Contests with Endogenous and Stochastic Entry*

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Abstract

This paper studies imperfectly discriminatory contests with costly and endogenous entries. A fixed pool of potential bidders may enter a contest to compete for an indivisible prize. Entry incurs an irreversible fixed cost. They then bid for the prize after entry. The game exemplifies a two-dimensional discontinuous game (Dasgupta and Maskin, 1986). We establish that there exists a symmetric equilibrium in the entry-bidding game, where all potential bidders enter with a probability. We further identify the conditions for the existence (non-existence) of symmetric equilibrium with pure-strategy bidding after entry. Based on the equilibrium result, we explore three main issues on optimal contest design. First, we investigate how the level of accuracy in the winner selection mechanism (i.e. the level of discriminatory power in Tullock rent-seeking contests) affects the expected overall bid. We find the relationship is non-monotonic. The contest designer may benefit from a noisier contest, which elicits the optimal amount of overall bid. Second, we study whether the contest designer should exclude potential bidders. Our analysis reveals that the contest designer prefers to limit its size by inviting only a subset of them for participation. Finally, we establish that with convex effort cost and endogenous entry there is no loss of generality to consider contest with nondisclosure of number of actual contestants for optimal design.

JEL Nos: C7, D8.

Keywords: Contest Design; Endogenous Entry; Entry Cost; Stochastic Entry

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1 Introduction

Economic agents are often involved in contests. They expend costly effort to compete for a limited number of prizes; while their investments, however, are usually non-refundable regardless of their win or loss. A wide variety of economic activities exemplify such competitions, including rent seeking, lobbying political campaigns, R&D race, competitive procurement, college admissions, organizational hierarchy and internal labor market, etc. Due to the ubiquity of such phenomenon, contests have spawned a huge body of economic studies. This enormous literature delineates economic agents' strategic behaviour in contests from diverse perspectives, and has identified various institutional elements in contest design that affect their incentives to bid.

The majority of the literature on contests has focused on the settings where a fixed number (n) of bidders participate. Under this fixed-n paradigm, the existing literature typically abstracts away from bidders' ex ante choices whether to participate in contests, but focuses on their post-entry activities, while assuming that the actual number of active participants is commonly known. In this paper, we complement the literature by examining explicitly a setting where bidders strategically decide whether to participate in a contest. They enter the contest randomly, so the actual number of participants is uncertain. In addition, we assume that the actual number of entrants is not known by the participants.[1]

As noted by Konrad (2009), a bidder often bears a nontrivial (fixed) entry cost, which can be either explicitly sunk resource, or simply foregone outside opportunities. It merely allows him to participate while does not relate directly to winning. To provide an analogy of this point, while an air ticket paves the way for American tennis star Venus Williams to arrive at the courts of the Australian Open, it does not contribute to her winning the championship. Similarly, to race in a R&D tournament, a research company may need to acquire necessary laboratory equipment, to gather project-specific information, or to turn down other profitable tasks, while its prospect of win depends more critically on its subsequent creative effort. In our setting, a fixed pool of potential bidders decide first whether to participate and then sink their bids after entering the contest. He weighs his expected payoff in future competition against the entry cost, and participates if and only if the former (at least) offsets the latter. With nontrivial entry costs, we show that a symmetric mixed-strategy equilibrium emerges, where each potential bidder enters with the same probability, and they adopt the same (possibly mixed) bidding strategy upon entry. This entry-bidding game complements and enriches the existing literature in several aspects. We elaborate upon its distinct flavors as follows.

First, the strategy of each potential bidder involves two elements in a contest with en-

[1] We show later in the paper that optimal contest design entails nondisclosure of number of actual contestants.

dogenous entry: (1) whether to enter; and (2) how to bid upon his entry. This entry-bidding game exemplifies a discontinuous game with two-dimensional actions (Dasgupta and Maskin, 1986). The game distinguishes itself from standard contests that are typically identified as uni-dimensional games (Baye et al, 1996 and Alcalde and Dahm, 2010), where a player's strategy involves only his bidding action.[2] The existence of symmetric equilibria in this extended setting has yet to be established formally in the literature. We embark on the equilibrium existence theorem of Dasgupta and Maskin (1986) in two-dimensional discontinuous games, and establish the existence of a symmetric mixed-strategy equilibrium.[3]

To our knowledge, our analysis provides the first application of the existence theorem within the contest literature. It deserves to be noted that stochastic entry complicates the analysis substantially and it necessitates the application of Dasgupta and Maskin (1986)'s general results on multi-dimensional discontinuous games. The conventional approach to establish equilibrium existence in contests (Baye et al, 1996 and Alcalde and Dahm, 2010) utilitizes Dasgupta and Maskin (1986)'s theorem on uni-dimensional discontinuous games. Its scope of applications does not include the settings where the number of active players is uncertain.[4]

Second, the bidding behavior in contests with stochastic participation has yet to be explored thoroughly. As well known in the literature, a bidder's payoff maximization problem contest becomes irregular when the contest success function is excessively elastic to effort, e.g. when the discriminatory parameter r in a Tullock contest exceeds certain boundary. Within fixed-n paradigm, previous studies have identified the conditions for the existence of pure-strategy bidding equilibria. Stochastic entries complicate the analysis tremendously. A participant is uncertain about the actual number of competitors. He chooses his bid to maximize his expected payoff, which is a weighted sum of his payoffs under all possible contingencies. Each term in the weighted sum can be an irregular function of his bid. It prevents

[2] As widely recognized in contest literature (Baye et al, 1994, and Alcalde and Dahm, 2010), a well defined contest success function (e.g., Tullock contest) can be discontinuous at origin, i.e., when all bidders bid zero.

[3] One should note that setting our two-dimensional strategy space of (entry, effort) cannot be reduced to a setting with single dimensional strategy of effort with a positive fixed cost. In our two dimensional setting, if no one enters the contest, no one wins. If everyone enters but exerts zero effort, every one incurs the entry cost and wins with equal chance. In the single dimensional setting, if everyone exerts zero effort, no one incurs a cost but they win the contest with the equal chance.

[4] To solve for the entry-bidding equilibrium, the traditional approach in auction literature proceeds in two steps. In the first step, for each given (symmetric) entry probability, one shows the existence of symmetric bidding equilibrium and solves for the bidders' equilibrium payoffs. In a second step, a break-even condition characterizes the equilibrium entry. This approach loses its bite in our setting. The problem in step 1 (bidding equilibrium when potential bidders enter with fixed probabilities) is solvable in an auction setting; while it is not the case in ours. The existing results on equilibrium existence in contests do not apply to contests with random entry and an uncertain number of active players.

us to readily discern the general property of his overall expected payoff function. The problem is further exacerbated by the fact that entry probabilities are endogenously determined, which determine the weights in the summation. We establish a sufficient condition (an upper bound for r in Tullock contests) under which participating bidders choose to (not to) randomize their bids upon entry. This result allows us to derive equilibrium bidding strategy in this game and facilitates further analysis on contest design.

Third, endogenous entry yields rich implications on contest design. In this study, we follow the mainstream literature by searching for the mechanisms that maximize expected overall bid in the contest. We primarily investigate three issues: (1) whether the contest designer would definitely prefers a more precise winner selection mechanism; (2) whether the contest designer should exclude potential bidders, and invite only a subset of them to participate in the competition; and (3) whether the contest designer could improve optimal design by disclosing the actual number of participating bidders when she can observe it.

We focus on Tullock contest and regard the discriminatory parameter r as a measure of the level of noise in the winner selection mechanism. A greater r implies that a higher bid can be translated into a higher likelihood of winning more effectively, thereby increasing the marginal return to one's bid. The conventional wisdom tells that a greater r provides higher-powered incentives and intensifies competition. We demonstrate, nevertheless, that the expected overall bid does not vary monotonically with the size of r. A contest with a smaller r can paradoxically elicit more effort. An immediate trade-off is triggered when r is raised. A more precise contest incentivizes each participant to bid more; while the overheated competition leaves lesser rent to participants, thereby discouraging entries. Moreover, contestants' (mixed) entry strategies affect the expected overall bids indefinitely. More active entry expands the contest and tends to amplify the overall supplies of bids; while it leads individual participants to bid more prudently, as they anticipate more potential competitors and a lesser likelihood of winning. The optimum requires the contest organizer to balance out these diverse and possibly conflicting forces.

Based on the contest design with optimal discriminatory parameter r,[5] we investigate whether the contest designer gets better off when there is a larger pool of potential bidders. Without endogenous entry, the conventional wisdom in contest literature tells that the overall bid always increases with the number of bidders. However, our analysis reveals the opposite: with a larger pool of potential bidder, the contest designer would elicit lesser effort. Hence, the contest designer prefers to limit its size by inviting only a subset of them for participation. Our results thus espouses the merit of exclusion in our setting.

Finally, we establish that there is no loss of generality to consider contests with nondisclosure of number of actual entrants for the optimal design. This result is virtually driven by the endogeneity of entry and convexity of effort costs of contestants.

[5] Optimal r is contingent on number of potential contestants.

The rest of the paper proceeds as follows. We discuss the relation of our paper to the relevant literature in the rest of this section. In section 2, the model is set up. Then we establish our main results on equilibrium existence. Optimal contest design is explored in Section 3. Section 4 concludes the paper.

1.1 Relation to Literature

Our paper complements the literature on contests and auctions, as well as standard oligopolistic competitions in various aspects. In what follows, we discuss the links to the three strands of literature respectively.

1.1.1 Contests

Our paper primarily belongs to the literature on equilibrium existence in contests. Szidarovszky and Okuguchi (1997) establish the existence of pure-strategy equilibria when contestants have concave production function. The existence and properties of the equilibria remains a nagging problem for contests with more general technologies. Baye, Kovenock and de Vries (1996) establish the existence of mixed-strategy equilibria in Tullock contests when r exceeds two when there is finite grid in bidding space. Alcalde and Dahm (2010) further the literature by showing that under a wide class of contest success functions there exist all-pay auction equilibria.[6] Both of the two studies apply the result of Dasgupta and Maskin (1986) on uni-dimensional discontinuous game. Our paper contributes to this literature by introducing bidders' choices of entry and allowing the number of active bidders to be stochastic. These new flavours enrich our analysis by forming a two-dimensional discontinuous game, and provide a novel application of the general result of Dasgupta and Maskin (1986) in contest literature.

Our paper is also linked to the relatively thin literature on contests with stochastic participation. The majority of studies in this strand of literature assume exogenous entry patterns. Myerson and Wärneryd (2006) examined a contest with an infinite number of potential entrants. Münster (2006), Lim and Matros (2009) and Fu, Jiao and Lu (2010) assumed a finite pool of potential contestants, with each contestant entering the contest with a fixed and independent probability.[7] Our paper is more closely linked to the pioneering study by Higgins, Shughart, and Tollison (1985). They study a contest in which each rent seeker bears a fixed entry cost, and he randomly participates in equilibrium. Our paper allows for a more general setting and provides a formal account of equilibrium existence in entry-bidding game. We fur-

[6] Wang (2010) also characterizes the equilibrium in two-player asymmetric Tullock contest when r is large.

[7] Münster (2006) focuses on the impact of players' risk attitudes on their effort supply. In contrast, Lim and Matros (2009) and Fu, Jiao and Lu (2010) consider risk-neutral contestants.

ther contribute to the literature by discussing the efficient design of contests with endogenous and stochastic entries.

Lim and Matros (2009) first study the issue of disclosing the number of contestants, where potential bidders enter with an exogenous probability. They demonstrate the independence of prevailing policy in Tullock contest with $r = 1$ and linear effort costs. Fu, Jiao and Lu (2010) further reveal that the optimal disclosure policy can depend on the characteristics of the production functions of contestants. The current paper illustrates the critical role played by the convexity of bidding cost function and endogeneity of entry.

Our analysis also complements this literature that explores the proper level of precision in evaluating bidding performance. The conventional wisdom tells that a precise contest incentivizes more aggressive bidding. A handful of studies, however, espouse low-powered incentives in contests, and demonstrate that a less "discriminatory" contest can improve efficiency. One salient example is provided by Lazear (1989), who argues that an excessively competitive environment leads workers to sabotage each other. A more popular strand of literature instead stresses the merit of a "handicapping" effect of imprecise performance evaluation mechanism in (two-player) asymmetric contests. When contestants differ in their abilities, a noisier contest balances the playfield. This effect encourages the weaker to bid more intensely, and further deters the stronger one from shirking. O'Keeffe, Viscusi and Zeckhauser (1984) are among the first to espouse this logic. This rationale is further elaborated upon by Che (2000), Nti (2004), Amegashie (2010), and Wang (2010). In contrast to these studies, our paper establishes the efficient effect of low-powered incentives in N-player symmetric contests, and stresses the trade-off between ex post bidding incentives and ex ante entry incentives.

