Characterization and Uniqueness of Equilibrium in Competitive Insurance *

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This paper studies subgame perfect equilibrium in the competitive insurance model of Rothschild and Stiglitz (1976). Finitely many firms compete by offering menus of contracts to an agent whose risk characteristics are private information. In the game considered, equilibrium existence follows from Dasgupta and Maskin (1986). We show that there is a unique symmetric equilibrium and characterize its properties.

The equilibrium is outcome-equivalent to the competitive equilibrium presented by Rothschild and Stiglitz, whenever the latter exists. Otherwise, it features randomization over a range of pairs of contracts that feature cross-subsidization. Strategies and welfare are continuous and monotone in the prior distribution over types, and the outcome converges to the perfect information allocations when the prior approaches its extreme values.

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1 Introduction

In an influential paper, Rothschild and Stiglitz (1976) (henceforth, RS) analyze a competitive insurance model with private risk information. Their most important conclusions are: (i) competitive equilibrium need not exist and (ii) when it does, there is a unique set of separating contracts that arise.

The goal of this paper is to provide an exhaustive game-theoretic analysis of their model. We follow Dasgupta and Maskin (1986) in modeling competition as a simultaneous offers game with finitely many firms. The agent has private information about having high or low risk of accident. For this game, Dasgupta and Maskin (86, Theorem 5) established existence of (at least) one symmetric equilibrium, possibly in mixed strategies. However, little is known about the properties and potential multiplicity of equilibria.

The findings of this paper are threefold: (i) it establishes uniqueness of the symmetric equilibrium, (ii) it solves explicitly for this equilibrium and (iii) using the characterization result, it derives properties and comparative statics of the equilibrium.

The first part of the paper focuses on pure strategy equilibria. We characterize the conditions under which the RS separating contracts are an equilibrium outcome of the game. It is shown that the most attractive deviation by a firm is always a new pair of separating contracts. Instead of considering all potential menu offers, one can focus on a critical set of separating contracts that generate zero expected profits in the market as a whole. The RS separating contracts fail to be an equilibrium if and only if there is a pair of contracts in this critical set that strictly attracts both types of agents. This requirement reduces to a local condition at the RS pair of contracts, which states that the prior probability of low-risk agents cannot be too high. This condition is new, as well as the characterization of the critical set of deviating pairs of contracts.

The competitive equilibrium concept considered in RS is different from the (game-theoretic) equilibrium concept considered here. Nevertheless, the RS pair of contracts is an equilibrium outcome of our game if and only if it is an equilibrium of the RS original model, provided that entering firms are allowed to propose pair of contracts.[1] Therefore, we provide exact characterization of equilibrium existence in the RS model. This condition is not related to pooling deviating contracts, as usually described in the literature.

We show that the RS pair of contracts is the unique equilibrium outcome, when firms play pure strategies. Therefore, any equilibrium involves mixed strategies whenever the RS pair of contracts cannot be sustained.

[1] In the definition of competitive equilibrium used in RS (Section I.4), outside firms are only allowed to offer a single contract. RS acknowledge that it might be profitable for an entering firm to offer a new pair of contracts that generates positive total profits in the market (Section II.3). However, their equilibrium concept does not take this into account (allowing a deviating firm to offer several contracts). This is a problem, since we show that the most profitable deviation is *always* a pair of contracts. We believe that the competitive equilibrium definition should take these deviations into account.

Even though equilibrium necessarily exists (Dasgupta and Maskin (1986)), there is no explicit construction of equilibrium valid for any prior distribution over types. In Section 4, we explicitly describe the strategy profile that constitutes the unique symmetric equilibrium, for any prior distribution over types. Naturally, if the RS separating contracts are an equilibrium outcome, equilibrium dictates that all firms offer these contracts.

If the RS separating contracts fail to be an equilibrium outcome, the equilibrium involves each firm randomizing over the set of critical contract pairs mentioned before. Offers in the support of equilibrium strategies have the following properties: (i) high-risk agents always receive a full insurance contract; (ii) low-risk agents always receive partial insurance, which leaves the high-risk agent indifferent between this contract and his own; (iii) all the menus of contracts in the support of the equilibrium strategy are ranked by attractiveness. The firm that delivers the most attractive menu of contracts will attract the customer, no matter what his type is. Equilibrium offers present cross-subsidization: they make a loss if the agent is high-risk and a positive profit if he is low risk. Moreover, firms always earn zero expected profits. In fact, each firm earns zero expected profits for any realization of its opponents' randomization. (The expectation is over the agent's type.)

The characterization of the equilibrium distribution of offers comes from a local condition that guarantees that, for any menu offer in the support of the equilibrium strategy, there is no local profitable deviation. We show that this condition implies there is no global profitable deviation by any single firm.

The equilibrium described is the unique symmetric one. Our uniqueness result hinges on showing that any symmetric mixed equilibrium features offers distributed in the same support. We do this by showing that an offer that delivers higher utility to high-risk agent also delivers higher utility to a low-risk agent. This result, coupled with a zero profits result, implies that equilibrium offers have to lie in the critical region of zero expected profits separating contracts analyzed before. These results reduce the region of possible equilibrium offers to a one-dimensional object. On a technical note, this uniqueness proof strategy is non-standard in the literature.

Exploiting our explicit characterization of the equilibrium, we analyze two relevant comparative statics exercises: with respect to the prior distribution and the number of firms. With respect to the prior probability over types, equilibrium offers have monotone comparative statics. If the probability of low-risk agents increases, firms make more attractive offers, in the sense of first order stochastic dominance (FOSD). Both agent types are better off. This means that, in a large market, a better composition is beneficial to all agents.

Equilibrium strategies also feature monotone comparative statics with respect to the number of firms, $N \geq 2$. The support of the equilibrium strategies does not change with the number of firms, but the distribution does. Welfare of both types decreases with the number of firms, a surprising result. The reason for this finding is: the distribution of the best offer among $N-1$ firms does not depend on N, which implies that every single firm makes worse offers as N increases. Hence the distribution of the best offer in the market decreases (FOSD) with N. However, it converges, as $N \to \infty$, to the equilibrium offer of a single firm in a duopoly. Each firm's offers converge to the worst pair of offers in the support, namely the pair of RS separating contracts. This result clarifies the impossibility of construction of mixed equilibrium when there are infinitely many firms and sheds light on the non-existence results for the competitive equilibrium concept considered in RS. All comparative statics are strict whenever the equilibrium involves mixed strategies.

Finally, equilibrium strategies are continuous with respect to the prior belief. More specifically, equilibrium outcome converges to the complete information outcome as the prior converges to the extreme points. When the probability of low risk agents converges to one, offers converge to mass point at actuarially fair full insurance allocation of the low risk agent. When the probability of low risk agent is sufficiently small, the RS pair of contracts are an equilibrium and almost all types consume the actuarially fair full insurance allocation of the high-risk agent.

Several papers have considered alternative models/equilibrium concepts that deal with non-existence problem in the RS model. Maskin and Tirole (1992) consider two alternative models: the model of an informed principal and a competitive model in which many uninformed firms offer mechanisms to the agent. In the informed principal model, the agent proposes a mechanism to the (uninformed) firm. The set of possible equilibrium outcomes always includes the RS pair of contracts. The equilibrium set consists of all incentive compatible allocations that Pareto dominate the RS allocation. This means that the unique equilibrium outcome distribution in our model is always contained in the equilibrium set of the informed principal model.

Maskin and Tirole (1992) also consider a competitive screening model in which finitely many firms simultaneously offer mechanisms to a privately informed agent. A mechanism is a game form, in which both the chosen firm and the agent choose actions. The equilibrium set of this model is always large: it contains any allocations that are incentive compatible and satisfy individual rationality for the agent and firms. Hence the equilibrium set always contains the unique equilibrium outcome distribution of our model. The distinguishing feature of their model is the richness of the strategy set. A firm might react to moves by its opponents by offering a mechanism that contains a subsequent move by it. In equilibrium, a firm can respond to a "cream skimming" attempt by an opponent by choosing to offer no insurance, if the mechanism allows for such move by the firm. In essence, their model does not contain a single simultaneous offer by the firms. The sharp contrast of their equilibrium set characterization to our uniqueness result illustrates the relevance richness assumptions regarding the mechanisms considered. In our model, firms are allowed to simultaneously offer menus of consumption contracts, that cannot be subsequently modified.

More recently, two more alternative models have been proposed. Dubey and Geanakoplos (2002) and Dubey, Geanakoplos, and Shubik (2005) consider a general equilibrium model in which agents trade shares of pools that combine the endowment of many agents. In their model, equilibrium always exists and it coincides with the separating contracts presented in RS. Guerrieri, Shimer, and

Wright (2010) consider a competitive search model, in which the chance of an agent getting a given insurance contract depends on the ratio of insurance firms offering and agents demanding it. They also show that equilibrium always exists and reduces to the Rothschild and Stiglitz pair of contracts in the two-types case. Both models have uniqueness results that depend on different sets of belief refinements (on pools that are never traded or contract options that are not offered in equilibrium, respectively).

An older literature deals with notions of reactive equilibrium that ‘solve’ the problem of existence of equilibrium. For example, Wilson (1977) restricts deviating contracts to be attractive, even after the incumbent firms are allowed to remove some of their contracts from the market. Riley (1979) allows incumbent firms to propose new contracts after a deviation. In these papers, equilibrium contracts might present cross-subsidization, as in our equilibrium.

Rosenthal and Weiss (1984) present an analysis of a competitive version of the Spence model that shares several common feature with ours. They characterize a mixed equilibrium of the model, whenever a pure equilibrium does not exist. They analyze comparative statics with respect to the number of firms as well, which are similar to ours.[2]

An important feature of the equilibrium characterized here is that the distribution of offers and the agent’s welfare depend on the prior probability of types. Offers get strictly better as the probability of the good type becomes large. In other words, the market composition of types matters for offers and welfare. In case different types have different outside options, this model allows for interesting adverse selection effects.

This phenomenon is absent in most of the literature. In Rothschild and Stiglitz (1976), the equilibrium is prior independent whenever it exists. We show that the restriction to prior probabilities for which competitive equilibrium exists is meaningful, since it is also the region for which the prior is not important for outcomes. Dubey and Geanakoplos (2002) and Guerrieri, Shimer, and Wright (2010) obtain equilibrium existence results that generate the RS separating contracts as an equilibrium outcome for any prior. This leads to a discontinuity of the equilibrium at the perfect information case in which all agents have low risk and there is efficient provision of insurance.

The paper is organized as follows. Section 2 formally describes the model. Section 3 discusses existence and uniqueness of pure strategy equilibria of the game. Section 4 describes a specific symmetric strategy profile and shows that it is an equilibrium. Section 5 shows that the constructed equilibrium is the unique symmetric one. Section 6 presents the comparative static results discussed above. Finally, Section 7 concludes and the Appendix contains the more technical proofs.

[2]They mention the existence of similar results for a special case of the Rothschild and Stiglitz model in a working paper that is not available.

2 Model

A single agent faces uncertainty regarding his future income. There are two possible states $\{0,1\}$ and his income in state 0 (1) is $y_0 = 0$ ($y_1 = 1$). The agent has private information regarding his risk type, which determines the probability of each state. For an agent of type $t \in \{h, l\}$, the probability of state 1 is p_t. Assume that $0 < p_l < p_h < 1$. This means that the h-type agent has higher expected income (and lower risk). The prior probability of type t is denoted μ_t. Define $\overline{p} \equiv \mu_l p_l + \mu_h p_h$. There are N identical firms $i = 1, \ldots, N$ which compete in offering menus of contracts.

We assume that the realization of the state is contractible. A contract is a vector $c = (c_0, c_1) \in \mathbb{R}^2_+$ that denotes the final consumption available to the agent in case any of the states is realized. Contracts are exclusive, as in Rothschild and Stiglitz. A menu of contracts is a compact subset of $\mathbb{R}^2_+$ denoted $\mathcal{M}^i$. The set of all compact subsets of $\mathbb{R}^2_+$ is defined as $\boldsymbol{M} \subseteq 2^{\mathbb{R}^2_+}$. A special case of a menu of contracts is a pair of contracts. We will show later that we can focus without loss on pairs of contracts, with each one of them targeted for one specific type.

Timing is as follows. All firms simultaneously offer menus of contracts $\mathcal{M}^i \in \boldsymbol{M}$. Nature draws the agent's type according to probabilities μ_l and μ_h. After observing his own type t and the complete set of contracts $\mathcal{M}^1, \ldots, \mathcal{M}^N$, the agent announces a choice $a \in \bigcup_i \left(\mathcal{M}^i \times \{i\}\right) \cup \{\emptyset\}$. A choice $a = (c, i)$ indicates that contract $c \in \mathcal{M}^i$ is chosen from firm i, while choice $a = \emptyset$ means that the agent chooses to get no contract (and will maintain his own income).

A final outcome of the game is $\left(\mathcal{M}^1, ..., \mathcal{M}^N, t, a\right)$ (everything is evaluated before the income realization is revealed). Given outcome $\left(\mathcal{M}^1, ..., \mathcal{M}^N, t, (c, i)\right)$, the realized profit by firm j is zero, if $j \neq i$, and otherwise is

$$\Pi\left(c \mid t\right) \equiv p_t\left(1 - c_1\right) - \left(1 - p_t\right) c_0.$$

The agents have instantaneous utility function $u\left(\cdot\right)$, which is strictly concave, increasing and continuously differentiable. Finally, the utility achieved by the agent is

$$U\left(c \mid t\right) \equiv p_t u\left(c_1\right) + \left(1 - p_t\right) u\left(c_0\right).$$

Given outcome $\left(\mathcal{M}^1, ..., \mathcal{M}^N, t, \emptyset\right)$ the realized profit by all firms is zero and the utility achieved by the agent is $U\left(y \mid t\right)$.

A (pure) strategy profile is a menu of contracts for each firm $\left(\mathcal{M}^i\right)_i$ and a choice strategy for the agent, which is a measurable function $s : \{h, l\} \times \left(\times_i \boldsymbol{M}\right) \to \left(\mathbb{R}^2_+ \times \{1, .., N\}\right) \cup \emptyset$ such that $s\left(t, \left(\mathcal{M}^i\right)_i\right) \in \bigcup_i \left(\mathcal{M}^i \times \{i\}\right) \cup \{\emptyset\}$.

A mixed acceptance rule is a Markov kernel[3]

$$s : \{h, l\} \times \left(\times_i \boldsymbol{M}\right) \to \Delta\left[\left(\mathbb{R}^2_+ \times \{1, \ldots, N\}\right) \cup \emptyset\right]$$

[3]The Markov kernel definition includes the requirement that, for any measurable set $A \subseteq \mathbb{R}^2_+ \times \{1, \ldots, N\}$, the functions $\left(t, \mathcal{M}^1, ..., \mathcal{M}^N\right) \longmapsto s\left(A \mid t, \mathcal{M}^1, ..., \mathcal{M}^N\right)$ is measurable.

with the restriction $s\left(t,\left(\mathcal{M}^i\right)_i\right)\left[\bigcup_i\left(\mathcal{M}^i\times\{i\}\right)\cup\{\emptyset\}\right]=1$ (with abuse of notation)

A mixed strategy profile is a probability measure over menus of contracts Φ_i for each firm i and a mixed acceptance rule s.[4] A mixed strategy profile defines a probability distribution over outcomes in the natural way, expected profits are defined by integrating realized profits across outcomes according to this probability distribution.

The equilibrium concept is subgame perfect equilibrium.[5] This means that (i) each firm i maximizes expected profits, given the strategies used by its opponents and the acceptance rule by the agent and (ii) the agent only chooses contracts that maximize his (interim) utility, i.e.,

$$(c,i)\in supp\left(s\left(t,\left(\mathcal{M}^i\right)_i\right)\right)\Rightarrow c\in\arg\max_{c\in\bigcup_i\mathcal{M}^i\cup\{y\}}U\left(c\mid t\right),$$

and

$$\emptyset\in supp\left(s\left(t,\left(\mathcal{M}^i\right)_i\right)\right)\Rightarrow U\left(y\mid t\right)\geq\max_{c\in\bigcup_i\mathcal{M}^i}U\left(c\mid t\right).$$

The optimization problem faced by the agent always has a solution because the set of available contracts, $\bigcup_i\mathcal{M}^i\cup\{y\}$, is compact.

