Large common value auctions with risk-averse bidders

Pauli Murto* Juuso Välimäki†

November 2014

Abstract

We analyze large symmetric auctions with conditionally i.i.d common values and risk averse bidders. Our main result characterizes the asymptotic equilibrium price distribution for the first- and second-price auctions. As an implication, we show that with constant absolute risk aversion (CARA), the second-price auction raises significantly more revenue than the first-price auction. While this ranking seems robust in numerical analysis also outside the CARA specification, we show by counterexamples that the result does not generalize to all risk averse utility functions.

1 Introduction

In common value auctions, winning conveys additional information of the other bidders's private signals. The exact information that the winner obtains depends on the auction format. At least since Milgrom & Weber (1982), the importance of these considerations has been recognized for the expected revenue. In this paper, we consider large common value auctions where a single object is sold to a large number of risk-averse bidders. It is known that with risk-neural bidders, first-price and second-price

*Department of Economics, Aalto University, and HECER pauli.murto@aalto.fi.
†Department of Economics, Aalto University, and HECER juuso.valimaki@aalto.fi.

auctions generate asymptotically the same expected revenue to the seller (Bali & Jackson (2002) and Kremer (2002)). We show here that such an asymptotic revenue equivalence result does not hold with risk-averse bidders.

Our main result is the characterization of the asymptotic probability distribution for the equilibrium price in the first- and second-price auction formats. We show that when the utility function takes the constant absolute risk aversion (CARA) form, the second-price auction generates significantly more revenue in expectation.

Our analysis relies on two observations applicable in the limit when the number of bidders increases towards infinity. First, we observe that the equilibrium inference reduces to a Bayesian learning problem about the parameter of an exponential distribution. To understand this, note that the bidding strategies in the symmetric Bayesian Nash Equilibrium of each of the two auction formats are strictly increasing, and therefore the realized price is a function of the highest signal within the bidder population in the first-price auction, and of the second highest signal in the second-price auction. In the limit with a large number of bidders, the information content of those order statistics is equivalent to information in exponential random variables with an unknown mean, and hence the bidders learn about the value of the object as if they observed such random variables. The true value of the object for sale remains uncertain as long as the most favorable signal is not perfectly informative. We have earlier used this same observation to compute equilibrium timing decisions in a social learning problem in Murto & Välimäki (2013).

Second, we show that in auctions with a large number of symmetric bidders the winner must be indifferent between winning and not winning. This requirement arises from an arbitrage condition: large auctions have many agents with similar information and therefore competition for the scarce resource drives out rents. This implies that the expected utility of buying the object at the equilibrium price, conditional on the information conveyed by the price, must be equal to the utility of not getting the object.

Combining these two observations, we can summarize our main finding as follows. The equilibrium price can be written as $p(Z)$, where Z is a random variable whose distribution depends on the auction format, and function $p(\cdot)$ gives the willingness to pay, i.e. price for which a buyer is indifferent between buying the object and not,

conditional on information inferred by the particular realization of Z. In the first-price auction, Z is an exponentially distributed random variable with unknown mean, while in the second price auction it is the sum of two conditionally i.i.d. exponential variables (with the same unknown mean). Since the unknown parameter depends on the true value of the object, the realization of Z leads to a new posterior by Bayes' rule. Being a deterministic function of Z, $p(Z)$ is itself a random variable whose distribution can be derived from that of Z. It is worth emphasizing that even conditional on the true value of the object, the equilibrium price remains random. This can be traced to our assumption that individual signals have bounded precision so that we are in an environment where full information aggregation in the large game limit is not possible, in contrast to e.g. Wilson (1977).

Our revenue ranking result stems from the difference in the informativeness of equilibrium price across the auction formats. Even though the interim expected payoff of each individual bidder converges to zero in both formats due to competitive bidding, the distribution of ex post gains matters as long as the bidders are risk-averse. More information leaves less uncertainty about ex-post gains, and since equilibrium price is more informative in the second-price auction than in the first-price auction, a CARA bidder is willing to pay more on average in that auction format. This implies a higher ex-ante expected revenue in the second-price auction in comparison to the revenue obtained in the first-price auction. We demonstrate numerically that this difference is significant and initially increasing in the degree of risk aversion. For extremely risk averse bidders, the willingness to pay converges to the lowest possible value of the object and hence the revenue difference between the auction formats vanishes. Although our statistical analysis is in the limit with a large number of bidders, the equilibrium strategies are defined for any finite auction, and therefore a straightforward continuity argument suffices to note that the ranking holds for a sufficiently large finite auction.

Based on the intuition about the informativeness of the equilibrium price, it seems natural to conjecture that our revenue ranking result should generalize beyond CARA preferences. However, as already shown in a different context by Milgrom & Weber (1982), an unambiguous relationship between average willingness to pay for an object and informativeness of a signal about its value exists only if utility is CARA. It is

therefore perhaps not surprising that we have not been able to formally extend the result to another class of utility functions. However, our numerical investigations seem to support the revenue ranking suggested by the CARA case. For reasonable parameter values, we show that the revenue ranking in the constant relative risk aversion (CRRA) remains qualitatively similarly to the CARA case. At the same time, we show by carefully specified examples that the revenue ranking result can indeed be reversed even in the CRRA case.

It is tempting to extend our analysis to more general auction formats. Within the common values model that we analyze, a problem arises in the first step of the analysis. It is by no means clear that formats such as all-pay auction have a monotonic pure-strategy equilibrium with a finite number of bidders. Hence our technique that relies on the extreme order statistics in a growing sample is no longer applicable in that case. The key requirement for generalizing the insights from the current paper is therefore that the finite version of the game should have a symmetric monotonic equilibrium in pure strategies.

This paper is related to the literature on information aggregation in large auctions. Wilson (1977) demonstrated how equilibrium price in a large auction can converge exactly to the true value of the object. However, as shown formally by Milgrom (1979), that result holds only under a special condition of the signal structure. This condition in essence requires that for any value realization there is a signal that can distinguish that value from all lower values. Our model does not allow any individual signal to be that informative. Like e.g. Pesendorfer & Swinkels (1997), we maintain that the likelihood ratio for any signal across different values is bounded, and this precludes full information aggregation in the auction formats that we study. Kremer (2002) works with the same statistical setup as we do with risk-neutral bidders. While he stresses the fact that equilibrium price distribution follows from inference based on order statistics in large samples, there is no explicit characterization of the limiting distribution. Using a simple result on extreme order statistics in large samples, we find an explicit characterization for the posterior belief on the value of the object. At the same time, we extend the analysis to risk-averse buyers, which makes the question of revenue comparison interesting in the first place; with risk neutral buyers expected revenue is the same in both auction formats.

Our revenue ranking result is quite different from existing papers that analyze the effect of risk aversion on expected revenue. It is useful to distinguish those results that are in the context of small auctions from asymptotic results such as ours. In the small auction context it has been shown that risk-aversion tends to favor first-price auction over second-price auction since the expected payment is less risky in the former (see e.g. Holt (1980), Matthews (1987), and Maskin & Riley (1984)). However, these results are in private values environments, and hence due to a quite different mechanism to ours. With private values, increasing the number of bidders leads to the convergence of the price to the maximal valuation in the bidder population, and as a result, the revenue difference between first- and second-price auctions disappears in the limit. In the case of large numbers of bidders Fibich & Gavious (2010) shows that in all-pay auctions, risk-aversity affects expected revenue in large auctions even when the valuations are private. The key observation is that in large auction the bidder with the highest signal faces (at the moment of bidding) a non-trivial risk of losing the auction. With all-pay auctions, bidders are always better off when winning, and as a consequence payoff relevant risk remains at the bidding stage. Our method of analysis based on extreme order statistics in fact provides a means to compute explicitly the expected revenues for private value all-pay auctions applicable to both risk-averse and risk-neutral bidders.