Our finding on efficient exclusion echoes a handful of pioneering studies by Baye, Kovenock and de Vries (1993), Taylor (1995), Fullerton and McAfee (1999), and Che and Gale (2003). Dasgupta (1990) studies a two-stage procurement. Competing R&D firms independently invest for cost reduction in the first stage. Their cost realization determines their equilibrium bids in the second stage. Dasgupta (1990) finds that in this setting limiting the number of competing firms may, but does not necessarily, benefit the principal. None of these studies involves an fixed entry cost that an entrant has to incur. Further, in our setting, an invited (potential bidder) may or may not eventually enter the subsequent contest as equilibrium entry is stochastic.

1.1.2 Oligopolistic Competition

Besides its apparent connection to contest literature, our paper echoes the argument of Shapiro (1986) on firms' behavior in oligopolistic markets. He shows that the more fierce Bertrand competition can be ex post more anti-competitive than an ex ante softer Cournot competition, as the latter restricts the contestability of the market and discourages entries. We focus on the issue of mechanism design in our particular context, and endogenize the level of subsequent competition as the strategic choice of the contest designer.

1.1.3 Auctions with Endogenous Entry

Our paper also bears a relation to the literature on auctions with endogenous entry. Myerson (1981) shows that a second-price or first-price auction with an optimal reserve price is revenue-maximizing when bidders bear zero entry costs. Samuelson (1985), Menezes and Monteiro (2000) and Lu (2009) require that bidders sink entry costs to participate in the auction. Levin and Smith (1994), Shi (2009), Lu (2010) and Moreno and Wooders (2010) allow bidders to make costly investment to learn their valuations of the object for sale. These studies all conclude that the revenue-maximizing auction requires a lower reservation price than that of Myerson (1981). A higher reserve price strengthens *ex post* incentive of bidding on the one hand; while it discourages the *ex ante* incentives of entry or information acquisition. Our study departs subtly from the auction literature in two main aspects. First, the auction design problem addresses an adverse-selection problem: bidders possess private information about their own types and therefore the efficient mechanism screens heterogeneous bidder. Our contest design problem nevertheless concerns itself primarily with a moral hazard problem: the type of players is commonly known; while the efficient mechanism sets out to incentivize effort supply. Second, the insights obtained from auction literature do not carry over to our setting. The auction literature shows that a weaker *ex ante* incentive, i.e. a lower reserve price than the benchmark of Myerson (1981), is always in demand whenever entry or information acquisition is costly. By way of contrast, the optimum in our setting could involve either a weaker (i.e. a smaller r) or a stronger (i.e. a bigger r) *ex ante* incentive than that in the zero-entry-cost benchmark.

Shortlisting and exclusion have long been recognized as one important element in designing auctions with costly entry. In a setting of Samuelson (1985) where bidders who know their private values but bear entry costs, Lu (2009) identifies sufficient conditions under which the optimal revenue to increase or decrease with the number of potential bidders. Levin and Smith (1994) let potential bidders make costly investment to discover their valuations of the object. They establish that the optimal revenue decreases with the number of potential bidders to the extent that the information acquisition costs lead to mixed-strategy entry. Our setting resembles that of Sameulson (1985) and Lu (2009); while the results run in contrast to them: Lu (2009) finds that shortlisting is not necessarily optimal, but we find that the contest designer can always elicit higher overall bid by excluding potential bidders. The finding, however, echoes Levin and Smith (1994), despite the differing settings.

The optimal disclosure policy has also been examined in auctions with stochastic number of bidders. McAfee and McMillan (1987) and Levin and Ozdenoren (2004) consider a setting of exogenous stochastic entry and show that the expected revenue is independent of the disclosure policy when bidder are risk neutral. Our paper allows for endogenous entry and concludes that concealment elicits a higher overall bid if the bidding cost function is convex.

2 Model and Analysis

We consider a two-stage game. A fixed pool of $M(\geq 2)$ identical risk-neutral potential bidders demonstrate interest in a contest with a winner's purse $v > 0$. In the first stage, potential bidders simultaneously decide whether or not to participate. In the second stage, all participants simultaneously submit their bids. A winner is selected and awarded the prize.

2.1 Winner Selection Mechanism

We model the competition explicitly as a Tullock contest. Suppose that $N \geq 2$ potential bidders enter the contest. They simultaneously submit their bids $x_i, i = 1, 2, ..., N$, to compete for the prize v. The probability of a participating bidder i winning the prize is given by

$$p_N(x_i, \mathbf{x}_{-i}) = \frac{x_i^r}{\sum_{j=1}^{N} x_j^r}, \text{ if } N \geq 2, \text{ and } \sum_{j=1}^{N} x_j^r > 0. \tag{1}$$

If all participants submit zero bid, the winner is randomly picked from set of the participants. To the extent that only one bidder enters, he automatically receives the prize v, regardless of his bid. In the event that nobody enters the contest, the designer keeps the prize.

A bid x_i costs a bidder $c(x_i)$, with $c'(\cdot) > 0$ and $c''(\cdot) \geq 0$. For the sake of tractability, we assume that the bidding cost function takes the form $c(x_i) = x_i^{\alpha}$, with $\alpha \geq 1$.

It should be noted that our main theorem on equilibrium existence in the entry-bidding game applies to contests with more general success functions and cost functions, which will be discussed in more detail later in the paper.

2.2 Entry

In the first stage of the game, potential bidders simultaneously decide whether to participate in the contest. Each participant has to sink a fixed cost $\Delta > 0$ if he enters. Entry is irreversible, and the cost Δ cannot be recovered. We impose the following regularity condition on the model.

Assumption 1 $\frac{v}{M} < \Delta < v$.

The assumption requires that the entry cost Δ is nontrivial but not prohibitively high. First, no entry is triggered if it costs more than the winner's purse. Second, the analysis becomes relatively trivial when entry involves little cost, in which case the institutional elements of the contest do not affect bidders' entry incentives significantly. Because of this assumption, there does not exist any symmetric equilibria where potential bidders participate in the contest with certainty.

With a bit abuse of notations, we define two cutoff probabilities, which are used repeatedly throughout the analysis.

Definition 1 *Let* $\bar{q} \in (0,1)$ *be the unique solution to* $(1-(1-q)^M)v - Mq\Delta = 0$*, and* $q_0 \in (0,1)$ *be the unique solution to* $(1-q)^{M-1}v - \Delta = 0$*.*

Comparing the two cutoffs yields the following.

Lemma 1 $q_0 < \bar{q}$*.*

Proof. See Appendix. ■

In our main analysis, we assume that each participating bidder does not know the actual number N of participants. This setting yields a two-dimensional discontinuous game and demands a more sophisticated analysis. Two remarks are in order. First, entry often involves hidden action, which cannot be readily observed or verified by other parties. Second, one may view the observaility of N as an institutional element, which is to be chosen strategically by the contest designer. In Section 3.2, we assume that the contest designer is able to observe N and she chooses the disclosure policy of the contest. We show that a contest would in general elicit lesser bid when N is to be disclosed.

2.3 Existence of Symmetric Equilibrium

A bidder i's behavioral strategy is an order pair $(q_i, \mu_i(x_i))$, where q_i is the probability he enters the contest, and x_i is the bid he would submit upon entry. We allow him to randomize on his bids. The probability distribution $\mu_i(x_i)$ depicts his strategy of bidding conditional on his entry. It reduces to a singleton when the participant does not randomize in placing his bid.

If potential bidders play mixed-strategy in entry stage, each participant is uncertain about the set of his competitors when placing his bid. He bids based on his rational belief about others' entry pattern. The solution concept of subgame perfect equilibrium would not apply, because participants possess only imperfect information and no proper subgame exists after the entry stage. We use the concept of Bayesian Nash equilibrium to solve the game. A Bayesian Nash Equilibrium is a combination of the pair strategies $\times_{i=1}^{M}(q_i, \mu_i(x_i))$. It requires

(1) every participant i updates his belief on the entry outcome by Bayesian rule; and (2) that $(q_i, \mu_i(x_i))$ maximizes his expected payoff based on his belief and others' strategy profile $\{(\mathbf{q}_j, \boldsymbol{\mu}_\mathbf{j}(\mathbf{x_j}))\}\Big|_{j\neq i}$.

With nontrivial entry cost ($\frac{v}{M} < \Delta$), there always exist asymmetric equilibria, where a subset of potential bidders stay inactive regardless, while the others enter either randomly or deterministically. Focusing on asymmetric equilibria simplifies the analysis. These equilibria, however, do not provide a comprehensive account of the game, as they essentially concern themselves with only a subset of players in the game. Throughout this paper, we focus on symmetric equilibrium.

We search for the symmetric equilibrium of the game where all potential bidders play the same strategy $(q^*, \mu^*(x))$. As aforementioned, a potential bidder's payoff can be discontinuous as the contest success function is discontinuous at origin (see Baye, Kovenock and de Vries, 1996, and Alcalde and Dahm, 2010), i.e. when all participants bid zero. However, the conventional approach in establishing equilibrium existence in contest (based on Dasgupta and Maskin, 1986) does not apply in this setting. The strategy of each player involves two elements. Furthermore, the equilibrium existence of Dasgupta and Maskin (1986) for uni-dimensional game does not concern itself with games with uncertain number of players. This game, however, can be viewed as a two-dimensional discontinuous game (Dasgupta and Maskin, 1986). We apply the general result of shown by Dasgupta and Maskin (1986) for a multi-dimensional strategy space to establish the existence of symmetric equilibria.

Theorem 1 *(i) For any $r > 0$, a symmetric equilibrium $(q^*, \mu^*(x))$ exists. In the equilibrium, each potential bidder enters with a probability $q^* \in (0, \bar{q})$ and his bid follow a probability distribution $\mu^*(x)$. (ii) Each potential bidder receives an expected payoff of zero in the equilibrium.*

Proof. See Appendix. ■

To our knowledge, Theorem 1 and its proof, within the contest literature, provide the first application of Dasgupta and Maskin's (1986) equilibrium existence result on two-dimensional discontinuous games. A few remarks are in order. First, the equilibrium existence result apply to broader contexts. The proof of the theorem does not rely on the specific properties of Tullock success functions and the functional form for bidding costs. For instance, we can readily adapt the result to contests with more broadly defined success functions, such as those in Alcalde and Dahm (2010), by redefining the discontinuity set slightly. We explicitly adopt Tullock technologies to economize on the presentation and facilitate subsequent discussion on contest design. Second, our analysis has yet to provide a more informative account on the equilibrium bidding behaviors, which remains one of the central concerns in contest literature. In this entry-bidding game, a participating bidder may randomize on his bid x_i in the equilibrium. It remains to explore when each participant would (not) mix his bids. We establish the relevant conditions for pure or mixed bidding strategies subsequently.

2.4 Existence of Equilibrium with Pure-Strategy Bidding

Suppose that a symmetric equilibrium with pure-strategy bidding exists. Consider an arbitrary potential bidder i who has entered the contest. Suppose that all other potential bidders play a strategy (q, x) with $x > 0$.[8] He chooses his bid x_i to maximize his expected payoff

$$\pi_i(x_i|\,\mathbf{x},\mathbf{q}) = \sum_{N=1}^{M} C_{M-1}^{N-1} q^{N-1}(1-q)^{M-N}[\frac{x_i^r}{x_i^r+(N-1)x^r}v - x_i^{\alpha}]. \tag{2}$$

Evaluating $\pi_i(x_i|\,\mathbf{x},\mathbf{q})$ with respect to x_i yields

$$\frac{d\pi_i(x_i|\,\mathbf{x},\mathbf{q})}{dx_i} = \sum_{N=1}^{M} C_{M-1}^{N-1} q^{N-1}(1-q)^{M-N}\frac{(N-1)rx_i^{r-1}x^r v}{[x_i^r+(N-1)x^r]^2} - \alpha x_i^{\alpha-1}. \tag{3}$$

The (pure) bidding strategy in such an equilibrium, if it exists, can be solved for by the first order condition $\frac{d\pi_i(x_i)}{dx_i} = 0$ and the symmetry condition $x_i = x$.

Lemma 2 *Suppose that a symmetric equilibrium with pure-strategy bidding exists. In such an equilibrium, each potential bidder enters the contest with a probability q^*, which satisfies*

$$\sum_{N=1}^{M} C_{M-1}^{N-1} q^{*N-1}(1-q^*)^{M-N}\frac{v}{N}(1-\frac{N-1}{N}\frac{r}{\alpha}) = \Delta. \tag{4}$$

Each participating bidder places a bid of

$$x^* = [\sum_{N=1}^{M} C_{M-1}^{N-1} q^{*N-1}(1-q^*)^{M-N}\frac{N-1}{N^2}\frac{rV}{\alpha}]^{\frac{1}{\alpha}} \tag{5}$$

The expected overall bid of the contest obtains as

$$x_T^* = Mq^*x^* = Mq^*[\sum_{N=1}^{M} C_{M-1}^{N-1} q^{*N-1}(1-q^*)^{M-N}\frac{N-1}{N^2}\frac{rV}{\alpha}]^{\frac{1}{\alpha}}. \tag{6}$$

Proof. See Appendix. ■

Lemma 2 depicts the main properties of a symmetric equilibrium with pure-strategy bidding, if it exists. We call equation (4) the break-even condition of the entry-bidding game with pure-strategy bidding. It determines the equilibrium entry probability q^* in such an equilibrium. The properties of the break-even condition lead to the following.

