Network Extension

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Abstract

In a model of strategic network formation, the endogenously formed network is built around a pre-existing network. We envisage that the pre-existing or core network is publicly provided. Strategic network formation is decentralized: Players act in their private interest and bear the costs when adding links to the pre-existing network. We study how the pre-existing network affects existence of Nash equilibria and efficiency of Nash equilibrium outcomes: It can foster or prohibit existence of Nash equilibria. It can improve or worsen equilibrium welfare. Special attention is paid to an insider-outsider model where society is partitioned into several groups and links within a group (between insiders) are much cheaper than links across groups (between outsiders). We also investigate the relationship between different notions of efficient networks, Pareto optimal networks on the one hand and welfare maximizing networks on the other hand; present equilibrium existence results; examine the effect of efficient publicly provided networks; and design and analyze a subscription game for the public provision of a network.

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1 Introduction

The rapidly growing importance of networks and network analysis is now recognized in many fields, for instance in artificial intelligence, biology, business and finance, computer science, economics, electrical engineering, neuroscience, sociology, and physics. Network analysis can be focused on network topology, network utilization, network formation, or the co-evolution of network utilization and network formation. Within game theory, several strands of literature on network creation (network formation, network design) have emerged. A number of recent contributions have treated social and economic networks as the outcome of a network formation game. The players of the game constitute the nodes of the network to be formed. In the purely noncooperative approach of Bala and Goyal (2000), addition and deletion of links are unilateral decisions of the player from whom the respective links originate.[1] The player's strategy is a specification of the set of agents with whom he forms links. The costs of link formation are incurred only by the player who initiates the link. The formed links define the network.

I adopt the basic paradigm of Bala and Goyal that a network is the outcome of a game in strategic form where the players are the nodes of the network and unilaterally form links. The benchmark model is the two-way flow connections model à la Galeotti, Goyal, and Kamphorst (2006) that incorporates cost and value heterogeneity. A player receives valuable information from others not only through direct links, but also via indirect links. The player incurs the cost of the direct links she initiates. The solution concept is Nash equilibrium. In general, the term Nash network refers to a network arising as the Nash equilibrium outcome of a network formation game. I shall use **Nash network** as a synonym for Nash equilibrium, since in our game (as in the literature in the tradition of Bala and Goyal (2000)) a strategy profile can be identified with the resulting network.

Frequently, networks are not designed de novo, but rather built around and upon pre-existing infrastructures and networks — which imposes constraints on network formation. In response to this observation, the subsequent analysis goes beyond the basic model of Bala and Goyal (2000), Gale-

[1]Pairwise stability à la Jackson and Wolinsky (1996) treats addition of a link in a network as a bilateral decision by the two players involved, whereas severance of a link constitutes a unilateral decision.

otti, Goyal, and Kamphorst (2006), Haller, Kamphorst and Sarangi (2007), and others by incorporating internal constraints on network formation. Before we proceed to the latter, let us briefly consider the opposite, external constraints. In a network formation game, players may be subject to an exogenous *external constraint*, given by a pre-existing network or infrastructure $\mathfrak{g}$. In that case, a player can only form direct links to his immediate neighbors in $\mathfrak{g}$. The ultimately formed network g has to be a subnetwork of $\mathfrak{g}$: $g \subseteq \mathfrak{g}$. **The endogenous network is built within $\mathfrak{g}$**. For example, the exogenous network could reflect geographical, legal or language barriers. If two persons are not neighbors broadly defined, then a link between them may be implausible, because it is impossible or to no avail. Alternatively, the exogenous network could be a physical infrastructure, like fiber-optical cables, which determines the set of feasible individual links. Infeasible could simply mean exorbitantly costly. Network formation with an exogenous external constraint has been investigated in Baron *et al.* (2006).

Here I am considering network formation games where players are subject to an (exogenous or endogenous) *internal constraint*, again given by a network or infrastructure $\mathfrak{g}$. Players can only create links in addition to those already existing in $\mathfrak{g}$. Consequently, the ultimately formed network g encompasses $\mathfrak{g}$: $g \supseteq \mathfrak{g}$. **The endogenous network is built around $\mathfrak{g}$**. For instance, the internet rests on major links, so-called "backbones", around which the rest of the network is constructed.

In the sequel, I treat $\mathfrak{g}$ as variable and study its effect on the architecture and efficiency of the resulting Nash networks, that is the networks that are Nash equilibrium outcomes of the strategic network formation game. I consider four possibilities. First, a non-trivial $\mathfrak{g}$ can prove efficiency enhancing, leading to an efficient outcome while an inefficient network may result when $\mathfrak{g}$ is the empty network with no links. Second, to the opposite, a non-trivial $\mathfrak{g}$ can prove efficiency reducing. Third, a non-trivial $\mathfrak{g}$ can be stabilizing, yielding existence of a Nash network in situations where Nash networks do not exist when $\mathfrak{g}$ is the empty network. Finally, under different circumstances, a non-trivial $\mathfrak{g}$ can prove destabilizing. I am going to exemplify all four possible effects. Furthermore, existence of Nash networks will be addressed. I shall analyze in more detail the effects of internal constraints in the special case of the insider-outsider model of Galeotti, Goyal, and Kamphorst (2006).

From a descriptive and positive perspective, introduction of internal constraints allows one to study endogenous network formation when only part of the network is created de novo. From a normative viewpoint, performing comparative statics with $\mathfrak{g}$ as exogenous variable provides the basis for making $\mathfrak{g}$ an endogenous public choice. One may be able to determine an optimal choice of a publicly provided infrastructure $\mathfrak{g}$. Alternatively, one might consider the possibility that $\mathfrak{g}$ is determined in a decentralized manner, like the rest of the network.

The next section contains a benchmark model without an internal constraint, that is, no pre-existing or publicly provided infrastructure or network. In Section 3, internal constraints are incorporated into the model. Section 4 deals with existence of Nash equilibria with and without external constraints and the effect of efficient infrastructures on equilibrium outcomes. Section 5 is devoted to the stabilizing or destabilizing effects of a publicly provided infrastructure. Section 6 is devoted to the welfare effects of a publicly provided infrastructure. In Section 7, we explore the impact of a publicly provided backbone infrastructure in the insider-outsider model of Galeotti, Goyal, and Kamphorst (2006). In Section 8, the question how to choose or fund an infrastructure $\mathfrak{g}$ is briefly addressed. Section 9 concludes.

2 Benchmark Model

We first introduce a benchmark model, where an internal constraint is absent. It serves two purposes. On the one hand, it constitutes the foundation of the more general model. One the other hand, it allows to assess the effects of internal constraints. The benchmark model is the two-way flow connections model à la Galeotti, Goyal, and Kamphorst (2006) that incorporates cost and value heterogeneity. We adopt the notation of Haller, Kamphorst and Sarangi (2007) for the case of perfectly reliable links.

Let $n \geq 3$. $N = \{1, \ldots, n\}$ denotes the set of players with generic elements i, j, k. N also constitutes the set of nodes of the network to be formed. For ordered pairs $(i, j) \in N \times N$, the shorthand notation ij is used and for non-ordered pairs $\{i, j\} \subset N$ the shorthand $[ij]$ is used. The symbol $\subset$ for set inclusion permits equality. The model is specified by two families of parameters, indexed by ij, with $i \neq j$:

- Cost parameters $c_{ij} > 0$.
- Value parameters $V_{ij} > 0$.

In case $c_{ij} \neq c_{kl}$ ($V_{ij} \neq V_{kl}$) for some $ij \neq kl$, the model exhibits **cost (value) heterogeneity**; otherwise, it exhibits **cost (value) homogeneity**. Following Derks and Tennekes (2009), we say that costs are **owner-homogeneous** if for each player i, there exists $c_i > 0$ such that $c_{ij} = c_i$ for all $j \neq i$. This condition is also considered in Galeotti (2006), Galeotti *et al.* (2006), Billand *et al.* (2008), and Derks and Tennekes (2009).

We only consider pure strategies. A pure strategy for player i is a vector $g_i = (g_{i1}, \ldots, g_{ii-1}, g_{ii+1}, \ldots, g_{in}) \in \{0,1\}^{N\setminus\{i\}}$. The set of all pure strategies of agent i is denoted by $\mathcal{G}_i$. It consists of 2^{n-1} elements. The joint strategy space is given by $\mathcal{G} = \mathcal{G}_1 \times \cdots \times \mathcal{G}_n$.

There is a one-to-one correspondence between the set of joint strategies $\mathcal{G}$ and the set of all directed graphs or networks with vertex set N. Namely, to a strategy profile $g = (g_1, \ldots, g_n) \in \mathcal{G}$ corresponds the graph $(N, E(g))$ with edge or link set $E(g) = \{(i,j) \in N \times N |\, i \neq j,\, g_{ij} = 1\}$. In the sequel, we shall identify a joint strategy g and the corresponding graph and use the terminology **directed graph** or **directed network** g. Since our aim is to model network formation, $g_{ij} = 1$ is interpreted to mean that a direct link between i and j is initiated by player i (edge ij is formed by i) whereas $g_{ij} = 0$ means that i does not initiate the link (ij is not formed). Regardless of what player i does, player j can set $g_{ji} = 1$, i.e., initiate a link with i, or set $g_{ji} = 0$, i.e., not initiate a link with i.