The paper is organized as follows: Section 2 sets up the basic model. Section 3 establishes the basic statistical properties of equilibrium learning. Section 4 shows that second-price auction dominates first-price auction in terms of expected revenue for CARA utilities, and demonstrates that this difference is significant. Section 5 discusses how our revenue ranking result extends beyond CARA utilities. Section 6 concludes.

2 Setup

A single object is for sale. Let V denote the common (random) value of the object. There are n bidders, and prior to participating in an auction, each bidder $i \in \{1, ..., n\}$ observes a signal $\theta_i \in [0, 1]$ distributed according to a joint density $g(\theta_i, v)$ that is identical across the bidders. Conditional on $V = v$, θ_i is independent

of θ_j for $i \neq j$. For each $v \in (0, \infty)$, there is a continuous and bounded conditional density function $g(\theta_i | v)$. A high signal realization is favorable news about the value of the object, which we formalize by requiring strict monotone likelihood ratio property (MLRP):

$$\frac{g(\theta' | v')}{g(\theta | v')} > \frac{g(\theta' | v)}{g(\theta | v)}$$

whenever $\theta' > \theta$ and $v' > v$.

The prior density on V is denoted by $\rho(v)$, and has a finite mean.[1] Since we are ultimately interested in large auctions, our interest will focus around players with very high signals and we define

$$\gamma_v := g(1 | v) < \infty.$$

By strict MLRP, γ_v is strictly increasing in v. Note that this also implies that $\gamma_v > 0$ for all $v > 0$, and hence even the most favorable signal possible, $\theta_i = 1$, does not rule out any realization of v. In other words, information content of each individual signal realization is bounded. This has the important consequence that even when n grows, equilibrium price cannot reveal perfectly v (see Wilson (1977) and Milgrom (1979) for the opposite case).[2]

After observing θ_i, each i chooses a bid $b_i(\theta_i)$. Bidder i wins with positive probability only if $b_i \geq b_j$ for all j. We consider first-price (FPA) and second-price (SPA) auction formats so that the winning bidder pays the k^{th} highest bid $b^{(k)}$ for $k \in \{1, 2\}$, respectively. The losing bidders do not make any payments. The players maximize their expected utility, and we let $u\left(v - b^{(k)}\right)$ denote the utility of the bidder who receives the good. The utility of not receiving the good is $u(0)$. We assume that $u(\cdot)$ is a continuous, strictly increasing and (weakly) concave function.

The bidders do not know each others' realized bids when preparing their own bids. For the case of risk-neutral bidders, the form of the symmetric equilibrium strategies

[1] We assume in most of the paper that this density is finite and continuous for all $v > 0$. However, nothing changes substantially if we assume a discrete probability distribution for v, as we will do in Section 5.2.

[2] As pointed out in Wilson (1977), risk aversion would play no role in a model where equilibrium price is fully revealing.

is documented in Milgrom & Weber (1982). A similar construction holds for the symmetric case with risk-averse bidders. For our purpose, the key requirement is that bids be strictly increasing in signals. Summarizing from earlier literature, we state the following:

Proposition 1 *In both first- and second-price auction formats, a unique equilibrium strategy profile exists in the class of symmetric and strictly increasing bid functions.*

Proof. For the second price auction, Theorem 3.1 of Milgrom (1981) gives existence and characterization of a strictly increasing symmetric equilibrium. Uniqueness in the class of symmetric bid functions is given in the Theorem by Levin & Harstad (1986). Alternatively, the proof of Pesendorfer & Swinkels (1997) in the risk neutral case can be adapted for the present case. For the first price auction, the result can be found in McAdams (2007). ■

For a game with n players, we denote the equilibrium bid function by $b_n^1(\theta)$ for the first price auction and $b_n^2(\theta)$ for the second price auction. We are interested in the properties of the model as n increases towards infinity. In an equilibrium where bidders use strictly monotonic strategies, the realized price is a function of the highest or second highest signal in the bidder population. As a result, the price distribution in large auctions can be derived from the statistical properties of the highest order-statistics in large samples. We turn to that next.

3 Price distribution

Consider a random sample $\{\theta_1, ..., \theta_n\}$ of realized signals. Since the equilibrium bidding strategies are strictly increasing in the bidders' signals, the order statistics $\theta_n^{(1)}$ and $\theta_n^{(2)}$ determine the equilibrium prices $P_n^1 = b_n^1\left(\theta^{(1)}\right)$ and $P_n^2 = b_n^2\left(\theta^{(2)}\right)$ for the first- and second-price auctions, respectively. We aim to derive the probability distributions for those prices. In order to do that, we need to characterize the information that winning in a large auction entails.

Under our signal density assumptions $\theta_n^{(1)}$ and $\theta_n^{(2)}$ converge in probability to a degenerate random variable at the upper bound of the support regardless of the true

value v. Therefore, to uncover the information content of winning, we must find a scaling for the signals that varies appropriately in n.

Consider first the informational content of the event that bidder i wins the auction after observing signal $\theta_i = \theta < 1$. Since the signals are assumed to be independent conditional on V, the probability of winning when $V = v$ is given by:

$$\Pr\{\theta_{n-1}^{(1)} \leq \theta \,|V = v\} = G\left(\theta \,|v\right)^{n-1}.$$

Under the assumptions that we have made, $G\left(\theta \,|v\right) < 1$ for all $\theta < 1$. Hence, for a fixed $\theta < 1$, this probability converges to zero as n grows. Consider therefore the probability of the event $\{\theta_n^{(1)} \leq \left(1 - \frac{z}{n}\right)\}$ as n grows. We have:

$$G\left(1 - \frac{z}{n} \,|v\right) = 1 - \frac{z\gamma_v}{n} + o\left(\frac{1}{n}\right),$$

where $\gamma_v = g\left(1 \,|v\right)$. Hence

$$\lim_{n\to\infty} \Pr\{\theta_n^{(1)} \leq 1 - \frac{z}{n} \,|V = v\} = \lim_{n\to\infty} \left(1 - \frac{z\gamma_v}{n}\right)^n = e^{-\gamma_v z}.$$

Denoting

$$Z_n^{(1)} := n \cdot \left(1 - \theta_n^{(1)}\right), \tag{1}$$

we can write this as

$$\lim_{n\to\infty} \Pr\{Z_n^{(1)} > z \;\; |V = v\} = e^{-\gamma_v z}.$$

This equation says that $Z_n^{(1)}$ converges in distribution to an exponential random variable with parameter γ_v as $n \to \infty$. Hence, (**??**) gives the scaling that we are looking for: since $Z_n^{(1)}$ is a deterministic function of $\theta_n^{(1)}$, its realized value gives exactly the same information as $\theta_n^{(1)}$, for each n. Moreover, since its probability distribution converges to an exponential distribution as $n \to \infty$, we note that the information content of $\theta_n^{(1)}$ converges to that of an exponential random variable as well.

A similar computation shows that if we let $Z_n^{(2)} := n \cdot \left(1 - \theta_n^{(2)}\right)$, then

$$\lim_{n\to\infty} \Pr\{Z_n^{(2)} - Z_n^{(1)} > z \,|V = v\} = e^{-\gamma_v z},$$

and furthermore $Z_n^{(2)} - Z_n^{(1)}$ is independent of $Z_n^{(1)}$ conditional on $V = v$.