3 Pure Strategy Equilibrium

In this section, we analyze existence and uniqueness of pure strategy equilibrium of the game. First, we introduce some notation that is important in the subsequent analysis.

First of all, since firms, as it turns out, "compete away" all profit opportunities, they offer a set of contracts that break even in the market as a whole. Also, the l-type will always receive a full insurance allocation (i.e., $c_1=c_0$) while the h-type will receive a partial insurance allocation (i.e., $c_1>c_0$) that provides some insurance while screening for h-types. This implies that the set of contracts that arise in equilibrium lies in a restricted locus, which we characterize now.

For any $k\in[p_l,\overline{p}]$, we define $\gamma\left(k\right)=\left(\gamma_1\left(k\right),\gamma_0\left(k\right)\right)$ to be the partial insurance contracts that could be offered to the h-type together with full insurance offer (k,k) to the l-type. These offers leave the l-type indifferent between the two contracts (otherwise the firm could offer a more profitable contract to the h-type) and generate zero total profits, if these are the most attractive contracts

[4] We endow $\boldsymbol{M}$ with the Borel sigma algebra induced by the open balls in the Hausdorff metric. We will only use two properties from this sigma algebra: (i) it contains any single contract and (ii) the function that leads to the best available utility to a l-type (h-type) agent must be measurable.

[5] In this game, perfect Bayesian equilibrium is outcome equivalent to subgame perfection. Considering a game tree in which the firms act sequentially, each subgame perfect equilibrium has a corresponding PBE with the same strategy profiles and firms' beliefs, about the earlier firms' play, given directly by equilibrium strategies. Notice that the agent, moving last, sustains perfect information because he knows all the offers and his type as well.

In this game, the concept of Nash equilibrium allows the agent to behave "irrationally" to sets of offers off the equilibrium path. This enables many additional "collusive" equilibria. In fact, we can sustain any individually rational allocation as a Nash equilibrium outcome.

available to both types. Formally, we define the following (set-valued) function $\gamma : [p_l, \overline{p}] \to 2^{\mathbb{R}^2_+}$ by

$$\gamma(k) \equiv \left\{ c \in \mathbb{R}^2_{++} \;\middle|\; \begin{array}{c} U(c \mid l) = u(k); \\ \mu_h \Pi(c \mid h) + \mu_l (p_l - k) = 0; \\ c_1 \geq c_0. \end{array} \right\}.$$

Lemma 1. *For any $k \in [p_l, \overline{p}]$, $\gamma(k)$ is a singleton, that is, there exists a unique $c \in \mathbb{R}^2_+$ such that $c \in \gamma(k)$.*

Proof. Let us define $\zeta = \sup\{c_1 | \exists c_0 \text{ such that } U(c \mid l) = u(p_l)\}$. The strict concavity of u implies that $\zeta > 1$ ($\zeta = \infty$ is possible). Consider the path $\iota : I = [0, w] \to \mathbb{R}^2$ that starts at (k, k) and moves along the indifference curve of $U(\cdot \mid l)$ by increasing c_1, i.e., $\iota_1(t) = k + t$ ($w = \infty$ if $\zeta = \infty$). Let total profit generated by point t in the path, when the l-type consumes (k, k) and the h-type consumes $\iota(t)$, be denoted as $\pi(t)$. We know that $\pi(0) \geq 0$ because $k \leq \overline{p}$. If $\zeta < \infty$, continuity implies that

$$\pi(w) \leq \mu_h p_h (1 - \zeta) < 0.$$

If $\zeta = \infty$, it follows that $\lim_{t \to \infty} \pi(t) = -\infty$. Therefore in both cases continuity of $\pi(t)$ implies that there is t_0 such that $\pi(t_0) = 0$. It also follows from concavity of $u(\cdot)$ that $\pi'(t) > 0$ for all $t > 0$, which means that $\pi(t_0) = 0$ for at most one point t_0. $\square$

From now on, we will refer to $\gamma(\cdot)$ as a single-valued function. The locus of $\{\gamma(k) \mid k \in [p_l, \overline{p}]\}$ is illustrated in Figure 1.

RS characterize a unique pair of contracts that can arise in a competitive environment. This pair of contracts is separating, provides actuarially fair full insurance allocation to the l-type and partial insurance to the h-type. Since this pair of contracts generates zero total profits, they can be described through the γ function. We define them formally.

Definition 1. The Rothschild-Stiglitz (RS) contracts are the pair

$$\left(c^{RS,l}, c^{RS,h}\right) \equiv \left((p_l, p_l), \gamma(p_l)\right).$$

First, notice that the key condition for the existence of pure strategy equilibria in this model relates to γ. Assume that all firms offer the RS pair of contracts. The best possible deviation from the separating RS equilibrium is not a pooling contract, but another separating pair of contracts. The reason is that whatever utility is being obtained by the l-type individual can be provided with lower cost by also offering a contract with the certainty equivalent consumption for this type. This second contract does not interfere with the choice of the h-type agent, since he has a preference leaned towards partial insurance contracts (compared to the l-type). The condition under which there is no profitable deviation is that there is no profitable separating deviation that offers a "no risk" contract to the l-type. This is equivalent to the statement that there is no pair of contracts that contains a risk free contract for the low type and partial insurance contract (to be used by the high type) that breaks even and attracts the high type with strictly higher utility.

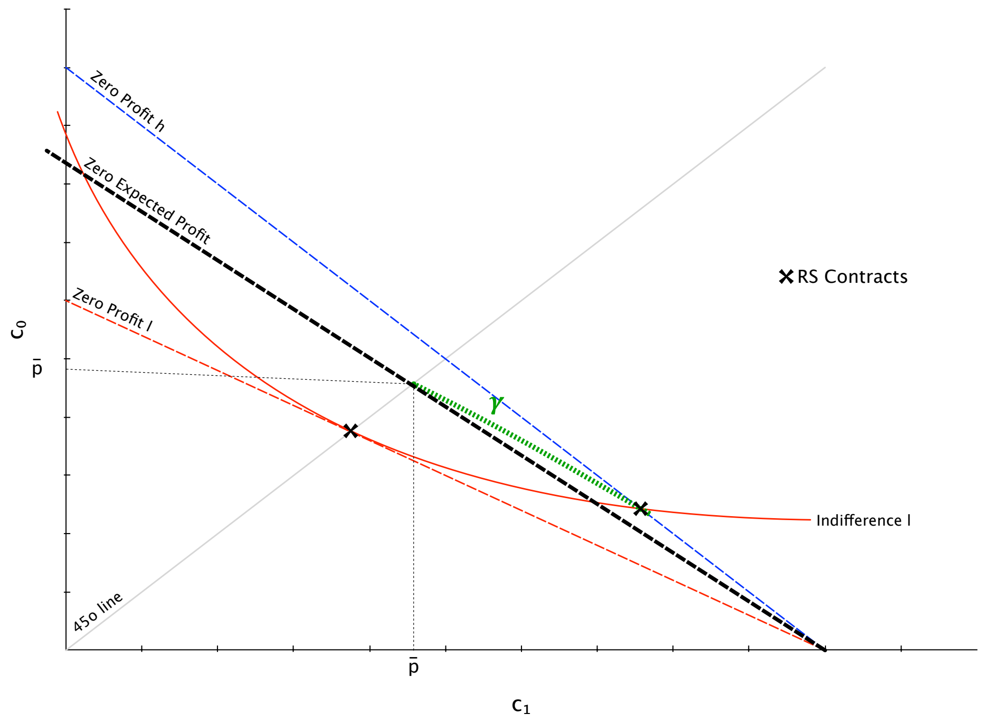

Figure 1: The $\gamma(\cdot)$ locus and the RS contracts

We say that the RS contracts are an equilibrium outcome if there is an equilibrium such that all possible equilibrium outcomes involve type l choosing contract $c^{RS,l}$ or type h choosing contract $c^{RS,h}$. The next lemma shows that the RS are an equilibrium outcome if and only if there is no $k \in [p_l, \overline{p}]$ such that the menu offer $\{(k,k), \gamma(k)\}$ attracts both types of agents.

Lemma 2. *The RS contracts are an equilibrium outcome if, and only if*

$$U(\gamma(k) \mid h) \leq U\left(c^{RS,h} \mid h\right),$$

for all $k \in [p_l, \overline{p}]$.

Proof. (if) Consider the following equilibrium: (i) all firms offer pair of contracts $\left(c^{RS,l}, c^{RS,h}\right)$, (ii) type t accepts contract $c^{RS,t}$ (from any firm) whenever it leads to the highest attainable continuation payoff (we are making a restriction on the indifference points) and (iii) otherwise, t-type agent's choice s is any arbitrary selection from his best response set. First notice that all firms make zero profits.

Given that utility level $U(p_l \mid l)$ is always available to the l-type, there is no offer that generates profit from this type. Now consider an alternative offer of single contract $\widetilde{c}$ that attracts and makes profit out of the h-type, i.e., $U(\widetilde{c} \mid h) > U\left(c^{RS,h} \mid h\right)$ and $\Pi(\widetilde{c} \mid h) > 0$. By construction of the RS contracts, this means that $U(\widetilde{c} \mid l) > U\left(c^{RS,l} \mid l\right)$ and the l-type accepts the new offer for sure.

The most profitable overall offer that offers $\widetilde{c}$ also includes $u^{-1}(U(\widetilde{c} \mid l)) = \widetilde{k}$, the certainty equivalent of $\widetilde{c}$ to the l-type. By offering $\left\{\widetilde{c}, \left(\widetilde{k}, \widetilde{k}\right)\right\}$, the firm attracts the h-type (who accepts $\widetilde{c}$) and minimizes the loss from the acceptance of the l-type (who accepts $\left(\widetilde{k}, \widetilde{k}\right)$). But then both $\widetilde{c}$ and $\gamma\left(\widetilde{k}\right) = \left(\gamma_1\left(\widetilde{k}\right), \gamma_0\left(\widetilde{k}\right)\right)$ are on the l-indifference curve containing $\left(\widetilde{k}, \widetilde{k}\right)$ and $U(\widetilde{c} \mid h) > U\left(c^{RS,h} \mid h\right) \geq U\left(\gamma\left(\widetilde{k}\right) \mid h\right)$. This means that $\widetilde{c}_1 > \gamma_1\left(\widetilde{k}\right)$ and $\widetilde{c}_0 < \gamma_0\left(\widetilde{k}\right)$, which implies that $\Pi(\overline{c} \mid h) < \Pi\left(\gamma\left(\overline{k}\right) \mid h\right)$. Therefore the total profit obtained by offer $\left\{\widetilde{c}, \left(\widetilde{k}, \widetilde{k}\right)\right\}$ is

$$\mu_l\left(p_l - \widetilde{k}\right) + \mu_h \Pi(\widetilde{c} \mid h) < \mu_l\left(p_l - \widetilde{k}\right) + \mu_h \Pi\left(\gamma\left(\widetilde{k}\right) \mid h\right) = 0,$$

by definition of $\gamma(\cdot)$.

(only if) Consider $k \in (p_l, \overline{p})$ such that $U(\gamma(k) \mid h) > U\left(c^{RS,h} \mid h\right)$. Notice that, $U(\gamma(k) \mid l) = u(k) \Rightarrow U(\gamma(k) \mid h) > u(k)$. Therefore, we can find $\varepsilon > 0$ so that $U(\gamma(k) - (\varepsilon, \varepsilon) \mid h) > \max\left\{U\left(c^{RS,h} \mid h\right), u(k)\right\}$. Now consider offering the pair of contracts $((k,k)(\gamma(k) - (\varepsilon, \varepsilon)))$. This contract is accepted by the h-type for sure and generates strictly positive profits. $\square$

This deviation is illustrated in the shaded area in Figure 2. Notice that in the plotted example, there is no pooling deviation, but the RS contracts still cannot be sustained as equilibrium outcome. The inexistence of a pooling deviation follows from the fact that any contract that generates higher utility to the h-type than $c^{RS,h}$, to the right of the h-type indifference curve, generates a loss when offered to both types since it pays more than contracts on the zero expected profits line.

Understanding equilibria in the competitive insurance model depends on the properties of $\gamma(\cdot)$. More specifically, we must understand for what region of $[p_l, \overline{p}]$ the function $U(\gamma(\cdot) \mid h)$ is increasing, since this is the region of potentially profitable deviations (in the case of pure equilibria) and is crucial to the analysis of mixed equilibria as well. The next lemma implies that this function increases in an interval. In the case where the RS contracts can be sustained, this interval reduces to the point p_l.

Lemma 3. *$U(\gamma(\cdot) \mid h)$ is strictly quasi-concave on the range $[p_l, \overline{p}]$. In particular, it is single-peaked.*

Proof. In the Appendix. $\square$

Define $\overline{k}$ to be the peak of $U(\gamma(\cdot) \mid h)$ on $[p_l, \overline{p}]$. Lemma 3 establishes the existence of a well defined peak. Notice that the RS contracts are an equilibrium outcome iff $\overline{k} = p_l$. We shall define the set R to be the relevant range, i.e.,

$$R \equiv \left[p_l, \overline{k}\right].$$

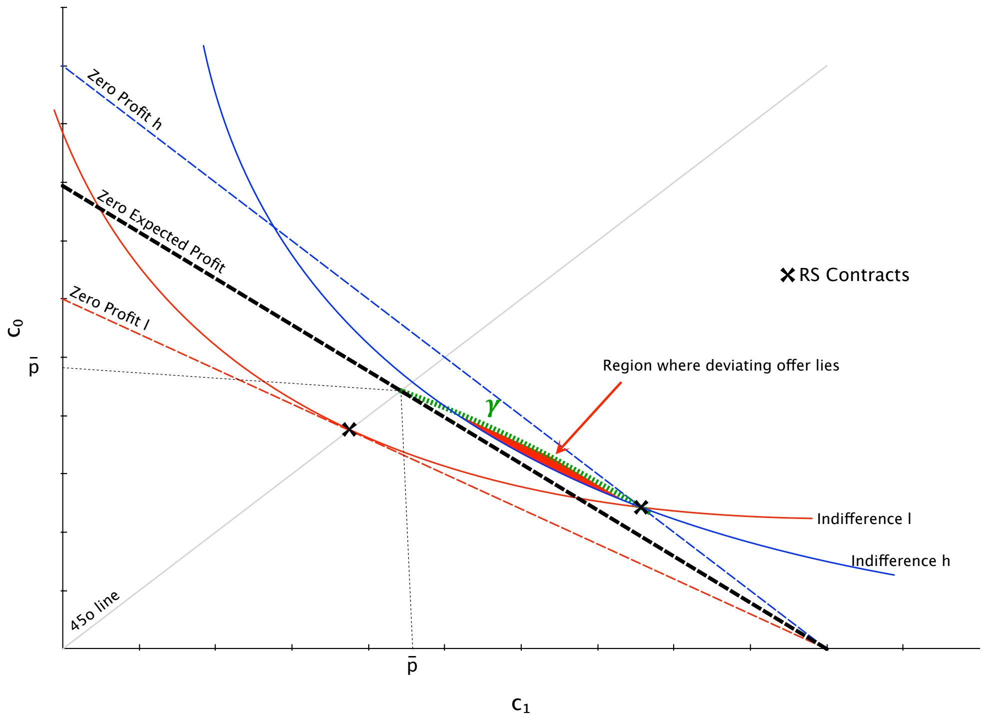

Figure 2: Deviating offers to h-type

Lemma 3 implies that finding improvements in utility along the $\gamma(\cdot)$ curve amounts to the existence of local utility improvements around $k = p_l$. This yields a necessary and sufficient condition for the existence of pure equilibria that depends on local variation around c^{RS}.

As k increases, the allocation provided to the l-type agent, namely (k, k), is more attractive and generates a higher loss. Therefore, by definition, the profit generated by allocation $\gamma(k)$ increases with k. However, the utility obtained by the h-type from $\gamma(k)$ might increase with k because the allocation $c^{RS,h}$ is inefficient.

Corollary 1. *The RS contracts are an equilibrium outcome if, and only if,*

$$u'\left(c^{RS,l}\right)\left[\frac{1}{u'\left(c_1^{RS,h}\right)} - \frac{1}{u'\left(c_0^{RS,h}\right)}\right] \leq \frac{\mu_l}{\mu_h}\left[\frac{1-p_l}{1-p_h} - \frac{p_l}{p_h}\right]. \tag{1}$$

Proof. Follows directly from $\left.\frac{\partial U(\gamma(k)|h)}{\partial k}\right|_{k=p_l} \leq 0$. $\square$

This result gives us an exact characterization, in terms of primitives under which the RS pair of contracts can arise as an equilibrium outcome in our model. This characterization, as well as the focus on the set of contracts defined by the γ function, are not present in the literature.

