*),a^*}$, and IC$_{M(a^*),a^*}$ are necessary and sufficient for G-IC$_{a^*}$ for any $a^* \in (\underline{a}, \overline{a})$.*

The FOA can now be modified; the maximization problem in Section 3 must be appended with two additional constraints, namely IC$_{m(a^*),a^*}$ and IC$_{M(a^*),a^*}$. In fact, it can be shown that as long as $p_1''(a) \neq 0$ for all $a \in [\underline{a}, \overline{a}]$, the ratio in (16) is monotonic (increasing or decreasing) if $\frac{p_2''(a)}{p_1''(a)}$ is monotonic. The proof is omitted, but is available on request. In this case, L-IC$_{a^*}$, IC$_{\underline{a},a^*}$, and IC$_{\overline{a},a^*}$ are necessary and sufficient for G-IC$_{a^*}$ for all $a^* \in (\underline{a}, \overline{a})$. That is, the agent must be prevented only from deviating to the highest and lowest action.

Corollary 3 *Assume $k = 2$, $p_1''(a) \neq 0$, and that $\frac{p_2''(a)}{p_1''(a)}$ is monotonic for all $a \in [\underline{a}, \overline{a}]$. Then, L-IC$_{a^*}$, IC$_{\underline{a},a^*}$, and IC$_{\overline{a},a^*}$ are necessary and sufficient for G-IC$_{a^*}$ for any $a^* \in (\underline{a}, \overline{a})$.*

Next, a procedure is outlined that can be applied when the assumption in Proposition 9 is not satisfied. Fix some $a^* \in (\underline{a}, \overline{a})$. Then, break the problem into smaller parts by partitioning $[\underline{a}, \overline{a}]$ into three disjoint subsets. The intention is to ensure that the agent has no incentive to deviate to any of these sets. One set, A_+, consists of all actions for which $\left[p_1^L(a|a^*) - p_1(a)\right] > 0$. In the second set, A_-, actions satisfy $\left[p_1^L(a|a^*) - p_1(a)\right] < 0$. Any member of the final set of actions, A_0, satisfies $\left[p_1^L(a|a^*) - p_1(a)\right] = 0$. Starting with the latter set, assume for simplicity that $\left[p_2^L(a|a^*) - p_2(a)\right]$ has a constant sign on A_0. Then, (15) reveals that if IC$_{\widehat{a},a^*}$ is satisfied for one arbitrary member $\widehat{a}$ of A_0 (with $\widehat{a} \neq a^*$), then IC$_{a,a^*}$ is satisfied for all other members of A_0 as well.[33] Consider now the set A_+. Extending the previous analysis, derive the arg inf and the arg max of the ratio in (17) and (18) on the closure of A_+. If there is no incentive to deviate to any of these actions, then there is no incentive to deviate to any other action in A_+.[34] A similar exercise can be applied to set A_-. In summary, G-IC$_{a^*}$ is ensured if L-IC$_{a^*}$ is satisfied and there is no incentive to deviate to any $a \in A_0$ or to the arg inf or the arg max of the ratio in (16) on the closures of A_+ and A_-.

Assuming F has a differentiable density and that the only other constraint on the problem is the participation constraint, Ke (2012b) has recently demonstrated that the optimal contract – regardless of the validity of the FOA – can be described by

$$\frac{1}{v'(w(x))} = \lambda + \mu l(x|a^*) + \gamma \left(1 - \frac{f(x|\widehat{a})}{f(x|a^*)}\right), \tag{19}$$

where $\gamma \geq 0$ is a multiplier to a non-local incentive compatibility constraint. If $\gamma > 0$, then the agent is indifferent between $\widehat{a}$ and a^*, meaning that this non-local incentive

[33] If $\left[p_2^L(a|a^*) - p_2(a)\right]$ does not have a constant sign on A_0, then G-IC$_{a^*}$ necessitates $C_2 = 0$, which in turn implies that G-IC$_{a^*}$ can be satisfied only if either A_+ or A_- is empty. The constraint that $C_2 = \int v(w(x))\,(g_2(x) - h(x))\,dx = 0$ can be added to the principals maximization problem.