We summarize these results in the following proposition. The proof for the part concerning $Z_n^{(2)} - Z_n^{(1)}$ (and an immediate generalization to any positive integer k) is given in Murto & Välimäki (2013).[3]

Proposition 2 *Let* $Z_n^{(k)} := n \cdot \left(1 - \theta_n^{(k)}\right)$ *denote the scaled* k^{th} *order statistic in the sample* $\{\theta_1, ..., \theta_n\}$, $k = 1, 2$. *Then, the vector* $\left[Z_n^{(1)}, Z_n^{(2)} - Z_n^{(1)}\right]$ *converges in distribution to a vector of two independent exponential random variables with parameter* γ_v.

For our purpose the key message of this proposition is that as n increases towards infinity, the information content of winning an auction converges to the information content of observing one or two independent exponentially distributed random variables with an unknown parameter γ_v. Intuitively, $\theta_n^{(1)}$ carries the information available to the winner in the first-price auction, and with appropriate scaling we see that this corresponds to one exponential random variable with unknown mean γ_v. Similarly, $\theta_n^{(2)}$ carries the additional information that the second highest bid incorporates into the price of the second-price auction, and with an appropriate scaling this corresponds to an additional exponential random variable with the same unknown mean.

Notice that the first and second order statics remain imperfectly informative as the sample size increases. This is a consequence of our assumption on signal distribution with a compact support and continuous and strictly positive conditional density: even the highest possible signal conveys only a limited amount of good news. Such a signal assumption is made for example in Pesendorfer & Swinkels (1997) and Kremer (2002) while an opposite case that leads to fully informative extreme order statistics appears in e.g. Wilson (1977) and Milgrom (1981).

To anticipate the revenue comparisons to be coverd in Sections 4 and 5, note that the posterior on V conditional on $Z^{(1)}$ is quite different from the posterior conditional on $Z^{(2)}$. For risk-averse bidders, these different posterior distributions induced by the different auction formats imply different monetary lotteries conditional on winning

[3]For a more general treatment of the distribution of extreme statistics, see e.g. the textbook de Haan & Ferreira (2006).

in the auction. Hence it is not surprising that the expected payoffs and the expected revenue to the seller turn out to be different across different auction formats.

The main insight of this paper is that in the large-game limit, the equilibrium price distribution is pinned down by two observations. First, inference is equivalent to exponential learning as formalized by Proposition 2 above. Second, competition dissipates the bidders' information rents. So far we have only discussed the former observation without linking it to bidding strategies. To formalize the latter observation, we define *willingness to pay* under exponential learning:

Definition 1 *The willingness to pay $b^k(z)$ is the highest price that a buyer is willing to pay, conditional on $Z^{(k)} = z$ where $Z^{(k)} \sim Gamma(k, \gamma_v)$. That is, $b^k(z)$ is the solution to*

$$\mathbb{E}_V[u\left(V - b^k(z)\right) \left| Z^{(k)} = z\right] = u(0). \tag{2}$$

Since u is strictly increasing, this solution must be unique if it exists. For existence, we need some additional assumption. For now, we assume for this purpose that the domain of $u(\cdot)$ is the entire real line. This implies that arbitrarily high bids are feasible. Since u is strictly increasing and V has a finite mean, we have $\mathbb{E}_V[u(V-b)\left|Z^{(k)} = z\right] < u(0)$ for high enough b, which in turn guarantees the existence of a solution to (2). In Section 5 we will discuss an alternative assumption to guarantee existence of $b^k(z)$ for constant relative risk aversion utility function, which has a domain bounded from below.

When $z < z'$, the posterior density on v based on z dominates posterior based on z' in the sense of first-order stochasic dominance. This implies that $b(z)$ is a strictly decreasing function that attains its maximum value at $z = 0$. This just says that a low realization of z is good news about the value, and hence the willingness to pay is at its highest when z is at its lowest. At signal $\theta = 1$ or equivalently at $z = 0$, winning the auction gives no further information. Since we have assumed bounded likelihood ratios, the bidder with this signal is still uncertain about the value of V. For any signal $\theta < 1$, we have $z_n := n(1-\theta) \to \infty$ as $n \to \infty$. Our assumption that γ_v increases in v then implies that in the limit as n grows, winning with a signal $\theta < 1$ implies that the posterior distribution on V converges to a point mass on $V = 0$.

To compute explicitly the expected utility in (2), one first derives the posterior density $g_k(v|z)$ of v conditional on $Z^{(k)} = z$ by Bayes' rule. The density of z conditional on v is given by $\frac{(\gamma_v)^k}{\Gamma(k)} z^{k-1} e^{-(\gamma_v)z}$, and therefore, Bayes' rule gives:

$$g_k(v|z) = \frac{\frac{(\gamma_v)^k}{\Gamma(k)} z^{k-1} e^{-(\gamma_v)z} \rho(v)}{\int_0^\infty \frac{(\gamma_v)^k}{\Gamma(k)} z^{k-1} e^{-(\gamma_v)z} \rho(v)\, dv}.$$

The expected value of $u\left(V - b^k(z)\right)$ can therefore be computed as follows:

$$\mathbb{E}_V[u\left(V - b^k(z)\right) \left| Z^{(k)} = z\right] = \int_0^\infty u\left(v - b^k(z)\right) g_k(v|z)\, dv.$$

We next give our main result, which states that the equilibrium price in k^{th}-price auction converges to the willingness to pay conditional on the random variable $Z^{(k)}$, $k \in \{1, 2\}$, as the number of bidders becomes large. In other words, in large auctions, equilibrium inference is based on exponential learning and all the rents are competed away.

Proposition 3 *As $n \to \infty$,*

1. The price realization $P_n^1 = b_n^1\left(\theta_n^{(1)}\right)$ in a first-price auction (FPA) converges in distribution to $b^1\left(Z^{(1)}\right)$, where $Z^{(1)} \sim Exp(\gamma_v)$.

2. The price realization $P_n^2 = b_n^2\left(\theta_n^{(2)}\right)$ in a second-price auction (SPA) converges in distribution to $b^2\left(Z^{(2)}\right)$, where $Z^{(2)} \sim Gamma(2, \gamma_v)$.

Proof. Consider first the first-price auction. We show that for every $z > 0$, $\lim_{n\to\infty} b_n^1\left(1 - \frac{z}{n}\right) = b^1(z)$.

Fix a $z > 0$. Suppose that $\limsup_{n\to\infty} b_n^1\left(1 - \frac{z}{n}\right) > b^1(z)$. Then, by Proposition 2 we can pick a large enough n' such that

$$\mathbb{E}_V[u\left(V - b_{n'}^1\left(1 - \frac{z}{n}\right)\right) \left| \theta^{(1)} = \theta\right] < \mathbb{E}_V[u\left(V - b^1(z)\right) \left| Z^{(1)} = z\right] = u(0),$$

which means that $b_{n'}^1\left(1 - \frac{z}{n'}\right)$ is not a best-response in a game with n' players, and hence cannot be an equilibrium bid. Since this is a contradiction, we have:

$$\limsup_{n\to\infty} b_n^1\left(1 - \frac{z}{n}\right) \leq b^1(z).$$

Suppose next that $\liminf_{n\to\infty} b_n^1\left(1-\frac{z}{n}\right) < b^1(z)$. Let $\theta_{-i,n}^{(1)} := \max\{\theta_1, ..., \theta_{i-1}, \theta_{i+1}, ..., \theta_n\}$. Using again Proposition 2, we can find a $\delta > 0$ and an increasing sequence of positive integers $\{n_k\}_{k=1}^{\infty}$ such that for every k, we have