It is worth mentioning that the competitive equilibrium concept considered by Rothschild and Stiglitz (1976) is different from ours. They define a competitive equilibrium to be a pair of contracts that generate nonnegative profits and leave no possibility of profits by an outside firm (Section I.4). It is easy to check that the primitive condition under which the RS pair of contracts is a competitive equilibrium coincides with the condition under which it is an equilibrium outcome in the game considered here. Therefore our paper presents an exact condition for competitive equilibrium existence in the competitive insurance model of Rothschild and Stiglitz. This condition complements the classical analysis of the model, and demonstrates that the focus on pooling deviating contracts to illustrate the equilibrium existence problem (as in RS) is misleading.

We have characterized necessary and sufficient conditions for the implementation of the RS pair of contracts as equilibrium outcome. Therefore, we know that equilibrium leads to different equilibrium outcomes whenever condition (1) fails. The equilibrium existence result of Dasgupta and Maskin (86, Theorem 5) implies that there always is a symmetric equilibrium of this game, potentially involving mixed strategies by the firms. We show that mixed strategies are also necessary to equilibrium whenever RS contracts cannot be sustained as an equilibrium outcome. This follows since any equilibrium that involves pure strategies by the firms generates as outcome the RS contracts.

Proposition 1. *In any equilibrium in which firms play pure strategies, the agent of type* $t \in \{l, h\}$ *accepts contract* $c^{RS,t}$ *with probability one.*

This result allows for (potentially) mixed acceptance rule by the agent and asymmetric strategies. This result implies that, whenever (1) fails, any equilibrium will necessarily involve mixed strategies by the firms. Indeed the equilibrium constructed in Section 4 is mixed whenever (1) does not hold, and in Section 5 we show that it is the unique symmetric equilibrium of the game.

Proposition 1 differs in two dimensions from the uniqueness of competitive equilibrium in Rothschild and Stiglitz (1976). First, this paper considers as potential menu offers any compact set of contracts, while Rothschild and Stiglitz consider fixed contract consumed by each agent type. Second, the assumption of a finite number of firms seems to leave open the possibility for collusive equilibria with positive profits. An outside firm would offer a new contract as soon as there is a chance to earn positive profits in the market. In the example of a duopoly, the existence of a profit opportunity does not mean that one of the firms can profitably deviate: they have to consider the effect of a deviation on consumers already attracted to their menu. Proposition 1 deals with these issues.

4 Complete Equilibrium Characterization

The goal of this section is to characterize a (potentially) mixed symmetric equilibrium of the game. Whenever condition (1) is satisfied, which means that the RS contracts are an equilibrium outcome, the equilibrium constructed coincides with the standard pure strategy equilibrium characterized in the literature. However, whenever condition (1) fails, and there is no pure strategy equilibrium of the game, our equilibrium features mixing among a continuum of pairs of contracts that generate zero expected profits.

In the first part of this section we assume that $N = 2$. We show, in the end of this section, how to adjust the equilibrium to the case $N > 2$.

The equilibrium has the following feature: every firm mixes over menu offers

$$\boldsymbol{M}^R \equiv \{\{(k,k), \gamma(k)\} \mid k \in R\}.$$

We define the menu offer $\mathcal{M}^k \equiv \{(k,k), \gamma(k)\}$ as k-offer, since the offers in the strategy support are parametrized by $k \in R$. The distribution over k-offers is smooth with distribution given by $k \to F(k)$ (and density $f(\cdot)$). Call any equilibrium in this class a smooth equilibrium. The equilibrium structure is illustrated in Figure 3, where the support of contract pairs is shown, as well as the region R of possible l-type consumption.

We complete the description of equilibria as follows: the distribution F has to be such that there is no local gain from making an offer around the offer $\mathcal{M}^k$, for any $k \in R$. This is the local condition established in Definition 2. Below, we will consider a special kind of deviations: alternative pairs of contracts that generate the same utility for the l-types, but change the utility delivered to h-types. This first order condition determines the complete distribution of offers.

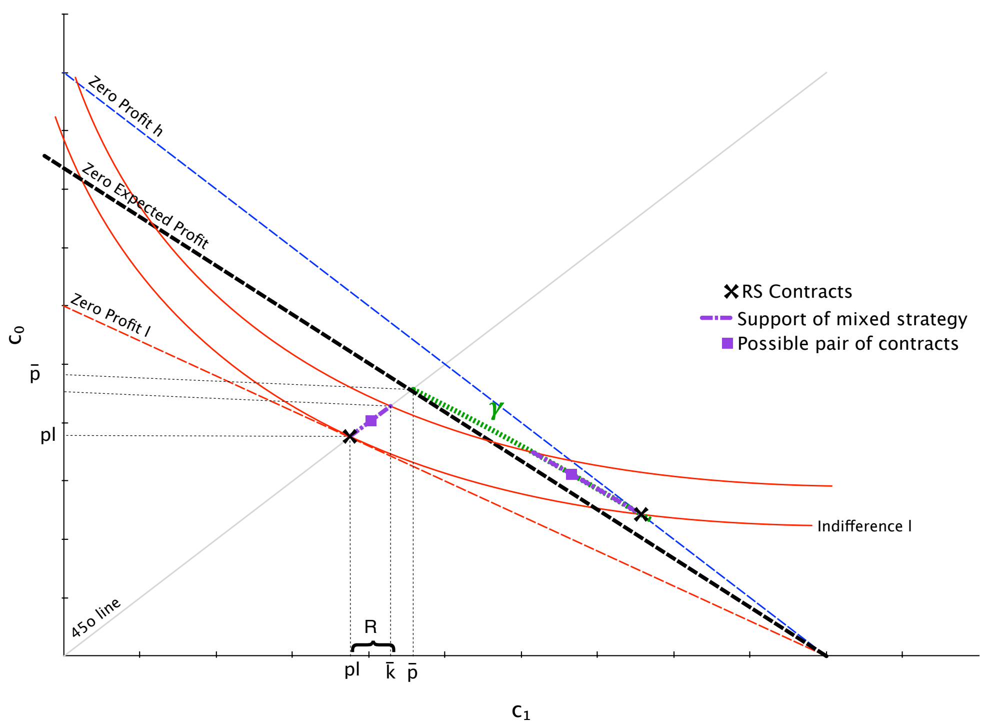

Figure 3: Mixed equilibrium

However, we still need to check that there are no global profitable deviations. This is done by checking that the marginal loss of moving away from the prescribed contract is increasing. This completes the proof that the strategy profile proposed is (part of) an equilibrium.

The intuition behind the local deviation we will consider is as follows. Since both types have different preferences, we can construct a perturbation around offer $\{(k,k), \gamma(k)\}$ altering the contract targeted to the h-type such that (i) the new offer is not more attractive to the l-type and (ii) it increases the utility offered to the h-type. The trade-off between the original offer and the considered perturbation is that the new offer, delivering higher utility the h-type agent, is accepted with higher probability while generating less profits whenever accepted. The following equation implies that, at the optimal contracts, these two margins are balanced:

$$\begin{aligned} f(k)\left[\frac{\partial U(\gamma(k)\mid h)}{\partial k}\right]^{-1}\left[\frac{(1-p_l)}{p_l}-\frac{(1-p_h)}{p_h}\right]\Pi(\gamma(k)\mid h) \\ = F(k)\left[\frac{(1-p_l)}{p_l}\frac{1}{u'(\gamma_1)}-\frac{(1-p_h)}{p_h}\frac{1}{u'(\gamma_0)}\right], \end{aligned} \tag{2}$$

for any $k \in R$.

The gain from higher acceptance depends on the mass around point k and the profit obtained at this point, so that $f(k)\Pi(\gamma(k)\mid h)$ is present in the LHS of the ODE. The cost of offering a lower profit contract depends on the total probability of the offer being accepted, $H^h(k)$, which is represented in the RHS of the equation. The remaining terms are the distortion factors that pin down by how much h-type utility and profits change, since the perturbed offers have to leave the l-type indifferent.

Definition 2. Define $F : [p_l, \overline{k}] \to [0,1]$ as as a smooth equilibrium distribution if it is a solution to the ODE (2) with condition $F(\overline{k}) = 1$.

After rearranging, the ODE becomes

$$\frac{f(k)}{F(k)} = \phi(k), \tag{3}$$

where we define

$$\phi(k) \equiv \frac{\left[\frac{\partial U(\gamma(k)|h)}{\partial k}\right]\left[\frac{(1-p_l)}{p_l}\frac{1}{u'(\gamma_1)}-\frac{(1-p_h)}{p_h}\frac{1}{u'(\gamma_0)}\right]}{\left[\frac{(1-p_l)}{p_l}-\frac{(1-p_h)}{p_h}\right]\Pi(\gamma(k)\mid h)}.$$

Lemma 4. *The ODE (3) has a unique solution. Moreover, F is given by*

$$F(k) = \exp\left[-\int_k^{\overline{k}} \phi(z)\,dz\right].$$

F puts no mass at point p_l, i.e.,

$$F\left(p_l\right)=0.$$

Proof. Integration of (3) implies that

$$1=F\left(\overline{k}\right)=F\left(k\right)\exp\left[\int_{k}^{\overline{k}}\phi\left(z\right)dz\right].$$

Finally notice that $\Pi\left(\gamma\left(k\right)\mid h\right)=\frac{\mu_l}{\mu_h}\left(k-p_l\right)$. This means that $\phi\left(z\right)$ is of the order of $\frac{1}{k-p_l}$ around p_l. Then we know that

$$\lim_{k\to p_l^+}\int_{k}^{\overline{k}}\phi\left(z\right)dz=\infty.$$

This implies that $F\left(p_l\right)=\lim_{k\to p_l^+}F\left(k\right)=0$. □

We extend the definition of the distribution function $F\left(\cdot\right)$ as follows. Whenever $R=\{p_l\}$, it has total mass probability at the point $\{p_l\}$; and whenever $R=\left[p_l,\overline{k}\right]$ is a nondegenerate interval, it is the solution to the ODE (3) in the interval R, extended as $F\left(k\right)=0$, for any $k<p_l$, and $F\left(k\right)=1$, for any $k>\overline{k}$. Now we are in a position to state the main result of this section, characterizing equilibrium in this model. The proof of this proposition constitutes the rest of the section.

Proposition 2. *There exists a symmetric equilibrium such that: (i) every firm randomizes over $\boldsymbol{M}^R$ with distribution over offers $\mathcal{M}^k$ given by $F\left(\cdot\right)$, (ii) after observing menu offers $\left(\mathcal{M}^{k_1},\mathcal{M}^{k_2}\right)$ with $k_i>k_j$ the agent chooses according to*

$$\begin{aligned} s\left(l,\left(\mathcal{M}^{k_1},\mathcal{M}^{k_2}\right)\right) &= \left(k_i,k_i\right), \\ s\left(h,\left(\mathcal{M}^{k_1},\mathcal{M}^{k_2}\right)\right) &= \gamma\left(k_i\right). \end{aligned}$$

After observing offers that are not of the form $\left(\mathcal{M}^{k_1},\mathcal{M}^{k_2}\right)$, the agent chooses any arbitrary selection from his best response set.[6]

Clearly, in the strategy profile described above, the agent is always choosing a contract in his best response set. Now we need to check that there does not exist a profitable deviation by a single firm.

We only need to consider menu deviations of the form $\left\{\left\{u^{-1}\left[U\left(c\mid l\right)\right],c\right\}\mid\Pi\left(c\mid h\right)>0 \text{ and } c_1>c_0\right\}$. As argued before, any offer that attracts the h-type with positive probability (which means it delivers utility at least $U\left(c^{RS,h}\mid h\right)$) and makes positive profits will necessarily attract l-types with positive probability (they receive utility as low as $U\left(c^{RS,l}\mid l\right)$ in equilibrium). This means that the cheapest offer that menu offer c to the h-type involves offering the certainty equivalent of c to the l-type, i.e., $u^{-1}\left[U\left(c\mid l\right)\right]$. This pair of contracts minimizes the loss from the l-type potentially

[6] With the restriction that s is still a mixed strategy, as defined in Section 2.

accepting a contract offer from the menu. From now on, we denote deviating menu offers by c, the consumption level that targets the h-type agents.

We proceed as follows: (i) characterize, for each utility u_l, the h-type allocation c^h that maximizes revenue within $\{c \mid U(c \mid l) = u_l\}$; (ii) show that any offer that delivers consumption $k \in R$ to the l type is worse than the equilibrium offer $\mathcal{M}^k$, and (iii) show that offers that involve offering $k \notin R$ are not optimal.

Fix a consumption c for the high type, and let $\overline{u} = U(c \mid l)$. We will consider the marginal revenue gain from moving along the l-indifference curve $\{c' \mid U(c' \mid l) = \overline{u}\}$ by increasing consumption of good 1 by ε and reducing consumption of good 0 by $\xi(\varepsilon)$. This amounts to increasing the utility offered to the h-type, so that he accepts the new offer with higher probability, but in case this offer is accepted it generates less profits.

This function $\xi(\cdot)$ satisfies

$$U((c_0 - \xi(\varepsilon), c_1 + \varepsilon) \mid L) = U(c \mid L) = \overline{u}, \tag{4}$$

which implies

$$\xi'(\varepsilon) = \frac{p_l u'(c_1^\varepsilon)}{(1-p_l)\, u'(c_0^\varepsilon)},$$

where $c^\varepsilon = (c_0 - \xi(\varepsilon), c_1 + \varepsilon)$.

Let us define $H^h(\cdot)$ to be the distribution function of highest equilibrium utility delivered to the h-type, i.e.,

$$H^h(U(\gamma(k) \mid h)) \equiv F(k).$$

The profit from offering such an allocation is

$$\text{Profit from l} + \mu_h \underbrace{Pr[U(\tilde{c} \mid h) \leq U(c^\varepsilon \mid h)]}_{H^h[U(c^\varepsilon \mid h)]} \Pi(c^\varepsilon \mid h),$$

where $\tilde{c}$ is the random h-type offer drawn according to equilibrium strategy F.

Notice that whenever $U(c \mid h) \in \left(U(\gamma(p_l) \mid h), U\left(\gamma\left(\overline{k}\right) \mid h\right)\right] \equiv U^R$, the expected profit function is smooth and differentiable. The derivative of this function by ε at $\varepsilon = 0$ is

$$\begin{aligned}
&\mu_h H^h[U(c \mid h)]\left[-p_h + \frac{p_l u'(c_1)}{(1-p_l)\,u'(c_0)}(1-p_h)\right] \\
&\qquad + \mu_h h^h[U(c \mid h)]\left[p_h u'(c_1) - \frac{p_l u'(c_1)}{(1-p_l)\,u'(c_0)}(1-p_h)\,u'(c_0)\right]\Pi(c \mid h) \\
&\qquad = \mu_h (1-p_h)\, u'(c_1)\, H^h[U(c^\varepsilon \mid h)]\left\{\begin{array}{c} \left[\frac{p_l}{(1-p_l)}\frac{1}{u'(c_0)} - \frac{p_h}{(1-p_h)}\frac{1}{u'(c_1)}\right] \\ +\frac{h^h[U(c|h)]}{H^h[U(c|h)]}\left[\frac{p_h}{(1-p_h)} - \frac{p_L}{(1-p_L)}\right]\Pi(c \mid h) \end{array}\right\}.
\end{aligned} \tag{5}$$

We will denote this function as $M(c)$. This function describes the marginal gain from a firm offering a new consumption that increases the utility of h-types, while maintaining the same expected profits from the l-type agents.

Remark. The ODE (3) is defined so that $M(\gamma(k)) = 0$, for any $k \in R$. This means that, by construction, there are no local from deviating offer c^{ε}.

The following lemma shows that the local condition implies global conditions. For example, if an offer c delivers utility $U(c \mid l) \in U^R$ to the l-type and utility $U(c \mid h) < U(\gamma(k) \mid h)$ to the h-type. This implies that $\gamma_1(k) > c_1$. Lemma 5 implies that $M(c) > 0$, which means that offers that deliver higher utility to the h-type are more profitable. Since this is true for any offer that generates utility lower than $U(\gamma(k) \mid h)$ to the h-type agent, offer $\gamma(k)$ is better than any such offer.