[34] Since these actions are in the closure of A_+, they may in fact belong to A_0.

compatibility constraint is binding. However, Ke (2012b) does not pin-point $\widehat{a}$. If the assumptions in Proposition 9 are satisfied, then $\widehat{a}$ is either $m(a^*)$ or $M(a^*)$.

Next, consider the general case in which $k \geq 1$. Here, I will impose more structure on the problem. First, assume that the contract must be monotonic (increasing) for some exogenous reason, as in Innes (1990). There may be actions that can be implemented with a non-monotonic contract, but not with a monotonic contract. Due to the constraint on the feasible contracts, such actions can be ignored.

Second, assume that G_1 first order stochastically dominates G_2, which in turn first order stochastically dominates G_3, and so on, with H being dominated by $G_1, G_2..., G_k$. One interpretation is that technologies with lower subscripts are more productive, and that it would be beneficial to the principal if more weight is shifted onto these technologies. In the following, I will use the term *ordered technology* to refer to this assumption. By (11), any ordered technology implies that any monotonic contract translates into non-negative C_i's that can themselves be ordered, or $C_1 \geq C_2 \geq ... \geq C_k \geq 0$.

To proceed, note that (10) can be rewritten as

$$EU^L(a|a^*) - EU(a) = \left[\sum_{i=1}^{k} p_i(a^*)C_i + (a - a^*)\sum_{i=1}^{k} p_i'(a^*)C_i\right] - \sum_{i=1}^{k} p_i(a)C_i, \quad (20)$$

where the first term is the tangent line to the (endogenously determined) function $\sum p_i(a)C_i$. This leads to the third assumption, namely that the function $\sum p_i(a)K_i$ is first convex (on a possibly empty interval) and then concave (on a possibly empty interval) for all $(K_1, K_2, ..., K_k)$ with the property that $K_1 \geq K_2 \geq ... \geq K_k \geq 0$.[35] I will use the term *S-shaped technology* to refer to this assumption. Incidentally, $F(x|a)$ can be written as

$$F(x|a) = -\sum_{i=1}^{k} p_i(a)\left(H(x) - G_i(x)\right) + H(x),$$

where $H(x) - G_1(x) \geq H(x) - G_2(x) \geq ... \geq H(x) - G_k(x) \geq 0$ due to the assumption that the technology is ordered. Thus, the joint assumption that the technology is both ordered and S-shaped means that $F(x|a)$ has a reverse S-shape as a function of a. In his conclusion, Conlon (2009a) notes that a very accurate signal will be described

[35] This assumption is satisfied if e.g. $p_i'''(a) \leq 0$ for all a and all i.

by a distribution function which has this property for at least some x. Finally, the S-shaped technology can also be thought of as a generalization of the environment in Example 3. In that example, the non-local IC constraint with "bite" was $\text{IC}_{\underline{a},a^*}$.

Lemma 5 *Fix $a^* \in (\underline{a}, \overline{a})$. Assume the technology is ordered and S-shaped. Then, any monotonic contract is G-IC_{a^*} if and only if L-IC_{a^*} and $IC_{\underline{a},a^*}$ are satisfied.*