Benefits. A link between agents i and j potentially allows for **two-way (symmetric) flow of information**. Accordingly, the benefits from network g are derived from its closure $\overline{g} \in \mathcal{G}$, defined by $\overline{g}_{ij} := \max\{g_{ij}, g_{ji}\}$ for $i \neq j$. Moreover, a player receives information from others not only through direct links, but also via indirect links. To be precise, information flows from player j to player i, if i and j are linked by means of a path in $\overline{g}$ from i to j. A **path** of length m in $f \in \mathcal{G}$ from player i to player $j \neq i$, is a finite sequence $i_0, i_1, \ldots, i_m$ of pairwise distinct players such that $i_0 = i$, $i_m = j$, and $f_{i_k i_{k+1}} = 1$ for $k = 0, \ldots, m-1$. Let us denote

$$N_i(f) = \{j \in N |\, j \neq i, \text{ there exists a path in } f \text{ from } i \text{ to } j\},$$

the set of other players whom player i can access or "observe" in the network f. Information received from player j is worth V_{ij} to player i. Therefore, player i's benefit from a network g with perfectly reliable links and two-way flow of information is (as in Galeotti, Goyal, and Kamphorst (2006)):

$$B_i(g) = B_i(\overline{g}) = \sum_{j \in N_i(\overline{g})} V_{ij}.$$

Notice that $\overline{g}$ belongs to the set $\mathcal{H} = \{h \in \mathcal{G} | h_{ij} = h_{ji} \text{ for } i \neq j\}$. In turn, there is a one-to-one correspondence between the elements of $\mathcal{H}$ and the non-directed networks (graphs) with node set N. Namely, for $h \in \mathcal{H}$ and $i \neq j$, $[ij]$ is an edge of the corresponding non-directed network if and only if $h_{ij} = h_{ji} = 1$. In the sequel, we shall identify h with the corresponding non-directed network. In that case, the notation $[ij] \in h$ stands for "$[ij]$ is an edge of h", that is h is given by its set of edges. Accordingly, for $k \in \mathcal{H}$, $k \subset h$ means that k is a subnetwork of h.

Costs. Player i incurs the cost c_{ij} when she initiates the direct link ij, i.e., if $g_{ij} = 1$. Hence i incurs the total costs

$$C_i(g) = \sum_{j \neq i} g_{ij} c_{ij}$$

when the network g is formed.

Payoffs. Player i's payoff from the strategy profile g is the net benefit

$$\Pi_i(g) = B_i(g) - C_i(g). \tag{1}$$

Nash Networks. Given a network $g \in \mathcal{G}$, let g_{-i} denote the network that remains when all of agent i's links have been removed so that $g_{-i} \in \mathcal{G}_{-i} \equiv \prod_{j \neq i} \mathcal{G}_j$. Clearly $g = g_i \oplus g_{-i}$ where the symbol $\oplus$ indicates that g is formed by the union of links in g_i and g_{-i}. A strategy g_i is a **best response** of agent i to g_{-i} if

$$\Pi_i(g_i \oplus g_{-i}) \geq \Pi_i(g'_i \oplus g_{-i}) \text{ for all } g'_i \in \mathcal{G}_i.$$

Let $BR_i(g_{-i})$ denote the set of agent i's best responses to g_{-i}. A network $g = (g_1, \ldots, g_n)$ is said to be a **Nash network** if $g_i \in BR_i(g_{-i})$ for each i, that is if g is a Nash equilibrium of the strategic game with normal form $(N, (\mathcal{G}_i)_{i \in N}, (\Pi_i)_{i \in N})$. A strict Nash network is one where agents are playing strict best responses.

Some Graph-theoretic Concepts. We now introduce some definitions of a more graph-theoretic nature. The network with no links is called the **empty network** and will be denoted e. A network g is said to be **connected** if there is a path in $\overline{g}$ between any two agents i and j. A connected network g is **minimally connected**, if it is no longer connected after the deletion of *any* link.

A set $C \subset N$ is called a **component** of g if there exists a path in $\overline{g}$ between any two different agents i and j in C and there is no strict superset C' of C for which this holds true. For each network g, the components of g form a partition of the player set (node set, vertex set) N into non-empty subsets. Each isolated point $i \in N$ in g, that is a player or node i with $g_{ij} = g_{ji} = 0$ for all $j \neq i$, gives rise to a singleton component $\{i\}$. In particular, the components of the empty network are the sets $\{i\}, i \in N$. N is the only component of g if and only if g is connected. If C is a component of the network g, we denote by g^C the network induced by g on the set of nodes C, that is $g^C_{ij} = g_{ij}$ for $i, j \in C$, $i \neq j$. A network g is **minimal**, if g^C is minimally connected for every component C of g. Minimally connected networks are both connected and minimal.

Efficient Networks. Let $W_0 : \mathcal{G} \to \mathbb{R}$ be defined as $W_0(g) = \sum_{i=1}^{n} \Pi_i(g)$. A network $\widehat{g}$ is **efficient (in the narrow traditional sense)** if $W_0(\widehat{g}) \geq W_0(g)$ for all $g \in \mathcal{G}$. Efficiency is a major performance criterion for network designers or planners and plays a prominent role in the traditional network literature. It is most attractive for cost-benefit analysis when payoffs are monetary and side-payments between players are feasible. Efficiency constitutes an important benchmark for network performance even when network formation is decentralized and structured as a strategic game. In economics, the term "efficiency" is often used in a broader sense, as a synonym for Pareto optimality. An efficient network in the traditional sense is necessarily Pareto optimal. That is, there is no other network that provides a higher payoff to some player(s) and no lower payoff to any player. But not every Pareto optimal network has to be efficient in the traditional sense as the following example demonstrates.

Example 1 (Efficiency and Social Welfare)
Let $N = \{1, 2, 3\}$. Further set $V_{ij} = 1$ for all $i \neq j$; $c_{2j} = 1.6$ and $c_{j2} = 1.8$ for $j \neq 2$; and $c_{ij} = 3$ otherwise. Then the empty network and $g' = \{12, 32\}$, the periphery-sponsored star with center 2, are the two Nash networks. Neither one is efficient in the traditional sense. The center-sponsored star with center 2, $\widehat{g} = \{21, 23\}$ is the only efficient network in the traditional sense. However, g' is Pareto optimal and maximizes the utilitarian social welfare function $W(g) = \Pi_1(g) + 2\Pi_2(g) + \Pi_3(g)$. □

In the example, the Pareto optimal network g' does not maximize the social welfare function W_0 with equal weights, but it maximizes another utilitarian social welfare function on $\mathcal{G}$. This raises the question whether in general, for every Pareto optimal network $g' \in \mathcal{G}$, there exists a social welfare function $W : \mathcal{G} \to \mathbb{R}$ of the form $W(g) = \sum_{i=1}^{n} a_i \Pi_i(g)$ so that $(a_1, \ldots, a_n) \gg 0$ and g' maximizes W on $\mathcal{G}$.

Pareto optimal networks are minimal, but not necessarily minimally connected. We obtain some partial answers in case all Pareto optimal networks are minimally connected:

Proposition 1 *Suppose all Pareto optimal networks are minimally connected.*

(a) *In case of owner-homogeneous costs, there exists a utilitarian social welfare function W^* such that all minimally connected networks maximize W^* and are Pareto optimal.*

(b) *If costs are not owner-homogeneous, there can exist Pareto optimal networks that do not maximize any utilitarian social welfare function.*

(c) *If costs are totally heterogeneous, that is, $c_{ij} \neq c_{kl}$ for $ij \neq kl$, then for generic utilitarian welfare weights, the corresponding utilitarian social welfare function has a unique maximizer.*

PROOF. If g is minimally connected and $W(g) = \sum_{i=1}^{n} a_i \Pi_i(g)$ is a utilitarian social welfare function, then

$$W(g) = \sum_{i=1}^{n} a_i \sum_{j \neq i} V_{ij} - \sum_{i=1}^{n} a_i \sum_{j \neq i} g_{ij} c_{ij}. \tag{2}$$

Hence maximization of $W(g)$ on the set of minimally connected networks amounts to minimization of $\sum_{i=1}^{n} a_i \sum_{j \neq i} g_{ij} c_{ij}$ on $\mathcal{G}$ subject to the constraint that g is minimally connected. In the literature on combinatorial optimization, computer science, and operations research, the latter is known as the **minimum spanning tree problem** with weights $w_{ij} = a_i c_{ij}$.

(a) In case of owner-homogeneous costs, for each player i, there exists $c_i > 0$ such that $c_{ij} = c_i$ for all $j \neq i$. Set $a_i^* = \prod_{j \neq i} c_j$ for each $i \in N$ and $b = \prod_i c_i$. Take the utilitarian social welfare function $W^*(g) = \sum_{i=1}^{n} a_i^* \Pi_i(g)$. Then $a_i^* c_i = b$ for all i and $W^*(g) = \sum_{i=1}^{n} a_i^* \sum_{j \neq i} V_{ij} - (n-1)b$ for all minimally connected networks g. If there exists a network g with $W^*(g) > \sum_{i=1}^{n} a_i^* \sum_{j \neq i} V_{ij} - (n-1)b$, then there exists a maximizer of W^* that is not minimally connected but Pareto optimal, contradicting the assumption that all Pareto optimal networks are minimally connected. Hence all minimally connected networks are maximizers of W^* and Pareto optimal.

(b) See Example 2 below.