$$\begin{aligned} & \mathbb{E}_V[u\left(V - b_{n_k}^1\left(1-\frac{z}{n_k}\right)\right) \Big| \ \theta_i = 1-\frac{z}{n_k}, \theta_{-i,n_k}^{(1)} \leq 1-\frac{z}{n_k}] \\ > \ & \mathbb{E}_V[u\left(V - b^1(z)\right) \big| Z^{(1)} = z] + \delta = u(0) + \delta, \end{aligned}$$

and

$$\Pr\left(\theta_{-i,n_k}^{(1)} \leq 1-\frac{z}{n_k} \Big| \ \theta_i = 1-\frac{z}{n_k}\right) > \delta$$

for some $\delta > 0$. By the continuity of $g(\theta \,|v\ )$, there is an $\varepsilon > 0$ such that

$$\mathbb{E}_V[u\left(V - b_{n_k}^1\left(1-\frac{z}{n_k}\right)\right) \Big| \ \theta_i = 1-\varepsilon, \theta_{-i,n_k}^{(1)} \leq 1-\frac{z}{n_k}] > u(0) + \frac{\delta}{2},$$

and

$$\Pr\left(\theta_{-i,n_k}^{(1)} \leq 1-\frac{z}{n_k} \mid \theta_i = 1-\varepsilon\right) > \frac{\delta}{2},$$

so that any player with signal $\theta > 1-\varepsilon$ will make an expected payoff strictly higher than

$$\left(1-\frac{\delta}{2}\right)\cdot u(0) + \frac{\delta}{2}\cdot\left(u(0)+\frac{\delta}{2}\right) = u(0) + \frac{\delta^2}{4}$$

by bidding $b_{n_k}^1(1-z/n_k)$. Letting k be arbitrarily large (so that $n_k \to \infty$), the number of such players goes to infinity, and therefore their expected joint surplus explodes. This is a contradiction and it follows that $\liminf_{n\to\infty} b_n^1\left(1-\frac{z}{n}\right) \geq b^1(z)$. Since we showed above that $\limsup_{n\to\infty} b_n^1\left(1-\frac{z}{n}\right) \leq b^1(z)$, it must be that

$$\limsup_{n\to\infty} b_n^1\left(1-\frac{z}{n}\right) = \liminf_{n\to\infty} b_n^1\left(1-\frac{z}{n}\right) = \lim_{n\to\infty} b_n^1\left(1-\frac{z}{n}\right) = b^1(z).$$

Since the result holds for all z, this implies that

$$b_n^1\left(\theta^{(1)}\right) \xrightarrow{d} b^1\left(Z^{(1)}\right),$$

where $\xrightarrow{d}$ denotes convergence in distribution.

Consider next the second-price auction. As shown in Milgrom (1981), the equilibrium bid $b_n^2(\theta)$ is the unique solution to

$$\mathbb{E}_V[u\left(V - b_n^2(\theta)\right) \Big| \theta^{(1)} = \theta^{(2)} = \theta] = u(0). \tag{3}$$

Consider two independent exponential random variables $Z^{(1)} \sim Exp(\gamma_v)$ and $Z^{(2)} - Z^{(1)} \sim Exp(\gamma_v)$. Then, for a fixed $z > 0$, we have (by Proposition 2):

$$\begin{aligned} & \lim_{n\to\infty} [\, \mathbb{E}_V[u\left(V - b_n^2\left(1 - \frac{z}{n}\right)\right) \Big| \theta^{(1)} = \theta^{(2)} = 1 - \frac{z}{n}] - \\ & -\mathbb{E}_V[u\left(V - b_n^2\left(1 - \frac{z}{n}\right)\right) \big| Z^{(1)} = z, Z^{(2)} - Z^{(1)} = 0]\,] \\ = \;& 0. \end{aligned} \tag{4}$$

By (3), we have for all n

$$\mathbb{E}_V[u\left(V - b_n^2\left(1 - \frac{z}{n}\right)\right) \Big| \theta^{(1)} = \theta^{(2)} = 1 - \frac{z}{n}] = u(0), \tag{5}$$

and by direct computation (or directly using the memoryless property of exponential random variables), we have

$$\begin{aligned} & \mathbb{E}_V[u\left(V - b_n^2\left(1 - \frac{z}{n}\right)\right) \big| Z^{(1)} = z, Z^{(2)} - Z^{(1)} = 0] \\ = \;& \frac{\int_0^\infty u\left(V - b_n^2\left(1 - \frac{z}{n}\right)\right)(\gamma_v) e^{-(\gamma_v)z}(\gamma_v)\rho(v)\,dv}{\int_0^\infty (\gamma_v) e^{-(\gamma_v)z}(\gamma_v)\rho(v)\,dv} \\ = \;& \frac{\int_0^\infty u\left(V - b_n^2\left(1 - \frac{z}{n}\right)\right)\frac{(\gamma_v)^2}{\Gamma(2)} z e^{-(\gamma_v)z}\rho(v)\,dv}{\int_0^\infty \frac{(\gamma_v)^2}{\Gamma(2)} z e^{-(\gamma_v)z}\rho(v)\,dv} \\ = \;& \mathbb{E}_V[u\left(V - b_n^2\left(1 - \frac{z}{n}\right)\right) \big| Z^{(2)} = z]. \end{aligned} \tag{6}$$

Combining (4) - (6), we have

$$\lim_{n\to\infty} \mathbb{E}_V[u\left(V - b_n^2\left(1 - \frac{z}{n}\right)\right) \big| Z^{(2)} = z] = u(0).$$

Since the willingness to pay $b^2(z)$ is the unique solution to

$$\mathbb{E}_V[u\left(V - b^2(z)\right) \big| Z^{(2)} = z] = u(0),$$

it follows that

$$\lim_{n\to\infty} b_n^2\left(1-\frac{z}{n}\right) = b^2(z).$$

Since the result holds for all z, this implies that

$$b_n^2\left(\theta^{(2)}\right) \xrightarrow{d} b^2\left(Z^{(2)}\right).$$

■

3.1 Discussion

In the risk neutral case, the willingness to pay is

$$b^k(z) = \mathbb{E}_V[V \left| Z^{(k)} = z\right],$$

so that we get immediately that: $P_n^k \to \mathbb{E}_V[V \left| Z^{(k)} = z\right]$. Since $\mathbb{E}[\mathbb{E}_V[V \left| Z^{(2)}\right] \left| Z^{(1)} = z\right] = \mathbb{E}_V[V \left| Z^{(1)} = z\right]$, we see that asymptotic revenue equivalence holds as a consequence of the iterated law of expectations. Indeed, it is well known that in competitive auctions as defined in Kremer (2002), risk-neutral bidders obtain a zero expected payoff. This generalizes in our paper to the risk-averse case: the winner is always indifferent between getting the object and not. For a risk averse bidder, $\mathbb{E}[u(V-p)]$ depends also on the higher moments of V and as a result, the wilingness to pay changes if the posterior on V changes.