Proof. L-IC_{a^*} and $\text{IC}_{\underline{a},a^*}$ are obviously necessary for G-IC_{a^*}. For sufficiency, assume that the contract is monotonic and that L-IC_{a^*} and $\text{IC}_{\underline{a},a^*}$ are satisfied. Since the contract is monotonic and the technology is ordered, $C_1 \geq C_2 \geq ... \geq C_k \geq 0$. Thus, the second assumption on the technology can be invoked. By L-IC_{a^*}, G-IC_{a^*} is satisfied if and only if (20) is always non-negative, or, stated differently, if a^* is on the concave hull of $\sum p_i(a)C_i$. However, since $\sum p_i(a)C_i$ is S-shaped, a^* is on the concave hull of $\sum p_i(a)C_i$ if and only if this function's tangent line through a^* lies above the function itself at $\underline{a}$. In other words, (20) is non-negative for all a if and only if it is non-negative at $\underline{a}$. Thus, $\text{IC}_{\underline{a},a^*}$ is sufficient. ■

The lemma implies that for all the actions that can be implemented with a monotonic contract, L-IC_{a^*} and $\text{IC}_{\underline{a},a^*}$ are sufficient. It is thus tempting to modify the FOA by simply adding the constraint $\text{IC}_{\underline{a},a^*}$, which would yield a contract of the form in (19). However, it must then be verified that this candidate is monotonic. Indeed, Grossman and Hart (1983, Example 1) demonstrate that optimal contracts need not be monotonic when more than one incentive compatibility constraint binds, even if MLRP holds. In fact, it was this observation that lead them to the spanning condition in the first place.

8 Conclusion

In this paper, a new approach to the moral hazard problem has been suggested. The approach is based on reformulating the problem in terms familiar to any economist. In particular, standard results from the theory of choice under uncertainty can be invoked to prove new and old results.

The new approach permits a unified proof of Rogerson's (1985) and Jewitt's (1988) one-signal justifications of the FOA. Indeed, the insights gained from reformulating the problem makes it possible to derive other sufficient conditions. Similarly, in the multi-signal model, the justifications provided by Jewitt (1988) and Conlon (2009a) can be explained with a common methodology. It is important to note that there are several different ways in which one-signal results can be extended into higher dimensions. The orthant orders form the basis of some tractable alternatives. One distinct advantage of the justifications based on the lower orthant order and the lower orthant-concave order is that they are robust to the inclusion of more independent signals.

The second part of the paper focused on environments in which the FOA is not generally valid. Local versions of Rogerson's and Jewitt's conditions were presented, thereby allowing subsets of actions for which L-IC implies G-IC to be identified. Next, a more specific model was considered. Though the spanning condition looks simple and dates back to Grossman and Hart (1983), the first full characterization of its solution is given here. Mirrlees' (1999) famous counterexample can also be solved using the techniques presented here. A simpler counterexample was also presented. When the spanning condition is relaxed, it turns out to be sometimes possible to identify the critical non-local incentive compatibility constraints. Determining these relevant non-local constraints seems to be a necessary first step to characterizing optimal contracts when the FOA is not valid. A more systematic analysis of this issue is left for future research.

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Rogerson	Jewitt	Third set of conditions
$F_{aa}(x\|a) \geq 0,\ \forall x, a$ $\Downarrow$ Any nondecreasing and L-IC contract is G-IC	$\int_{\underline{x}}^{x} F_{aa}(y\|a)dy \geq 0,\ \forall x, a$ $\Downarrow$ Any nondecreasing, concave, and L-IC contract is G-IC	$\int_{\underline{x}}^{x} \int_{\underline{x}}^{z} F_{aa}(y\|a)dydz \geq 0,\ \forall x, a$ and $\int_{\underline{x}}^{\overline{x}} F_{aa}(y\|a)dy \geq 0,\ \forall a$ $\Downarrow$ Any nondecreasing, concave, positively skewed, and L-IC contract is G-IC
FOSD	SOSD	TOSD
$G(x) \leq H(x),\ \forall x$ $\Updownarrow$ $E_G\left[u(x)\right] \geq E_H\left[u(x)\right]$ for any nondecreasing $u(x)$	$\int_{\underline{x}}^{x} G(y)dy \leq \int_{\underline{x}}^{x} H(y)dy,\ \forall x$ $\Updownarrow$ $E_G\left[u(x)\right] \geq E_H\left[u(x)\right]$ for any nondecreasing and concave $u(x)$	$\int_{\underline{x}}^{x} \int_{\underline{x}}^{z} G(y)dydz \leq \int_{\underline{x}}^{x} \int_{\underline{x}}^{z} H(y)dydz,\ \forall x$ and $\int_{\underline{x}}^{\overline{x}} G(y)dy \leq \int_{\underline{x}}^{x} H(y)dy$ $\Updownarrow$ $E_G\left[u(x)\right] \geq E_H\left[u(x)\right]$ for any nondecreasing, concave, and positively skewed $u(x)$.