(c) If $c_{ij} \neq c_{kl}$ for $ij \neq kl$, then $a_i c_{ij} \neq a_k c_{kl}$ for $ij \neq kl$ unless $a_i = a_k \cdot c_{kl}/c_{ij}$. Hence with the exception of $(a_1, \ldots, a_n) \gg 0$ belonging to a finite number of hyperplanes in $\mathbb{R}^n$, the weights $w_{ij} = a_i c_{ij}$ satisfy $w_{ij} \neq w_{kl}$ for $ij \neq kl$. But if the weights w_{ij} are distinct across pairs ij, the minimum spanning tree is unique. See, for instance, Property 2 and its proof in Gallager *et al.* (1983). ■

Example 2 (Pareto Optimality and Social Welfare)
In this example, all Pareto optimal networks are minimally connected but some do not maximize any utilitarian social welfare function.

Let $N = \{1, 2, 3, 4, 5\}$. Further set $V_{ij} = 8$ for all $i \neq j$; $c_{41} = c_{51} = 2, c_{42} = c_{52} = 4, c_{43} = 6, c_{53} = 5$; and $c_{ij} = 40$ otherwise. In this example, there exist at least nine Pareto optimal networks and all Pareto optimal networks are minimally connected. First, let us show that every Pareto optimal network is minimally connected. As an immediate consequence of the definitions, a Pareto optimal network is minimal. It remains to be shown that it is connected. If g is a network and $i \in \{1, 2\}$ and $j \in \{3, 4, 5\}$ belong to different components of g, then $\Pi_i(g \oplus \{ij\}) > \Pi_i(g)$ and $\Pi_{i'}(g \oplus \{ij\}) \geq \Pi_{i'}(g)$ for all $i' \in N$. Hence in this example, a Pareto optimal network can have only one component, that is it is connected.

Let us focus next on four specific Pareto optimal networks. We label these networks g^b, g^d, g^p, g^q such that $x = (\Pi_4(g^x), \Pi_5(g^x))$ for $x = b, d, p, q$. In each of the four networks, both 4 and 5 initiate a link to 1. Moreover, for $j = 2, 3$

either 4 or 5 initiates a link to j. In all four networks, players $j = 1, 2, 3$ don't initiate any links and receive payoffs $\Pi_j(g^x) = 32$. The payoff pairs $x = (\Pi_4(g^x), \Pi_5(g^x))$ are $b = (20, 30), d = (24, 26), p = (26, 25), q = (30, 21)$. The five center sponsored stars constitute additional Pareto optimal networks.

Now consider an arbitrary utilitarian social welfare function W with weights $(a_1, \ldots, a_n) \gg 0$.

- If $a_4 < a_5$, then
$a_4 \cdot b_4 + a_5 \cdot b_5 = 20 \cdot a_4 + 30 \cdot a_5 > 24 \cdot a_4 + 26 \cdot a_5 = a_4 \cdot d_4 + a_5 \cdot d_5$; hence, $W(g^b) > W(g^d)$ and g^d does not maximize W.

- If $a_4 \geq a_5$, then
$a_4 \cdot p_4 + a_5 \cdot p_5 = 26 \cdot a_4 + 25 \cdot a_5 > 24 \cdot a_4 + 26 \cdot a_5 = a_4 \cdot d_4 + a_5 \cdot d_5$; hence, $W(g^p) > W(g^d)$ and g^d does not maximize W.

It follows that g^d does not maximize W in any case. Thus, g^d proves to be a Pareto optimal network that does not maximize any utilitarian social welfare function. □

3 Network Formation under Internal Constraints

Internal constraints, given by a pre-existing network or infrastructure $\mathfrak{g} \in \mathcal{G}$, can be incorporated into a model in two ways: 1. via restrictions on players' strategy choices or 2. via a modification of the players' payoff functions. We opt for the second way, which proves more tractable in our context. For $g, h \in \mathcal{G}$, $h \oplus g \in \mathcal{G}$ denotes the network whose set of links is the union of the links in h and the links in g. Given $\mathfrak{g}$, we define for each $i \in N$ a payoff function $\Pi_i(\mathfrak{g}; \cdot) : \mathcal{G} \to \mathbb{R}$ by

$$\Pi_i(\mathfrak{g}; g) = B_i(\mathfrak{g} \oplus g) - C_i(g) \text{ for } g \in \mathcal{G}. \qquad (3)$$

This formulation treats $\mathfrak{g}$ free of costs which is of course implausible. Under the separable cost assumption of the connections model, the cost of providing $\mathfrak{g}$ can be neglected as long as merely the formation of a network g given $\mathfrak{g}$ is considered. We shall account for the cost of $\mathfrak{g}$ when we are going to perform comparative statics and welfare analysis.

In the extended model, one ought to distinguish between a Nash equilibrium g^* given the infrastructure $\mathfrak{g}$ and the outcome $\mathfrak{g} \oplus g^*$, the entire available network. Notice that $\mathfrak{g}$ and g^* are disjoint because of $c_{ij} > 0$ for all ij.

4 Existence and Optimality of Nash Networks under Internal Constraints

Bala and Goyal (2000) outlined a constructive proof of the existence of Nash networks in the case of cost and value homogeneity. Subsequently, along the same lines, existence of Nash networks has been shown by Haller, Kamphorst and Sarangi (2007) in the case of cost homogeneity, allowing for value heterogeneity. An analogous argument demonstrates existence of Nash networks when costs are homogeneous and there is a pre-existing network or infrastructure $\mathfrak{g} \in \mathcal{G}$. Closer inspection of the proof shows that cost homogeneity can be replaced by the weaker assumption of owner-homogeneity of costs. Owner-homogeneous costs are also invoked in the main existence result for one-way information flow models: Proposition 2 in Billand *et al.* (2008) and Proposition 1 in Derks and Tennekes (2009).

Proposition 2 *Consider a strategic model of network formation with payoff functions* $\Pi_i(\mathfrak{g}; g)$, $g \in \mathcal{G}$, $i \in N$. *Suppose that costs are owner-homogeneous. Then there exists a Nash network* g^*.

PROOF. First we proceed under the assumption that the pre-existing network or infrastructure $\mathfrak{g} \in \mathcal{G}$ is minimal and are going to show that there exists a Nash network g^* with the property that $\mathfrak{g} \oplus g^*$ is minimal.

Beginning with the empty network, we construct a network g^* such that $\mathfrak{g} \oplus g^*$ is minimal and g^* is Nash in the network formation game with payoffs $\Pi_i(\mathfrak{g}; g)$, $g \in \mathcal{G}$, $i \in N$. By assumption, the empty network e has the property that $\mathfrak{g} \oplus e$ is minimal and that no player benefits from deleting a link.

Next let g be any network with the property that $\mathfrak{g} \oplus g$ is minimal and that no player benefits from deleting a link. Since $\mathfrak{g} \oplus g$ is minimal, a link ik in g connects i with the members of k's component in $\mathfrak{g} \oplus g_{-i}$. By assumption, i does not gain from simply severing that link. Because of owner-homogeneity of link costs, player i does not strictly prefer to replace that link with a link to another member of k's component in $\mathfrak{g} \oplus g_{-i}$. Consequently, there remain two possibilities: either (a) g is Nash or (b) some player is better off by sponsoring an additional link. In the latter case, suppose that player i is better off sponsoring the additional link ij and denote $g' = g \oplus \{ij\}$. Since player i is better off sponsoring the extra link ij, i and j belong to different components of $\mathfrak{g} \oplus g$. Hence $\mathfrak{g} \oplus g'$ is also minimal. Moreover, adding the link ij makes the existing links more valuable (or at least not less valuable). Therefore, no player benefits from deleting a link in g'.

We have shown so far: If g is a network with the property that $\mathfrak{g} \oplus g$ is minimal and that no player benefits from deleting a link and if g is not Nash, then adding a suitably chosen link to g creates a larger minimal network g' with the property that $\mathfrak{g} \oplus g'$ is also minimal and that no player benefits from deleting a link in g'.

Now let us begin with the empty network and label it g^0. In case g^0 is Nash, we are done. Otherwise, by the previous argument, there exists a network g^1 with one link and the property that $\mathfrak{g} \oplus g^1$ is minimal and that no player benefits from deleting a link. In case g^1 is Nash, we are done. Otherwise, there exists a network g^2 with two links and the property that $\mathfrak{g} \oplus g^2$ is minimal and that no player benefits from deleting a link, etc. Since a minimal network with n nodes has at most $n - 1$ links, in finitely many steps, say k steps with $0 \leq k \leq n - 1$, a network g^k is reached which has k links, is Nash and has the property that $\mathfrak{g} \oplus g^k$ is minimal.

Finally, let $\mathfrak{g} \in \mathcal{G}$ be arbitrary. Deleting redundant links in $\mathfrak{g}$ if any exist, one obtains a network $\mathfrak{g}'$ that is minimal and has the same components as $\mathfrak{g}$. Then $\Pi_i(\mathfrak{g}'; g) = \Pi_i(\mathfrak{g}; g)$ for all $g \in \mathcal{G}$ and $i \in N$. Therefore, the Nash networks of the game with payoff functions $\Pi_i(\mathfrak{g}'; g)$, $g \in \mathcal{G}$, $i \in N$, and of the game with payoff functions $\Pi_i(\mathfrak{g}; g)$, $g \in \mathcal{G}$, $i \in N$, coincide. We have shown that the first game has a Nash network — which, consequently, is a Nash network of the second game as well. ■

Proposition 2 improves upon the existence result of Haller, Kamphorst and Sarangi (2007) in two respects: Cost homogeneity is replaced by the weaker assumption of owner-homogeneity of costs. $\mathfrak{g} = e$ is replaced by an arbitrary $\mathfrak{g}$. Examples 4 and 5 demonstrate that Nash networks need not exist if the assumptions of Proposition 2 are not met.