Lemma 5. *Consider consumption allocations c^A, c^B such that $U\left(c^A \mid h\right) = U\left(c^B \mid h\right) \in U^R$, $c_1^A > c_1^B$ and $M\left(c^A\right) \geq 0$, then we have that*

$$M\left(c^B\right) > 0.$$

Similarly, if we find c^A, c^B such that $U\left(c^A \mid h\right) = U\left(c^B \mid h\right) \in U^R$, $c_1^A < c_1^B$ and $M\left(c^A\right) \leq 0$, then we have that

$$M\left(c^B\right) < 0.$$

Proof. Follows directly from equation (5) defining $M(c)$. □

Lemma 5 shows that the local conditions imposed by the definition of distribution F are enough to guarantee that no global deviation exists. Therefore, we have shown that no firm has a profitable deviation.

Lemma 6. *At the strategy profile described in Proposition 2, no firm has a profitable deviation.*

Proof. We index offers by the consumption level it generates to the l-type and the level of utility it generates to h-type. An offer (k, u) is an offer that includes (k, k) for the l-type as well as another offer c^h to the h-type such that $U\left(c^h \mid h\right) = u$ and $U\left(c^h \mid l\right) = u(k)$.

1) First, consider $k \in R$.

1.a) Offering (k, u) with $u < U^R$ is not attractive because it generates a loss by the l-type and generates zero out of the h-type (he never accepts this offer).

1.b) Offering (k, u) with $u > U^R$ is less attractive then offering $\left(k, \max U^R\right)$ (also accepted with probability 1 by the h-type and generates more profit).

1.c) Offering (k, u) with $u \in U^R$ but $u > U(\gamma(k) \mid h)$. Call the corresponding offer to the high type c^h. In that case there is an on-path offer $\left(\tilde{k}, \tilde{c}^h\right)$ with $\tilde{c}_1^h < c_1^h$. Then from Lemma 5 we know that

$$M\left(\tilde{c}^h\right) = 0 \Rightarrow M\left(c^h\right) < 0.$$

This means that there is a strict gain from reducing u.

1.d) Offering (k,u) with $u \in U^R$ but $u < U(\gamma(k) \mid h)$. By a symmetric argument from (1.c) we can see that $M\left(c^h\right) > 0$, and therefore it is optimal to increase u.

This means that for $k \in R$, the best offers are indeed of the form $(k, \gamma(k))$.

2) Now let us consider offering $k < R$. In this case the only offers that attract the h-type with positive probability generate a loss (view graph).

3) Consider offers with $k > R$.

3.a) Offering (k,u) with $u < U^R$. In this case all the l-types accept the offer and none of the h-types accept it, so a loss is made for sure.

3.b) Offering (k,u) with $u > U^R$. The l-types accept this offer for sure, as well as the h types. However, we know that $U(\gamma(k) \mid h) < u$. So this offers generates an ex-ante expected loss.

3.c) Offering (k,u) with $u \in U^R$. Call the implied offer to the high type c^h. In that case there an on-path offer $\left(\widetilde{k}, \widetilde{c}^h\right)$with $\widetilde{c}_1^h > c_1^h$. Then from Lemma 5 we know that

$$M\left(\tilde{c}^h\right) = 0 \Rightarrow M\left(c^h\right) > 0.$$

Since this is true for any $u \in U^R$, we know that, for a given $k > R$, the best available offer is $\left(k, \max U^R\right)$ that gets accepted with probability one by both types. However, we know that $U(\gamma(k) \mid h) < \max U^R$, and this means that this offer makes an expected loss as well. $\square$

The case $N > 2$

In the analysis of the duopoly case, we have shown that we can find a distribution over the set of menu offers

$$\boldsymbol{M}^R = \left\{\left\{(k,k), \gamma(k)\right\} \mid k \in R\right\},$$

such that each of the firms finds it equally optimal to offer any menu offer $\{k, \gamma(k)\}$, for $k \in R$.

This support $\boldsymbol{M}^R$ has the following special property: $U((k,k) \mid l) = u(k)$ and $U(\gamma(k) \mid h)$ are both strictly increasing in k, for $k \in R$. This means that if firm $i = 1$ faced two firms 2 and 3 that were choosing menu offers $\{(k,k), \gamma(k)\}$ according to continuous distributions F_2 and F_3, the relevant random variable for firm 1 is $k_{23} = \max\{k_2, k_3\}$ which determines the only relevant threat to their offers. The distribution of this variable is given by $F(k) = F_1(k)\, F_2(k)$. This allows us to adapt the arguments above, by interpreting $-i$'s offer as the best offer among $N-1$ firms.

Proposition 3. *In the game with N firms, the following is a symmetric equilibrium: every firm randomizes over $\boldsymbol{M}^R$ with distribution over offers $\mathcal{M}^k$ given by $F_i(\cdot)$, where*

$$F_i(k) = F(k)^{\frac{1}{N-1}}.$$

The choice strategy s is the same as established in Proposition 2.

The equilibrium described has the following properties. First, whenever there is randomization (which occurs if condition (1) fails), ties occur with zero probability: there is always a firm that offers M^{k_i} such that $k_i > \max_{j \neq i} k_j$. The agent gets a contract from this firm, independent of which type is realized. If the type is l, the agent ends up with contract (k_i, k_i). If the agent is of type h, he chooses contract $\gamma(k_i)$. Second, whenever the pure equilibrium with the RS contracts exists, $R = \{p_l\}$ and the support of strategies reduces to the RS contracts.

The equilibrium is continuous on the prior μ_h, and the equilibrium outcome converges to the perfect information outcomes as the prior converges to the extreme values 0 and 1. It also presents monotone comparative statics with respect to the number of firms. These comparative statics are discussed in more detail in Section 6.

5 Uniqueness

In this section, we prove that the equilibrium described above the unique symmetric equilibrium of the game.

Our proof strategy is as follows. First, our first result in this section characterizes some basic properties of any symmetric equilibrium, namely, that all menu offers are essentially equivalent to a pair of offers and that all firms have zero expected profits. Then, we characterize the distribution of utilities obtained by each agent type implied by the offer distribution. We show that such distribution has no gaps and no mass points, except for the utility levels generated by the pair of RS contracts.

Finally, by looking at local optimality conditions at the equilibrium offers of the same nature as in Section 4, we are able to show that, relative to a fixed on-path offer, offers that are more attractive to the h-type agent (by generating higher utility) are necessarily more attractive to the l-type agent. This special ordering of the equilibrium offers, together with zero profits, implies that the pairs of contracts that arise in equilibrium lie in the region $\mathbf{M}^R$ characterized in Section 4. The reason for this result is that, given equilibrium offers, any firm either delivers the most attractive offer for both types or it is unattractive for both types. Zero profits implies then that the firm must break even (in interim terms) in both these situations. After showing that the support of offers has to contain the whole set $\mathbf{M}^R$, uniqueness follows from uniqueness of the solution of the ODE determined by local deviations.

We define a *symmetric equilibrium* as an equilibrium strategy profile $(\phi_1, \ldots, \phi_N, s)$ such that $\phi_1 = \ldots = \phi_N$. Notice that our definition of symmetric equilibrium allows for acceptance rules by the agent.

Now, we state our first characterization result for symmetric equilibrium. The proof is presented in the appendix.

Proposition 4 (Necessary conditions)**.** *Consider a symmetric equilibrium. For any menu* $\mathcal{M} \in \mathbf{M}$

offered on-path, the following hold:
(i) there is only one contract, namely $c^{\mathcal{M},t}$*, that is accepted with positive probability by type* $t \in \{l, h\}$*,*
(ii) firms earn zero expected profits,
(iii) profits obtained from l*-type agents are always nonpositive, i.e.,* $\Pi\left(c^{\mathcal{M},l} \mid l\right) \leq 0$*,*
(iv) $\Pi\left(c^{\mathcal{M},t} \mid h\right) \geq 0$*, which means that* $c^{\mathcal{M},t}$ *lies in a compact set, and*
(v) l*-type agents only accept full insurance allocations and are indifferent between* $c^{\mathcal{M},l}$ *and* $c^{\mathcal{M},h}$*, i.e.,*

$$c_1^{\mathcal{M},l} = c_0^{\mathcal{M},l}$$

and

$$U\left(c^{\mathcal{M},l} \mid l\right) = U\left(c^{\mathcal{M},h} \mid l\right).$$

Dasgupta and Maskin (1986) have a similar characterization result.[7] Their result states that conditions (i)-(v) hold in any equilibrium of this game, focusing on the case $N = 2$ and on symmetric acceptance rules by the agent. Therefore, in comparison with their result, ours is more restrictive, by focusing on symmetric equilibrium, while more general in allowing for N firms and (potentially) non-symmetric acceptance rules.

It follows from condition (v) in Proposition 4 that we can index all offers by the allocation offered to the h-type, c (the l-type will be offered a full insurance allocation that makes him indifferent). Denote as $\tilde{c}$ as the offer made by a fixed firm in equilibrium, which is distributed [8]according to Φ.

For any menu offer $\mathcal{M} \in \mathbf{M}$, define $u^{\mathcal{M},t} \equiv \max_{c \in \mathcal{M}} U(c \mid t)$. Also, denote as H^t the implied distribution over utility available to type $t \in \{l, h\}$ from $N - 1$ offers distributed according to ϕ (being offered by all opponents of a given firm). Formally

$$H^t(u) = \left[\phi\left\{\mathcal{M} \in \mathbf{M} \mid u^{\mathcal{M},t} \leq u^t\right\}\right]^{N-1}.$$

We will also denote H^t as a general measure (instead of a cumulative distribution), in those cases we use notation $H^t A$ for the measure of set A implied by H^t.

We start by showing that, whenever mass points exist on this measure, they will be concentrated in one specific point. We start by looking for mass points in the utility offered to the l-type.

Lemma 7 (l-utility mass points)**.** *Suppose* $H^l\{u_0\} > 0$*, then* $u_0 = u(p_l)$*.*

Proof. Suppose that we have $H^l\{u_0\} > 0$ for some $u_0 > u(p_l)$ (firms only make losses on l). Consider an offer c on the support of Φ and firm i such that: (i) $U(c \mid l) = u_0$ and (ii) if the opponent is offering something that delivers $u_l = u_0$, then his c offer is accepted with at least $\frac{1}{2}$

[7]For a formal statement of their claim and the proof, the reader is directed to a working paper that is not available.
[8]Φ is the distribution of $c^{\mathcal{M},h}$ implied by ϕ, the equilibrium distribution of menus.

probability. So profits are at most

$$\mu_l \left\{ H^l\left(-\infty, u_0\right) + H^l\left\{u_0\right\} \frac{1}{2} \right\} \left[p_l - u^{-1}\left(U\left(c \mid l\right)\right)\right]$$
$$+\mu_h H^h\left(U\left(c \mid h\right)\right) \Pi\left(c \mid h\right),$$

but by offering $c + \varepsilon(1, -K)$ for some $K \in \left(\frac{p_l u'(c_1)}{(1-p_l)u'(c_0)}, \frac{p_h u'(c_1)}{(1-p_h)u'(c_0)}\right)$ attracts more h-types and less l-types, so that when $\varepsilon > 0$ is arbitrarily small the firm gains

$$\mu_l H^l\left(-\infty, u_0\right) \left[p_l - u^{-1}\left(U\left(c \mid l\right)\right)\right] + \mu_h H^h\left(U\left(c \mid h\right)\right) \Pi\left(c \mid h\right),$$

which is strictly better than offer c. $\square$

Now we characterize mass points on the utility offered to the h-type.

Lemma 8 (h-utility mass points)**.** *Suppose* $H^h\{u_0\} > 0$, *then* $c \in supp\,(\Phi) \wedge U\left(c \mid h\right) = u_0 \Rightarrow \Pi\left(c \mid h\right) = 0$.

Proof. Suppose that we have $H^h\{u_0\} > 0$. Then we can find an offer $c \in supp\,(\Phi)$ and firm i such that: (i) $U\left(c \mid h\right) = u_0$, (ii) $\Pi\left(c \mid h\right) > 0$ and that (iii) if the opponent is offering something that delivers $u_h = u_0$, then his c offer is accepted with probability less than 1 by h. So profits is strictly less than

$$\mu_l H^l\left(-\infty, U\left(c \mid l\right)\right) \left[p_l - u^{-1}\left(U\left(c \mid l\right)\right)\right] + \mu_h H^h\left(u_0\right) \Pi\left(c \mid h\right),$$

but by offering $c + \varepsilon(1, -K)$ for some $K \in \left(\frac{p_l u'(c_1)}{(1-p_l)u'(c_0)}, \frac{p_h u'(c_1)}{(1-p_h)u'(c_0)}\right)$ attracts more h-types and less l-types, so that when ε arbitrarily small the firm's expected profit is approximately this. So there is a profitable deviation. $\square$

Using these two statements together, and also the fact that firms get zero expected utility in equilibrium, informs us about what can arise in equilibrium.

Corollary 2 (Allocation mass points)**.** *The only possible point mass in* Φ *is* $c^{RS,h} = \gamma\left(p_l\right)$.

Proof. If there is a mass point of Φ at c, this means that $H^l\{U\left(c \mid l\right)\} > 0$ and $H^h\{U\left(c \mid h\right)\} > 0$. By the lemmata above we then know that $U\left(c \mid l\right) = u\left(p_l\right)$ and that $\Pi\left(c \mid h\right) = 0$. $\gamma\left(p_l\right)$ is the only allocation that satisfies this (with $c_1 \geq c_0$). $\square$

Given this consequence, we can also establish that there is only one mass point for the utility of h-type.

Corollary 3. *Suppose* $H^h\{u_0\} > 0$, *then* $u_0 = U\left(\gamma\left(p_l\right) \mid h\right)$.

Proof. Consider $c \in supp(\Phi)$ such that $U(c \mid h) = u_0$. Following Lemma 8, we know that $\Pi(c \mid h) = 0$. But for a given u_0, there is only one allocation c_0 that attains utility u_0 and generates zero profit. So this means that $H^h\{u_0\} \geq \Phi(\bar{c})$. We know that the only mass point of Φ is $\gamma(p_l)$. $\square$

We have proved that there are no mass points besides the RS allocation. Now we also must establish that there is no gap in utility levels.

Lemma 9 (No gap)**.** *There is no gap in the H^h distribution, i.e., if there exist $u_0 < u_1$ such that $H^h(u_0) = H^h(u_1) < 1$, then $H^h(u_0) = 0$.*

Proof. Suppose that we can find such u_0 and u_1 as in the lemma and such that $H^h(u_0) > 0$. Consider the offer that delivers u "right above" the gap (which means that $H^h(u) - H^h(u_0)$ is arbitrarily small). Reducing this offer by (δ, δ) strictly increases profits (it might reduce the probability that a l type accepts it and it almost does not affect the probability that a h-type accepts it). $\square$

Since the menu offer $\mathcal{M}^{\overline{k}}$ must always generate zero profits, the support of equilibrium utilities must contain $U\left(\gamma\left(\overline{k}\right) \mid h\right)$, the upper bound of $U^R = \{U(\gamma(k)) \mid k \in R\}$ (defined in Section 4). If the utility obtained by the h-type agent is always uniformly below this level, an offer slightly less attractive than $\mathcal{M}^{\overline{k}}$ generates positive profits.

Lemma 10. *For any $\varepsilon > 0$,*

$$H^h\left(U\left(\gamma\left(\overline{k}\right) \mid h\right) - \varepsilon\right) < 1.$$

Proof. Offering $\gamma\left(\overline{k}\right)$ and $\overline{k}$ for the h and l-types, respectively, generates zero expected profit if those are the best offer for both types. If the statement in the lemma fails, for some $\delta > 0$ small, we could offer $c' = \gamma\left(\max U^R\right) - (\delta, \kappa(\delta))$ such that $U(c' \mid l) = u\left(\overline{k}\right)$ that generates more profit out of the h-types, which means that it generates strictly positive expected profits (even if the l type accepts his offer for sure). $\square$

We denote the support of H^h by $D = [\underline{u}, \overline{u}]$. We know it is an interval by lemma 9, we also know that $\underline{u} \geq U(\gamma(p_l) \mid h)$[9] and that $\overline{u} \geq \max U^R$. Let us denote also $D_0 \equiv (\underline{u}, \overline{u}]$ (which excludes the potential mass point). We know that it only has (potentially) a mass point at $U(\gamma(p_l) \mid h)$ and is continuous otherwise. We know then that H^h is differentiable almost everywhere (Lebesgue). When differentiable, let h^h denote its derivative.