It may appear confusing that in the second-price auction where the price is determined by the second highest order statistic, the winner is indifferent between getting the object and not irrespective of the realization of her own normalized signal. This is because the value of the object conditional on the second highest order statistic is the same as the value of the object conditional on both the first and the second order statistics. This is an immediate consequence of the memoryless property of the underlying exponential distributions.[4]

We may use the result on extreme order statistics and large numbers of bidders in other settings as long as existence of a symmetric equilibrium with strictly monotonic

[4]Since the inference is based on two independent exponential random variables $Z^{(1)}$ and $Z^{(2)} - Z^{(1)}$, only their sum matters.

bidding strategies is guaranteed. For example in an all-pay auction with independent private values, we can follow the same steps to calculate the distribution of equilibrium bids (except that in this simpler case, we need not carry out the step of computing the distribution on V conditional on winning the auction). With risk averse bidders, the expected revenue to the sellers falls below the risk-neutral case as shown in Fibich & Gavious (2010). This follows from the fact that the bidder with the highest signal has a non-degenerate probability of winning and losing and therefore the willingness to pay of a risk-averse bidder (and consequently her bid) is less than the expected value of the implied lottery. Note that all-pay auctions in common values environment are problematic, since a symmetric monotonic equilibrium easily fails to exist.

4 Revenue comparison with CARA preferences

We next explore the implications of the asymptotic price distributions derived in Proposition 3 for the expected revenue. In this section we restrict the utility function to have constant absolute risk aversion (CARA). We first prove that the second-price auction is superior to the first-price auction in terms of expected revenue. We then compute explicitly the price distributions for a parametric example and illustrate how the revenue difference between the auction formats varies in the coefficient of risk aversion.

The CARA utility function is given by:

$$u(x) = -\frac{e^{-\eta x}}{\eta},$$

where η is the coefficient of absolute risk aversion. The utility of a bidder that receives the object of value V at price p is[5]

$$\mathbb{E}_V\left(u\left(V-p\right)\right) = \mathbb{E}_V\left(-\frac{e^{-\eta(V-p)}}{\eta}\right) = -\frac{e^{\eta p}\mathbb{E}_V\left(e^{-\eta V}\right)}{\eta}.$$

[5]Since initial wealth plays no role with CARA, we normalize that to zero.

The willingness to pay (2) is then equalized to the price if:

$$-\frac{e^{\eta b^k(z)}\mathbb{E}_V\left(e^{-\eta V}\left|Z^{(k)}=z\right.\right)}{\eta}=u\left(0\right)=-\frac{1}{\eta}.$$

Solving this for $b^k\left(z\right)$, we have:[6]

$$b^k\left(z\right)=-\frac{\log\mathbb{E}_V\left(e^{-\eta V}\left|Z^{(k)}=z\right.\right)}{\eta}. \tag{7}$$

We next state that in expectation, the second-price auction raises more revenue than the first-price auction. The key to this result is the observation that the equilibrium price in the second-price auction is more informative than the price in the first-price auction. In particular, recall that information in the FPA corresponds to one exponentially distributed random variable, while in the SPA it corresponds two such independent random variables. By the law of iterated expectation, the additional information in the SPA relative to FPA induces no change in the expectation of $e^{-\eta V}$, but since $-\log\left(\cdot\right)$ is a convex function, Jensen's inequality implies that an additional signal induces on expectation an upward shift in the willingness to pay. This is formalized in the proof below.

Proposition 4 *Suppose that all the buyers have an identical CARA utility function. Then, the expected revenue in the symmetric equilibrium of the second price auction is at least as high as the expected revenue in the symmetric equilibrium of the first price auction in the limit model where* $n\to\infty$.

Proof. For clarity, we use a subscript to denote the random variable with respect to which an expectation is taken. From (7), the equilibrium price in a first-price auction for a normalized first order statistic $z^{(1)}$ is given by:

$$P\left(z^{(1)}\right)=-\frac{\log\left(\mathbb{E}_V[e^{-\eta V}\left|Z^{(1)}=z^{(1)}\right.]\right)}{\eta}.$$

[6]Note that in the limit $\eta\downarrow 0$, we get the risk neutral case:

$$b^k\left(z\right)=\mathbb{E}_V\left(V\left|Z^{(k)}=z\right.\right).$$

Let us now write the expected price for the second-price auction conditional on $z^{(1)}$ (using (7) with $k = 2$):

$$\mathbb{E}_{Z^{(2)}}[P\left(Z^{(2)}\right)\left|Z^{(1)}=z^{(1)}\right.] = \mathbb{E}_{Z^{(2)}}[-\frac{\log\left(\mathbb{E}_V[e^{-\eta V}\left|Z^{(2)}=z^{(2)}\right.]\right)}{\eta}\left|Z^{(1)}=z^{(1)}\right.].$$

Noting that $-\log(\cdot)$ is a convex function, we have by Jensen's inequality:

$$\mathbb{E}_{Z^{(2)}}[-\frac{\log\left(\mathbb{E}_V[e^{-\eta V}\left|Z^{(2)}=z^{(2)}\right.]\right)}{\eta}\left|Z^{(1)}=z^{(1)}\right.] \geq -\frac{1}{\eta}\log\left(\mathbb{E}_{Z^{(2)}}[\mathbb{E}_V[e^{-\eta V}\left|Z^{(2)}=z^{(2)}\right.]\left|Z^{(1)}=z^{(1)}\right.]\right)$$

By the law of iterated expectation,

$$\mathbb{E}_{Z^{(2)}}[\mathbb{E}_V[e^{-\eta V}\left|Z^{(2)}=z^{(2)}\right.]\left|Z^{(1)}=z^{(1)}\right.] = \mathbb{E}_V[e^{-\eta V}\left|Z^{(1)}=z^{(1)}\right.].$$

Combining all of the equations above, we have:

$$\mathbb{E}_{Z^{(2)}}[P\left(Z^{(2)}\right)\left|Z^{(1)}=z^{(1)}\right.] \geq -\frac{1}{\eta}\log\mathbb{E}_V[e^{-\eta V}\left|Z^{(1)}=z^{(1)}\right.] = P\left(z^{(1)}\right).$$

In words, conditional on a realization of $Z^{(1)}$, the expected price in SPA is higher than in FPA. Since this holds across all possible realizations of $Z^{(1)}$, it follows that the ex-ante expected revenue in SPA exceeds the expected revenue in FPA. ■

4.1 Example

To illustrate that the revenue difference between the auction formats is actually significant, we consider a parametric example specified as follows. First, we assume that the conditional signal density is given by

$$g(\theta\,|v) = v\theta^{v-1}, \tag{8}$$

which implies that $\gamma_v = v$. Second, we let the prior $\rho(v)$ be a Gamma distribution with parameters (α, β). The choice of Gamma prior is convenient, because it is a conjugate prior for the inference in our model.

In the Appendix we derive the following explicit formula for the equilibrium bid functions in the two auction formats:

$$b^k(z) = \frac{(\alpha+k)}{\eta}\log\left(1+\frac{\eta}{\beta+z}\right),\ k=1,2. \tag{9}$$

As we show in the Appendix, (9) implies an intuitive comparative statics result: the bid function in both auction formats (and hence revenue to the auctioneer) is decreasing in the parameter of risk aversion η. We also derive in the appendix explicit formulas for the expected revenues $\mathbb{E}\left(b^k\left(Z^{(k)}\right)\right)$, $k = 1, 2$, and show that $\mathbb{E}\left(b^2\left(Z^{(2)}\right)\right) \geq \mathbb{E}\left(b^1\left(Z^{(1)}\right)\right)$ confirming Proposition 4.

To illustrate the significance of the revenue difference between the auction formats, we depict in Figure 1 the expected revenues $\mathbb{E}\left(b^k\left(Z^{(k)}\right)\right)$ as functions of the coefficient of risk aversion η. The second price auction raises substantially more revenue. It can also be seen that this difference is increasing in η. It should be noted, however, that as η is further increased towards infinity, the expected revenues in both cases converge to 0 so that the difference eventually disappears (this happens well outside the range of this figure).