[8] It is impossible to have all participating bidders to bid zero in an equilibrium with pure bidding action. When all others bid zero, a participating bidder would win the prize with probability one if places an infinitely small positive bid.

Lemma 3 *For any $r > 0$, there exists a unique $q^* \in (0, \overline{q})$ that satisfies the break-even condition (4). Hence, x^* is also uniquely determined for the given r. Furthermore, q^* strictly decreases with r.*

Proof. See Appendix. ∎

Lemma 3 verifies the unique correspondence between r and (q^*, x^*). The symmetric equilibrium with pure bidding strategy must be unique for each given r, if it exists. However, the strategy profile $(\mathbf{q}^*, \mathbf{x}^*)$ may not constitute an equilibrium.

When all others enter with a probability q and bid x if enter, a participating bidder i chooses his bid x_i to maximize his expected payoff $\pi_i(x_i|\, \mathbf{x}, \mathbf{q})$, which is the weighted sum of $\pi_i^N(x_i|\, \mathbf{x}, \mathbf{q}) = \frac{x_i^r}{x_i^r + (N-1)x^r} v - x_i^\alpha$ for all possible N. Note that $\pi_i^N(x_i|\, \mathbf{x}, \mathbf{q}) = \frac{x_i^r}{x_i^r + (N-1)x^r} v - x_i^\alpha$ is simply his expected payoff when he enters a contest in which $N-1$ other bidders have also entered and each bids x. We graphically illustrate $\pi_i^N(x_i|\, x, q)$ and $\pi_i(x_i|\, x, q)$ by Figure 1.

The equilibrium analysis is trivial when $r \le 1$. In that case, the maximization of $\pi_i^N(x_i|\, \mathbf{x}, \mathbf{q})$ is a well-behaved concave program, so is the maximization of $\pi_i(x_i|\, \mathbf{x}, \mathbf{q})$. In this case, the hypothetical equilibrium bid x^*, as given by (5), must maximize $\pi_i(x_i|\, \mathbf{x}^*, \mathbf{q}^*)$ and a strategy profile with all playing (q^*, x^*) must constitute the unique symmetric equilibrium. As well known in contest literature, the maximization of $\pi_i^N(x_i|\, x, q)$, however, is not a regular problem for a large r, as $\pi_i^N(x_i|\, \mathbf{x}, \mathbf{q})$ is no longer globally concave. The irregularity is exacerbated tremendously in a contest with stochastic and endogenous entry. It is difficult to draw a general conclusion on the properties of the payoff function $\pi_i(x_i|\, \mathbf{x}, \mathbf{q})$, which is the weighted sum of a series of not necessarily concave functions. Furthermore, the weights $\sum_{N=1}^{M} C_{M-1}^{N-1} q^{N-1}(1-q)^{M-N}$ ultimately depends on the size of entry probability q, which is determined endogenously. This fact further complicates our analysis. The general results that are obtained from contests with deterministic participation cannot be carried over.

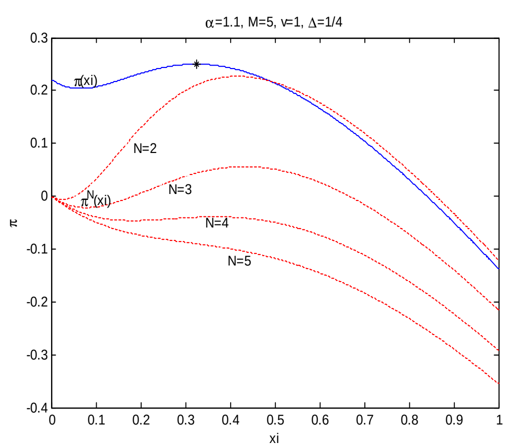

Figure 1

In subsequently analysis, we derive the upper (lower) bound of r which guarantees the existence (non-existence) of a symmetric equilibrium with pure-strategy bidding. Recall by Lemma 3 the unique correspondence between r and (q^*, x^*). Consider a contest with a given r. Define

$$\tilde{\pi}_i(x_i) = \pi_i(x_i|\mathbf{x}^*, \mathbf{q}^*) = \sum_{N=1}^{M} C_{M-1}^{N-1} q^{*N-1}(1-q^*)^{M-N}(\frac{x_i^r}{x_i^r + (N-1)x^{*r}}v) - x_i^{\alpha}, \qquad (7)$$

which is a bidder i's expected payoff in the contest when all other bidders play the strategy (q^*, x^*), as given by Lemma 2. Clearly, $\tilde{\pi}_i(x_i)$ is continuous on $[0, \infty)$. The next result depicts an important property of the payoff function.

Lemma 4 *When $r \in (1, \alpha\frac{M-1}{M-2}]$, x^* is the unique inner local maximizer of $\tilde{\pi}_i(x_i)$, i.e. $\tilde{\pi}_i(x^*) > \tilde{\pi}_i(x), \forall x \in (0, \infty)\backslash\{x^*\}$. There exists a unique $x_m \in (0, x^*)$ such that $\tilde{\pi}_i(x_i)$ decreases on $[0, x_m]$, increases on $[x_m, x^*]$, and then decreases on $[x^*, \infty)$.*

Proof. See Appendix. ■

Lemma 4 depicts the property of $\tilde{\pi}_i(x_i)$ when r remains in the range $(1, \alpha\frac{M-1}{M-2}]$. We define $\alpha\frac{M-1}{M-2}$ as $+\infty$ when $M = 2$. Although it is not longer globally concave, x^* is its unique maximizer for $x \in (0, \infty)$. However, x^* has yet to be established as the global maximizer: the equilibrium requires the boundary condition $\tilde{\pi}_i(x^*) \geq \tilde{\pi}_i(0)$ hold. Recall that x^* is uniquely determined the break-even condition (4) for each given r. A participating bidder's expected payoff in the contest, if bidding x^*, amounts to exactly Δ. However, the bidder can automatically receive a reserve payoff $(1-q^*)^{M-1}v$ from the contest by bidding zero: with a probability $(1-q^*)^{M-1}$, all other potential bidders stay out of the contest, and a rent of $(1-q^*)^{M-1}v$ will accrue to him as long as he participates. Hence, the bidder has an incentive to bid x^* if and only if $(1-q^*)^{M-1}v \leq \Delta$, which requires

$$\sum_{N=2}^{M} C_{M-1}^{N-1} q^{*N-1}(1-q^*)^{M-N}\frac{v}{N}(1 - \frac{N-1}{N}\frac{r}{\alpha}) \geq 0. \qquad (8)$$

The implication of condition (8) is straightforward: bidding $x^*(> 0)$ must generate nonnegative additional return (when all others bid x^*) in excess of the reservation payoff. An upper bound of r is required to make sure that each bidder is sufficiently rewarded by bidding x^*. Recall the cutoff $q_0 \in (0, \overline{q})$ depicted by Definition 1, which uniquely satisfies $(1-q_0)^{M-1}v = \Delta$. The unique correspondence between r and (q^*, x^*), as determined by the break-even condition (4), allows us to obtain the following cutoff of r.

Definition 2 *Define $r_0 \in (\alpha(1 + \frac{1}{M-1}), 2\alpha]$ to be the unique solution to $\sum_{N=1}^{M} C_{M-1}^{N-1} q_0^{N-1}(1 - q_0)^{M-N}\frac{v}{N}(1 - \frac{N-1}{N}\frac{r_0}{\alpha}) = \Delta$.*

By Lemma 3, with $r > r_0$, q^* must fall below q_0. A bidder, when bidding x^*, receives $\tilde{\pi}_i(x^*) = \Delta$. He would strictly prefer to bid zero, because his expected payoff when bidding zero, $(1-q^*)^{M-1}v$, must be strictly more than Δ. In other words, x^* is not the best response of player i. The following is immediate.

Theorem 2 *A symmetric equilibrium with pure-strategy bidding does not exist if $r > r_0$.*

When $r > r_0$, the strategy profile $(\mathbf{q}^*, \mathbf{x}^*)$ would not constitute an equilibrium. The prevailing symmetric equilibria must involved randomized bidding strategies. Following Lemma 4 and Definition 2, we define a cutoff of r with a bit abuse of notation.

Definition 3 *Define $\overline{r} \triangleq \min(r_0, \alpha\frac{M-1}{M-2})$.*

The previous analysis leads to the following.

Theorem 3 *For each $r \in (0, \overline{r}]$, the strategy profile $(\mathbf{q}^*, \mathbf{x}^*)$, as characterized by Lemma 2, constitutes the unique symmetric equilibrium with pure-strategy bidding of the entry-bidding game.*

When r is bounded from above by both r_0 and $\alpha\frac{M-1}{M-2}$, a unique symmetric equilibrium with pure-strategy bidding emerges. The condition $r \in (0, \overline{r}]$ guarantees (1) that the payoff function $\tilde{\pi}_i(x_i)$ is well behaved, in the sense that its curve reaches a unique peak at x^* for $x \in (0, \infty)$; and (2) that the boundary condition $\tilde{\pi}_i(x^*) \geq \tilde{\pi}_i(0)$ is met. We then conclude that x^* is the global maximizer when r is subject to both upper bounds. The strategy profile $(\mathbf{q}^*, \mathbf{x}^*)$ are established as the unique equilibrium accordingly.

Theorem 3 imposes a (conservative) upper limit on r for the existence of such an equilibrium. It should be noted that $r \leq \alpha(1 + \frac{1}{M-2})$ is sufficient but not necessary to establish x^* as the local maximizer of $\tilde{\pi}_i(x_i)$ for $x > 0$. Analytical difficulty prevents us from fully characterizing the property of $\tilde{\pi}_i(x_i)$ when r exceeds $\alpha(1+\frac{1}{M-2})$. It remains less than explicit how the equilibrium would behave if $\alpha(1 + \frac{1}{M-2}) < r_0$ and $r \in (\alpha(1 + \frac{1}{M-2}), r_0]$.

More definitive conclusion can be drawn only in more specific contexts with small numbers of potential bidders.

Corollary 1 *When M is small, i.e. $M = 2, 3$, a symmetric equilibrium with pure-strategy bidding exists if and only if $r \leq r_0$.*

In these instances, $\alpha(1 + \frac{1}{M-2}) > r_0$ regardless of v and Δ. Whenever r falls below r_0, it automatically satisifies the condition $r \leq \alpha(1 + \frac{1}{M-2})$, which guarantees that x^* maximizes $\tilde{\pi}_i(x_i)$. We do not have to explore the property of $\tilde{\pi}_i(x_i)$ when x_i exceeds the cutoff $\alpha(1+\frac{1}{M-2})$, as pure-strategy bidding would not arise in that case. However, we are unable to draw general conclusion analytically when M is larger, which may possibly leads $\alpha(1+\frac{1}{M-2})$ to exceed r_0.

We resort to numerical exercise to obtain further insight on the properties of the expected payoff function $\tilde{\pi}_i(x_i)$. We normalize v to unity. Our simulation is run over a large set of the parameters (α, M), which span the entire space of $[1,2] \times \{4,5,\ldots,100\}$. For given (α, M), we let r vary over the entire range of $(\alpha(1+\frac{1}{M-2}), r_0)$ if $\alpha(1+\frac{1}{M-2}) < r_0$, and let Δ vary over the interval $(\frac{1}{M}, 1)$. Our simulation, however, reports that x^* uniquely maximizes $\tilde{\pi}_i(x_i)$ in all these cases. We plot $\tilde{\pi}_i(x_i)$ in our simulations. In all resultant figures, the curve is regularly shaped as described by Lemma 4. It reaches a single peak at x^*. Figure 2 provides one example of them.

We observe from our simulation results, with no exception, that all $\tilde{\pi}_i(x_i)$ demonstrates the property depicted by Lemma 4, and is uniquely maximized by x^* for all $r \leq r_0$, despite that r exceeds $\alpha(1+\frac{1}{M-2})$. The strategy profile (q^*, x^*) constitutes the unique symmetric equilibrium with pure-strategy bidding in all the simulated settings. We therefore propose the following conjecture.

Conjecture 1 *A symmetric equilibrium with pure bidding exists if and only if* $r \leq r_0$.

We are unable to prove it analytically. However, all of our numerical exercises lend support to the claim. We leave it to future studies and we will also attempt it in further work despite its technical difficulty.