Lemma 11 (Local optimality)**.** *If $c \in supp(\Phi)$ and H^h is differentiable at $U(c \mid h) \in D_0$, then it*

[9] This follows since any offers have to generate no profit from l-types, which means that $U(c \mid l) \geq U(p_l \mid l)$ for any $c \in supp(\Phi)$; and any offers generate no losses from the h-type, which means $\Pi(c \mid h) \geq 0$ for any $c \in supp(\Phi)$.

satisfies

$$H^h \left[U\left(c \mid h\right)\right]\left[\frac{p_l}{\left(1-p_l\right)}\frac{1}{u'\left(c_0\right)} - \frac{p_h}{\left(1-p_h\right)}\frac{1}{u'\left(c_1\right)}\right] + h^h\left[U\left(c \mid h\right)\right]\left[\frac{p_h}{\left(1-p_h\right)} - \frac{p_l}{\left(1-p_l\right)}\right]\Pi\left(c \mid h\right) = 0. \quad (6)$$

Proof. Similar to the derivation of condition (5). Consider offer around c that delivers same utility level to l-type and changes utility offered to the h-type. Notice that for c' close to c, u_l has no mass point at $U\left(c \mid l\right)$, since otherwise $U\left(c' \mid l\right) = u\left(p_l\right)$ and $\max\left\{U\left(\hat{c} \mid h\right) \mid U\left(c \mid l\right) = u\left(p_l\right) \wedge \Pi\left(c \mid h\right) \leq 0\right\} = U\left(\gamma\left(p_l\right)\right)$. □

A direct corollary of this Lemma is that, for all such utility levels, there is at most one offer that delivers this utility in equilibrium.

Corollary 4. *Suppose $U \in supp\left(H^h\right)$ and H^h is differentiable at U, then there is only one offer $\chi\left(U\right) \in supp\left(\Phi\right)$ such that*

$$U\left(c \mid h\right) = U.$$

Proof. Same logic as proof of Lemma 5. The equality 6 cannot hold for two different points on the same indifference curve of h-type. □

So we know how almost all levels of h-type utility u_h are “delivered”:

(i) If there is a mass point at $U\left(\gamma\left(p_l\right) \mid h\right) = \underline{u}$, then the only offer that delivers this is $\gamma\left(p_l\right)$. As for $U > \underline{u}$, we know that for almost all U (Lebesgue) there is only one $\chi\left(U\right)$ in the support of Φ that deliver this utility. But since H^h is absolutely continuous with respect to Lebesgue measure, for H^h restricted to D_0 (proved in Section 8.3), this means that for H^h-almost all $U > \underline{u}$ there is only one $\chi\left(U\right) \in supp\left(\Phi\right)$ that delivers this utility.

(ii) If there is no mass at $U\left(\gamma\left(p_l\right) \mid h\right) = \underline{u}$. By the same argument as above, we know that for H^h-almost all U, there is only one $\chi\left(U\right) \in supp\left(\Phi\right)$ that delivers this utility.

The offer function $\chi\left(.\right)$ is already pinned down and defined H^h-almost everywhere. However we must find a measurable extension of it to D. Consider $h^h\left(\cdot\right)$ to be a measurable extension of the derivative of H^h over D_0 (the Radon-Nikodym derivative). For any $u \in D_0$ let $\chi\left(u\right)$ be the solution to equation (6) (it always has a solution if $H^h\left(U\right) > 0$). Also let $\chi\left(U\left(\gamma\left(p_l\right) \mid h\right)\right) := \gamma\left(p_l\right)$.

Intuitively, Φ is generated by the following algorithm: draw U according to $\widetilde{H} \equiv \left[H^h\right]^{\frac{1}{N-1}}$, and then offer $\chi\left(U\right)$. Formally, $\Phi = \left[H^h\right]^{\frac{1}{N-1}}\chi^{-1}$ or Φ is the push forward of the $\chi\left(\cdot\right)$ function.

Denote as $\pi\left(U_h, U_l\right)$ as the expected profit of making offer that delivers utility U_l to the l-type and U_h to the h-type (the two indifference curves cross only once, this means that $\pi\left(\cdot,\cdot\right)$ is a well defined function).

Lemma 12 (Marginal gains)**.** *For any fixed $U_l > u(p_l)$, $\pi(\cdot, U_l) : D_0 \to \mathbb{R}$ is continuous and differentiable almost everywhere (Lebesgue). Moreover, whenever it is differentiable, the derivative is*

$$M(U_h, U_l) \equiv$$

$$\mu_h (1-p_h) u'(c_1) H^h [U(c \mid h)] \left\{ \begin{array}{c} \left[\frac{p_l}{(1-p_l)} \frac{1}{u'(c_0)} - \frac{p_h}{(1-p_h)} \frac{1}{u'(c_1)}\right] \\ + \frac{h^h[U(c|h)]}{H^h[U(c|h)]} \left[\frac{p_h}{(1-p_h)} - \frac{p_l}{(1-p_l)}\right] \Pi(c \mid h) \end{array} \right\},$$

where c is such that $U(c \mid h) = U_h$ and $U(c \mid l) = U_l$. Additionally, for any (U_l, U_h) such that $U_h \in D_0$:

$$\pi(U_h + \Delta, U_l) = \pi(U_h, U_l) + \int_{U_h}^{U_h + \Delta} M(u, U_l)\, du.$$

The first part of the statement is quite similar to Lemma 11, since we are considering the same potential deviation at an equilibrium offer. The second part is technical, and is postponed to Section 8.3 (together with the proof of absolute continuity of $H^h(\cdot)$).

This next lemma states that differentiability only depends on differentiability of H^h, so it is a condition only in U_h. It also states that local optimality has implication for global deviations, as in Lemma 5.

Lemma 13. *For any $u \in D_0$ such that $H^h(u) > 0$, if $\pi(\cdot, U_l)$ is differentiable at u, then $\pi(\cdot, U_l + \Delta)$ also is. The function $M(U_h, \cdot)$ is strictly increasing at $(u, U(\chi(u) \mid l))$, for almost all $u \in D_0$. Moreover, if $\Delta > 0$ and $M(U_h, U_l) \geq 0$, then*

$$M(U_h, U_l + \Delta) > 0,$$

and similarly, if $\Delta < 0$ and $M(U_h, U_l) \leq 0$, then

$$M(U_h, U_l + \Delta) < 0.$$

Proof. The first part of the lemma follows from the fact that differentiability of $\pi(\cdot, U_l)$ reduces to differentiability of H^h.

From Proposition 4, we know that for almost all offers c, $c_1 \geq c_0$. Define the utility level delivered by c as (U_h, U_l), and let c^Δ denote the consumption level that generates utility $(U_h, U_l + \Delta)$, for types h and l, respectively.

Therefore, we know that c^Δ satisfies $\frac{dc_0^\Delta}{d\Delta} > 0$ and $\frac{dc_1^\Delta}{d\Delta} < 0$. Also notice that, since $c_1 \geq c_0$, contract c^Δ increases the insurance level offered to the h-type and keeps his utility constant. This implies more profits, i.e., $\frac{d\Pi\left(c^\Delta | h\right)}{d\Delta} \geq 0$.

By definition, the function $M(U_h,\cdot)$ is differentiable, and

$$\frac{dM(U_h,U_l+\Delta)}{d\Delta}\equiv$$

$$\mu_h(1-p_h)H^h[U(c\mid h)]\left\{\begin{array}{c}\frac{p_l}{(1-p_l)}\frac{d\left[u'\left(c_1^{\Delta}\right)/u'\left(c_0^{\Delta}\right)\right]}{d\Delta}\\+\frac{h^h[U(c|h)]}{H^h[U(c|h)]}\left[\frac{p_h}{(1-p_h)}-\frac{p_l}{(1-p_l)}\right]\left[\frac{du'\left(c_1^{\Delta}\right)}{d\Delta}\Pi\left(c^{\Delta}\mid h\right)+u'\left(c_1^{\Delta}\right)\frac{d\Pi\left(c^{\Delta}|h\right)}{d\Delta}\right]\end{array}\right\},$$

which is strictly positive. □

A corollary of this Lemma is the following. For a fixed utility level of the h-type U_h, offers that generate higher utility to the l-type present a higher marginal gain from h-type attractive offers. The implication of these is that, in the set of optimal menu offers, higher utility to the h-type agents have to be paired with higher utility offers to the l-type agent.

Corollary 5. *The function $U\mapsto U(\chi(U)\mid l)$, defined over D_0, is non-decreasing.*

Proof. Consider $U_h^0<U_h^1$ in D_0. Assume, by way of contradiction, that

$$U_l^0\equiv U\left(\chi\left(U_h^0\right)\mid l\right)>U\left(\chi\left(U_h^1\right)\mid l\right)\equiv U_l^1.$$

This means that

$$\pi\left(U_h^1,U_l^1\right)-\pi\left(U_h^0,U_l^1\right)=\int_{U_h^0}^{U_h^1}M\left(z,U_l^1\right)dz\geq 0.$$

But then, by Lemma 13,

$$\pi\left(U_h^1,U_l^0\right)-\pi\left(U_h^0,U_l^0\right)=\int_{U_h^0}^{U_h^1}M\left(z,U_l^0\right)dz>\int_{U_h^0}^{U_h^1}M\left(z,U_l^1\right)dz\geq 0.$$

This is a contradiction with optimality of both $\left(U_h^0,U_l^0\right)$. □

Since the only mass point of the l-type utility is $u(p_l)$, which is only (potentially) generated by offer $\gamma(p_l)$, we know that the function $\chi(\cdot)$ is strictly increasing.

Lemma 14. *The function $U\rightarrow U(\chi(U)\mid l)$ (defined over D_0) is strictly increasing.*

Proposition 5. *All equilibrium offers lie in the set $\{\gamma(k)\mid k\in R\}$.*

Proof. Follows from the fact that the attractiveness of offers of both types increase together in the support, which means that they have to generate zero profit if they are the most attractive offers for both types. So we know that they lie in $\{\gamma(k)\mid k\in[p_l,\overline{p}]\}$. The second part follows form the fact that the utility delivered to both types has to increase together, which only occurs when $k\in R$ (see existence proof). □

It only remains to be proved that $D = \left[U\left(\gamma\left(p_l\right) \mid h\right), U\left(\gamma\left(\overline{k}\right) \mid h\right)\right]$, that is, the support is not a strict subset of $\{\gamma(k) \mid k \in R\}$.

Proposition 6. $D = \left[U\left(\gamma\left(p_l\right) \mid h\right), U\left(\gamma\left(\overline{k}\right) \mid h\right)\right]$.

Proof. Suppose that $D = \left[U, U\left(\gamma\left(\overline{k}\right) \mid h\right)\right]$ for some $U > U\left(\gamma\left(p_l\right) \mid h\right)$. Also let U_l denote the utility generated by the lowest offer. Then we can find allocation c' such that

$$\begin{aligned} U\left(c' \mid h\right) &> U; \\ U\left(c' \mid h\right) &< U_L; \\ \Pi\left(c' \mid h\right) &> 0. \end{aligned}$$

(We can accomplish this by considering c' sufficiently close to $\chi(U)$.) This offer generates strictly positive expected profits. □

We only needed to fix the support, so that the ODE defining the distribution on Section 4 is a necessary condition as well. The distribution F defined in Section 4 is the only solution to the ODE defining optimality of contracts on D.

Proposition 7. *The equilibrium characterized in Proposition 2 is the only symmetric equilibrium.*

Proof. Distribution F, characterized in Lemma 4 is the only function that satisfies (i) $F\left(\overline{k}\right) = 1$ and (ii) $M\left(U\left(\gamma(k) \mid h\right), (k, k)\right) = 0$, for all $k \in R$. □

6 Comparative statics

Since the equilibrium construction is independent of the number of firms N and the prior distribution over types (μ_h, μ_l), we are able to analyze the comparative statics with respect to these primitives of the model.

6.1 Number of firms

First, consider the number of firms in the market. These results are reminiscent of the ones presented in Rosenthal and Weiss (1984) for the Spence model. Since the distribution of the best offer among any $N-1$ firms is independent of N, it follows that as more firms are present in the market, each firm will pursue a less aggressive strategy, i.e., with offers that are less attractive to both types of the agent, in the sense of first order stochastic dominance.

Let $F^{(n)}$ denote the equilibrium distribution over $k \in R$ that defines the equilibrium distribution over contracts $\left\{\mathcal{M}^k \mid k \in R\right\}$.

Proposition 8. *The distribution $F^{(N)}$ first-order stochastically dominates $F^{(N+1)}$. In case $\overline{k} > p_l$, the dominance is strict. Additionally, $F^{(N)}$ converges weakly to the RS l-type consumption $c^{RS,l} = p_l$, as $N \to \infty$.*

Proof. Just notice that

$$F^{(N)} = F(k)^{\frac{1}{N-1}} \leq F(k)^{\frac{1}{N}} = F^{(N+1)}.$$

In case $\overline{k} > p_l$, there is continuous mixing over $\left[p_l, \overline{k}\right]$, so that the inequality is strict for any $k \in \left(p_l, \overline{k}\right)$.

Finally, if the distribution F is a point mass at p_l, the convergence of $F^{(n)}$ is trivial. In case F is continuous on $\left[p_l, \overline{k}\right]$, notice that for any $k \in \left(p_l, \overline{k}\right)$

$$F(k) \in (0,1) \Rightarrow F(k)^{\frac{1}{N-1}} \to^{N\to\infty} 1.$$

□

Since the utility provided by offer $\mathcal{M}^k$ is increasing in k, for both types h and l, we know that the utility delivered by a single firm decreases as N increases. However, for higher number of firms, the agent is sampling a higher number of offers, so that the overall effect seems unclear. However, since the distribution of the best offer among any $N-1$ firms is fixed, it follows that the distribution of the best among N firms is lower (first-order stochastic dominance) and converges to F. Let $\overline{F}^{(N)}$ be the distribution of $\max_{i=1,\ldots,N} k_i$, where each k_i is distributed according to $F^{(N)}$.

Proposition 9. *The distribution $\overline{F}^{(N)}$ first-order stochastically dominates $\overline{F}^{(N+1)}$. Moreover,*

$$\overline{F}^{(N)} \to^{w} F,$$

as $N \to \infty$.

Proof. Just notice that $\overline{F}^{(N)} = \left(F^{(N)}\right)^N = \left(F^{(N)}\right)^{N-1} F^{(N)} = FF^{(N)}$. Also, in case $\overline{k} > p_l$, for any $k \in \left(p_l, \overline{k}\right)$ we know that $F(k) \in (0,1)$ and then

$$\overline{F}^{(N)}(k) = F(k)\, F^{(N)}(k) \to F(k).$$

□

The consequence of the propositions above is that, when the RS pure equilibrium fails to exist, the model with a continuum of firms cannot be simply taken as the limit of a model with N firms, as $N \to \infty$. The problem is that as the number of firms grow, each firm provides worse offers. However, they get worse "slowly" so that the best offer among N firms converges to a nondegenerate distribution F. In the case of a continuum of firms, there is no way to obtain a nondegenerate distribution for the best offer among all firms with independent symmetric randomization across firms.

An important characteristic of the mixed equilibrium constructed is that an outside firm, facing equilibrium offers in the market, can obtain positive expected profits.

Consider the duopoly case. An outside firm (called firm 3) faces two competing offers (from firms 1 and 2) distributed according to F, so that the most attractive competing offer is distributed according to $G = F^2$. If an outside firm considers any offer $\mathcal{M}^k$, for $k \in R$, it would have zero expected profits (by definition of $\gamma(\cdot)$). However, since firms 1 and 2 have zero expected gains from the local deviation around $\mathcal{M}^k$ described in (4), when facing competing offer distributed according to F, firm 3 has a strict gain from such deviation around $\gamma(k)$ that attracts h-type agents with higher probability.