< Figure 1 to be inserted here >

5 Revenue ranking with other utility functions

One might wonder whether Proposition 4 extends to an arbitrary risk averse utility function. Indeed, we have shown that in the second-price auction the equilibrium price reflects an additional informative signal in comparison to the first-price auction price. One might intuitively expect risk-averse bidders to like this reduction of risk, and therefore to pay more on average for the object. However, as already shown in a related context by Milgrom & Weber (1982), the average willingness to pay for an object increases upon receiving an *arbitrary* additional signal if and *only if* utility is CARA. Therefore, if Proposition 4 were to be generalized beyond CARA preferences, the argument would have to be based on the specific informational setup in our model.

We have not been able to provide general revenue ranking results outside of CARA class. Instead, we will demonstrate below numerically that the revenue ranking that we have established for CARA seems to hold well within constant relative risk aversion (CRRA) utility functions, given gamma prior on V. However, we then show with two counterexamples that the revenue ranking can be reversed either by changing

the prior distribution for V, or by moving outside of the CRRA specification.

5.1 Numerical analysis of CRRA and gamma prior

With constant relative risk aversion (CRRA), the utility of a buyer who buys the object of value V at price p is

$$u(V-p) = \frac{(w_0 + V - p)^{1-\gamma}}{1-\gamma}, \tag{10}$$

whereas the utility of not getting the object is

$$u(0) = \frac{(w_0)^{1-\gamma}}{1-\gamma},$$

where γ is the coefficient of relative risk aversion. Note that unlike in the case of CARA the initial wealth matters, and therefore we must also parameterize the utility function with initial wealth level w_0.

Note that the domain of $u(\cdot)$ is $(-w_0, \infty)$, which means that bids higher than w_0 are infeasible. Therefore, we need an additional joint assumption on the model parameters to guarantee existence of willingness to pay function, i.e. existence of a solution to (2) for all $z > 0$. It is easy to show that this is guaranteed whenever w_0 is high enough, and/or $\rho(v)$ is pessimistic enough.[7]

The full generalization of Proposition 4 to CRRA class would require that the second-price auction dominates first-price auction for all model specifications including any choice of prior $\rho(v)$. Section 5.2 below shows that such a generalization with an arbitrary prior $\rho(v)$ does not hold.

With $\rho(v)$ restricted to gamma distribution (as in Section 4.1), the unambiguous revenue ranking result seems at least initially plausible. However, even though we

[7] In essence, we need $\mathbb{E}_V[u(V-b) \,|\, Z^{(k)} = z] < u(0)$ for the highest possible bid $b = w_0$. Requiring this for realization $z = 0$ in the second price auction (the most optimistic posterior possible in our model), gives the exact condition for the existence of willingness to pay function:

$$\mathbb{E}_V[\frac{(V)^{1-\gamma}}{1-\gamma} \,\Big|\, Z^{(2)} = 0] < u(0) = \frac{(w_0)^{1-\gamma}}{1-\gamma}.$$

have not been able to produce numerically a single counterexample, we have not been able to prove such a result analytically either. For an illustration, Figure 2 shows the expected revenues in the two auction formats as functions of the coefficient of relative risk aversion γ. The prior distribution is the same as in the illustration of CARA case in Figure 1. The additional parameter needed for the CRRA case is the initial wealth, for which we have used here $w_0 = 3$.[8] The figure is strikingly similar to Figure 1: the expected revenue is higher with the second-price auction, and the difference increases in the degree of risk aversion.

< Figure 2 to be inserted here >

5.2 Counterexample: CRRA with discrete prior

We show here that a different prior distribution for V may reverse the revenue ranking within CRRA class. We assume that V can take only two values. Without further loss of generality, assume that $V \in \{0, 1\}$. We let $\pi_0 \in (0, 1)$ denote the prior probability that $V = 1$, and let π denote some unspecified posterior probability. We allow an arbitrary prior only ruling out the trivial cases where there is no uncertainty, i.e. where $\pi_0 = 0$ or $\pi_0 = 1$.

Modifying slightly Definition 1, we denote by $b(\pi)$ the willingness to pay for the object given an arbitrary posterior belief π. Given CRRA with relative risk aversion coefficient γ and initial wealth w_0, $b(\pi)$ is implicitly defined for each π by

$$\pi \cdot \frac{(w_0 + 1 - b(\pi))^{1-\gamma}}{1-\gamma} + (1-\pi) \cdot \frac{(w_0 - b(\pi))^{1-\gamma}}{1-\gamma} = \frac{(w_0)^{1-\gamma}}{1-\gamma}.$$

The notable feature is that with some parameter values the resulting bid function is concave, as can be easily checked numerically. Figure 3 shows this function for parameter values $\gamma = 1.5$, $w_0 = 0.1$ (in this case $b(\pi)$ is strictly concave throughout the unit interval). The concavity of $b(\pi)$ leads directly to the superiority of first-price auction in terms of expected revenue. To see intuitively why this is the case, note that $b(\pi)$ gives the selling price given the posterior π inferred in equilibrium.

[8] The figure remains qualitatively unchanged with other feasible values of w_0. For w_0 low enough, the bid function $b^k(z)$ fails to exist.

When $b(\pi)$ is everywhere concave, any additional information on the value of the object hurts the seller on expectation. As we have seen, second-price auction delivers more information than the first-price auction, therefore it follows that in this case the first-price auction results in higher extected revenue.[9]

< Figure 3 to be inserted here >

The discrete probability distribution that puts a positive probability for the very lowest possible value realization is crucial for the concavity of the willingness to pay function. Note that $b(\pi) \to w_0$ as $\pi \to 1$, but $b(1) = 1 > w_0$ so that there is a discontinuity in $b(\pi)$ at $\pi = 1$.

5.3 Counterexample: kinked utility function

As the counterexample above relied on the discrete prior, it is useful see that we can also reverse the revenue ranking by moving beyond CRRA class in terms of preferences. In this example we allow an arbitrary prior density $\rho(v)$ with support $[0, \infty)$.

We will construct a piecewise linear (concave) utility function. As a preliminary step, assume that bidders are risk-neutral so that $u(x) = x$. In that case, the equilibrium prices in the first and second price auctions are given by random variables P^1 and P^2:

$$\begin{aligned} P^1 &= \mathbb{E}\left(V \left| Z^{(1)}\right.\right) \text{ and} \\ P^2 &= \mathbb{E}\left(V \left| Z^{(2)}\right.\right), \end{aligned}$$

[9]Formally, let π_1 denote the posterior conditional on the realization of the normalized first order statistic $Z^{(1)}$. Hence, the realized price in the first-price auction is given by $b(\pi_1)$ according to Proposition 3. Let π_2 denote the posterior conditional on the realization of $Z^{(2)}$ so that the realized price in the second-price auction is given by $b(\pi_2)$. By the law of iterated expectation (noting that $\pi_k = \mathbb{E}\left(V \left| Z^{(k)}\right.\right)$), we have $\mathbb{E}_{Z^{(2)}}(\pi_2 | \pi_1) = \pi_1$. Since $b(\cdot)$ is strictly concave, we have by Jensen's inequality $\mathbb{E}_{Z^{(2)}}(b(\pi_2) | \pi_1) < b(\pi_1)$. Since this holds for any realization of $Z^{(1)}$, the ex-ante expected value of $b(\pi_2)$ must be lower than that of $b(\pi_1)$, in other words, ex-ante selling price is lower in the second-price auction than in the first-price auction.

respectively, where $Z^{(1)} \sim Exp(\gamma_v)$ and $Z^{(2)} \sim Gamma(2, \gamma_v)$. Let us denote by $\overline{P}^1$ and $\overline{P}^2$ the highest possible price realization in these auctions:

$$\begin{aligned} \overline{P}^1 &: = \mathbb{E}\left(V \left| Z^{(1)} = 0\right.\right), \\ \overline{P}^2 &: = \mathbb{E}\left(V \left| Z^{(2)} = 0\right.\right). \end{aligned}$$

Hence, the support of $V - P^k$ is given by $\left[-\overline{P}^k, \infty\right)$, $k = 1, 2$. Note that $Z^{(2)} = 0$ is a stronger positive signal on V than $Z^{(1)} = 0$, and therefore we have $\overline{P}^2 > \overline{P}^1$. Hence, the support of $V - P^k$ is wider in the second-price auction than in the first-price auction. Nevertheless, as bidders are risk neutral, the expected price must be the same in both auctions.