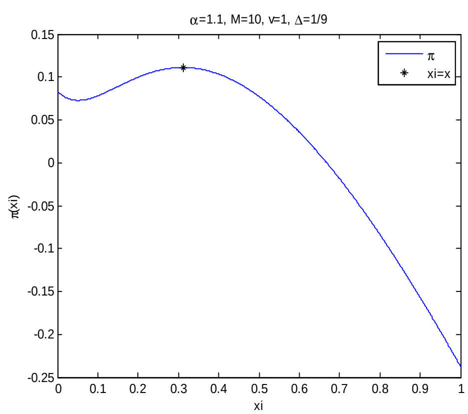

Figure 2

3 Contest Design

Bidders's equilibrium behaviors may depend critically on the institutional elements of the contest. Central to the contest literature is the question of how the rules of the contest affect equilibrium bidding. As argued by Gradstein and Konrad (1999), "... contest structures result from the careful consideration of a variety of objectives, one of which is to maximize the effort of contenders." Based on the equilibrium analysis, we follow in vein of this literature to discuss the optimal design of the contest that maximizes overall bid. Specifically, we consider three main issues: (1) the optimal level of accuracy in winner selection mechanism (the proper size of r in Tullock contests); (2) the efficiency implications of shortlisting and exclusion; and (3) the optimal disclosure policy.

3.1 Optimal Accuracy: Choice of r

In a Tullock contest, the parameter r arguably reflects the "discriminatory power" or the level of precision of the winner selection mechanism in the contest. With a greater r, one's win depends more on the quality of his bid, rather than other noisy or random factors. The level of precision in a contest is largely subject to the autonomous choice of the contest designer. For instance, the designer can modify the judging criteria of the contest to suit her strategic goals, e.g. adjusting the weights of subjective component in contenders' overall ratings. Alternatively, she can vary the composition of judging committees (experts vs. non-experts).

Following the literature (e.g. Nti, 2004), we let r be chosen strategically by the contest designer. We then consider a three-stage game. The designer chooses r and annonces publicly in the beginning. Then the entry-bidding game follows. In the subsequent analysis, we investigate how the size of r affect the equilibrium bids.

Before we proceed, we consider the benchmark case of a contest with a fixed number M of participant. A larger r increases the marginal return of a bid and further incentivizes bidders. It is well known in the contest literature that both individual bid and overall bid strictly increase with r whenever a pure-strategy equilibrium exists, i.e. $r \in [0, \alpha\frac{M}{M-1}]$. This conventional wisdom, however, loses its bite in our setting.

3.1.1 Optimum

A contest with endogenous and costly entry involves tremendously more extensive strategic trade-offs. A more discriminatory contest forces contestants to bid more on the one hand; while the increasing dissipation of rent limits entry on the other. As revealed by Lemma 3, q^* would strictly decreases with r in the symmetric equilibrium with pure-strategy bidding.

This trade-off, however, does not exhaust the intricacy involved in the determination of optimal r. An additional trade-off is triggered at a differing layer. More extensive participation (i.e. a higher q^*) does not necessarily improve the supply of bids in the contest. On the one hand, the contest engages more bidders, which amplify the sources of contribution and tends to increase the overall bid. On the other hand, each participant would bid less, as they anticipate more intense competition and therefore expect less reward. The overall effect remains less than explicit and has yet to be explored more formally.

Consider an arbitrary entry-bidding game where potential bidders enter with a probability q^* in a symmetric equilibrium. The prize v is given away with a probability $1-(1-q^*)^M$. Hence, bidders win an expected overall rent of $[1-(1-q^*)^M]v$; while they on average incur entry cost $Mq^*\Delta$. The following fundamental equality must hold in this symmetric equilibrium:

$$[1-(1-q^*)^M]v \equiv Mq^*(\Delta + E(x^\alpha)). \tag{9}$$

The equality allows us to identify unambiguously the expected overall bidding cost incurred by the bidders in the equilibrium without explicitly solving it: it further leads to

$$Mq^*E(x^\alpha) = [1-(1-q^*)^M]v - Mq^*\Delta.$$

The convex cost function ($\alpha \geq 1$) further implies that the expected overall bid ($Mq^*E(x)$) must be bounded from above:

$$(Mq^*E(x)) = Mq^*E[(x^\alpha)^{\frac{1}{\alpha}}] \leq Mq^*[E(x^\alpha)]^{\frac{1}{\alpha}}. \tag{10}$$

By the fundamental equality (9), we obtain

$$(Mq^*E(x)) \leq [Mq^*]^{\frac{\alpha-1}{\alpha}}\{[1-(1-q^*)^M]v - Mq^*\Delta\}^{\frac{1}{\alpha}}. \tag{11}$$

Regardless of the equilibrium bidding strategy upon entry, RHS of (11) imposes an upper limit on the overall bid an equilibrium with entry probability q^* could possibly elicit. The upper limit $[Mq^*]^{\frac{\alpha-1}{\alpha}} \times \{[1-(1-q^*)^M]v - Mq^*\Delta\}^{\frac{1}{\alpha}}$ are reached, i.e., $(Mq^*E(x)) = [Mq^*]^{\frac{\alpha-1}{\alpha}} \times \{[1-(1-q^*)^M]v - Mq^*\Delta\}^{\frac{1}{\alpha}}$, if and only if (1) bidders play a pure bidding strategy; or (2) participants randomize their bids but $\alpha = 1$.

Define

$$\overline{x_T}(q) \triangleq (Mq)^{\frac{\alpha-1}{\alpha}}\left\{[1-(1-q)^M]v - Mq\Delta\right\}^{\frac{1}{\alpha}} \tag{12}$$

with $q \in (0,1)$. The function exhibits the following important property.

Lemma 5 *(i) There exists a unique $\widehat{q} \in (q_0, \bar{q})$, which uniquely maximizes $\overline{x_T}(q)$;*

(ii) The function $\overline{x_T}(q)$ strictly increases with q when $q \in (0, \widehat{q})$, and strictly decreases when $q \in (\widehat{q}, 1)$.

Proof. See Appendix. ■

As stated by Lemma 5, the function $\overline{x_T}(q)$ varies nonmonotonically with q and it is uniquely maximized by $\widehat{q} \in (0,1)$. The overall bid that can be possibly elicited from the contest would never exceed $\overline{x_T}(\widehat{q})$. With a bit abuse of notation, let us define the following.

Definition 4 *Define* $\overline{x_T}^* \equiv \overline{x_T}(\widehat{q})$*, which indicates the maximum amount of the overall bid a contest can elicit.*

By the previous argument, the contest can never elicit an overall bid that exceeds $\overline{x_T}^*$. The key to the design problem unfolds in Lemma 5: the "first best" $\overline{x_T}^*$ can be achieved in a symmetric equilibrium with an entry probability $\widehat{q}$, if (1) participants play a pure bidding strategy in the equilibrium; or (2) participants randomize their bids but $\alpha = 1$. The result thus prompts is to explore how to set r properly to induce the "first best".

The exact forms of equilibrium bidding strategies in equilibria with mixed bidding remain unknown. It is difficult to pin down the correspondence between (large-sized) r and the properties of the equilibrium induced by the parameter. We focus on the possibility of inducing the "first best" in equilibria with pure-strategy bidding. Recall by Lemma 3 the unique correspondence between r and the entry probability in the prevailing symmetric equilibrium with pure-strategy bidding. The equilibrium with entry probability q^* is determined by the break-even condition $v\sum_{N=1}^{M} C_{M-1}^{N-1} q^*(1-q^*)^{M-N}[\frac{1}{N} - \frac{N-1}{N^2}\frac{r}{\alpha}] = \Delta$. We highlight the following cutoff.

Definition 5 *Let* $r(\widehat{q})$ *be the unique solution of* r *to*

$$v\sum_{N=1}^{M} C_{M-1}^{N-1}\widehat{q}(1-\widehat{q})^{M-N}[\frac{1}{N} - \frac{N-1}{N^2}\frac{r}{\alpha}] = \Delta. \tag{13}$$

The following is immediate.

Theorem 4 *(i)* $r(\widehat{q}) \leq r_0$*. (ii) Whenever* $r(\widehat{q}) \leq \alpha(1+\frac{1}{M-2})$*, the contest designer can elicit the "first best"* $\overline{x_T}^*$ *by setting* $r = r(\widehat{q})$*. It induces a symmetric equilibrium with pure-strategy bidding. Potential bidders enter the contest with a probability* $\widehat{q}$ *in the symmetric equilibrium.*

Proof. See Appendix. ■

Setting r to $r(\widehat{q})$ could allow the contest designer to elicit the "first best" overall bid $\overline{x_T}^* \equiv \overline{x_T}(q)$. Because $r(\widehat{q}) \in (0, r_0)$, whenever $r(\widehat{q})$ falls below $\alpha(1+\frac{1}{M-2})$, it satisfies the sufficient condition $r \leq \overline{r}$. A symmetric equilibrium with pure-strategy bidding is induced by $r(\widehat{q})$, where potential bidders enter the contest with a probability exactly $\widehat{q}$. The optimum $r(\widehat{q})$ balances the various competing effects as we discussed above. By Lemma 5, equilibrium bid would decrease when r deviates from it.

3.1.2 Discussion

Our analysis has been limited so far. The global optimality of $r(\widehat{q})$ is conditioned on that it also leads to pure-strategy bidding. It remains to explore to what extent $r(\widehat{q})$ could robustly induce pure-strategy bidding. Because $r(\widehat{q}) \leq r_0$, pure-strategy bidding must arise as long as $r(\widehat{q})$ falls below $\alpha(1+\frac{1}{M-2})$. A definitive conclusion can be drawn in contests with small pools of potential participants.

Corollary 2 *When the contest is small, i.e., $M = 2, 3$, $r(\widehat{q}) \leq \alpha(1 + \frac{1}{M-2})$ must hold, and a symmetric equilibrium with pure-strategy bidding can always be induced by setting $r = r(\widehat{q})$.*

The condition $r(\widehat{q}) \leq \alpha(1 + \frac{1}{M-2})$ is less certain when M is large. We further check its robustness through numerical exercise. The condition is found to hold over a large parameter space, and ample incidents can be observed. In Figure 3, we provides three sets of results we obtain from numerical exercises. In each set of computation, we hold constant two of the three critical environmental parameters $(M, \alpha, \frac{\Delta}{v})$, and let the other vary. We then compute $r(\widehat{q})$ and $\alpha(1 + \frac{1}{M-2})$ and compare them in the figures.

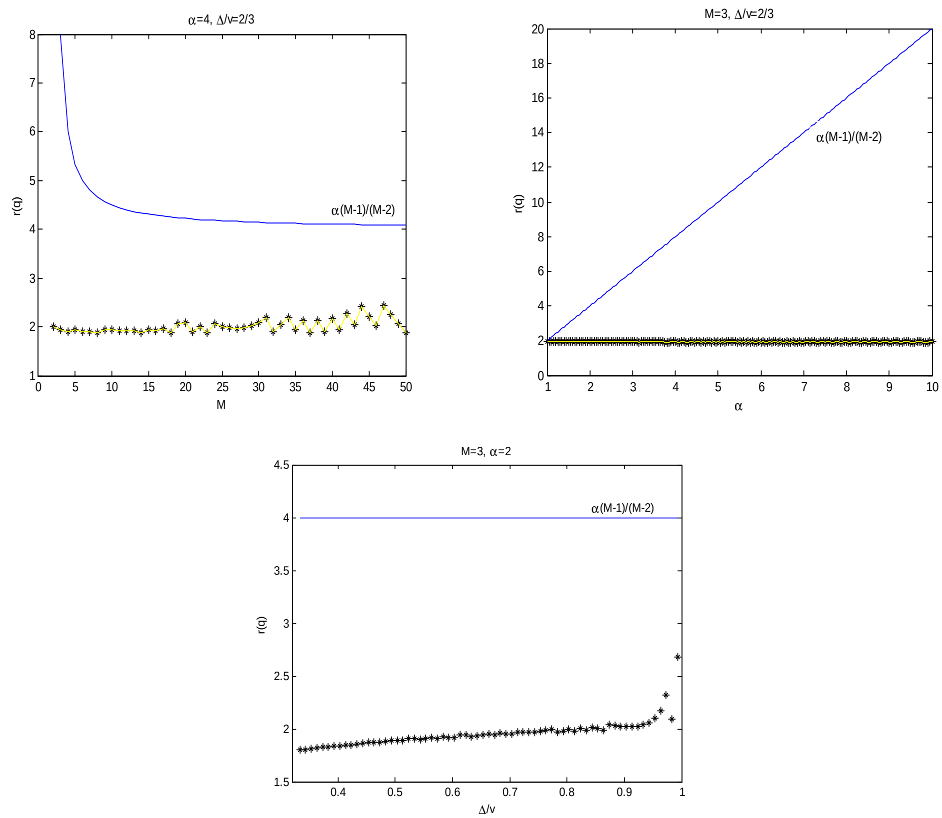

Figure 3

We observe incidents of $r(\widehat{q}) > \alpha(1+\frac{1}{M-2})$ as well. However, recall that $r \leq \alpha(1+\frac{1}{M-2})$ is a sufficient but not necessary condition for pure-strategy bidding. It should be noted that pure-strategy bidding can still be induced by $r \in (\alpha(1 + \frac{1}{M-2}), r_0]$, as evidenced by our simulation results. The numerical exercises in Section 2, which are conducted over a large parameter space, also apply to this contest design problem. These results show that all $r \leq r_0$ lead to pure-bidding equilibria. Hence, in all the simulated settings, we can elicit the "first best" by setting $r = r(\widehat{q})$, although it may exceeds $\alpha(1+\frac{1}{M-2})$. Based on these observations, we propose the following conjecture, which is implied by Conjecture 1.