It is quite surprising that firm 3, facing two competing offers distributed according to F, can obtain higher expected profits than a firm facing a single competing offer distributed according to F. In most competitive settings, such as auctions, a player always benefits from less aggressive offer distribution from its competitors. In this model, cross-subsidization between contracts means that the relative frequency with which an offer attracts both types is the important feature.

Hence, in the market with $N > 2$ firms the following is not (part of) an equilibrium: the first N' firms deliver offers according to equilibrium strategies of the market with N' firms and the remaining $N - N'$ firms are inactive, where $2 \leq N' < N$.

6.2 Prior Distribution

One of the main advantages of considering an extensive form version of Rothschild and Stiglitz's model is to obtain equilibrium existence for all prior distributions. Here we consider how equilibrium strategies change with the prior probability of the h-type. The equilibrium distribution of offers continuously changes with the prior distribution, converging to the RS contracts as $\mu_h \to 0$ (it actually becomes a pure strategy whenever condition (1) holds) and converging to the full information full insurance offer (p_h, p_h) as $\mu_h \to 1$. More specifically, this means that both agents benefit from a better pool of agents. This result is in conflict with the ones obtained by Dubey and Geanakoplos (2002) and Guerrieri, Shimer, and Wright (2010), who consider extensions of the Rothschild-Stiglitz model. In both papers, equilibria generate the RS pair of contracts as final outcome for any prior distribution. We will consider a fixed number N of firms, and we will define $F^{(\mu_h)}$ as the equilibrium distribution of offers by a single firm. Also, we will denote as $\overline{F}^{(\mu_h)}$ the equilibrium distribution of the best offers, i.e., the distribution of $k = \max\{k_1, \ldots, k_N\}$.

Proposition 10. *If $\mu_h' > \mu_h$, then $F^{(\mu_h')}$ first-order stochastically dominates $F^{(\mu_h)}$ and $\overline{F}^{(\mu_h')}$ first-order stochastically dominates $\overline{F}^{(\mu_h)}$. When $\mu_h \to 1$, then $F^{(\mu_h)}$ converges to a point mass at p_h. When $\mu_h \to 0$, then $F^{(\mu_h)}$ converges to a point mass at p_l. Moreover, the function $\mu_h \longmapsto F^{(\mu_h)}$ is continuous (weak-convergence).*

7 Conclusion

In this paper, we consider a competitive insurance model in which a finite number of firms simultaneously offer menus of contracts to an agent with private information regarding his risk type. We show that there always exists a unique symmetric equilibrium. This equilibrium features firms offering the separating contracts analyzed in Rothschild and Stiglitz (1976), whenever they can be sustained as an equilibrium outcome. When this is not the case, which occurs if the prior probability of low-risk agents is too high, firms randomize over a set of separating pairs of offers. Firms obtain zero expected profits in equilibrium.

The equilibrium features monotone comparative statics with respect to the prior over types and the number of firms. As the probability of h-type agents grows, firms offer more attractive offers. This shift is strict whenever the equilibrium involves mixed strategies by the firms. Regarding the number of firms, the distribution of the best offer in the market and agent's welfare decrease as it grows. The distribution of the best offer converges to the mixed strategy of a single firm in duopoly.

The equilibrium is continuous with respect to the prior. As a consequence, equilibrium outcomes converge to the perfect information allocation when the prior converges to both extremes. This result is at odds with the usual analysis of the RS model that only focuses on the region of prior probabilities for which a pure equilibrium exists, as well as some recent papers that modify the RS model and obtain equilibrium outcomes that do not depend on the prior (see Guerrieri, Shimer, and Wright (2010); Dubey and Geanakoplos (2002)).

Due to the implicit nature of the equilibrium construction, we are not able to obtain clear comparative statics results with respect to preferences. Numerical exercises suggest that higher constant risk aversion leads to more aggressive offers by the firms and reduces the set of priors for which a pure equilibrium exists.

An interesting question is whether this equilibrium construction is possible for an arbitrary number of types, because the equilibrium existence problem presented by Rothschild and Stiglitz becomes more severe as the number of types increases. For the limiting case of a continuum of types, competitive equilibrium never exists (see Riley (1979, 2001)). In the two types model considered here, every optimal pair of contracts has to satisfy one local optimality condition, which is used to characterize the equilibrium distribution. If there are n potential types, there are $n-1$ such local conditions. All of these conditions have to be simultaneously satisfied at any n-tuple offered in equilibrium. In the case of two types, the region of offers is given by γ and corresponds to the pairs of separating contracts that generate expected zero profits and is a one dimensional object. In the case of n risk types, it is a $n-1$ dimensional object, namely tuples that provide full insurance to the lowest type, leave any given type indifferent between his allocation and the next higher type and generate zero expected profits. The first $n-2$ local conditions characterize the one dimensional region in which the randomization occurs. The local condition connected to the highest type would characterize the equilibrium distribution. So far, we have no proof that this construction indeed works for the n types case.

The analysis presented here is important in order to better understand the classical results on competitive insurance such as non-existence of equilibrium, uniqueness and the welfare impact of private information.

However, there are issues with the interpretation of equilibrium in the insurance market when it involves mixed strategies. The equilibrium presented in Rothschild and Stiglitz (1976) can be seen as a stable set of contracts in a dynamic market. The analogous interpretation is problematic for the mixed strategy equilibrium presented in this paper, since firms eventually "learn" their opponents' policy menus. A systematic analysis of a dynamic competitive insurance model is important, and fully understanding the equilibrium properties in the static model considered here are a key step in this direction.

8 Appendix

Proof of Lemma 3. Continuous differentiability of $u(\cdot)$ implies that γ is continuous and differentiable. Just notice that

$$\gamma_1'(k) = \frac{u'(k) + \frac{\mu_l}{\mu_h}\frac{(1-p_l)}{(1-p_h)}u'(\gamma_0)}{p_l u'(\gamma_1) - (1-p_l)\frac{p_h}{(1-p_h)}u'(\gamma_0)} < 0$$

and

$$\gamma_0'(k) = \frac{u'(k) + \frac{\mu_l}{\mu_h}\frac{p_l}{p_h}u'(\gamma_1)}{(1-p_l)u'(\gamma_0) - p_l\frac{(1-p_h)}{p_h}u'(\gamma_1)} > 0;$$

Therefore, for $K \equiv \left[\frac{p_h}{(1-p_h)}\frac{1}{u'(\gamma_1)} - \frac{p_l}{1-p_l}\frac{1}{u'(\gamma_0)}\right]^{-1}(1-p_l)^{-1} > 0$,

$$\begin{aligned}
\frac{\partial U(\gamma(k) \mid h)}{\partial k} &= p_h u'(\gamma_1)\gamma_1'(k) + (1-p_h)u'(\gamma_0)\gamma_0'(k) \\
&= \begin{array}{c} p_h u'(\gamma_1)\frac{u'(k)+\frac{\mu_l}{\mu_h}\frac{(1-p_l)}{(1-p_h)}u'(\gamma_0)}{p_l u'(\gamma_1)-(1-p_l)\frac{p_h}{(1-p_h)}u'(\gamma_0)} \\ -(1-p_h)u'(\gamma_0)\frac{\frac{p_h}{(1-p_h)}u'(k)+\frac{p_l}{(1-p_h)}\frac{\mu_l}{\mu_h}u'(\gamma_1)}{p_l u'(\gamma_1)-\frac{p_h}{(1-p_h)}(1-p_l)u'(\gamma_0)} \end{array} \\
&= K\left\{\begin{array}{c} u'(k)\,p_h\left[\frac{1}{u'(\gamma_1)} - \frac{1}{u'(\gamma_0)}\right] \\ -\frac{\mu_L}{\mu_h}(1-p_l)\left[\frac{p_h}{(1-p_h)} - \frac{p_l}{(1-p_l)}\right] \end{array}\right\}
\end{aligned} \tag{7}$$

Now notice that

$$
\begin{aligned}
u'(k+\Delta) &\leq u'(k); \\
\frac{1}{u'(\gamma_1(k+\Delta))} - \frac{1}{u'(\gamma_0(k+\Delta))} &< \frac{1}{u'(\gamma_1(k))} - \frac{1}{u'(\gamma_0(k))}.
\end{aligned}
$$

This means that $\frac{\partial U(\gamma(k+\Delta)|h)}{\partial k} > 0 \Rightarrow \frac{\partial U(\gamma(k)|h)}{\partial k} > 0$. □

Proof of Proposition 10. Since we will consider variation in μ_h explicitly, we will acknowledge dependence of each variable on μ_h through the notation, as in $\overline{k}^{\mu_h}$.

First notice that, for each $k \in [p_l, \overline{p}]$, $\gamma^{\mu_h}(k)$ is the solution $\gamma = (\gamma_1, \gamma_0)$ (with $\gamma_1 \geq \gamma_0$) to the following system:

$$
A_1(\gamma, \mu_h, k) \equiv \left[\begin{array}{c} (1-\mu_h)[p_l - k] + \mu_h \Pi(\gamma \mid h) \\ U(\gamma \mid l) - u(k) \end{array} \right] = 0.
$$

Then, since $\frac{\partial A_1}{\partial \gamma}$ is full rank and $A_1(\cdot)$ is clearly continuously differentiable at (γ, μ_h, k), it follows that $\gamma^{\mu_h}(k)$ is continuously differentiable in (μ_h, k). It is simple to show that $\frac{\partial \gamma_1^{\mu_h}(k)}{\partial \mu_h} > 0$ and $\frac{\partial \gamma^{\mu_h}(k)}{\partial \mu_h} < 0$. From (7), we know that $\overline{k}^{\mu_h}$ is defined implicitly as the solution to the following equation:

$$
A_2(k, \mu_h) \equiv u'(k) p_h \left[\frac{1}{u'(\gamma^{\mu_h}(k))} - \frac{1}{u'(\gamma^{\mu_h}(k))} \right] - \frac{\mu_l}{\mu_h}(1-p_l) \left[\frac{p_h}{(1-p_h)} - \frac{p_l}{(1-p_l)} \right] = 0.
$$

But we already showed that $\gamma^{\mu_h}(k)$ is continuously differentiable in (μ_h, k), and therefore so is A_2. Since $\frac{\partial A_2(k,\mu_h)}{\partial k} < 0$, $\overline{k}^{\mu_h}$ is continuously differentiable. Moreover, it follows from $\frac{\partial A_2(k,\mu_h)}{\partial \mu_h} > 0$ that $\frac{\partial \overline{k}^{\mu_h}}{\partial \mu_h} > 0$. Finally, we mention that $A_2\left(\overline{k}^{\mu_h}, \mu_h\right) = 0$ for all $\mu_h \to (0,1)$ implies that $\lim_{\mu_h \to 1} \overline{k}^{\mu_h} = p_h$ (since in the limit both types receive a full insurance allocation that generates zero expected profits).

Finally, notice that, after substitution, we have that

$$
\phi^{\mu_h}(k) \equiv \frac{\left(\frac{1-p_h}{p_h}\right)\left(\frac{1-p_l}{p_l}\right) \left\{ \begin{array}{c} u'(k) \frac{p_h}{(1-p_l)} \left[\frac{1}{u'\left(\gamma_1^{\mu_h}\right)} - \frac{1}{u'\left(\gamma_0^{\mu_h}\right)} \right] \\ -\frac{(1-\mu_h)}{\mu_h} \left[\frac{p_l}{(1-p_l)} - \frac{p_l}{(1-p_l)} \right] \end{array} \right\}}{\left[\frac{(1-p_l)}{p_l} - \frac{(1-p_h)}{p_h} \right] \Pi(\gamma^{\mu_h}(k) \mid h)}. \tag{8}
$$

For any $(\mu_h, k) \in (0,1) \times [p_l, \overline{p}]$, $\Pi(\gamma^{\mu_h}(k) \mid h)$ satisfies

$$
(1-\mu_h)[p_l - k] + \mu_h \Pi(\gamma^{\mu_h}(k) \mid h) = 0. \tag{9}
$$

This implies that $\frac{\partial \Pi(\gamma^{\mu_h}(k)|h)}{\partial \mu_h} < 0$. This implies that, for any $k \in \left[p_l, \overline{k}^{\mu_h}\right]$,

$$\frac{\partial \phi^{\mu_h}}{\partial \mu_h} > 0.$$

Now consider $\mu_h' > \mu_h$. In case $\overline{k}^{\mu_h} = p_l$ the result is trivial; otherwise we have from the ODE (3) satisfied by $\left[F^{(\mu_h)}\right]^{N-1}$, for any $k \in \left[p_l, \overline{k}^{\mu_h}\right)$

$$\begin{aligned}\left[\frac{F^{(\mu_h)}(k)}{F^{(\mu_h')}(k)}\right]^{N-1} &= \exp\left[\int_k^{\overline{k}^{\mu_h'}} \phi^{\mu_h'}(z)\,dz - \int_k^{\overline{k}^{\mu_h}} \phi^{\mu_h}(z)\,dz\right] \\ &= \exp\left[\int_{\overline{k}^{\mu_h}}^{\overline{k}^{\mu_h'}} \phi^{\mu_h'}(z)\,dz + \int_k^{\overline{k}^{\mu_h}} \left[\phi^{\mu_h'}(z) - \phi^{\mu_h}(z)\right] dz\right] > 1.\end{aligned}$$

Monotonicity of $\overline{F}^{(\mu_h)}$ follows from the fact that $\overline{F}^{(\mu_h)} = \left[F^{(\mu_h)}\right]^N$.

Now for a fixed $\mu_h \in (0,1)$, let's consider $\mu_{h,n} \to^{n\to\infty} \mu_h$ monotonically. Monotone convergence implies that for any $k \in \left[p_l, \overline{k}^{\mu_h}\right)$

$$\int_k^{\overline{k}^{\mu_h}} \left[\phi^{\mu_{h,n}}(z) - \phi^{\mu_h}(z)\right] dz \to 0.$$

And continuity of $\overline{k}^{\mu_h}$ implies that

$$\int_{\overline{k}^{\mu_h}}^{\overline{k}^{\mu_{h,n}}} \phi^{\mu_h'}(z)\,dz \to 0.$$

Therefore, it follows that, for any $k \in \left[p_l, \overline{k}^{\mu_h}\right)$

$$\frac{F^{(\mu_h)}(k)}{\overline{F}^{(\mu_{h,n})}(k)} \to 1,$$

as $n \to \infty$.

Finally, we will show that implies that $F^{(\mu_h)}(k) \to 0$ for any $k \in [p_l, p_h]$. We have already shown that $\overline{k}^{\mu_h} \to p_h$. Now, from equation (8), it follows that for any $k \in [p_l, p_h]$, $\lim_{\mu_h \to 1} \phi^{\mu_h}(k) = \infty$. this follows because that (9) implies that $\Pi(\gamma^{\mu_h}(k) \mid h) \to 0$ (the denominator converges to zero). Also, for any $k \in [p_l, p_h)$, we have that $\lim_{\mu_h \to 1} U(\gamma^{\mu_h}(k) \mid l) = u(k)$ so that

$$\lim_{\mu_h \to 1}\left[\frac{1}{u'\left(\gamma_1^{\mu_h}(k)\right)} - \frac{1}{u'\left(\gamma_0^{\mu_h}(k)\right)}\right] > 0,$$

otherwise we would have that $\lim_{\mu_h \to 1} \gamma^{\mu_h} = \overline{\gamma}$ satisfied $\Pi(\overline{\gamma} \mid h) = 0$ and $\overline{\gamma}_1 = \overline{\gamma}_0$, which implies

$\overline{\gamma} = (p_h, p_h)$. (The numerator is bounded.)

Pointwise monotone convergence of $\phi^{\mu_h}(k)$ implies that, for any $k \in [p_l, p_h)$,

$$\int_k^{\overline{k}^{\mu_h}} \phi^{\mu_h}(z)\, dz \nearrow \infty.$$

This implies that $F^{(\mu_h)}(k) = 0$, for any $k \in [p_l, p_h)$. The convergence result for $\mu_h \to 0$ follows trivially from Corollary (1). $\square$

8.1 Proof of Proposition 1

Let the following be an equilibrium:

- Offers $\mathcal{M}_1^*, \ldots, \mathcal{M}_N^* \in \mathbf{M}$;
- acceptance rule $s : \{l, h\} \times (\times_i \mathbf{M}) \to \Delta\left(\left(\mathbb{R}_+^2 \times \{1, \ldots, N\}\right) \cup \{\emptyset\}\right)$.