Clearly, if $\mathfrak{g}$ is not minimal and g^* is a Nash network of the game with payoff functions $\Pi_i(\mathfrak{g}; g)$, $g \in \mathcal{G}$, $i \in N$, then $\mathfrak{g} \oplus g^*$ is not minimal either. For if a link ij is redundant in $\mathfrak{g}$, then it is still redundant in $\mathfrak{g} \oplus g^*$. On the other hand, if $\mathfrak{g}$ is minimal and g^* is a Nash network of the game with payoff functions $\Pi_i(\mathfrak{g}; g)$, $g \in \mathcal{G}$, $i \in N$, then $\mathfrak{g} \oplus g^*$ is minimal as well. To see this, define for $g \in \mathcal{G}$ a cycle in $\overline{g}$ as a path in $\overline{g}$ whose endpoints are directly linked. To be precise, if $i_0, i_1, \ldots, i_m$ is a path in $\overline{g}$ and $i_m i_0 \in \overline{g}$, then $i_0, i_1, \ldots, i_m, i_0$ constitutes a cycle in $\overline{g}$. Now obviously g is minimal if and only if $\overline{g}$ has no cycles. If $\mathfrak{g}$ is minimal, then $\overline{\mathfrak{g}}$ has no cycles. Suppose $\mathfrak{g} \oplus g^*$ is not minimal. Then $\overline{\mathfrak{g} \oplus g^*}$ has a cycle $i_0, i_1, \ldots, i_m, i_0$. Since $\overline{\mathfrak{g}}$ has no cycles, there must be some $k \in \{0, \ldots, m\}$ with $i_k i_{k+1} \notin \overline{\mathfrak{g}}$ where we set $m + 1 = 0$. Then $i_k i_{k+1} \in g^*$ or $i_{k+1} i_k \in g^*$. Hence player i_k or player i_{k+1} forms a redundant link in g^*, contradicting the fact that g^* is a Nash network.

What happens if one starts with a Pareto optimal infrastructure $\mathfrak{g}$ rather than just some arbitrary one? The short answer is that nothing happens: One ends up with the equilibrium outcome $\mathfrak{g} \oplus e = \mathfrak{g}$.

Proposition 3 *Consider a strategic model of network formation with payoff functions* $\Pi_i(\mathfrak{g}; g)$, $g \in \mathcal{G}$, $i \in N$. *Suppose that the pre-existing network or infrastructure* $\mathfrak{g} \in \mathcal{G}$ *is Pareto optimal. Then the empty network is a strict Nash network and the only Nash network.*

PROOF. Let $\mathfrak{g}$ be Pareto optimal. Consider the strategic model of network formation with payoff functions $\Pi_i(\mathfrak{g}; g)$, $g \in \mathcal{G}$, $i \in N$. If g^* is a Nash network, then for each $j \in N$, $\Pi_j(\mathfrak{g}; g^*) = \Pi_j(\mathfrak{g}; g^*_{-j} \oplus g^*_j) \geq \Pi_j(\mathfrak{g}; g^*_{-j}) \geq \Pi_j(\mathfrak{g}; e)$, hence $\Pi_j(\mathfrak{g} \oplus g^*) = \Pi_j(\mathfrak{g}; g^*) - C_j(\mathfrak{g}) \geq \Pi_j(\mathfrak{g}; e) - C_j(\mathfrak{g}) = \Pi_j(\mathfrak{g})$.

Suppose now that for some i, $g_i \neq 0_i$ is a best response against e_{-i} where 0_i denotes the empty network in $\mathcal{G}_i$ and is encoded by the vector $(0, \ldots, 0) \in \{0, 1\}^{N \setminus \{i\}}$ and e denotes the empty network in $\mathcal{G}$. Then player i incurs some positive cost of forming the links in g_i. Therefore, for g_i to be a best response against e_{-i}, i must have sufficiently large benefits from forming the links in g_i. Hence, g_i must link i to at least one component C of $\mathfrak{g}$ to which i does not belong. Consequently, creating the links in g_i increases the payoffs of players $k \in C$ so that for each of these players, $\Pi_k(\mathfrak{g} \oplus g_i) > \Pi_k(\mathfrak{g})$ while for all other players j (including i), $\Pi_j(\mathfrak{g}; g_i) \geq \Pi_j(\mathfrak{g}; 0_i)$ and, consequently, $\Pi_j(\mathfrak{g} \oplus g_i) = \Pi_j(\mathfrak{g}; g_i) - C_j(\mathfrak{g}) \geq \Pi_j(\mathfrak{g}; 0_i) - C_j(\mathfrak{g}) = \Pi_j(\mathfrak{g})$. This contradicts the Pareto optimality of $\mathfrak{g}$. Therefore, 0_i is the unique best response against e_{-i} for all i. This shows the first claim.

Suppose next that $g^* \neq e$ is a Nash network. Then choose a player i with $g^*_i \neq 0_i$. Player i incurs some positive cost of forming the links in

g_i^*. Therefore, for g_i^* to be a best response against g_{-i}^*, i must have sufficiently large benefits from forming the links in g_i^*. Hence, g_i^* must link i to at least one component C of $\mathfrak{g} \oplus g_{-i}^*$ to which i does not belong. Specifically, creating the link ik to player $k \in C$ increases k's payoff by at least V_{ki} so that $\Pi_k(\mathfrak{g}; g^*) = \Pi_k(\mathfrak{g}; g_{-k}^* \oplus g_k^*) \geq \Pi_k(\mathfrak{g}; g_{-k}^*) > \Pi_k(\mathfrak{g}; e)$ and, therefore, $\Pi_k(\mathfrak{g} \oplus g^*) = \Pi_k(\mathfrak{g}; g^*) - C_k(\mathfrak{g}) > \Pi_k(\mathfrak{g}; e) - C_k(\mathfrak{g}) = \Pi_k(\mathfrak{g})$. For all players $j \neq k$ (including i), $\Pi_j(\mathfrak{g} \oplus g^*) \geq \Pi_j(\mathfrak{g})$. This contradicts the Pareto optimality of $\mathfrak{g}$. Therefore, $g^* \neq e$ cannot be a Nash network. This shows the second claim. ■

If a Pareto optimal network $\mathfrak{g}$ is already in place and Pareto optimal outcomes are desired, then there is no point to add further links to $\mathfrak{g}$. More importantly, the result shows that in equilibrium, no additional links — which might undo Pareto optimality — are formed. Proposition 3 is reminiscent of the second welfare theorem for standard pure exchange economies: *If the endowment allocation is Pareto optimal, then a no-trade equilibrium results and — in case there are multiple equilibria — equilibrium welfare is identical across all competitive equilibria.* Likewise, if the infrastructure already in place is Pareto optimal, then nothing occurs in equilibrium. Notice, however, that a publicly provided or pre-existing infrastructure $\mathfrak{g}$ is not necessarily Pareto optimal. Most of the subsequent analysis is devoted to this more plausible and more intriguing case.

5 (De)stabilizing Effects of Internal Constraints

According to Proposition 3, a Pareto optimal pre-existing or publicly provided network is always stabilizing. It gives rise to the empty network as a strict Nash network. But then, it renders private network formation obsolete. However, a Pareto optimal pre-existing or publicly provided infrastructure is not necessarily to be expected. In a similar vein, a connected pre-existing or publicly provided network would yield the empty network as a strict Nash network and the only Nash network. Therefore, it would always prove stabilizing and render private network formation obsolete. Yet again, such an infrastructure need not materialize. We begin with an instructive example where a single publicly provided link gives rise to the existence of a unique Nash network (which is non-empty and strict Nash) whereas Nash networks do not exist without the link.

Example 4 (Stabilizing Effect)
The basic example[2] constitutes a 4-player game with cost heterogeneity and value homogeneity. It exhibits non-existence of a Nash network. However, a unique Nash equilibrium (which is strict Nash) exists after introduction of a single publicly provided link.

Let $n = 4$ and $V_{ij} = V > 0$ for all ij. Suppose $c_{1k} > 3V$ for all $k \neq 1$; $c_{23} = c_{24} > 3V$ and $c_{21} < V$; $V < c_{34} < c_{32} < 2V < 3V < c_{31}$; $2V < c_{42} < 3V < c_{41} = c_{43}$. Then the unique best reply of player 1 to any network is to add no links at all. The unique best reply of player 2 to any network g_{-2} in which he does not observe player 1 is to add a link to player 1 only. Players 3 and 4 will never have a link to player 1 as part of their best reply. Moreover, in a best reply player 4 will never initiate a link to player 3.

Now let us take those best replies for granted and consider best responses regarding the remaining links 32, 34, and 42. If player 4 initiates link 42, then player 3's best response is to initiate link 34 and not 32, and in turn player 4's best response is not to form link 42. If player 4 does not initiate link 42, then player 3's best response is to form link 32 and not 34, against which player 4's best response is to initiate link 42. Hence there do not exist any mutual best responses. Therefore, a Nash network does not exist.