Conjecture 2 *The first best overall effort $\overline{x_T}^*$ can always be induced in a unique symmetric equilibrium with pure-strategy bidding by setting r to $r(\widehat{q})$.*

3.1.3 Comparison to Benchmark Cases

We now compare our results to those in previous literature. We consider two benchmark settings.

Our results run in sharp contrast to the conventional wisdom in contest literature. In a contest with a fixed number M of participants, a higher r provides stronger incentives to bidders, and elicits strictly higher bids whenever a pure-strategy equilibrium prevails. Equilibrium overall bid is maximized when $r = \alpha(1+\frac{1}{M-1})$. The size of r in our setting, however, triggers substantially richer strategic trade-offs and affects the resultant equilibrium bid non-monotonically. The optimum must balance out various competing forces. The optimal size of the parameter could either fall above or below the benchmark $\alpha(1+\frac{1}{M-1})$. In the left panel of Figure 4, the sample demonstrates the incidents of optimal "soft" incentives, with $r(\widehat{q}) < \alpha(1+\frac{1}{M-1})$. In the right panel, the results illustrate the possibility of the opposite, where $r(\widehat{q}) \in (\alpha(1+\frac{1}{M-1}), \alpha(1+\frac{1}{M-2}))$.

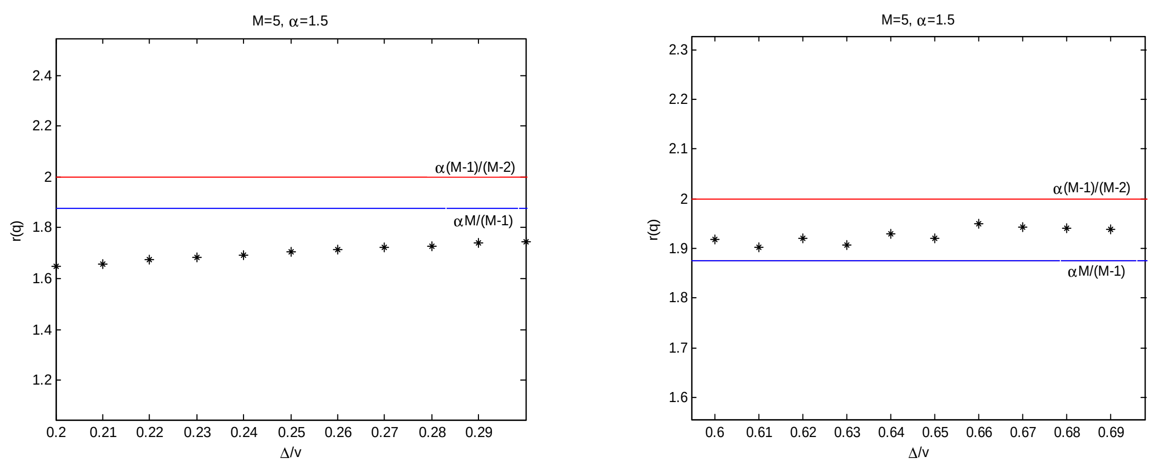

Figure 4

Our results can also contrast those of the related studies in auction literature. A number of studies have been devoted to the optimal design of auctions with costly entry or costly information acquisition. These studies, including Menezes and Monteiro (2000) and Lu (2009), Levin and Smith (1994), Shi (2009), Lu (2010) and Moreno and Wooders (2010), all espouse the efficiency effect of a "soft" incentive: the optimal reserve price is always lower than that in a free-entry setting. With free entry, the level of subsequent competition does not affect bidder's incentive to participate. In our setting, if entry does not involve fixed cost, all the M

potential bidders will participate. The conventional wisdom in contest literature would apply, such that $r = \alpha(1 + \frac{1}{M-1})$ would emerge as the optimum. As we discussed above, the contest does not necessarily requires a lower-powered incentive mechanism than the benchmark level under free entry, i.e. $\alpha(1 + \frac{1}{M-1})$.

3.2 Efficient Exclusion

The equilibrium analysis also allows us to investigate another classical question in the literature on contest design. With a fixed number n of bidders, a Tullock contest elicits an overall bid of $\alpha r \frac{n^2}{n-1}$ whenever a pure-strategy equilibrium exists. The amount of overall bid strictly increases with the number of bidders n. A handful of studies, including Baye, Kovenock and de Vries (1993), Taylor (1995), Fullerton and McAfee (1999), and Che and Gale (2003), demonstrate that a contest designer may benefit from narrowing the slate of potential prize winners and exclude a subset of contestants. This strand of literature conventionally focuses on heterogenous players and concerns themselves with selecting bidders of proper types. None of these studies involve stochastic and endogenous entry. In what follows, we demonstrate that exclusion can improve the efficiency of the contest in our setting despite that the potential bidders are symmetric.

Consider our basic setting where M potential bidders are interested in participating in the competition. We now allow the contest designer to invite only a subset of these bidders for participation. The invited bidders then decide whether to participate in the contest after they observe the rules of the contest, i.e. the size of r set by the contest designer.

Let M' be an arbitrary positive integer. Define $M_0 \triangleq \min(M' | \frac{v}{M'} < \Delta)$ and we assume $M_0 < M$. Recall that the amount of overall bid in a given contest is bounded from above by the first best $\overline{x_T}^*$. The first best overall bid is achieved when r is set to $r = r(\widehat{q})$, and $r(\widehat{q})$ leads to pure-strategy bidding. The first best $\overline{x_T}^*$ varies with M, the number of potential bidders. Let $\overline{x_T}^*(M')$ be the first best bid of a contest with M' potential bidders. We show that it exhibits the following property.

Lemma 6 *$\overline{x_T}^*$ strictly decreases with M' for all $M' \geq M_0$.*

Proof. See Appendix. ■

Lemma 6 shows that the first best $\overline{x_T}^*$ of a contest strictly declines if it involves a larger pool of potential bidders. Although inviting more bidders may engage more participants to contribute their bid, each of them would enter less often and bid less (if they enter) anticipating a more intense competition. Again, we allow the contest designer to set r strategically. Let $r(\widehat{q}(M'))$ be the unique solution to (13) in a contest with M' potential bidders. This result prompts us to conclude the following.

Theorem 5 *Whenever $r(\widehat{q}(M_0)) \leq \alpha(1 + \frac{1}{M_0-2})$, the contest designer would not invite more than M_0 contestants.*

Theorem 5 demonstrates that exclusion improves bidding efficiency. Whenever the condition $r(\widehat{q}(M_0); M_0) \leq \alpha(1 + \frac{1}{M_0-2})$ is met, the contest designer would get strictly better off by excluding $M - M_0$ potential bidders from the contest. By inviting M_0 of them, and setting r to $r(\widehat{q}(M_0))$, it elicits an overall bid $\overline{x_T}^*(M_0)$, which, by Lemma 6, is unambiguously more than what she can possibly achieves if she engages a greater number of potential bidders. Our result thus provides an alternative rationale for shortlisting and exclusion in a setting with homogeneous bidders but costly and endogenous entry.

Theorem 5 shows that the optimal number of invited bidders must not exceed M_0. It provides only an upper bound for the possible optimum; while it does not pin down exactly how many bidders should be invited in the optimum. When the contest designer invites less than M_0 potential bidders, the overall bid of the contest can elicit would change indefinitely, and the efficiency of the contest may either improve or suffer.[9]

The analysis for a contest with less than M_0 potential bidders is beyond the scope of the current paper, as Assumption 1 does not hold. The alternative context in fact renders an even more handy equilibrium analysis. Most of the analysis in the current setting would not lose its bite after slight alteration. Although it is not difficult to characterize the property of the equilibrium in the alterative setting, a general conclusion on the exact optimum M^* is difficult. First,the optimization problem requires comparison across integers. Second, the discontinuity in the optimization problem is further exacerbated when the number of invited bidders hypothetically drops further from M_0. Bidders behave qualitatively differently across the two contexts. The comparison depends sensitively on the specific settings of (v, Δ).

3.3 Optimality of Contest with Nondisclosure

Our analysis so far assumes that the actual participation rate N is unknown to bidders. In this section, we investigate whether allowing disclosure of number of actual contestants could improve the optimal contest design. Assuming that the actual participation is observable to the contest designer, we explore whether the designer can benefit from disclosing the realization of N to participating bidders before they place their bids.

Let the contest designer commit to her disclosure policy prior to the entry-bidding game. Upon learning the disclosure policy, bidders enter and bid. Denote by d the policy that commits to announcing the true realization of N to participating bidders and by c the policy that concealing the actual N. Participants learn N before they bid if and only if policy d

[9] Examples in specific settings are available from the authors upon request, which demonstrate that the overall bid may either decrease or increase.

is chosen. The actual number of participants N then becomes common knowledge upon its realization. Under policy c, our basic setting remains.

Under policy d, the analysis on the entry-bidding game is simplified substantially. It essentially reduces to a uni-dimensional game. Each contest after the entry stage is a proper subgame. The existence theorem of Dasgupta and Maskin (1986) for uni-dimensional discontinuous games allows us to verify the existence of equilibrium in every possible subgame.[10] We establish the existence of symmetric equilibria in this game.

Theorem 6 *For any given $r > 0$, there exist symmetric subgame perfect equilibria $(q_d^*, \{x_N^*, N = 1, 2, ..., M\})$ in the entry-bidding game. All potential bidders enter with a probability $q_d^* \in (0, 1)$, and play a (pure or mixed) bidding strategy x_N^* in each subgame with N entrants. Each potential bidder receives zero expected payoff in the entry bidding game.*

Proof. See Appendix. ■

We denote by (r, t) to denote a contest with a discriminatry parameter r and a disclosure policy t, with $t = c, d$. Further, we denote by $x_T^*(r, t)$ the expected overall bid in the contest (r, t).

Theorem 7 *Suppose that $r(\widehat{q})$ (as identified in Definition 5) can induce a symmetric equilibrium with pure-strategy bidding under policy c. A contest $(r(\widehat{q}), c)$ dominates any contest (r, d) regardless of r, i.e. $x_T^*(r(\widehat{q}), c) \geq x_T^*(r, d), \forall r \in (0, \infty)$.*

Proof. See Appendix. ■

Theorem 7 states that policy d (that discloses the number of participating bidders) would not elicit more efficient bidding when r can be set by the contest designer. The logic underlying Theorem 7, to a large extent, reflects the general argument of Myerson (1982). The amount of overall bid a contest can possibly elicit can never exceed the first best $\overline{x_T}^*$, regardless of the prevailing disclosure policy. Hence, when a contest $(r(\widehat{q}), c)$ can successfully achieve the first best, it must (at least weakly) dominate all other possible mechanisms.

Theorems 4 and 7 lead to following result.

Corollary 3 *When $r(\widehat{q}) \leq \alpha(1 + \frac{1}{M-2})$, a contest with $r = r(\widehat{q})$ and policy c must dominate all other contests, regardless of the size of r and the prevailing disclosure policy.*

[10] Under policy c, the theorem for uni-dimensional game does not apply as the bidding game involves an uncertain number of bidders.

4 Concluding Remarks

In this paper, we provide a thorough account of contests with endogenous and stochastic entries. We show the existence of a symmetric mixed-strategy equilibrium in which potential bidders randomly enter. We also provide a sufficient condition under which participants engage in pure bidding actions. We further apply these equilibrium results to exploring optimal institutional elements in contest rules, and we demonstrate that analysis in this setting adds substantially to the existing knowledge on contest design.

Our study is one of the first steps to investigate the subtle and rich strategic interaction in contests with endogenous entries. Our analysis, however, reveals the enormous possibilities for future studies. Due to the constraints in analytical capacities, the two open conjectures post a challenge to future research on contests, which will also be attempted by the authors, despite the technical difficulty.

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Appendix

Proof of Lemma 1

Proof. Let $f_1(q) = [1-(1-q)^M]v - Mq\Delta$, and $f_2(q) = (1-q)^{M-1}v - \Delta$. $\bar{q}$ (> 0) is defined as $f_1(\bar{q}) = 0$. The first order derivative of $f_1(q)$ is $f_1'(q) = Mf_2(q)$, which is a decreasing function of q. $f_1'(q)$ is positive when $q = 0$,and it is negative when $q = 1$.

q_0 is defined as $f_2(q_0) = 0$. Therefore, $f_1(q)$ increases on $[0, q_0]$, and decreases from $[q_0, 1)$. $f_1(q)$ thus has two zero points, i.e. $\{0, \bar{q}\}$, and $q_0 < \bar{q}$. ■

Proof of Theorem 1

Proof. Part (i) Existence of symmetric equilibria: Consider the following extended game. There are M contestants who simultaneously choose their two-dimensional actions, which are denoted by $a_i = (a_{i1}, a_{i2}) = (q_i, x_i) \in A$, $i = 1, 2, ..., M$, where the uniform action space $A = [0, 1] \times [0, v^{1/\alpha}]$ is nonempty, convex and compact.