Let $s_i\left(t, (\mathcal{M}_i)_i\right) \in \Delta\mathbb{R}_+^2$ as the the marginal probability of acceptance of firm i's contracts, i.e., for any measurable set $A \subseteq \mathbb{R}_+^2$ it is defined as follows

$$s_i\left(A \mid t, (\mathcal{M}_i)_i\right) = s\left(A \times \{i\} \mid t, (\mathcal{M}_i)_i\right).$$

Expected profits for firm i, from offer $\mathcal{M}_i$, is the following

$$\begin{aligned}\pi_i(\mathcal{M}_i) &= \mu_h\left[\int \Pi(c \mid h)\, s_i\left(dc \mid h, \mathcal{M}_i, \overline{\mathcal{M}}_{-i}\right)\right] + \mu_l\left[\int \Pi(c \mid l)\, s_i\left(dc \mid l, \mathcal{M}_i, \overline{\mathcal{M}}_{-i}\right)\right] \\ &= \pi_i^h(\mathcal{M}_i) + \pi_i^l(\mathcal{M}_i).\end{aligned}$$

Consequences of equilibrium as as follows. The acceptance rule always has to satisfy the following:

$$(c, i) \in supp\left(s\left(t, (\mathcal{M}_i)_i\right)\right) \Rightarrow c \in \arg\max_{c \in \bigcup_i \mathcal{M}^i \cup \{y\}} U(c \mid t).$$

Firms maximize profits, i.e.,

$$\pi_i\left(\overline{\mathcal{M}}_i\right) \geq \pi_i(\mathcal{M}_i),$$

for all $\mathcal{M}_i \in \mathbf{M}$.

Define u^h and u^l to be the equilibrium utility delivered to each type. It is simply given by the most attractive contract in $(\cup_i \mathcal{M}_i) \cup \{(1, 0)\}$, which is a compact set. Similarly define u_i^h and u_I^l as the best utility available in $\overline{\mathcal{M}}_i$ for types h and l respectively.

Lemma 15. *For any i and $t \in \{l, h\}$, $\pi_i^t\left(\overline{\mathcal{M}}_i\right) \geq 0$. As a consequence, $u_i^t \leq u(p_t)$ for all i.*

Proof. By way of contradiction, consider firm i_0 that has $\pi_i^t\left(\overline{\mathcal{M}}_i\right) < 0$ for $t \in \{l,h\}$, so that its contracts are accepted by the t-type with strictly positive probability in equilibrium. Since $\pi_i\left(\overline{\mathcal{M}}_i\right) \geq 0$, we know that $\pi_i^{t^c}\left(\overline{\mathcal{M}}_i\right) > 0$ for $t^c \neq t$. First, this implies that in equilibrium the probability that firm i_0's contract is accepted by t^c-type is $\varepsilon > 0$. Second, firm i_0 necessarily offers contracts that generate u^h and u^l, i.e., $u_{i_0}^h = u^h$ and $u_{i_0}^l = u^l$.

Consider contract

$$\begin{aligned} c^* &= \arg\max_{c\in\mathbb{R}_+^2} \Pi\left(c \mid t^c\right) \\ & \quad s.t. \begin{cases} U\left(c \mid t^c\right) \geq u^{t^c}; \\ U\left(c \mid t\right) \leq u^t. \end{cases} \end{aligned}$$

Notice that $\Pi\left(c^* \mid t^c\right) > \pi_{i_0}^{t^c}\left(\overline{\mathcal{M}}_i\right) > 0$. Take any firm $i_1 \neq i_0$. Equilibrium profits for firm i_1 are at most

$$\mu_{t^c}\left(1-\varepsilon\right)\Pi\left(c^* \mid t^c\right).$$

However, firm i_1 can offer $\{c'\}$, where c' is close enough to c^* such that: (i) $U\left(c' \mid t^c\right) > u^{t^c}$, (ii) $U\left(c' \mid t\right) < u^t$ and (iii) profits are sufficiently close to $\Pi\left(c^* \mid t^c\right)$, i.e.,

$$\Pi\left(c' \mid t^c\right) - \Pi\left(c^* \mid t^c\right) > -\varepsilon\Pi\left(c^* \mid t^c\right).$$

Therefore, optimality of s implies that

$$\pi_{i_1}\left(\left\{c'\right\}\right) > \mu_{t^c}\left(1-\varepsilon\right)\Pi\left(c^* \mid t^c\right).$$

□

Now we conclude with the main proposition.

Proposition (1)**.** *The equilibrium outcome is $\left(c^{RS,l}, c^{RS,h}\right)$, i.e., the only contract accepted with positive probability by type $t \in \{l,h\}$ is $c^{RS,t}$.*

Proof. First, define $c^{*,t}$ as follows:

$$\begin{aligned} c^{*,t} &\equiv \arg\max_{c\in\mathbb{R}_+^2} \Pi\left(c \mid t\right) \\ & \quad s.t. \begin{cases} U\left(c \mid t\right) \geq u^{t}; \\ U\left(c \mid t^c\right) \leq u^{t^c}. \end{cases} \end{aligned} \tag{10}$$

Notice that $\Pi\left(c^{*,t} \mid t\right) \geq \pi_i^t\left(\overline{\mathcal{M}}_i\right) \geq 0$ for all i and t.

Suppose now that $\Pi\left(c^{*,l} \mid l\right) > 0$. Take firm i_0 that has contract sold to l-type agent with probability below $\frac{1}{N}$. By single-crossing of preferences we can construct c'^h and c'^l such that: (i) both types prefer the c' allocation to c^*, i..e, $U\left(c'^h \mid h\right) > U\left(c^{*,h} \mid h\right)$ and $U\left(c'^l \mid l\right) > U\left(c^{*,l} \mid l\right)$; (ii) types self select into their own contracts, i.e., $U\left(c'^t \mid t\right) > U\left(c'^{t^c} \mid t\right)$ for $t, t^c \in \{l, h\}$ and finally (iii) contract c' is close enough to c^* such that

$$\mu_h\left[\Pi\left(c'^h \mid h\right) - \Pi\left(c^{*,h} \mid h\right)\right] + \mu_l\left[\Pi\left(c'^l \mid l\right) - \Pi\left(c^{*,l} \mid l\right)\right] > -\frac{1}{N}\mu_l\Pi\left(c^{*,l} \mid l\right)$$

$$\Downarrow$$

$$\mu_h\Pi\left(c'^h \mid h\right) - +\mu_l\Pi\left(c'^l \mid l\right) >$$

$$\Pi\left(c^{*,h} \mid h\right) + \left(1 - \frac{1}{N}\right)\mu_l\Pi\left(c^{*,l} \mid l\right) \geq \pi_{i_0}\left(\overline{\mathcal{M}}_i\right).$$

So that firm i_0 would have a profitable deviation.

An analogous argument works in case $\Pi\left(c^{*,h} \mid h\right) > 0$. Therefore we conclude that $\Pi\left(c^{*,t} \mid t\right) = 0$ for both types $t \in \{l, h\}$. The construction of $c^{*,t}$ involves maximization (10) that has unique maximizer. This means that the equilibrium outcome has to be such that only contract $c^{*,t}$ is accepted with positive probability by type $t \in \{l, h\}$.

We illustrate three characteristics of the allocation $c^{*,t}$. First, notice that it is necessarily the case that $c_1^{*,h} \geq c_0^{*,h}$. If $c_1^{*,h} < c_0^{*,h}$, then $u^{-1}\left[U\left(c^{*,h} \mid h\right)\right]$ generates more profit then $c^{*,h}$, a contradiction. Second, we know that $\Pi\left(c^{*,h} \mid h\right) = 0$. Third, notice that $U\left(c^{*,h} \mid l\right) = u^l$ because this restriction does not bind in the maximization (10) then zero profit implies that $c^{*,h} = (p_h, p_h)$ a contradiction with Lemma 15.

We finally conclude that $c^{*,l} = (p_l, p_l)$. If this is not the case, then it follows from maximization (10) that $U\left(c^{*,l} \mid h\right) = u^h$ and in this case $c^{*,l}$ and $c^{*,h}$ are on the same l-type and h-type indifference curves implying that $c^{*,l} = c^{*,h}$. This is a contradiction with the zero profit condition for $c^{*,t}$ for both types. The fact that $c^{*,h}$ leaves l-type agent indifferent between this allocation and (p_l, p_l) and that it generates zero profits implies that $c^{*,h} = c^{RS,h}$. □

8.2 Proof of Proposition 4

Proof. Consider equilibrium offer distribution $\phi \in \Delta(\mathbf{M})$ followed by the firms and acceptance rule $s : \{l, h\} \times (\times_i \mathbf{M}) \to \Delta\left(\left(\mathbb{R}_+^2 \times \{1, \ldots, N\}\right) \cup \{\emptyset\}\right)$.

Let $s_i\left(t, (\mathcal{M}_i)_i\right) \in \Delta\mathbb{R}_+^2$ denote the the marginal probability of acceptance of firm i's contracts, i.e., for any measurable set $A \subseteq \mathbb{R}_+^2$ it is defined as follows

$$s_i\left(A \mid t, (\mathcal{M}_i)_i\right) = s\left(A \times \{i\} \mid t, (\mathcal{M}_i)_i\right).$$

The expected profit firm i obtains by offering $\mathcal{M}_i$ is

$$\begin{aligned}\pi_i(\mathcal{M}_i) &= \int \left\{ \begin{array}{c} \mu_h \left[\int \Pi(c \mid H)\, s_i(dc \mid h, \mathcal{M}_i, \mathcal{M}_{-i})\right] \\ +\mu_l \left[\int \Pi(c \mid l)\, s_i(dc \mid l, \mathcal{M}_i, \mathcal{M}_{-i})\right] \end{array} \right\} d\left(\times_{-i}\phi(\mathcal{M}_{-i})\right). \\ &= \pi_i^h(\mathcal{M}_i) + \pi_i^l(\mathcal{M}_i).\end{aligned}$$

The two implications of equilibrium are: (i) acceptance rule always has to satisfy the following,

$$(c,i) \in supp\left(s\left(t, (\mathcal{M}_i)_i\right)\right) \Rightarrow c \in \arg \max_{c \in \bigcup_i \mathcal{M}^i \cup \{y\}} U(c \mid t);$$

and (ii) firms maximize profits, i.e.,

$$\int \pi_i(\mathcal{M})\, d\phi(\mathcal{M}) \geq \pi_i(\mathcal{M}_i),$$

for all $\mathcal{M}_i \in \mathbf{M}$.

Consider an on-path menu offer $\mathcal{M} \in \mathbf{M}$. For $t \in \{l, h\}$, define $u^{\mathcal{M},t} \equiv \max\{U(c \mid t) \mid c \in \mathcal{M}\}$ and $c^{\mathcal{M},t}$ as follows (denote t^c as type other than t):

$$\begin{aligned} c^{\mathcal{M},t} &\equiv \arg\max_{c \in \mathbb{R}^2_+} \Pi(c \mid t) \\ & \quad s.t. \begin{cases} U(c \mid t) = u^{\mathcal{M},t}; \\ U(c \mid t^c) \leq u^{\mathcal{M},t^c}. \end{cases} \end{aligned} \tag{11}$$

Strict concavity of $u(\cdot)$ implies that the maximizer above is uniquely defined. Any contract $c \in \mathcal{M}$ accepted with positive probability by type t generates (at most) profit $\Pi\left(c^{\mathcal{M},t} \mid t\right)$, since it lies in the constrained set of the optimization problem. It will generate strictly less profits if $c \neq c^{\mathcal{M},t}$.

Also notice that, for any $\mathcal{M}$, $\left(c^{\mathcal{M},l}, c^{\mathcal{M},l}\right)$ satisfies:

1. $c_1^{\mathcal{M},h} \geq c_0^{\mathcal{M},h}$ and $c_1^{\mathcal{M},l} \leq c_0^{\mathcal{M},l}$;
2. for some $t \in \{l, h\}$, $c_1^{\mathcal{M},t} = c_0^{\mathcal{M},t}$ and $U\left(c^{\mathcal{M},t} \mid t\right) = U\left(c^{\mathcal{M},t^c} \mid t\right) = u^{\mathcal{M},t}$.

In the ensuing proof, we will also use the following fact: if $\left(c^l, c^h\right)$ are the limit of a sequence of pairs of contracts $\left(c^{\mathcal{M}_n,l}, c^{\mathcal{M}_n,l}\right)$ offered on path, then the pair $\left(c^l, c^h\right)$ also satisfies these properties.

Part 1. Whenever a firm offers menu $\mathcal{M}$, type t only accepts contract $c^{\mathcal{M},t}$ with positive probability.

Suppose that $\Pi\left(c^{\mathcal{M},t} \mid t\right) > 0$ and $\Pi\left(c^{\mathcal{M},t^c} \mid t^c\right) \leq 0$. By single crossing, monotonicity and continuity, for any $\varepsilon > 0$ we can find contracts $\left(c^{\epsilon,t}, c^{\epsilon,t^c}\right)$ such that: (i)$\left\|c^{\epsilon,t} - c^{\mathcal{M},t}\right\| \leq \varepsilon$ and $\left\|c^{\epsilon,t^c} - c^{\mathcal{M},t^c}\right\| \leq \varepsilon$, (ii) $U\left(c^{\varepsilon,t} \mid t\right) > \max\left\{U\left(c^{\mathcal{M},t} \mid t\right), U\left(c^{\epsilon,t^c} \mid t\right)\right\}$ and (iii) $U\left(c^{\mathcal{M},t^c} \mid t^c\right) > U\left(c^{\varepsilon,t^c} \mid t^c\right) > U\left(c^{\varepsilon,t} \mid t^c\right)$.

For a fixed firm i, let us define H_i^t as the implied distribution of utility obtained by an t-type from offers $\mathcal{M}_{-i}$, distributed according to $\times_{-i}\phi$. Similarly, the probability that an offer from firm i is accepted by type t, given offer $\mathcal{M}$, is denoted[10] $P_i^{\mathcal{M},t}$. It follows from optimality of choice rule s that

$$\lim_{\eta \searrow 0} H_i^t\left(u^{\mathcal{M},t} - \eta\right) \leq P_i^{\mathcal{M},t} \leq H_i^t\left(u^{\mathcal{M},t}\right).$$

As $\varepsilon \to 0$, total expected profit obtained by firm i offering menu $\left\{c^{\varepsilon,t}, c^{\varepsilon,t^c}\right\}$ converges to

$$\mu_t H_i^t\left(u^{\mathcal{M},t}\right)\Pi\left(c^{\mathcal{M},t} \mid t\right) + \mu_{t^c}\lim_{\eta \searrow 0} H_i^{t^c}\left(u^{\mathcal{M},t^c} - \eta\right)\Pi\left(c^{\mathcal{M},t^c} \mid t^c\right),$$

which is greater than the profits obtained by offer $\mathcal{M}$. It generates strictly more profits in case an offer different from $c^{\mathcal{M},t}$ is accepted with positive probability by t-type agent from firm i.

An analogous construction can be made for the cases (i) $\Pi\left(c^{\mathcal{M},t} \mid t\right) > 0$ and $\Pi\left(c^{\mathcal{M},t^c} \mid t^c\right) > 0$ and (ii) $\Pi\left(c^{\mathcal{M},t} \mid t\right) \leq 0$ and $\Pi\left(c^{\mathcal{M},t^c} \mid t^c\right) \leq 0$.

Part 2. Firms make zero profits.

Assume that a firm, namely i, obtains expected profits $\pi_0 > 0$. Define as $\underline{u}^t$ the lowest utility delivered to t-type agent in equilibrium, i.e.,

$$\underline{u}^t = \inf\left\{u \in \mathbb{R} \mid \phi\left(\left\{\mathcal{M} \in \mathbf{M} \mid u^{\mathcal{M},t} \leq u\right\}\right) > 0\right\}.$$

We want to prove that there exists a pair of contracts $\left(\underline{c}^l, \underline{c}^h\right)$ such that: (i) $U\left(\underline{c}^h \mid h\right) = \underline{u}^h$, (ii) $\Pi\left(\underline{c}^h \mid h\right) = 0$, (iii) $\underline{c}_1^h > \underline{c}_0^h$, (iv) $\underline{c}_1^l = \underline{c}_0^l = u^{-1}\left(U\left(\underline{c}^h \mid l\right)\right)$ and (v) $U\left(\underline{c}^l \mid l\right) \geq \underline{u}^l$. We will prove this statement separately for the two possible cases.