Table 1: Rogerson, Jewitt and stochastic dominance.

Note: $F(\cdot|a)$ is the distribution over outcomes given action a. In the third column, a positively skewed utility function, $u(x)$, is one for which $u'(x)$ is non-negative, decreasing, and convex.

Rogerson	Jewitt	Third set of conditions
$F_{aa}(x\|a) \geq 0,\ \forall x, a.$	$\int_{\underline{x}}^{x} F_{aa}(y\|a)dy \geq 0,\ \forall x, a.$	$\int_{\underline{x}}^{x} \int_{\underline{x}}^{z} F_{aa}(y\|a)dydz \geq 0,\ \forall x, a$ and $\int_{\underline{x}}^{\overline{x}} F_{aa}(y\|a)dy \geq 0,\ \forall a.$
$\omega'(z) > 0$ $l_x(x\|a) \geq 0$	$\omega'(z) > 0,\ \omega''(z) \leq 0$ $l_x(x\|a) \geq 0,\ l_{xx}(x\|a) \leq 0$	$\omega'(z) > 0,\ \omega''(z) \leq 0,\ \omega'''(z) \geq 0$ $l_x(x\|a) \geq 0,\ l_{xx}(x\|a) \leq 0,\ l_{xxx}(x\|a) \geq 0$

Table 2: Justifying the first order approach, Part I.

Rogerson (1-icx)	2-icx	3-icx
$\overline{F}_{aa}(x\|a) \leq 0,\ \forall x, a.$	$\int_{x}^{\overline{x}} \overline{F}_{aa}(y\|a)dy \leq 0,\ \forall x, a.$	$\int_{x}^{\overline{x}} \int_{z}^{\overline{x}} \overline{F}_{aa}(y\|a)dydz \leq 0,\ \forall x, a$ and $\int_{\underline{x}}^{\overline{x}} \overline{F}_{aa}(y\|a)dy \leq 0,\ \forall a.$
$\omega'(z) > 0$ $l_x(x\|a) \geq 0$	$\omega'(z) > 0,\ \omega''(z) \geq 0$ $l_x(x\|a) \geq 0,\ l_{xx}(x\|a) \geq 0$	$\omega'(z) > 0,\ \omega''(z) \geq 0,\ \omega'''(z) \geq 0$ $l_x(x\|a) \geq 0,\ l_{xx}(x\|a) \geq 0,\ l_{xxx}(x\|a) \geq 0$

Table 3: Justifying the first order approach, Part II.

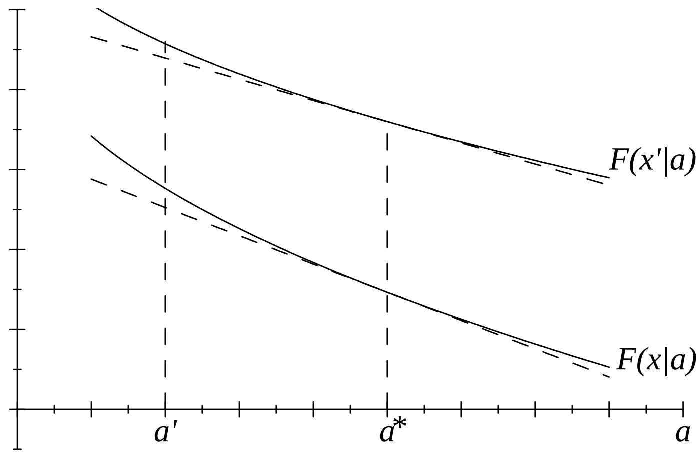

Figure 1: A new approach to moral hazard.