If the link 42 is publicly provided, i.e., $\mathfrak{g} = \{42\}$, then the network $g^* = \{21, 34\}$ is the unique Nash equilibrium and is strict Nash. Thus, public provision of $\mathfrak{g}$ has a stabilizing effect. □

Proposition 2 has two immediate consequences, Propositions 4 and 5. The first one yields a stabilization result: Suppose owner-homogeneity of costs is violated and a Nash equilibrium does not exist in the absence of a publicly provided infrastructure, but violation of owner-homogeneity of costs only occurs for a small number of nodes and links. Then there exists an infrastructure $\mathfrak{g}$ that involves no other nodes or links such that in the strategic model of network formation with payoff functions $\Pi_i(\mathfrak{g}; g)$, $g \in \mathcal{G}$, $i \in N$, there exists a Nash network.

[2]Example 2 of Haller, Kamphorst and Sarangi (2007).

Proposition 4 *Consider a strategic model of network formation with payoff functions* $\Pi_i(\mathfrak{g}; g)$, $g \in \mathcal{G}$, $i \in N$. *Suppose that*

(i) *the pre-existing network or infrastructure* $\mathfrak{g} \in \mathcal{G}$ *is minimal;*

(ii) *there exist* $c_1 > 0, \ldots, c_n > 0$ *such that* $c_{ij} = c_i$ *for all* $ij \notin \overline{\mathfrak{g}}$.

Then there exists a Nash network g^* *with the property that* $\mathfrak{g} \oplus g^*$ *is minimal.*

PROOF. In the proof of Proposition 2, replace owner-homogeneity by condition (ii). ■

In Section 6, we obtain Proposition 7, a stabilization result for a class of insider-outsider models. Furthermore, we obtain an irrelevance result for network formation games with owner-homogeneous costs:

Proposition 5 *Suppose that costs are owner-homogeneous. Then the public provision of a minimal infrastructure* $\mathfrak{g}$ *proves irrelevant for the existence of Nash networks, that is, it neither fosters nor impedes the existence of Nash networks.*

PROOF. Proposition 2 asserts existence of a Nash network in the presence of a minimal infrastructure $\mathfrak{g}$. Proposition 2 also applies when $\mathfrak{g}$ is the empty network. ■

Finally, we observe that the public provision of an infrastructure can have a destabilizing effect.

Example 5 (Destabilizing Effect)
Let us first consider the 3-player game[3] with player set $N = \{1, 2, 3\}$; strategy sets $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$; joint strategy space $\mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_2 \times \mathcal{G}_3$; cost and value parameters satisfying

$V_{12} > c_{12} > c_{13} > V_{13}; c_{21} = c_{23} > V_{21} + V_{23}; c_{31} > V_{31} + V_{32} > c_{32} > V_{32};$

and corresponding payoff functions $\Pi_i(g)$, $g \in \mathcal{G}$.
In Nash equilibrium:

— player 2 will not sponsor any links;

[3] Communicated by Jurjen Kamphorst via email dated July 12, 2004.

— player 3 will not sponsor a link to player 1; he will sponsor a link to player 2 only if player 1 and player 2 are connected and players 1 and 3 are not;

— player 1 will not sponsor any links if in g_{-1} he is connected to player 2; otherwise he will sponsor a link to player 2 if players 2 and 3 are unconnected, and to player 3 if players 2 and 3 are connected.

These conditions cannot be satisfied simultaneously in pure strategies. Hence the game does not have a Nash network.

Let us next consider a 4-player game with player set $N' = \{1,2,3,4\}$; strategy sets $\mathcal{G}'_1, \mathcal{G}'_2, \mathcal{G}'_3, \mathcal{G}'_4$; joint strategy space $\mathcal{G}' = \mathcal{G}'_1 \times \mathcal{G}'_2 \times \mathcal{G}'_3 \times \mathcal{G}'_4$;
cost parameters

$c'_{12} = 50, c'_{13} = 30, c'_{14} = 80;\ c'_{21} = c'_{23} = c'_{24} = 80;$
$c'_{31} = 80, c'_{32} = 40, c'_{34} = 80;\ c'_{41} = c'_{42} = c'_{43} = 80;$

value parameters

$V'_{12} = 45, V'_{13} = 10, V'_{14} = 10;\ V'_{21} = V'_{23} = V'_{24} = 10;$
$V'_{31} = 20, V'_{32} = 20, V'_{34} = 10;\ V'_{41} = V'_{42} = V'_{43} = 10;$

and corresponding payoff functions $\Pi'_i(g')$, $g' \in \mathcal{G}'$.
Then $g^* = e$ is a Nash network of this 4-player game in the absence of a publicly provided infrastructure. Now suppose that the publicly provided infrastructure $\mathfrak{g} = \{42\}$ is installed. Then the links 14, 24, 34, 41, and 43 will not be formed in a Nash equilibrium. The 3-player game with player set $N = \{1,2,3\}$ and payoff functions $\Pi_i(g) = \Pi'_i(\mathfrak{g}; g)$, $g \in \mathcal{G}$, $i \in N$, can be represented by cost parameters c_{ij} and value parameters V_{ij} that satisfy the inequalities in the 3-player game we started with. Hence that 3-player game cannot have Nash equilibria. It follows that the 4-player game with payoff functions $\Pi'_i(\mathfrak{g}; g'), g' \in \mathcal{G}'$, does not have a Nash equilibrium either. □

6 Welfare Effects of Internal Constraints

We commence with an instructive example where a publicly provided link — which would not be created by any player — does not alter the set of Nash equilibria, yet constitutes a strict Pareto improvement.

Example 6 (Welfare Improvement)
Let $N = \{1, \ldots, 8\}$, $K = \{1, 2, 3, 4\}$, $L = \{5, 6, 7, 8\}$. Further set

$$V_{ij} = 1 \text{ for all } i \neq j;$$

$$c_{ij} = \begin{cases} 0.8 & \text{if } i \neq j,\ i, j \in K, \\ 0.8 & \text{if } i \neq j,\ i, j \in L, \\ 16 & \text{if } i \in K, j \in L, \\ 16 & \text{if } i \in L, j \in K. \end{cases}$$

In the absence of a publicly provided infrastructure, i.e., in case $\mathfrak{g} = \emptyset$, a Nash network g^* has two minimally connected components, one consisting of the members of K and the other one consisting of the members of L.

Next suppose that the link $ij = 48$ is publicly provided, that is $\mathfrak{g} = \{48\}$. Then the set of Nash equilibria remains the same, but the total benefit is increased by $8 \times 4 = 32$, at a cost of 16 for the link 48. This constitutes a strict Pareto improvement if there is a way to charge each player a cost contribution of 2 for the link 48. □

Example 6 constitutes an instance of the insider-outsider model of Galeotti, Goyal, and Kamphorst (2006) and the welfare improvement observed in Example 6 is an instance of Proposition 6(b). Rather than increasing welfare per se, as is often the case in the insider-outsider model, the benefit of a publicly provided infrastructure may consist in the elimination of inefficient equilibria:

Example 7 (Elimination of Inefficient Networks)
Let $N = \{1, 2, 3\}$, $V_{ij} = 2$ for $i \neq j$ and $c_{ij} = 3$ for $i \neq j$. In case $\mathfrak{g} = \emptyset$, the empty network and periphery-sponsored stars are the only Nash networks and the empty network is the only strict Nash network. In the empty network, every player receives a payoff of 0. In a periphery-sponsored star, the center receives a payoff of 4 and each of the other two players receives a payoff of 1.

Now suppose that the link $ij = 12$ is publicly provided, that is $\mathfrak{g} = \{12\}$. Then $g^* = \{31\}$ and $g^{**} = \{32\}$ are the only Nash networks. Neither one is a strict Nash network. But both outcomes $\mathfrak{g} \oplus g^*$ and $\mathfrak{g} \oplus g^{**}$ are efficient. □

There are instances of the insider-outsider model where public provision of a backbone infrastructure is ineffective (Proposition 8) and other instances where it is harmful (Proposition 9). A publicly provided infrastructure can not only be too costly (as in Proposition 9), but can also induce players to choose costlier links, as the following example demonstrates:

Example 8 (Welfare Reduction via Cost Increase)
Let $N = \{1,2,3\}$, $V_{ij} = V = 24$ for all $i \neq j$, $c_{13} = 32, c_{23} = 28$, $c_{12} = 56, c_{32} = 40$, and $c_{21} = c_{31} = 80$.

The periphery-sponsored star with center 3 is the only (strict) Nash network in the absence of a publicly provided infrastructure.

Now suppose that the publicly provided infrastructure $\mathfrak{g} = \{12\}$ is installed. Then $g' = \{32\}$ is the only (strict) Nash network. The corresponding outcome is $\mathfrak{g} \oplus \{32\}$. In $\mathfrak{g} \oplus \{32\}$, each of the two links costs more than a link in the center-sponsored star with center 3. □

This example is reminiscent of Braess's paradox [Braess (1968), Braess *et al.* (2005)] that an extension of a road network may cause a redistribution of traffic which results in longer individual travel times. In the present example, installation of the public infrastructure causes players to switch to costlier links.