Now, modify the utility function so that there is a kink at some $\widetilde{v} \in \left(-\overline{P}^2, -\overline{P}^1\right]$:

$$u(x) = \begin{cases} x \text{ for } x \geq \widetilde{v} \\ \widetilde{v} - 2(\widetilde{v} - x) \text{ for } x < \widetilde{v}. \end{cases}$$

This modification increases the bidders' marginal utility at the low end of the support of $V - P^2$, and therefore to avoid negative expected utility, the equilibrium bid is lower than in the risk-neutral case for some signal realizations (in particular, this is the case for very high signal realizations). On the other hand, the utility function is still linear in the entire support of $V - P^1$, and therefore this modification has no effect on the bidding behavior in the first-price auction. It follows that the expected revenue is higher in the first-price auction than in the second-price auction.

Note that once we have a piecewise linear utility function with first-price auction strictly dominating second-price auction, we could modify this to get a smooth utility function in the decreasing absolute risk aversion (DARA) class without affecting the revenue ranking. To do this, one would keep the part of the utility function above the kink unchanged, and replace the part below the kink with a function that has decreasing absolute risk aversion and which smooth-pastes to the linear part at the kink. This simple observation sufficies to note that there is no hope in generalizing our revenue ranking to DARA class.

6 Conclusions

In this paper, we have characterized the equilibrium price distributions for large common value auctions with first-price and second-price formats. The key insight is that the information content of the equilibrium price can be expressed with exponentially distributed random variables. This allows easy computation for price distributions and expected revenues with any utility specification for the bidders.

The main difference between the two auction formats is that equilibrium price is strictly more informative in the second-price auction than in the first-price auction. This is the key to revenue differences across the auction formats with risk averse bidders. With CARA utility, the effect of more information is unambiguous: the expected revenue is strictly higher in the second-price auction. Generalizing this result beyond CARA turns out to be surprisingly tricky. On one hand, our numerical investigations indicate that it is quite hard to find cases where first-price auction generates more revenue than second-price auction. At the same time, by carefully constructed examples we show this to be possible. Our view is that second-price auction is likely to dominate first-price auction in typical cases, but it is not easy to find interesting sufficient conditions outside of the CARA class for such domination.

While we have restricted our analysis to auctions for a single object, it is very easy to generalize the results concerning the second-price auction to the sale of k identical objects by $k+1^{th}$ price auction. Existence of a symmetric monotonic equilibrium is established in Theorem 3.1 of Milgrom (1981). We can therefore write the price realization as $P_n^{k+1} = b_n^{k+1}\left(\theta^{(k+1)}\right)$. A straight-forward extension of the proof for Proposition 3 will then guarantee that the equilibrium price converges in distribution to $b^{k+1}\left(Z^{(k+1)}\right)$, where $b^{k+1}(\cdot)$ is the willingness to pay according to Definition 1 and $Z^{(k+1)} \sim Gamma\left(k+1, \gamma_v\right)$. Note that if we then let k grow large, we get full information aggregation as $Z^{(k+1)}$ becomes an arbitrarily precise signal about true γ_v as k increases. This is in line with the information aggregation result in the risk neutral case in Pesendorfer & Swinkels (1997).

Appendix

In this Appendix, we derive the bid functions given in (9) and the formulas for the expected revenues plotted in Figure 1.

First, we compute $\mathbb{E}\left(e^{-\eta V}\right)$, when $V \sim \text{Gamma}(\alpha, \beta)$:

$$\begin{aligned}\mathbb{E}\left(e^{-\eta V}\right) &= \int_0^\infty e^{-\eta v}\frac{\beta^\alpha}{\Gamma(\alpha)}v^{\alpha-1}e^{-\beta v}dv = \frac{\beta^\alpha}{(\beta+\eta)^\alpha}\int_0^\infty \frac{(\beta+\eta)^\alpha}{\Gamma(\alpha)}v^{\alpha-1}e^{-(\beta+\eta)v}dv \\ &= \frac{\beta^\alpha}{(\beta+\eta)^\alpha} = \left(1+\frac{\eta}{\beta}\right)^{-\alpha}. \qquad (11)\end{aligned}$$

Next, consider the Bayesian updating in the example. Given an observation z from Gamma(k, v), the posterior on v is also a Gamma distribution with parameters updated from (α, β) to (α', β') as follows:

$$\alpha' = \alpha + k,\ \beta' = \beta + z. \qquad (12)$$

Combining the expectation computed in (11) with the updating formulas (12) yields the formula (9) for the equilibrium bid function in k^{th} price auction:

$$\begin{aligned}b^k(z) &= -\frac{\log \mathbb{E}\left(e^{-\eta V}\left|Z^{(k)}=z\right.\right)}{\eta} \\ &= -\frac{\log\left(1+\frac{\eta}{\beta+z}\right)^{-(\alpha+k)}}{\eta} \\ &= \frac{(\alpha+k)}{\eta}\log\left(1+\frac{\eta}{\beta+z}\right), \qquad (13)\end{aligned}$$

for $k = 1, 2$. From this, we can directly derive the comparative statics result that the bid function (and hence revenue to the auctioneer) is decreasing in the parameter of risk aversion η. To see this, observe that for a fixed z, we have

$$\begin{aligned}\frac{db^k(z)}{d\eta} &= -\frac{(\alpha+k)}{\eta^2}\log\left(1+\frac{\eta}{\beta+z}\right)+\frac{(\alpha+k)}{\eta}\frac{1}{\beta+z+\eta} \\ &= \frac{(\alpha+k)}{\eta}\left(\frac{1}{\beta+z+\eta}-\frac{\log\left(1+\frac{\eta}{\beta+z}\right)}{\eta}\right) < 0,\end{aligned}$$

where the last inequality follows from the observation:

$$\log\left(1+\frac{\eta}{\beta+z}\right) > \frac{\frac{\eta}{\beta+z}}{1+\frac{\eta}{\beta+z}} = \frac{\eta}{\beta+z+\eta}.$$