Let $\mathbf{k} = (k_1, k_2, ..., k_i, ..., k_N)$ where k_i is either 0 or 1. Let K to be the set of all possible k. Similarly, we define $\mathbf{k}_{-i}$ and K_{-i}, $i = 1, 2, ..., M$.

Given action profile $\mathbf{a} = \{a_1, a_2, ..., a_M\}$ of the M players, the payoff of player i is defined as

$$U_i(\mathbf{a}) = q_i\{[\sum_{\mathbf{k}_{-i}\in K_{-i}} (\prod_{j\neq i} q_j^{k_j}(1-q_j)^{1-k_j}) \Pr(i|\mathbf{k}_{-i}, \mathbf{x})]v - x_i^\alpha - \Delta\}, i = 1, 2, ..., M,$$

where $\Pr(i|\mathbf{k}_{-i}, \mathbf{x}) = \frac{x_i^r}{x_i^r + \sum_{j\neq i} k_j x_j^r}$if $x_i^r + \sum_{j\neq i} k_j x_j^r > 0$, and $\Pr(i|\mathbf{k}_{-i}, \mathbf{x}) = \frac{1}{1+\sum_{j\neq i} k_j}$ if $x_i^r + \sum_{j\neq i} k_j x_j^r = 0$. Note that $\Pr(i|\mathbf{k}_{-i}, \mathbf{x})$ equals to the winning probability of an entrant i when the entry status of others is denoted by $\mathbf{k}_{-i}$ and players' effort is $\mathbf{x}$ if they enter.

Note that this game is a symmetric game as defined by Dasgupta and Maskin (1986) in their Definition 7. We will apply their Theorem 6* in Appendix to establish the existence of symmetric equilibrium in mixed strategy.

Following the notations on page 22 of Dasguspta and Maskin (1986). Let $Q = \{2\}, D(i) = 1$, and f_{ij}^1 to be an identity function. Following their $(A1)$ of page 22, we define $A^*(i) = \{\mathbf{a} \in A | \exists j \neq i, \exists k \in Q, \exists d, 1 \leq d \leq D(i)$ such that $a_{jk} = f_{ij}^d(a_{ik})\}$. The set of discontinuous point for $U_i(\mathbf{a})$ can be written as $A^{**}(i) = \{\mathbf{a} \in A | q_j x_j = 0, \forall j = 1, 2, ..., M; q_i > 0, x_i = 0; \exists j_0 \neq i$, such that $q_{j_0} > 0$ and $x_{j_0} = 0\}$. Clearly, $A^{**}(i) \subset A^*(i)$, since any element in $A^{**}(i)$ must satisfy the following conditions: For $k = 2 \in Q, \exists j_0 \neq i$, such that $x_{j_0} = f_{ij}^1(x_{ik})$, i.e. $a_{j_0 2} = f_{ij}^1(a_{i2})$. According to their Theorem 6*, we need to verify the following conditions hold.

First, as constructed above, $U_i(\mathbf{a})$ is continuous except on a subset $A^{**}(i)$ of $A^*(i)$, where $A^*(i)$ is defined by $(A1)$.

Second, clearly, we have $\sum_i U_i(\mathbf{a}) = v[1 - \prod_i (1 - q_i)] - \sum_i q_i(x_i^\alpha + \Delta)$, which is continuous and thus upper semi-continuous.

Third, $U_i(\mathbf{a})$ clearly is bounded on $A = [0, 1] \times [0, v^{1/\alpha}]$.

Fourth, we verify that Property (α^*) of page 24 is satisfied. Define B^2 as the unit circle with the origin as its center, i.e. $B^2 = \{\mathbf{e} = (q, x) \mid q^2 + x^2 = 1\}$. Pick up any continuous density function $v(\cdot)$ on B^2 such that $v(\mathbf{e}) = 0$ iff $e_1 \leq 0$ or $e_2 \leq 0$.Note that $U_i(a_i, \mathbf{a}_{-i})$ is continuous in a_{i1} and lower semi-continuous in a_{i2}. $\forall \mathbf{a} = (\bar{a}_i, \mathbf{a}_{-i}) \in A^{**}(i)$, clearly we have that for any $\mathbf{e}$ such that $v(\mathbf{e}) > 0$ (i.e. $min(e_1, e_2) > 0$), $\liminf_{\theta \to 0^+} U_i(\bar{a}_i + \theta \mathbf{e}, \mathbf{a}_{-i}) > U_i(\bar{a}_i, \mathbf{a}_{-i})$ as $\theta > 0, e_2 > 0$ and $q_i > 0, x_i = 0$ in $\bar{a}_i$. This leads to that $\int_{B^2} [\liminf_{\theta \to 0^+} U_i(\bar{a}_i + \theta \mathbf{e}, \mathbf{a}_{-i}) v(\mathbf{e}) d\mathbf{e}] > U_i(\bar{a}_i, \mathbf{a}_{-i}), \forall \bar{a}_i \in A_i^{**}(i), \mathbf{a}_{-i} \in A_{-i}^{**}(\bar{a}_i)$, where $A_i^{**}(i)$ is the collection of all $\bar{a}_i$ of player i that appear in $A^{**}(i)$, $A_{-i}^{**}(\bar{a}_i)$ is the collection of others' actions $\mathbf{a}_{-i}$ such that $\mathbf{a} = (\bar{a}_i, \mathbf{a}_{-i}) \in A^{**}(i)$. This confirms that Property (α^*) holds for the above game.

Thus according to Theorem 6* of Dasgupta and Maskin (1986), there exists a symmetric mixed strategy equilibrium. Without loss of generality, we use $\mu_1(q)$ to denote the equilibrium probability measure of action q, and use $\mu_2(x)$ to denote the equilibrium probability measure of action x.

Next we show that for any strategy profile of players $\{(\mu_{i1}(q_i), \mu_{i2}(x_i))\}$. The players' payoffs are same from strategy profile of players that is defined as $\{(E_{\mu_{i1}} q_i, \mu_{i2}(x_i))\}$.The expected utility of player i from profile $\{(\mu_{i1}(q_i), \mu_{i2}(x_i))\}$ is

$$\begin{aligned}
E_{\mathbf{a}}U_i(\mathbf{a}) &= E_{q_i}\{E_{\mathbf{q}_{-i}}E_{\mathbf{x}}[q_i \sum_{\mathbf{k}_{-i}\in K_{-i}} (\prod_{j\neq i} q_j^{k_j}(1-q_j)^{1-k_j}) \Pr(i|\mathbf{k}_{-i},\mathbf{x}))v - x_i^{\alpha} - \Delta]\} \\
&= E_{q_i}\{q_i E_{\mathbf{x}} E_{\mathbf{q}_{-i}}[\sum_{\mathbf{k}_{-i}\in K_{-i}} (\prod_{j\neq i} q_j^{k_j}(1-q_j)^{1-k_j}) \Pr(i|\mathbf{k}_{-i},\mathbf{x}))v - x_i^{\alpha} - \Delta]\} \\
&= E_{q_i}\{q_i E_{\mathbf{x}}[\sum_{\mathbf{k}_{-i}\in K_{-i}} (\prod_{j\neq i} (Eq_j)^{k_j}(1-Eq_j)^{1-k_j}) \Pr(i|\mathbf{k}_{-i},\mathbf{x}))v - x_i^{\alpha} - \Delta]\} \\
&= Eq_i \cdot E_{\mathbf{x}}[\sum_{\mathbf{k}_{-i}\in K_{-i}} (\prod_{j\neq i} (Eq_j)^{k_j}(1-Eq_j)^{1-k_j}) \Pr(i|\mathbf{k}_{-i},\mathbf{x}))v - x_i^{\alpha} - \Delta]\}, \forall i. \quad (14)
\end{aligned}$$

The above result means that given others take strategy $(E_{\mu_1}q, \mu_2(x))$, the same strategy is also the best strategy for player i. Otherwise, $(\mu_1(q), \mu_2(x))$ would not be the optimal strategy for player i when others take the same strategy $(\mu_1(q), \mu_2(x))$. Therefore, $(E_{\mu_1}q, \mu_2(x))$ is a *symmetric equilibrium* for the above game.

It is easy to see that $(q^*, \mu^*(x)) = (E_{\mu_1}q, \mu_2(x))$ is a symmetric equilibrium for our original game based on the way the new game is constructed. $U_i(\mathbf{a})$ equals player i's expected payoffs when he enters with probability q_i and exerts effort x_i when he enters, given that other bidder j enters with probability q_j and exerts effort x_j when he enters. This claim also holds when they adopt any other entry strategies with measure $\{\mu_{i1}(q), i = 1, 2, ..., M\}$ due to (14). According to (14), only the expected entry probabilities $\{E_{\mu_{i1}}q, i = 1, 2, ..., M\}$ count.

Note we must have $q^* = E_{\mu_1}q \in (0, 1)$. First, $q^* = E_{\mu_1}q = 0$ cannot be an entry equilibrium when $\Delta < v$ (Assumption 1). Second, $q^* = E_{\mu_1}q = 1$ cannot be an entry equilibrium when $\Delta > \frac{v}{M}$ (Assumption 1). The expected equilibrium payoff of players must be nonnegative. Thus we must have $(1-(1-E_{\mu_1}q)^M)v - M(E_{\mu_1}q)[\Delta + E_{\mu_2}x] \geq 0$. This leads to $(1-(1-E_{\mu_1}q)^M)v - M(E_{\mu_1}q)\Delta > 0$. Thus $q^* = E_{\mu_1}q < \bar{q}$ by Definition 1 and proof of Lemma 1.

Part (ii): The equilibrium payoff cannot be negative. When $q^* = E_{\mu_1}q \in (0, 1)$, we must have the equilibrium payoffs of player to be zero as otherwise it cannot be an equilibrium as the player would enter with probability 1 and earn a positive payoff. ■

Proof of Lemma 2

Proof. If a symmetric equilibrium with pure strategy bidding exists, according to the first order condition $\frac{d\pi_i(x_i)}{dx_i} = 0$ and the symmetry condition $x_i = x$.

x^* must solve

$$\sum_{N=1}^{M} C_{M-1}^{N-1} q^{N-1}(1-q)^{M-N} \frac{(N-1)rv}{N^2 x^*} - \alpha x^{*\alpha-1} = 0$$

which yields

$$x^* = [\sum_{N=1}^{M} C_{M-1}^{N-1} q^{N-1}(1-q)^{M-N} \frac{N-1}{N^2}\frac{rV}{\alpha}]^{\frac{1}{\alpha}}.$$

the equilibrium expected payoff is

$$\begin{aligned}\pi^*(x^*,q) &= \sum_{N=1}^{M} C_{M-1}^{N-1} q^{N-1}(1-q)^{M-N}\frac{v}{N} \\ &\quad -[\sum_{N=1}^{M} C_{M-1}^{N-1} q^{N-1}(1-q)^{M-N}\frac{N-1}{N^2}\frac{rv}{\alpha}] \\ &= \sum_{N=1}^{M} C_{M-1}^{N-1} q^{N-1}(1-q)^{M-N}\frac{v}{N}(1-\frac{N-1}{N}\frac{r}{\alpha}).\end{aligned}$$

By entering the contest and submit the bid x_i, a potential contestant i ends up with an expected payoff

$$u_i(x_i|x,q) = \pi_i(x_i|x,q) - \Delta.$$

By Theorem 1 (ii), each potential bidder receives an zero expected payoff in the equilibrium. Then $u^*(x^*,q^*) = \pi^*(x^*,q^*) - \Delta = 0$.