7 An Insider-Outsider Model and Backbone Infrastructures

Example 6 constitutes an instance of the insider-outsider model of Galeotti, Goyal, and Kamphorst (2006). In that model, the player population is partitioned into $m \geq 2$ non-empty subsets $N_k, k = 1, \ldots, m$. For $i \in N_k$ and $j \in N_l$, $c_{ij} = c_{ji} = f(|k-l|)$ where f is a nondecreasing function with $f(0) = c_L > 0$. Further $V_{ij} = V = 1$ for all $i \neq j$. We shall concentrate on the special case of equal group size, $\overline{n} > 1$, so that $|N_k| = \overline{n}$ for all k, and two cost levels, $f(0) = c_L > 0$ and $f(d) = c_H > c_L$ for $d \geq 1$. The general case does not yield noticeably different insights, but proves more tedious with respect to both analysis and exposition.

In Example 6, the publicly provided link $ij = 48$ serves as welfare enhancing backbone infrastructure and would not have been created by any of the players. More generally, costly intergroup links could serve as backbone infrastructure in the insider-outsider model. Such an infrastructure might be socially beneficial, but would not be created in the strategic network formation game if the private cost exceeded the private benefit of forming a link. Let us formally define a **backbone infrastructure** as a network $\mathfrak{g}$ with $m-1$ links and a component consisting of m nodes i_k, $k = 1, \dots, m$, such that $i_k \in N_k$ for $k = 1, \dots, m$. For instance, a star with m nodes i_k, $k = 1, \dots, m$, such that $i_k \in N_k$ for $k = 1, \dots, m$ will do. We first identify instances where public provision of a backbone infrastructure is beneficial, then instances where it is ineffective, and finally instances where it is wasteful. As a by-product, we get a further stabilization result, Proposition 7.

Proposition 6 *Consider the insider-outsider model of network formation with equal group size* $\overline{n} > 1$ *and two cost levels* $0 < c_L < c_H$.

(a) *Suppose* $0 < c_L < 1 < c_H < (m-1)\overline{n}$. *Then there does not exist an efficient strict Nash network in the absence of a publicly provided infrastructure. With a publicly provided backbone infrastructure* $\mathfrak{g}$, *there do exist strict Nash networks and each strict network* g^* *has an efficient outcome* $\mathfrak{g} \oplus g^*$.

(b) *Suppose* $0 < c_L < 1$ *and* $(m-1)\overline{n} < c_H < m\overline{n}^2$. *Then all Nash networks are inefficient in the absence of a publicly provided infrastructure. With a publicly provided backbone infrastructure* $\mathfrak{g}$, *the outcome* $\mathfrak{g} \oplus g^*$ *is efficient for every Nash network* g^*.

PROOF. (a) Suppose $0 < c_L < 1 < c_H < (m-1)\overline{n}$. Then $0 < c_L < 1 < c_H < m\overline{n}^2$ and by Galeotti *et al.* (2006, Proposition 4.2 (1)), the efficient networks are of the form g^{mc} where the members of each group are minimally connected via $\overline{n}-1$ links inside the group and there are $m-1$ links across groups so that the entire network is minimally connected. In the absence of a publicly provided infrastructure, there exist Nash networks of the form g^{mc}. But no strict Nash network is of the form g^{mc}, since

- either $c_H \leq \overline{n}$ in which case strict Nash networks do not exist, by Galeotti *et al.* (2006, Proposition 4.1 (2b))
- or $c_H > \overline{n}$ in which case strict Nash networks are unconnected center-sponsored stars, by Galeotti *et al.* (2006, Proposition 4.1 (2c)).

Suppose a backbone infrastructure $\mathfrak{g}$ is publicly provided. Then $\mathfrak{g} \oplus g^*$ is of the form g^{mc} for every Nash equilibrium g^* of the strategic game with payoff functions $\Pi_i(\mathfrak{g}; g)$, $g \in \mathcal{G}$, $i \in N$. Strict Nash equilibria of the game exist and are composed of m center-sponsored stars, one for each group N_k.

(b) Suppose $0 < c_L < 1$ and $(m-1)\overline{n} < c_H < m\overline{n}^2$. Then by Galeotti *et al.* (2006, Proposition 4.2 (1)), the efficient networks are of the form g^{mc} whereas in the absence of a publicly provided infrastructure, each Nash network has m minimal components $N_k, k = 1, \ldots, m$.

Suppose a backbone infrastructure $\mathfrak{g}$ is publicly provided. Then like in Example 6, the set of Nash equilibria does not change. But with the infrastructure $\mathfrak{g}$ in place, the outcome $\mathfrak{g} \oplus g^*$ is of the form g^{mc} for every Nash equilibrium g^*. ■

Next we obtain a stabilization result for a class of insider-outsider models as a corollary to Proposition 6(a).

Proposition 7 *Consider the insider-outsider model of network formation with equal group size* $\overline{n} > 1$ *and two cost levels* $0 < c_L < c_H$.
If $0 < c_L < 1 < c_H \leq \overline{n}$, *then strict Nash equilibria do not exist in the absence of a publicly provided infrastructure whereas public provision of a backbone infrastructure yields existence of strict Nash networks.*

PROOF. See proof of Proposition 6(a). ■

We have seen in Example 8 that public provision of a small piece of infrastructure need not be beneficial. In the context of the insider-outsider model, public provision of a backbone infrastructure is not always beneficial either. In some instances, public provision of a backbone infrastructure does not affect aggregate welfare whereas in other instances public provision of a backbone infrastructure proves wasteful.

Proposition 8 *Consider the insider-outsider model of network formation with equal group size* $\overline{n} > 1$ *and two cost levels* $0 < c_L < c_H < 1$. *Then public provision of a backbone infrastructure is ineffective.*

PROOF. Suppose $0 < c_L < c_H < 1$. Then all minimally connected networks are Nash and efficient in the absence of a publicly provided infrastructure. Some Nash networks are strict while most are not because by Galeotti *et al.* (2006, Proposition 4.1 (2a)), strict Nash networks are "generalized center-sponsored stars". Public provision of a backbone infrastructure does not affect welfare. ■

Proposition 9 *Consider the insider-outsider model of network formation with equal group size* $\overline{n} > 1$ *and two cost levels* $0 < c_L < c_H$. *If* $0 < c_L < 1$ *and* $c_H > m\overline{n}^2$, *then public provision of a backbone infrastructure is wasteful.*

PROOF. Suppose $0 < c_L < 1$ and $c_H > m\overline{n}^2$. Then in the absence of a publicly provided infrastructure, each Nash network has m minimal components $N_k, k = 1, \ldots, m$ — and is efficient. Public provision of a backbone infrastructure is wasteful, since its cost $(m-1)c_H$ exceeds its benefit $(m-1)m\overline{n}^2$. ■

8 On the Provision of Public Infrastructure

We have seen that the public provision of infrastructure followed by decentralized network formation can be beneficial or undesirable: it can be stabilizing or destabilizing, respectively; welfare improving or welfare reducing, respectively. In some instances, a pre-existing infrastructure ought to be removed or not to be used any longer. If the latter can be achieved at no cost, say by just leaving a cable in the ground, then a benevolent, omniscient, and omnipotent policy maker may do. Otherwise, the abandonment, removal, renewal, expansion, or replacement of a publicly provided infrastructure can be a formidable political challenge.

In the remainder of this section, I shall focus on the simpler task of creating a public infrastructure de novo, with the absence of a public infrastructure as the status quo. Policy makers as well as players compare a Nash network g^* of the benchmark model without a public infrastructure with an outcome $\mathfrak{g} \oplus g^{**}$ where $\mathfrak{g} \neq e$ is a candidate for a publicly provided infrastructure and g^{**} is a Nash network of the strategic game of network formation with payoff functions $\Pi_i(\mathfrak{g}; g)$, $g \in \mathcal{G}$, $i \in N$. To simplify the analytic task even further, I assume that

(i) the choice is between the empty infrastructure e and the proposed infrastructure project $\mathfrak{g} \neq e$;

(ii) all players anticipate the Nash equilibrium g^* in the subsequent network formation game if the public infrastructure e is in place;

(iii) all players anticipate the Nash equilibrium g^{**} in the subsequent network formation game if the public infrastructure $\mathfrak{g}$ is in place.