Let us next derive the actual price distributions and formulas for expected revenues. The equilibrium price in the first price auction is a random variable $b^1\left(Z^{(1)}\right)$, where $Z^{(1)} \sim Exp\left(v\right)$, and in the second price auction it is $b^2\left(Z^{(2)}\right)$, where $Z^{(2)} \sim$Gamma$(2,v)$. Note, however, that these probability distributions are conditional on the true value of v, which is unknown to the seller. Hence, as perceived by the seller with prior $v \sim$Gamma(α,β), $Z^{(1)}$ and $Z^{(2)}$ are compound random variables, whose probability density functions we can easily derive:

$$\begin{aligned}
\frac{d}{dz}\Pr\left(Z^{(1)}\leq z\right) &= \frac{d}{dz}\int_0^\infty \left(1-e^{-vz}\right)\cdot\frac{\beta^\alpha}{\Gamma\left(\alpha\right)}v^{\alpha-1}e^{-\beta v}dv \\
&= \int_0^\infty ve^{-vz}\cdot\frac{\beta^\alpha}{\Gamma\left(\alpha\right)}v^{\alpha-1}e^{-\beta v}dv \\
&= \frac{\alpha}{\beta}\left(\frac{\beta}{\beta+z}\right)^{\alpha+1},
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dz}\Pr\left(Z^{(2)}\leq z\right) &= \frac{d}{dz}\int_0^\infty \left(1-e^{-vz}-vz\cdot e^{-vz}\right)\cdot\frac{\beta^\alpha}{\Gamma\left(\alpha\right)}v^{\alpha-1}e^{-\beta v}dv \\
&= \int_0^\infty v^2ze^{-vz}\cdot\frac{\beta^\alpha}{\Gamma\left(\alpha\right)}v^{\alpha-1}e^{-\beta v}dv \\
&= \frac{\alpha\left(\alpha+1\right)}{\beta^2}\left(\frac{\beta}{\beta+z}\right)^{\alpha+2}\cdot z.
\end{aligned}$$

The price distribution is hence obtained by combining functional form (13) with these densities. Hence, the expected revenues in the two auction formats are:

$$\mathbb{E}\left(b^1\left(Z^{(1)}\right)\right) = \int_0^\infty \frac{(\alpha+1)}{\eta}\log\left(1+\frac{\eta}{\beta+z}\right)\frac{\alpha}{\beta}\left(\frac{\beta}{\beta+z}\right)^{\alpha+1}\cdot dz, \tag{14}$$

$$\mathbb{E}\left(b^2\left(Z^{(2)}\right)\right) = \int_0^\infty \frac{(\alpha+2)}{\eta}\log\left(1+\frac{\eta}{\beta+z}\right)\frac{\alpha\left(\alpha+1\right)}{\beta^2}\left(\frac{\beta}{\beta+z}\right)^{\alpha+2}\cdot z\cdot dz \tag{15}$$

To confirm that $\mathbb{E}\left(b^2\left(Z^{(2)}\right)\right) \geq \mathbb{E}\left(b^1\left(Z^{(1)}\right)\right)$, we can write the difference in the revenues between the two auctions as follows:

$$\mathbb{E}\left(b^1\left(Z^{(1)}\right)\right) - \mathbb{E}\left(b^2\left(Z^{(2)}\right)\right) = \tag{16}$$
$$\frac{\alpha(\alpha+1)}{\beta^2}\int_0^\infty \frac{\beta+z}{\eta}\log\left(1+\frac{\eta}{\beta+z}\right)\left(\frac{\beta}{\beta+z}\right)^{\alpha+2}\left(1-\frac{(\alpha+2)z}{\beta+z}\right)\cdot dz.$$

Letting

$$\xi(\eta,z) := \frac{\beta+z}{\eta}\log\left(1+\frac{\eta}{\beta+z}\right),$$

and

$$\zeta(z) := \left(\frac{\beta}{\beta+z}\right)^{\alpha+2}\left(1-\frac{(\alpha+2)z}{\beta+z}\right),$$

we can write the revenue difference as:

$$\frac{\alpha(\alpha+1)}{\beta^2}\int_0^\infty \xi(\eta,z)\,\zeta(z)\,dz.$$

In the risk-neutral limit as $\eta \to 0$, the function $\xi(\eta,z) \to 1$ uniformly. Simple algebra shows that

$$\int_0^\infty \zeta(z)\,dz = 0,$$

so that in the case of risk-neutral bidders, there is no difference in revenues across the auction formats, which is in line with the earlier results obtained under risk-neutral bidding.

For $\eta > 0$, $\xi(\eta,z)$ is increasing in z for all $z \geq 0$. By simple computation for the function $\zeta(\cdot)$, we have

$$(\alpha+2)\beta\zeta'(z) = (z(\alpha+1)-2\beta)\left(\frac{\beta}{\beta+z}\right)^{\alpha+1}.$$

Hence $\zeta(\cdot)$ has at most one local maximum or minimum. Combiming with this the information that

$$\zeta(0) > 0, \zeta(z) < 0 \text{ for } z > \frac{\beta}{\alpha+1},$$

and

$$\lim_{z\to\infty}\zeta(z) = 0,$$

we know that $\zeta(z)$ is quai-monotone in the sense that for all $z' > z$,

$$\zeta(z) < 0 \Longrightarrow \zeta(z') < 0.$$

It is well known (see e.g. Lemma 1 in Persico (2000)) that if i) $\zeta(z)$ is quasi-monotone in the above sense and ii)

$$\int_0^\infty \zeta(z)\, dz = 0,$$

Then for all increasing $\xi(\eta, z)$,

$$\int_0^\infty \xi(\eta, z)\, \zeta(z)\, dz \leq 0.$$

This implies the revenue ranking result:

$$\begin{aligned} \mathbb{E}\left(b^1\left(Z^{(1)}\right)\right) - \mathbb{E}\left(b^2\left(Z^{(2)}\right)\right) &= \\ \frac{\alpha(\alpha+1)}{\beta^2} \int_0^\infty \xi(\eta, z)\, \zeta(z)\, dz &\leq 0. \end{aligned}$$

References

Bali, Valentina & Matthew Jackson. 2002. "Asymptotic Revenue Equivalence in Auctions." *Journal of Economic Theory* 106(1):161 – 176.

de Haan, L. & A. Ferreira. 2006. *Extreme Value Theory: An Introduction*. Springer.

Fibich, G. & A. Gavious. 2010. "Large auctions with risk-averse bidders." *International Journal of Game Theory* 39(3):359–390.

Holt, Charles A., Jr. 1980. "Competitive Bidding for Contracts under Alternative Auction Procedures." *Journal of Political Economy* 88(3):pp. 433–445.

Kremer, I. 2002. "Information Aggregation in Common Value Auctions." *Econometrica* 70:1675–1682.

Levin, Dan & Ronald M. Harstad. 1986. "Symmetric bidding in second-price, common-value auctions." *Economics Letters* 20.

Maskin, Eric & John Riley. 1984. "Optimal Auctions with Risk Averse Buyers." *Econometrica* 52(6):1473–1518.

Matthews, Steven. 1987. "Comparing auctions for risk averse buyers: A buyer's point of view." *Econometrica* 55(3):633–46.

McAdams, David. 2007. "Uniqueness in symmetric first-price auctions with affiliation." *Journal of Economic Theory* 136(1):144 – 166.

Milgrom, P. 1979. "A Convergence Theorem for Competitive Bidding with Differential Information." *Econometrica* 47(3):679–88.

Milgrom, P.R. 1981. "Rational Expectations, Information Acquistion and Competitive Bidding." *Econometrica* 49:921–943.

Milgrom, P.R. & R.J. Weber. 1982. "A Theory of Auctions and Competitive Bidding." *Econometrica* 50:1089–1122.

Murto, P. & J. Välimäki. 2013. "Delay and Information Aggregation in Stopping Games with Private Information." *Journal of Economic Theory* 148(6):2404–2435.

Persico, N. 2000. "Information Acquisition in Auctions." *Econometrica* 68:135–148. mimeo, University of Pennsylvania.

Pesendorfer, W. & J. Swinkels. 1997. "The Loser's Curse and Information Aggregation in Common Value Auctions." *Econometrica* 65(6):1247–1282.

Wilson, R. 1977. "A Bidding Model of Perfect Competition." *Review of Economic Studies* 44:511–518.