The expected overall effort of the contest (x_T^*) obtains as

$$\begin{aligned}x_T^* &= Mq^*x^* \\ &= Mq^*[\sum_{N=1}^{M} C_{M-1}^{N-1} q^{*N-1}(1-q^*)^{M-N}\frac{N-1}{N^2}\frac{rV}{\alpha}]^{\frac{1}{\alpha}}.\end{aligned}$$

■

Proof of Lemma 3

Proof. By Lemma 2, q^* satisfies $F(q^*,r) = \sum_{N=1}^{M} C_{M-1}^{N-1} q^{*N-1}(1-q^*)^{M-N}\frac{v}{N}(1-\frac{N-1}{N}\frac{r}{\alpha}) - \Delta = 0$. Apparently, $F(q^*,r)$ is continuous in and differentiable with both arguments. We first claim that $F(q^*,r)$ strictly decreases with q^*. Define $\pi_N = \frac{v}{N}(1-\frac{N-1}{N}\frac{r}{\alpha})$. Taking its first order derivative yields

$$\begin{aligned}\frac{F(q^*,r)}{dq^*} &= \sum_{N=1}^{M} C_{M-1}^{N-1}[(N-1)q^{*N-2}(1-q^*)^{M-N} - (M-N)q^{*N-1}(1-q^*)^{M-N-1}]\pi_N \\ &= \sum_{N=1}^{M} C_{M-1}^{N-1}(N-1)q^{*N-2}(1-q^*)^{M-N}\pi_N - \sum_{N=1}^{M} C_{M-1}^{N-1}(M-N)q^{*N-1}(1-q^*)^{M-N-1}\pi_N \\ &= (M-1)\{\sum_{N=2}^{M} C_{M-2}^{N-2} q^{*N-2}(1-q^*)^{M-N}\pi_N - \sum_{N=1}^{M-1} C_{M-2}^{N-1} q^{*N-1}(1-q^*)^{M-N-1}\pi_N\} \\ &= (M-1)\sum_{N=1}^{M-1} C_{M-2}^{N-1} q^{*N-1}(1-q^*)^{M-N-1}\left(\pi_{N+1}-\pi_N\right),\end{aligned}$$

which is obviously negative because $\pi_N = \frac{1}{N}\left[1-\left(1-\frac{1}{N}\right)\frac{r}{\alpha}\right]v \geq 0$ and it monotonically decreases with N.

When all other potential contestants play $q=0$, a potential contestant receives a payoff $v-\Delta>0$, and he must enter with probability one. When all others play $q=\overline{q}$, a participating contestant receives negative expected payoff if he enters by Definition 1 and Lemma 1($(1-\overline{q})^{M-1}v<\Delta$), which cannot constitute an equilibrium either. Hence, a unique $q^* \in (0,\overline{q})$ must exist that solves $\pi^*(x^*,q)=\Delta$. Each potential contestant is indifferent between entering and staying inactive when all others play the strategy. This constitutes an equilibrium.

Moreover, $F(q^*,r)$ strictly decreases with r, since it also strictly decreases with q^*. The lemma is then verified. ■

Proof of Lemma 4

Proof. Denote $k_i = x_i^\alpha$, $k^* = x^{*\alpha}$, $t=\frac{r}{\alpha}\in(0,\frac{M-1}{M-2}]$, then $\tilde{\pi}_i(x_i)$ can be rewritten as

$$\tilde{\pi}_i(k_i) = \sum_{N=1}^{M} C_{M-1}^{N-1} q^{*N-1}(1-q^*)^{M-N}\frac{k_i^t}{k_i^t+(N-1)k^{*t}}v - k_i,$$

Evaluating π_i with respect to k_i yields

$$\frac{d\pi_i}{dk_i} = \sum_{N=1}^{M} C_{M-1}^{N-1} q^{*N-1}(1-q^*)^{M-N}\frac{(N-1)tk_i^{t-1}k^{*t}v}{[k_i^t+(N-1)k^{*t}]^2} - 1.$$

Note

$$k^* = \sum_{N=1}^{M} C_{M-1}^{N-1} q^{*N-1}(1-q^*)^{M-N}\frac{N-1}{N^2}tv.$$

To verify that k^* is the global maximizer of $\pi_i(k_i;q^*)$ given that all other participants exert the same effort. Define $p_i(k_i,\mathbf{k}_{-i};N) = \frac{k_i^t}{k_i^t+(N-1)k^{*t}}$. One can verify $\xi_N(k_i) = \left.\frac{\partial^2 p_i(k_i,\mathbf{k}_{-i};N)}{\partial k_i^2}\right|_{k_{-i}=k^*} = \frac{-(t+1)k_i^t+(t-1)(N-1)k^{*t}}{[k_i^t+(N-1)k^{*t}]^3}tk_i^{t-2}(N-1)k^{*t}$. It implies that $\Phi_N(k_i) = \left.\frac{\partial p_i(k_i,\mathbf{k}_{-i};N)}{\partial k_i}\right|_{k_{-i}=k^*}$ is not monotonic: It is positive if $k_i^t < \frac{t-1}{t+1}(N-1)k^{*t}$, and negative if $k_i^t > \frac{t-1}{t+1}(N-1)k^{*t}$. Clearly $\frac{t-1}{t+1}(N-1)\leq 1$ if and only if $t\leq\frac{N}{N-2}$. Because $t\leq 1+\frac{1}{M-2}$, we must have $\frac{t-1}{t+1}(N-1)<1$ for all $N\leq M$.

Let $\Phi(k_i) = \sum_{N=1}^{M} C_{M-1}^{N-1} q^{*N-1}(1-q^*)^{M-N}\frac{\partial p_i(k_i,\mathbf{k}_{-i};N)}{\partial k_i}|_{k_{-i}=k^*}$, and $\xi(k_i) = \sum_{N=1}^{M} C_{M-1}^{N-1} q^{*N-1}(1-q^*)^{M-N}\frac{\partial^2 p_i(k_i,\mathbf{k}_{-i};N)}{\partial k_i^2}|_{k_{-i}=k^*}$. The above results imply that $k_i^t > \frac{t-1}{t+1}(N-1)k^{*t}$ when $k_i = k^*$ for all $N\leq M$, which means that $\xi(k_i)|_{k_i=k^*}<0$. This leads to that $\left.\frac{d^2\tilde{\pi}_i(k_i)}{dk_i^2}\right|_{k_i=\mathbf{k}_{-i}=k^*} = v\,\xi(k_i)|_{k_i=k^*}<0$. Hence, $k_i=k^*$ must be at least a local maximizer of when $k_{-i}=k^*$.

Since when $k_i < [\frac{t-1}{t+1}]^{1/t}k^*$, $\xi_N(k_i)>0$ for all $N\leq M$, we have $\xi(k_i)>0$ when $k_i<[\frac{t-1}{t+1}]^{1/t}k^*$, which means that $\Phi(k_i)$ increases when $k_i<[\frac{t-1}{t+1}]^{1/t}k^*$. Similarly, $\xi(k_i)<0$ when

$k_i > [\frac{t-1}{t+1}(M-1)]^{1/t}k^*$, which means that $\Phi(k_i)$ decreases when $k_i > [\frac{t-1}{t+1}(M-1)]^{1/t}k^*$. We next show that there exists a unique $k' \in ([\frac{t-1}{t+1}]^{1/t}k^*, [\frac{t-1}{t+1}(M-1)]^{1/t}k^*)$ such that $\Phi(k_i)$ increases (decreases) if and only if $k_i < (>)\ k'$. For this purpose, it suffices to show that there exists a unique $k' \in ([\frac{t-1}{t+1}]^{1/t}k^*, [\frac{t-1}{t+1}(M-1)]^{1/t}k^*)$, such that $\xi(k') = 0$.

First, such k' must exist by continuity of $\xi(k_i)$. As have been revealed, $\xi(k_i) > 0$ when $k_i < [\frac{t-1}{t+1}]^{1/t}k^*$; and $\xi(k_i) < 0$ when $k_i > [\frac{t-1}{t+1}(M-1)]^{1/t}k^*$.

Second, the uniqueness of k' can be verified as below. We have

$$\begin{aligned} & \left.\frac{\partial^3 p_i(k_i, \mathbf{k}_{-i}; N)}{\partial k_i^3}\right|_{k_{-i}=k^*} \\ = \;& t(N-1)k^{*t}\left\{\begin{array}{c} (t-2)k_i^{t-3}\frac{-(t+1)k_i^t+(t-1)(N-1)k^{*t}}{[k_i^t+(N-1)k^{*t}]^3} \\ +k_i^{t-2}\frac{-t(t+1)k_i^{t-1}[k_i^t+(N-1)k^{*t}]-3tk_i^{t-1}[-(t+1)k_i^t+(t-1)(N-1)k^{*t}]}{[k_i^t+(N-1)k^{*t}]^4} \end{array}\right\} \\ = \;& \frac{t(N-1)k^{*t}k_i^{t-3}}{[k_i^t+(N-1)k^{*t}]^3}\left\{\begin{array}{c} (t-2)[-(t+1)k_i^t+(t-1)(N-1)k^{*t}] \\ +\frac{-t(t+1)k_i^t[k_i^t+(N-1)k^{*t}]-3tk_i^t[-(t+1)k_i^t+(t-1)(N-1)k^{*t}]}{[k_i^t+(N-1)k^{*t}]} \end{array}\right\} \\ = \;& \frac{t(N-1)k^{*t}k_i^{t-3}}{[k_i^t+(N-1)k^{*t}]^3}\left\{\begin{array}{c} (t-2)[-(t+1)k_i^t+(t-1)(N-1)k^{*t}] \\ +\frac{2tk_i^t}{[k_i^t+(N-1)k^{*t}]}[(t+1)k_i^t-(2t-1)(N-1)k^{*t}] \end{array}\right\}. \end{aligned}$$

Recall $\xi_N(k_i) = \frac{-(t+1)k_i^t+(t-1)(N-1)k^{*t}}{[k_i^t+(N-1)k^{*t}]^3}tk_i^{t-2}(N-1)k^{*t}$. We then have

$$\begin{aligned} & \left.\frac{\partial^3 p_i(k_i, \mathbf{k}_{-i}; N)}{\partial k_i^3}\right|_{k_{-i}=k^*} \\ = \;& (t-2)k_i^{-1}\xi_N(k_i) \\ & +\frac{2t^2(N-1)k^{*t}k_i^{2t-3}}{[k_i^t+(N-1)k^{*t}]^4}[(t+1)k_i^t-(2t-1)(N-1)k^{*t}]. \end{aligned}$$

We now claim $[(t+1)k_i^t - (2t-1)(N-1)k^{*t}]$ is negative for all $k_i \le [\frac{t-1}{t+1}(M-1)]^{1/t}k^*$. A detailed proof is as follows. From $k_i \le [\frac{t-1}{t+1}(M-1)]^{1/t}k^*$, we have $(t+1)k_i^t \le (t-1)(M-1)k^{*t}$. To show $(t+1)k_i^t-(2t-1)(N-1)k^{*t} < 0$, it suffices to show $(t-1)(M-1) < (2t-1)(N-1)$ when $N = 2$, which requires $t < 1 + \frac{1}{M-3}$. This holds as $t \le 1 + \frac{1}{M-2}$.

We thus have at any $k_i \in ([\frac{t-1}{t+1}]^{1/t}k^*, [\frac{t-1}{t+1}(M-1)]^{1/t}k^*)$ such that $\xi(k_i) = 0$, $\xi(k_i)$ must be locally decreasing, because $\frac{\partial \xi(k_i)}{\partial k_i} = (t-2)k_i^{-1}\sum_{N=1}^{M} C_{M-1}^{N-1}q^{*N-1}(1-q^*)^{M-N}\xi_N(k_i)+\sum_{N=1}^{M} C_{M-1}^{N-1}q^{*N-1}(1-q^*)^{M-N}A_N(k_i) = (t-2)k_i^{-1}\xi(k_i) + \sum_{N=1}^{M} C_{M-1}^{N-1}q^{*N-1}(1-q^*)^{M-N}A_N(k_i) = \sum_{N=1}^{M} C_{M-1}^{N-1}q^{*N-1}(1-q^*)^{M-N}A_N(k_i) < 0$ as $A_N(k_i) = \frac{2t^2(N-1)k^{*t}k_i^{2t-3}}{[k_i^t+(N-1)k^{*t}]^4}[(t+1)k_i^t-(2t-1)(N-1)k^{*t}] < 0$.

We are ready to show the uniqueness of k' by contradiction. Suppose that there exists more than one zero points k' and k'' with $k' \ne k''$ for $\xi(k_i)$. Because $\xi(k_i)$ must be locally decreasing, then there must exist at least another zero point $k''' \in (k', k'')$ at which $\xi(k_i)$ is locally increasing. Contradiction thus results. Hence, such a zero point k' of $\xi(k_i)$ must be unique.

Recall $\Phi(k_i)$ increases (decreases) if and only if $k_i < (>)\ k'$ and it reaches its maximum at k'. Note $\frac{\partial \tilde{\pi}_i(k_i)}{\partial k_i} = v\Phi(k_i) - 1$ and $\Phi(0) = 0$. Therefore $\frac{\partial \tilde{\pi}_i(k_i)}{\partial k_i}|_{k_i=0} < 0$. Thus $\frac{\partial \tilde{\pi}_i(k_i)}{\partial k_i}$ has exactly two zero points with the smaller one (k_s) being the local minimum point of $\tilde{\pi}_i(k_i)$. Note $k_i = k^*$ must be a zero point for $\frac{\partial \tilde{\pi}_i(k_i)}{\partial k_i}$ by definition. Since $k_i = k^*$ is a local maximum point of $\tilde{\pi}_i(k_i)$, it is higher than other zero point (k_s) of $\frac{\partial \tilde{\pi}_i(k_i)}{\partial k_i}$ which is a local minimum point of $\tilde{\pi}_i(k_i)$.