Then frequently, a version of the well known "subscription game" will yield selection and sufficient funding of the more efficient of the two infrastructures. In that game, the infrastructure $\mathfrak{g}$ gets built if the sum of the players' voluntary contributions s_i, $i \in N$, equals or exceeds the cost of constructing $\mathfrak{g}$. In that case, the players forfeit any contributions in excess of the construction cost. The infrastructure $\mathfrak{g}$ is not built, if the aggregate voluntary contributions fall short of the construction cost. In that case, the contributions are refunded. The actions in the following strategic game Γ represent the individual contributions s_i, $i \in N$, and the payoffs reflect the players' evaluations of the resulting outcomes in accordance with (i)–(iii). Let $\Gamma = (N, (S_i)_{i\in N}, (U_i)_{i\in N})$ where

$$\begin{array}{ll} s_i \in S_i = \mathbb{R}_+ \text{ for } i \in N; & \\ U_i(s_1, \ldots, s_n) = \Pi_i(g^*) & \text{if } \sum_{j\in N} s_j < \mathfrak{c}(\mathfrak{g}); \\ U_i(s_1, \ldots, s_n) = \Pi_i(\mathfrak{g}; g^{**}) - s_i & \text{if } \sum_{j\in N} s_j \geq \mathfrak{c}(\mathfrak{g}); \\ \mathfrak{c}(\mathfrak{g}) = \sum_{j\neq k} \mathfrak{g}_{jk} \cdot c_{jk}, & \text{the cost of constructing } \mathfrak{g}. \end{array}$$

To analyze Γ, set $\Delta_i = \Pi_i(\mathfrak{g}; g^{**}) - \Pi_i(g^*)$ for $i \in N$. Δ_i is i's equilibrium payoff differential between the ensuing network formation games $(N, (\mathcal{G}_i)_{i\in N}, (\Pi_i(\mathfrak{g}; \cdot))_{i\in N})$ and $(N, (\mathcal{G}_i)_{i\in N}, (\Pi_i)_{i\in N})$. Further set $\Delta = \sum_i \Delta_i$, $N_+ = \{i \in N : \Delta_i > 0\}$, and $\Delta_+ = \sum_{i\in N_+} \Delta_i \geq \Delta$. Then

$$W_0(\mathfrak{g} \oplus g^{**}) - W_0(g^*) = \Delta - \mathfrak{c}(\mathfrak{g}). \tag{4}$$

On efficiency grounds, $\mathfrak{g}$ should be built if $W_0(\mathfrak{g} \oplus g^{**}) > W_0(g^*)$ or equivalently, by (4), if $\Delta > \mathfrak{c}(\mathfrak{g})$. And $\mathfrak{g}$ should not be built if $W_0(\mathfrak{g}\oplus g^{**}) < W_0(g^*)$ (or equivalently, $\Delta < \mathfrak{c}(\mathfrak{g})$).

Proposition 10

(a) *Suppose* $W_0(\mathfrak{g} \oplus g^{**}) > W_0(g^*)$.

(aa) *A strict Nash equilibrium of* Γ *exists.*

(ab) $\sum_i s_i^* = \mathfrak{c}(\mathfrak{g})$ *in every strict Nash equilibrium* $(s_1^*, \ldots, s_n^*)$ *of* Γ.

(b) *Suppose* $W_0(\mathfrak{g} \oplus g^{**}) < W_0(g^*)$.

(ba) *If* $\Delta_i < \mathfrak{c}(\mathfrak{g})$ *for all* i*, then a Nash equilibrium of* Γ *exists.*

(bb) *If* $\Delta_i \geq 0$ *for all* i*, then* $\sum_i s_i^* < \mathfrak{c}(\mathfrak{g})$ *in every Nash equilibrium* $(s_1^*, \ldots, s_n^*)$ *of* Γ.

(bc) *If* $\Delta_+ > \mathfrak{c}(\mathfrak{g})$*, then there exists a strict Nash equilibrium* $(s_1, \ldots, s_n)$ *of* Γ *with* $\sum_i s_i = \mathfrak{c}(\mathfrak{g})$.

PROOF. (aa) Suppose $W_0(\mathfrak{g} \oplus g^{**}) > W_0(g^*)$. Then $\Delta > \mathfrak{c}(\mathfrak{g})$ and, consequently, $\Delta_+ \geq \Delta > \mathfrak{c}(\mathfrak{g}) > 0$. Set $s_i = (\mathfrak{c}(\mathfrak{g})/\Delta_+) \cdot \Delta_i$ for $i \in N_+$ and $s_i = 0$ for $i \notin N_+$. Then $s_i \in S_i$ for all i and $\sum_i s_i = \mathfrak{c}(\mathfrak{g})$.

For $i \notin N_+$, $U_i(s_1, \ldots, s_n) = \Pi_i(\mathfrak{g}; g^{**})$ and $s_i = 0$. Choosing $s_i' > 0$ given s_{-i} would yield payoff $\Pi_i(\mathfrak{g}; g^{**}) - s_i' < \Pi_i(\mathfrak{g}; g^{**})$. Hence s_i is i's unique best response against s_{-i}.

For $i \in N_+$, $U_i(s_1, \ldots, s_n) = \Pi_i(\mathfrak{g}; g^{**}) - s_i$ and $0 < s_i = (\mathfrak{c}(\mathfrak{g})/\Delta_+) \cdot \Delta_i < \Delta_i$. Choosing $s_i' > s_i$ given s_{-i} would yield payoff $\Pi_i(\mathfrak{g}; g^{**}) - s_i' < \Pi_i(\mathfrak{g}; g^{**}) - s_i$. Choosing $s_i' < s_i$ given s_{-i} would yield payoff $\Pi_i(g^*) = \Pi_i(\mathfrak{g}; g^{**}) - \Delta_i < \Pi_i(\mathfrak{g}; g^{**}) - s_i$. Hence s_i is i's unique best response against s_{-i}. This shows that $(s_1, \ldots, s_n)$ is a strict Nash equilibrium of Γ.

(ab) Suppose $W_0(\mathfrak{g} \oplus g^{**}) > W_0(g^*)$. Now let $(s_1^*, \ldots, s_n^*)$ be a strict Nash equilibrium of Γ. Then $\sum_i s_i^* = \mathfrak{c}(\mathfrak{g})$. Suppose not. If $\sum_i s_i^* > \mathfrak{c}(\mathfrak{g})$, then $U_i(s_1^*, \ldots, s_n^*) = \Pi_i(\mathfrak{g}; g^{**}) - s_i^*$ for all i and $s_j^* > 0$ for some j. If such a player reduces his contribution by a sufficiently small amount $\varepsilon_j > 0$, given s_{-j}^*, then $s_j^* - \varepsilon_j \geq 0$, $s_j^* - \varepsilon_j + \sum_{i \neq j} s_i^* \geq \mathfrak{c}(\mathfrak{g})$ and $U_j(s_j^* - \varepsilon_j; s_j^*) = \Pi_i(\mathfrak{g}; g^{**}) - (s_i^* - \varepsilon_j) > \Pi_i(\mathfrak{g}; g^{**}) - s_i^* = U_j(s_1^*, \ldots, s_n^*)$, contradicting the assumption that $(s_1^*, \ldots, s_n^*)$ is a Nash equilibrium of Γ. If $\sum_i s_i^* < \mathfrak{c}(\mathfrak{g})$, then $U_i(s_1^*, \ldots, s_n^*) = \Pi_i(g^*)$ for all i. If some player j increases his contribution by a sufficiently small amount $\varepsilon_j > 0$, given s_{-j}^*, then $s_j^* + \varepsilon_j \geq 0$, $s_j^* + \varepsilon_j + \sum_{i \neq j} s_i^* < \mathfrak{c}(\mathfrak{g})$ and $U_j(s_j^* - \varepsilon_j; s_j^*) = \Pi_j(g^*) = U_j(s_1^*, \ldots, s_n^*)$, contradicting the assumption that $(s_1^*, \ldots, s_n^*)$ is a strict Nash equilibrium of Γ. Thus $\sum_i s_i^* \neq \mathfrak{c}(\mathfrak{g})$ always leads to a contradiction. Hence as asserted, $\sum_i s_i^* = \mathfrak{c}(\mathfrak{g})$ has to hold.

(ba) Suppose $W_0(\mathfrak{g} \oplus g^{**}) < W_0(g^*)$ and $\Delta_i < \mathfrak{c}(\mathfrak{g})$ for all i. Consider $(s_1, \ldots, s_n) = (0, \ldots, 0)$. Then $\sum_i s_i = 0 < \mathfrak{c}(\mathfrak{g})$ and $U_i(s_1, \ldots, s_n) = \Pi_i(g^*)$ for all i. If some player j chooses $s'_j \in (0, \mathfrak{c}(\mathfrak{g}))$ given s_{-j}, then $s'_j + \sum_{i \neq j} s_i = s'_j < \mathfrak{c}(\mathfrak{g})$ and $U_j(s'_j, s_{-j}) = \Pi_i(g^*) = U_j(s_1, \ldots, s_n)$. If j chooses $s'_j \geq \mathfrak{c}(\mathfrak{g})$ given s_{-j}, then $s'_j + \sum_{i \neq j} s_i = s'_j \geq \mathfrak{c}(\mathfrak{g})$ and $U_j(s'_j, s_{-j}) = \Pi_j(\mathfrak{g}; g^{**}) - s'_j \leq \Pi_j(\mathfrak{g}; g^{**}) - \mathfrak{c}(\mathfrak{g}) < \Pi_j(\mathfrak{g}; g^{**}) - \Delta_j = \Pi_i(g^*) = U_j(s_1, \ldots, s_n)$. Hence s_j is a best response against s_{-j} for all j. This shows that $(0, \ldots, 0)$ is a Nash equilibrium of Γ.

(bb) Suppose $W_0(\mathfrak{g} \oplus g^{**}) < W_0(g^*)$ and $\Delta_i \geq 0$ for all i. Now let $(s_1^*, \ldots, s_n^*)$ be a Nash equilibrium of Γ. Suppose $\sum_i s_i^* \geq \mathfrak{c}(\mathfrak{g})$. Then $U_i(s_1^*, \ldots, s_n^*) = \Pi_i(\mathfrak{g}; g^{**}) - s_i^*$ for all i. By the same argument as in the proof of (ab), $\sum_i s_i^* > \mathfrak{c}(\mathfrak{g})$ can be ruled out. Further, $\Pi_i(g^*) = \Pi_i(\mathfrak{g}; g^{**}) - \Delta_i$ for each player i. Hence $s_i^* \leq \Delta_i$ has to hold for $(s_1^*, \ldots, s_n^*)$ to be a Nash equilibrium of Γ with $\sum_i s_i^* = \mathfrak{c}(\mathfrak{g})$. But then $\sum_i s_i^* \leq \sum_i \Delta_i = \Delta < \mathfrak{c}(\mathfrak{g})$ because of $W_0(\mathfrak{g} \oplus g^{**}) < W_0(g^*)$, in contradiction to $\sum_i s_i^* = \mathfrak{c}(\mathfrak{g})$. Hence to the contrary, $\sum_i s_i^* < \mathfrak{c}(\mathfrak{g})$ has to hold.