Note: Step 1: Fix a^*. For each x, construct the tangent line to $F(x|a)$ at a^* (moving horizontally). Step 2: For each $a \neq a^*$, like a', move vertically to trace out the cdf in the auxiliary and real problems. Here, F^L FOSD F (F^L lies always below F). Thus, any monotonic and L-IC$_{a^*}$ contract yields $EU(a^*) = EU^L(a^*|a^*) = EU^L(a'|a^*) \geq EU(a')$. Step 3: To validate the FOA, the conclusion in step 2 must hold regardless of a^*.

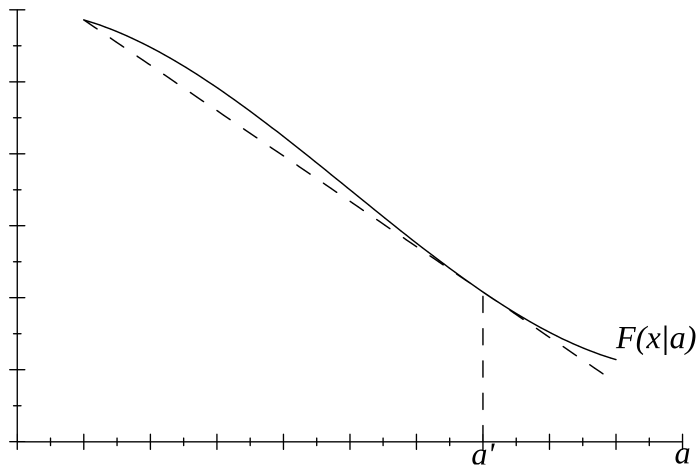

Figure 2: Comparing $F(x|a)$, $F^L(x|a,a')$, and $F^C(x|a)$.

Note: $F^C(x|a)$ coincides with the dashed line ($F^L(x|a,a')$) to the left of a', and with the solid curve ($F(x|a)$) to the right of a'. If $a^* \geq a'$ then $F(x|a) \geq F^L(x|a,a^*)$ for all a. If this holds for all x, then any monotonic and L-IC$_{a^*}$ contract is G-IC$_{a^*}$. If $a^* < a'$ then there are small a for which $F^L(x|a,a^*) > F(x|a)$. Then, when $F_a(x|a) < 0$, there are monotonic and L-IC$_{a^*}$ contracts that are not G-IC$_{a^*}$ (Lemma 4).

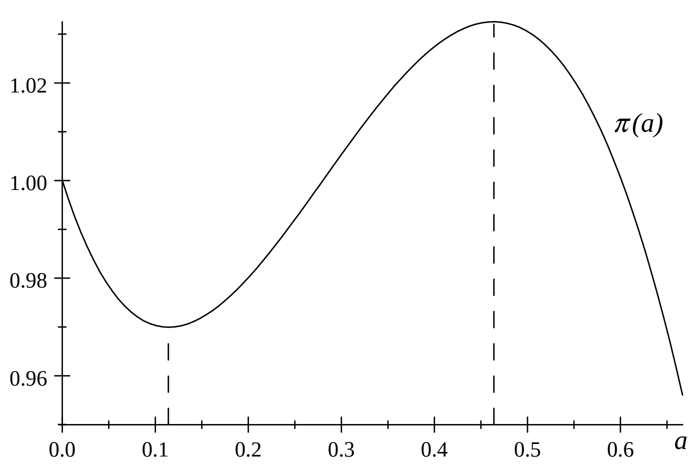

Figure 3: Simplified counterexample.