Note $x_m = (k_s)^{1/\alpha}$ is the unique local minimum of $\tilde{\pi}_i(x_i)$, and note $x^* = (k^*)^{1/\alpha}$ is the unique inner local maximum of $\tilde{\pi}_i(x_i)$. Note $x_m < x^*$. The results of Lemma 4 are shown. ■

Proof of Lemma 5

Proof. Define an increasing transformation of $\overline{x_T}(q)$:

$$\Psi(q) = [\overline{x_T}(q)]^\alpha = (Mq)^{\alpha-1}\left\{[1-(1-q)^M]v - Mq\Delta\right\}$$

Note that $\Psi(q)|_{q=0} = 0$;and $\Psi(q)|_{q=1} = M^{\alpha-1}(v - M\Delta) < 0$ since $\frac{v}{M} < \Delta$ (Assumption 1). We have

$$\frac{d\Psi(q)}{dq} = f(q)\, q^{\alpha-2} M^{\alpha-1},$$

where

$$f(q) = (\alpha-1)\underbrace{\left\{[1-(1-q)^M]v - Mq\Delta\right\}}_{f_1(q)} + Mq\underbrace{[(1-q)^{M-1}v - \Delta]}_{f_2(q)}.$$

We have

$$f'(q) = Mv(1-q)^{M-2}[\alpha - (M+\alpha-1)q] - \alpha M\Delta.$$

Note that $f'(0) = \alpha Mv - \alpha M\Delta > 0$, $f'(1) = -\alpha M\Delta < 0$ and $f'(q)$ decreases with $q \in (0, \frac{\alpha}{M+\alpha-1}]$. Clearly, $f'(q) < 0$ when $q \in [\frac{\alpha}{M+\alpha-1}, 1]$. Then there exists a unique $q_c \in (0, \frac{\alpha}{M+\alpha-1})$, such that $f'(q_c) = 0$. Which means q_c is the maximum point of $f(q)$. Since $f(0) = 0$, $f(q_c) > 0$ and $f(1) = (\alpha-1)v - \alpha M\Delta = \alpha(v - M\Delta) - v < 0$, then there must exist a unique $\widehat{q} \in (q_c, 1)$, such that $f(\widehat{q}) = 0$. Note that $f'(q) < 0$ on $(q_c, 1)$. Clearly, $f(q) > 0$ when $0 < q < \widehat{q}$;and $f(q) < 0$ when $\widehat{q} < q < 1$.

Since $\frac{d\Psi(q)}{dq}$ shares the same sign with $f(q)$, we have that $\frac{d\Psi(q)}{dq} > 0$ when $0 < q < \widehat{q}$; and $\frac{d\Psi(q)}{dq} < 0$ when $\widehat{q} < q < 1$. This implies $\widehat{q} = \arg\max_q \Psi(q)$, i.e. $\widehat{q} = \arg\max_q \overline{x_T}(q)$.

By the proof of Lemma 1, we know both $f_1(q)$ and $f_2(q)$ are positive when $q \in [0, q_0]$ and both are negative when $q > \bar{q}$. Thus the zero point ($\widehat{q}$) of $f(q)$ must falls in $[q_0, \bar{q}]$. ■

Proof of Theorem 4

Proof. Proof of Lemma 3 has shown that $F(q,r) = \sum_{N=1}^{M} C_{M-1}^{N-1} q^{N-1}(1-q)^{M-N}\frac{v}{N}(1-\frac{N-1}{N}\frac{r}{\alpha})-\Delta$ decreases with both q and r. Thus $F(q,r)=0$ uniquely defines r as a decreasing function of q. Since $F(q_0,r_0)=0$ and $\hat{q}>q_0$, we must have $r(\hat{q})<r_0$. Theorem 3 thus means that contest $r(\hat{q})$ would induce entry equilibrium $\hat{q}$ and pure-strategy bidding whenever $r(\hat{q}) \leq \alpha(1+\frac{1}{M-2})$. Since we have a pure-strategy bidding, an overall effort of $\overline{x_T}(\hat{q})$ clearly is induced at the equilibrium.

Consider any other $r \neq r(\hat{q})$. If r induces equilibrium entry $q(r)$ and pure-strategy bidding, then the total effort induced is $\overline{x_T}(q(r))$. Note that by Lemma 3, equilibrium $q(r)$ decreases with r. Thus $r \neq r(\hat{q})$ means $q(r) \neq \hat{q}$. $\overline{x_T}(q)$ is single peaked at $\hat{q}$ according to Lemma 5. Thus for any $r \neq r(\hat{q})$, we must have $\overline{x_T}(q(r)) < \overline{x_T}(\hat{q})$. If r induces equilibrium entry $q(r)$ and mixed-strategy bidding, then the total expected effort induced is strictly lower than $\overline{x_T}(q(r))$ when $\alpha>1$, based on the arguments deriving this boundary in Section 3.1. There the total effort induced must be strictly lower than $\overline{x_T}(\hat{q})$. ■

Proof of Lemma 6

Proof. By definition $\overline{x_T}^*\left(M'\right) = \overline{x_T}(\hat{q}(M');M')$.

By Envelope Theorem, $\frac{d\overline{x_T}(\hat{q}(M');M')}{dM'} = \frac{\partial \overline{x_T}(q;M')}{\partial M'}|_{q=\hat{q}(M')}$. Further,

$$
\begin{aligned}
& \frac{\partial \overline{x_T}(q;M')}{\partial M'}|_{q=\hat{q}(M')}. \\
= \ & \partial\left[(M'\widehat{q}(M'))^{\frac{\alpha-1}{\alpha}}\left\{[1-(1-\widehat{q}(M'))^{M'}]v - M'\widehat{q}(M')\Delta\right\}^{\frac{1}{\alpha}}\right]/\partial M' \\
= \ & \frac{\alpha-1}{\alpha}M'^{-\frac{1}{\alpha}}\left[\widehat{q}(M')\right]^{\frac{\alpha-1}{\alpha}}\left\{[1-(1-\widehat{q}(M'))^{M'}]v - M'\widehat{q}(M')\Delta\right\}^{\frac{1}{\alpha}} \\
& +\frac{1}{\alpha}(M'\widehat{q}(M'))^{\frac{\alpha-1}{\alpha}}\left\{[1-(1-\widehat{q}(M'))^{M'}]v - M'\widehat{q}(M')\Delta\right\}^{\frac{1}{\alpha}-1} \\
& \times[-(1-\widehat{q}(M'))^{M'}v\ln(1-\widehat{q}(M')) - \widehat{q}(M')\Delta]
\end{aligned}
$$

which has the same sign as

$$\lambda = (\alpha-1)\left\{[1-(1-\widehat{q}(M'))^{M'}]v - M'\widehat{q}(M')\Delta\right\} + M'[-(1-\widehat{q}(M'))^{M'}v\ln(1-\widehat{q}(M')) - \widehat{q}(M')\Delta].$$

Because $-\ln(1-\widehat{q}(M')) < \frac{\widehat{q}(M')}{1-\widehat{q}(M')}$, we have $M'[-(1-\widehat{q}(M'))^{M'}v\ln(1-\widehat{q}(M'))-\widehat{q}(M')\Delta] < \widehat{q}(M')[M'(1-\widehat{q}(M'))^{M'-1}v - M'\Delta]$. Hence, $\lambda < (\alpha-1)\left\{[1-(1-\widehat{q}(M'))^{M'}]v - M'\widehat{q}(M')\Delta\right\} + \widehat{q}(M')[M'(1-\widehat{q}(M'))^{M'-1}v - M'\Delta] = 0$ (by the definition of $\widehat{q}(M')$). We then have $\frac{d\overline{x_T}(\hat{q}(M');M')}{dM'} < 0$. ■

Proof of Theorem 6

Proof. We first show the following claim for a subgame with N players.

Claim: *For $N \leq M$ such that $\frac{N}{N-1} < \frac{r}{\alpha}$, there exists a symmetric mixed strategy equilibrium for the $N-$player subgame. The equilibrium payoff of a player π_N^d falls in $[0, \frac{v}{N})$.*

The proof of this claim replies on Theorem 6 of Dasgupta and Maskin (1986). The application of their Theorem 6 requires four conditions as has been pointed out by Baye et al (1994) who have shown the existence of a symmetric mixed-strategy equilibrium when $N = 2$ and effort costs are linear. However, when effort costs are nonlinear and $N > 2$, the proof is almost identical. Condition (i) requires that the discontinuity set S_i of player i's payoff is confined to a subset of a continuous manifold of dimension less than N. Let this manifold be defined as $A^*(i) = \{\mathbf{x}|x_1 = x_2 = ... = x_N\}$, which has a zero measure. The only discontinuity point of player i's payoff is $(0, 0, ..., 0) \in A^*(i)$. Thus condition (i) holds. Condition (ii) of this theorem requires that the sum of players' payoffs must be upper semi-continuous. From (2), we have that this sum is $v - \sum_i x_i^\alpha$, which is continuous and therefore upper semi-continuous. Condition (iii) requires that player i's payoff is bounded. This clearly holds as it falls in $[-v, v]$ when $x_i \in [0, v^{1/\alpha}]$. Note that a player never bids higher than $v^{1/\alpha}$. Condition (iv) requires that player i's payoff must be weakly lower semi-continuous. The only point one needs to check is the discontinuity point $(0, 0, ..., 0)$. At this point, player i's payoff is lower semi-continuous, and thus is weakly lower semi-continuous. Since all four conditions required are satisfied. The existence of a symmetric mixed-strategy equilibrium is guaranteed by Theorem 6 in Dasgupta and Maskin (1986).

In a symmetric equilibrium, every contestant wins the prize v with the same probability, and they incur positive effort costs.[11] Therefore, the equilibrium payoff must be lower than $\frac{v}{N}$.

We now introduce the definition of a symmetric entry equilibrium. Entry probability $q_d^* \in [0, 1]$ constitutes a symmetric entry equilibrium if and only if

$$\begin{aligned} \sum_{N=1}^{M} C_{M-1}^{N-1} q_d^{*N-1} (1 - q_d^*)^{M-N} \pi_N^d &= \Delta, \text{if } q_d^* \in (0, 1), \\ \pi_M^d &\geq \Delta, \text{ if } q_d^* = 1, \\ \pi_1^d &= v < \Delta, \text{ if } q_d^* = 0. \end{aligned}$$

We now are ready to show a symmetric entry equilibrium exists which must fall into $(0, 1)$.

Note that with Assumption 1, both $q_d^* = 1$ and $q_d^* = 0$ cannot be an entry equilibrium. The existence of symmetric entry equilibria depends on the existence of the solution of $\sum_{N=1}^{M} C_{M-1}^{N-1} q_d^{*N-1} (1 - q_d^*)^{M-N} \pi_N^d = \Delta$. Note the left hand side is continuous in q_d^*. When

[11] Clearly, exerting a zero effort is not an equilibrium.

$q_d^* = 0$, it is lower than the right hand side. When $q_d^* = 1$, it is higher than the right hand side. Therefore, there must exist $q_d^* \in (0, 1)$ such that $\sum_{N=1}^{M} C_{M-1}^{N-1} q_d^{*N-1}(1-q_d^*)^{M-N}\pi_N^d = \Delta$. ■

Proof of Theorem 7

Proof. First note that at any symmetric equilibrium when the number of bidders is disclosed, every bidder enjoys zero payoff. Therefore, we have $[1-(1-q_d^*)^M]v = Mq_d^*\{\Delta + E_N E[(x_N)^\alpha]\}$, i.e. $E_N E[(x_N)^\alpha] = [Mq_d^*]^{-1}[1-(1-q_d^*)^M]v - \Delta$, where x_N denotes the equilibrium individual effort in a subgame with N contestants. The expected total effort at the equilibrium is Mq_d^* $E_N[E(x_N)] = Mq_d^*\ E_N E\{[(x_N)^\alpha]^{1/\alpha}\} \leq Mq_d^*\ E_N\{E[(x_N)^\alpha]\}^{1/\alpha} \leq Mq_d^*\ \{E_N E[(x_N)^\alpha]\}^{1/\alpha} = [Mq_d^*]^{\frac{\alpha-1}{\alpha}} \times \{[1-(1-q_d^*)^M]v - Mq_d^*\Delta\}^{\frac{1}{\alpha}}$ as $\alpha \geq 1$. Note that the last expression is identical to the right hand side of (11). When $r(\widehat{q})$ induces entry $\widehat{q}$ and pure-strategy bidding while the number of bidders is concealed, the maximum of $[Mq_d^*]^{\frac{\alpha-1}{\alpha}} \times \{[1-(1-q_d^*)^M]v - Mq_d^*\Delta\}^{\frac{1}{\alpha}}$ is achieved with concealment policy. Therefore, any contest with number of bidders being disclosed is dominated by a contest $r(\widehat{q})$ with the number of bidders being concealed. ■