(bc) Suppose $\Delta_+ > \mathfrak{c}(\mathfrak{g})$. Then a strict Nash equilibrium $(s_1, \ldots, s_n)$ of Γ with $\sum_i s_i = \mathfrak{c}(\mathfrak{g})$ can be constructed as in the proof of (aa). ■

Remarks. 1. The restriction to strict Nash equilibria in Proposition 10(ab) is crucial. Take for instance the backbone infrastructure $\mathfrak{g} = \{48\}$ in Example 6. Then $W_0(\mathfrak{g} \oplus g^{**}) > W_0(g^*)$. But $(0, \ldots, 0)$ is a Nash equilibrium of Γ which is not strict. Under the assumptions of Proposition 10(bb), strict Nash equilibria do not exist and, therefore, restricting attention to strict Nash equilibria proves counter-productive. The outcome in Proposition 10(bc) is the public provision of the infrastructure $\mathfrak{g}$ even though aggregate welfare is diminished after the creation of the infrastructure. The beneficiaries from $\mathfrak{g}$, the members of N_+, are willing to shoulder the entire cost of construction while others are opposed to the creation of $\mathfrak{g}$. Such a scenario is conceivable, both in practice and in theory. In Example 8, consider $\mathfrak{g} = \{12\}$, $g^* = \{13, 23\}$ (the periphery-sponsored star with center 3), and $g^{**} = \{32\}$ so that $\mathfrak{g} \oplus g^{**}$ is the periphery-sponsored star with center 2. Then $\mathfrak{c}(\mathfrak{g}) = 56$, $\Delta = 20$, and $\Delta_+ = 60$, with $N_+ = \{1, 2\}$. Hence the assumptions of Proposition 10(bc) are met.

2. An equilibrium outcome of Γ, e or $\mathfrak{g}$, defines a subgame-perfect equilibrium outcome g^* or $\mathfrak{g} \oplus g^{**}$, respectively, of the two-stage game where Γ is played in the first stage and depending on the outcome of the first stage, $(N, (\mathcal{G}_i)_{i \in N}, (\Pi_i)_{i \in N})$ or $(N, (\mathcal{G}_i)_{i \in N}, (\Pi_i(\mathfrak{g}; \cdot))_{i \in N})$ is played in the sec-

ond stage. A second-stage game may have multiple equilibria. To obtain a well defined subscription game, I assume that the players expect g^* to prevail in $(N, (\mathcal{G}_i)_{i\in N}, (\Pi_i)_{i\in N})$ and $\mathfrak{g} \oplus g^{**}$ to prevail in $(N, (\mathcal{G}_i)_{i\in N}, (\Pi_i(\mathfrak{g};\cdot))_{i\in N})$. A similar dilemma of multiple second stage equilibria — that needs to be resolved by specific expectations — arises in two-stage resource allocation mechanism where the first stage is under the planner's control whereas the second stage consists of competitive exchange outside the planner's control. See Guesnerie (1995), Section 3.1.

3. The general approach taken in this section and the equilibrium analysis in Proposition 10 share the basic features of the model of Bagnoli and Lipman (1989), but differ in many details, in particular equilibrium selection. Here we deal with the special case $M = 1$ in Bagnoli and Lipman (1989). Bagnoli and McKee (2001) provide episodical empirical support as well as experimental support for that case. $M = 1$ in our context means a choice between e and $\mathfrak{g} \neq e$. $M > 1$ would mean to choose an infrastructure from among $M + 1$ different infrastructures e, $\mathfrak{g}(1), \mathfrak{g}(2), \ldots, \mathfrak{g}(M)$. The corresponding analysis of Bagnoli and Lipman (1989) shows that this case poses a more intricate mechanism design problem, which I leave to future research.

4. The "subscription model" or "subscription game" has been investigated numerous times, under a variety of names, both theoretically (most notably by Bergstrom, Blume, and Varian (1986)) and experimentally (by Andreoni (1988) and many others). It has become part of the textbook literature in public economics, e.g., Cornes and Sandler (1986) and Miles (1995). In the standard model, the aggregate contribution $\sum_i s_i$ determines the amount q of a public good: $q = \sum_i s_i$ or, more generally, $q = f(\sum_i s_i)$. It is a widely held wisdom among economists that the non-cooperative equilibrium outcome of the model is inefficient and in fact, there is under-provision of the public good. It was the basic insight of Bagnoli and Lipman and others that efficient equilibrium outcomes could obtain if the problem was selection of a discrete public project from a finite set rather than choosing an amount q from a continuum of possible amounts. With the exception of conclusion (bc), Proposition 10 further confirms their insight.

5. The idea to finance the cost of building a network via voluntary contributions also appears in Anshelevich *et al.* (2008). In their network formation game, each player i decides on her non-negative contribution $p_i([jk])$ to the cost $c_{[jk]}$ of each potential edge $[jk]$. Edge $[jk]$ is created if $\sum_i p_i([jk]) \geq c_{[jk]}$. In Bloch and Jackson (2007), players can subsidize other players to form specific links and can also bribe other players not to form particular links.

6. One could in principle start with an efficient public infrastructure $\mathfrak{g}$, which would imply $\Delta \geq \mathfrak{c}(\mathfrak{g})$ and existence of at least one (not necessarily strict) Nash equilibrium $(s_1, \ldots, s_n)$ of Γ with $\sum_i s_i \geq \mathfrak{c}(\mathfrak{g})$. By Proposition 3, however, efficiency of $\mathfrak{g}$ renders the private provision of a network g built around or upon $\mathfrak{g}$ obsolete. In addition, distrust and disbelief in big government could preempt huge public projects. For instance, in the insider-outsider model of Section 7, public provision of a backbone infrastructure might be acceptable while public provision of a larger infrastructure might not. Notice that a backbone infrastructure $\mathfrak{g}$ by itself is inefficient, but may lead to an efficient equilibrium outcome $\mathfrak{g} \oplus g^{**}$. There could also exist an economic reason — which would warrant a slight modification of the model — why very expensive public infrastructures should be avoided. A conceivable reason could be a "shadow cost of public funds" $\lambda > 0$ which could mean, for example, that 'distortionary taxation inflicts disutility \$$(1+\lambda)$ on taxpayers in order to levy \$1 for the state.' [4] In the present context, λ may simply account for administrative or overhead costs. Needless to say, the private sector may incur overhead costs as well, but possibly less than the public sector.

7. Mutuswami and Winter (2002) address the question "how a social planner can ensure the formation of an efficient network in a scenario where the costs of network formation are publicly known but an individual player's benefits from network formation are not known to him." They consider mechanisms where a planner decides upon the network and cost contributions, based on the desired links and cost contributions announced by the agents. They show that a mechanism can be designed that meets three criteria: efficiency, balanced budget, and equity. Their approach as it stands yields an efficient publicly provided network and (in view of Proposition 3) leaves no room for decentralized network formation.

[4] Laffont and Tirole (1993, p. 55).

9 Final Remarks

The foregoing analysis rests on two main premises. First, strategic network formation builds a network around or upon a core network or infrastructure. The latter imposes what I call internal constraints on network formation. Second, the core network is either pre-existing or publicly provided prior to the onset of strategic network formation.

Most of my investigation is devoted to the effects of a publicly provided infrastructure, with the absence of such an infrastructure as the benchmark case. Four possible effects are demonstrated: welfare or efficiency improvement and reduction, stabilization and destabilization. Welfare effects of a publicly provided backbone infrastructure are studied in more detail for the insider-outsider model of Section 7.

Section 8 is devoted to the choice and funding of a publicly provided infrastructure. The problem is confined to a binary choice between the absence of a public infrastructure (choice of the empty network e) and the public provision of a particular infrastructure $\mathfrak{g}$ at cost $\mathfrak{c}(\mathfrak{g})$. Everybody anticipates that the choice of e will lead to a specific Nash equilibrium outcome g^* in the subsequent network formation game whereas the choice of $\mathfrak{g}$ will yield the Nash equilibrium outcome $\mathfrak{g} \oplus g^{**}$. It turns out that the more efficient alternative is frequently chosen in an equilibrium of a suitable version of the subscription game.

There remain several important and promising directions for future research. First of all, comparative statics should be further developed: Determine the set and properties of Nash equilibrium outcomes of the strategic network formation game with a given core infrastructure. Find how they change when the infrastructure varies. This might be possible in some interesting cases. Second, in some instances, like in certain generalizations of the insider-outsider model, one might agree on an "optimal" public infrastructure. Further, the implementation of such a public project might be addressed beyond the simple approach taken in Section 8. As mentioned earlier, if a publicly provided infrastructure already exists, but is possibly deteriorating or outdated, then its abandonment, removal, renewal, expansion, or replacement can pose a formidable political — or sometimes technical — challenge. Finally, other benchmark models of strategic network formation ought to be considered. To conclude, a systematic study of internal constraints on network formation and the interplay between private and public network provision has just begun.

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