Case 1) $H^h\left(\underline{u}^h\right) = 0$. Consider sequence $\{\mathcal{M}_n\}_n$ such that $\mathcal{M}_n \in supp\,(\phi)$ and $H^h\left(U\left(c^{\mathcal{M}_n,h} \mid h\right)\right) = \frac{1}{n}$ (for n large enough). First notice that

$$\pi_0 = \pi_i^h\left(\mathcal{M}_n\right) + \pi_i^l\left(\mathcal{M}_n\right) \leq \pi_i^l\left(\mathcal{M}_n\right) + \frac{1}{n}p_h. \tag{12}$$

This means that almost all the profits will come from the l-type contract $c^{\mathcal{M}_n,l}$. Offering the certainty equivalent allocation to the l-type must not be profitable, which means that

$$H^l\left(U\left(c^{\mathcal{M}_n,l} \mid l\right)\right)\Pi\left(u^{-1}\left(U\left(c^{\mathcal{M}_n,l} \mid l\right)\right) \mid l\right) \leq H^l\left(U\left(c^{\mathcal{M}_n,l} \mid l\right)\right)\Pi\left(c^{\mathcal{M}_n,l} \mid l\right) + \frac{1}{n}p_h,$$

which can be rewritten as (remember that the certainty equivalent allocation is always the most profitable one, given a utility level)

$$0 \leq H^l\left(U\left(c^{\mathcal{M}_n,l} \mid l\right)\right)\left[\Pi\left(u^{-1}\left(U\left(c^{\mathcal{M}_n,l} \mid l\right)\right) \mid l\right) - \Pi\left(c^{\mathcal{M}_n,l} \mid l\right)\right] \leq \frac{1}{n}p_h. \tag{13}$$

[10] The probability $P_i^{\mathcal{M},t}$ may depend on the identity of the firm in a symmetric equilibrium, since the acceptance rules by the agent might be asymmetric.

Let $\left(\underline{c}^l, \underline{c}^h\right)$ be the limit of any convergent subsequence of contract pairs. It follows that $\Pi\left(\underline{c}^h \mid h\right) = 0$. In case $\Pi\left(\underline{c}^h \mid h\right) > 0$, by single crossing we can find allocation $\tilde{c}^h$ close to $\underline{c}^h$ such that (i) $U\left(\tilde{c}^h \mid h\right) > U\left(\underline{c}^h \mid h\right)$, (ii) $U\left(\tilde{c}^h \mid l\right) < U\left(\underline{c}^h \mid l\right)$ and (iii) $\Pi\left(\tilde{c}^h \mid h\right) > 0$. Also consider allocation $\tilde{c}^l_\varepsilon$ such that (i) $U\left(\tilde{c}^l_\varepsilon \mid l\right) > U\left(\underline{c}^l \mid l\right)$, (ii) $U\left(\tilde{c}^l_\varepsilon \mid h\right) < U\left(\underline{c}^l \mid h\right)$ and (iii) $\left\|\tilde{c}^l_\varepsilon - \underline{c}^l\right\| \leq \varepsilon$. Offering pair $\left(\tilde{c}^l_\varepsilon, \tilde{c}^h\right)$, for ε arbitrarily small, generates profit strictly above π_0 since the profit obtained from l-type converges to π_0 as $\varepsilon \to 0$ and the profit obtained form the h-type is strictly positive, since $H^h\left(U\left(\tilde{c}^h \mid h\right)\right)\Pi\left(\tilde{c}^h \mid h\right) > 0$.

Since, as shown, $\underline{c}^h_1 \geq \underline{c}^h_0$. It follows that $\underline{c}^h$ is the unique contract such that (i) generates zero profit, i.e., $\Pi\left(c \mid h\right) = 0$, (ii) generates utility $\underline{u}^h$, i..e., $U\left(c \mid h\right) = \underline{u}^h$ and (iii) has partial insurance, i.e., $c_1 \geq c_0$. Also, since the allocation $\underline{c}^l$ generates profits, we know that $\underline{c}^h_1 > \underline{c}^h_0$, otherwise the agent of type l would the full insurance contract of the h-type agent. $\underline{c}^h_1 > \underline{c}^h_0$ and the fact that $\underline{c}^h$ solves maximization problem (11) at utilities $\underline{u}^h$ and $U\left(\underline{c}^l \mid l\right)$ implies that

$$U\left(\underline{c}^l \mid l\right) = U\left(\underline{c}^h \mid l\right),$$

and $\underline{c}^l_1 = \underline{c}^l_0$.

Since $\underline{c}^l$ is a full insurance allocation, it follows that $\underline{c}^l = u^{-1}\left(U\left(\underline{c}^h \mid l\right)\right)$. The fact that $U\left(\underline{c}^l \mid l\right) \geq \underline{u}^l$ follows from continuity of $u\left(\cdot\right)$.

Case 2) $H^h\left(\underline{u}^h\right) > 0$.

Consider any on-path offer $\mathcal{M} \in \mathbf{M}$ and firm i such that (i) $u^{\mathcal{M},h} = \underline{u}^h$ and (ii) $P^{\mathcal{M},h}_i < \frac{1}{2}H^h\left(u^{\mathcal{M},h}\right)$. Since there is a tie in the utility offered to the h-type with positive probability, there is at least one firm that receives acceptance by the h-types with less than $\frac{1}{2}$ chance in case the tie occurs. Define $\underline{c}^h = c^{\mathcal{M},h}$ and $\underline{c}^l = c^{\mathcal{M},l}$. First, $\Pi\left(\underline{c}^h \mid h\right) = 0$. Suppose that $\Pi\left(\underline{c}^h \mid h\right) > 0$. In this case there exists an offer that generates strictly more profits than $\mathcal{M}$. Suppose that $\Pi\left(\underline{c}^l \mid l\right) \geq 0$ (an analogous argument can be made in case $\Pi\left(\underline{c}^l \mid l\right) < 0$). By single crossing, for any $\varepsilon > 0$ we can construct $\left(\tilde{c}^l_\varepsilon, \tilde{c}^l_\varepsilon\right)$ such that: (i) $U\left(\underline{c}^h \mid l\right) > U\left(\tilde{c}^h_\varepsilon \mid l\right)$; $U\left(\tilde{c}^h_\varepsilon \mid h\right) > U\left(\underline{c}^h \mid h\right)$, (ii) $U\left(\underline{c}^l \mid h\right) > U\left(\tilde{c}^l_\varepsilon \mid h\right)$ and $U\left(\tilde{c}^l_\varepsilon \mid l\right) > U\left(\underline{c}^l \mid l\right)$ and (iii) $\left\|\underline{c}^t - \tilde{c}^t_\varepsilon\right\| \leq \varepsilon$. As $\varepsilon \to 0$, the profit obtained from this offer converges to

$$H^l\left(U\left(\underline{c}^l \mid l\right)\right)\Pi\left(\underline{c}^l \mid l\right) + H^h\left(U\left(\underline{c}^h \mid h\right)\right)\Pi\left(\underline{c}^h \mid h\right).$$

This is a strict improvement for firm i, since acceptance by the h-type occurs with strictly higher probability. Therefore, for ε arbitrarily small, offer $\left(\tilde{c}^l_\varepsilon, \tilde{c}^h_\varepsilon\right)$ is a strict improvement from $\mathcal{M}$, a contradiction.

Now we conclude the proof of part 2. I claim that $U\left(\underline{c}^l \mid l\right) = \underline{u}^l$. Suppose that there exists another offer $\mathcal{M} \in supp\left(\phi\right)$ such that $U\left(c^{\mathcal{M},l} \mid l\right) < U\left(\underline{c}^l \mid l\right)$. This pair of contracts must deliver utility below $\underline{u}^h$ to the h-type agent, a contradiction with $\underline{u}^h$ being a lower bound. We know that one of the following holds: (i) $c^{\mathcal{M},l}_1 = c^{\mathcal{M},l}_0$ and $U\left(c^{\mathcal{M},l} \mid l\right) = U\left(c^{\mathcal{M},h} \mid l\right)$, or (ii) $c^{\mathcal{M},l}_1 \leq$

$c_0^{\mathcal{M},l}$, $U\left(c^{\mathcal{M},h} \mid h\right) = U\left(c^{\mathcal{M},l} \mid h\right)$ and $c_1^{\mathcal{M},h} = c_0^{\mathcal{M},h}$. If (i) is true, than $U\left(c^{\mathcal{M},h} \mid h\right) < U\left(\underline{c}^h \mid h\right)$, otherwise we would contradict the fact that $\underline{c}^h$ is the unique solution to problem (11) at utility levels $U\left(c^{\mathcal{M},h} \mid h\right)$ and $U\left(c^{\mathcal{M},l} \mid l\right)$. If (ii) is true, than necessarily $c_1^{\mathcal{M},h} < \underline{c}_1^l$ (because the l-type agent has lower utility). However, we know that $\underline{c}_1^l = u^{-1}\left(U\left(\underline{c}^h \mid l\right)\right) < u^{-1}\left(U\left(\underline{c}^h \mid h\right)\right)$ since $\underline{c}_1^h > \underline{c}_0^h$. This means that an agent of type h has lower utility at offer $\mathcal{M}$ than in $\underline{c}^h$, a contradiction.

Finally, we claim that firm i must make zero expected profits.

Case $H^l\left(\underline{u}^l\right) = 0$. If $H^h\left(\underline{u}^h\right) > 0$, consider offer $\mathcal{M}$ such that $U\left(c^{\mathcal{M},h} \mid h\right) = \underline{u}^h$. We already know that $\Pi\left(c^{\mathcal{M},h} \mid h\right) = 0$, but since $U\left(c^{\mathcal{M},l} \mid l\right) = \underline{u}^l$, it also follows that profits obtained from the l-type agents are zero as well.

If $H^h\left(\underline{u}^h\right) = 0$. Consider sequence of menu offers $\{\mathcal{M}_n\}_n$ such that $c^{\mathcal{M}_n,h} \to \underline{c}^h$. We already know that $c^{\mathcal{M}_n,l} \to \underline{c}^l$, therefore the profit obtained with this offer converges to zero, since acceptance probabilities go to zero.

Case $H^l\left(\underline{u}^l\right) > 0$. In this case, any offer $\mathcal{M}$ such that $U\left(c^{\mathcal{M},l} \mid l\right) = \underline{u}^l$, it must be the case that $\Pi\left(c^{\mathcal{M},l} \mid l\right) = 0$. Otherwise, one of the firms could improve by offering $\tilde{c}^l$ close to $c^{\mathcal{M},l}$ that generates utility above $\underline{u}^l$ to the l-type (and decreases utility obtained by the h-type). Since $\underline{u}^l < u\left(p_l\right)$, it is necessarily the case that $c_0^{\mathcal{M},l} > c_1^{\mathcal{M},l}$ (maximization (11) implies that $c_0^{\mathcal{M},l} \geq c_1^{\mathcal{M},l}$). Therefore it must be the case that the h-type allocation is of full insurance and leaves him indifferent between $c^{\mathcal{M},l}$ and $c^{\mathcal{M},h}$, i.e., $c_1^{\mathcal{M},h} = c_0^{\mathcal{M},h} = u^{-1}\left(U\left(c^{\mathcal{M},l} \mid h\right)\right)$. But generates utility strictly lower than $\underline{c}^h$, a contradiction since $U\left(\underline{c}^h \mid h\right) = \underline{u}^h$.

Part 3. $\Pi\left(c^{\mathcal{M},l} \mid l\right) \leq 0$, for all $\mathcal{M} \in supp\left(\phi\right)$.

Consider offering a single contract $c^{\varepsilon} \equiv \left(p_l - \varepsilon, p_l - \varepsilon\right)$, for some $\varepsilon > 0$. This contract generates profits whenever accepted because

$$\Pi\left(c^{\epsilon} \mid h\right) > \Pi\left(c^{\epsilon} \mid l\right) = \varepsilon > 0.$$

Therefore, it must be accepted with zero probability, for any $\varepsilon > 0$. This means that $H^l\left(u\left(p_l - \varepsilon\right)\right) = 0$, for any $\varepsilon > 0$. Therefore, all the offers accepted by the l-type agent generate at least utility $u\left(p_l\right)$. However, since the actuarially fair contract $\left(p_l, p_l\right)$ maximizes expected utility of the l-type among the contracts that generate nonnegative contracts, it follows that

$$\begin{gathered} \mathcal{M} \in supp\left(\phi\right) \\ \Downarrow \\ U\left(c^{\mathcal{M},l} \mid l\right) \geq u\left(p_l\right) \\ \Downarrow \\ \Pi\left(c^{\mathcal{M},l} \mid l\right) \leq 0. \end{gathered}$$

Part 4. $\Pi\left(c^{\mathcal{M},t} \mid h\right) \geq 0$.

First, assume that for some on-path offer $\mathcal{M} \in \mathbf{M}$ the following holds: $\Pi\left(c^{\mathcal{M},h} \mid h\right) < 0$. Consider a firm i such that $P_i^{\mathcal{M},h} > 0$. Since firms make nonnegative profits, we know that $P_i^{\mathcal{M},l} > 0$ and $\Pi\left(c^{\mathcal{M},l} \mid l\right) > 0$. This implies that $\Pi\left(c^{\mathcal{M},l} \mid h\right) > 0$. Then, firm i could strictly improve by just offering contract $c^{\mathcal{M},l} + (\varepsilon, \varepsilon)$, for $\varepsilon > 0$ sufficiently small. This offer would guarantee the same profits from the l-types and also generate at least no loss from the h-type agent.

Now consider the case $\Pi\left(c^{\mathcal{M},l} \mid h\right) < 0$ and firm i such that $P_i^{\mathcal{M},l} > 0$. Similarly, this implies that $P_i^{\mathcal{M},h} > 0$ and $\Pi\left(c^{\mathcal{M},h} \mid h\right) > 0$. We know that $c_0^{\mathcal{M},l} > c_1^{\mathcal{M},l}$, because if $c^{\mathcal{M},l}$ was a full insurance allocation the initial assumption implies that $c_1^{\mathcal{M},l} = c_0^{\mathcal{M},l} > p_h$, so that any contract that generates as much utility to the h-type as this one generates a loss. Therefore, we know that $c_1^{\mathcal{M},h} = c_0^{\mathcal{M},h} < p_h$ and $c_1^{\mathcal{M},h} = u^{-1}\left(U\left(c^{\mathcal{M},l} \mid h\right)\right)$. The fact that h has a preference for consumption in state 1 implies that $u^{-1}\left(U\left(c^{\mathcal{M},l} \mid h\right)\right) < u^{-1}\left(U\left(c^{\mathcal{M},l} \mid l\right)\right)$. This means that (i) $c^{\mathcal{M},h}$ attracts the l-type strictly less than $c^{\mathcal{M},l}$ and (ii) $c^{\mathcal{M},h}$ generates more profits than $c^{\mathcal{M},l}$, because

$$\Pi\left(c^{\mathcal{M},h} \mid l\right) > \Pi\left(u^{-1}\left(U\left(c^{\mathcal{M},l} \mid l\right)\right) \mid l\right) \geq \Pi\left(c^{\mathcal{M},l} \mid l\right).$$

Therefore offering single contract $\mathcal{M}' \equiv \left\{c^{\mathcal{M},h} + (\varepsilon, \varepsilon)\right\}$ for $\varepsilon > 0$ sufficiently small generates strictly more profits. It generates at least as much profits from the h-type. Also, it is accepted with smaller probability $P_i^{\mathcal{M}',l} < P_i^{\mathcal{M},l}$ by the l-type and generates more profits, whenever accepted.

Notice that this conclusion implies that the set of contracts that are accepted with positive probability lie in the compact set $\left\{c \in \mathbb{R}_+^2 \mid \Pi\left(c \mid h\right) \geq 0\right\}$.

Part 5. $c_1^{\mathcal{M},l} = c_0^{\mathcal{M},l}$.

Suppose that, for some on-path offer $\mathcal{M}$, we have that $c_0^{\mathcal{M},l} > c_1^{\mathcal{M},l}$. In this case we know that, necessarily, $c_1^{\mathcal{M},h} = c_0^{\mathcal{M},h}$ and $U\left(c^{\mathcal{M},h} \mid h\right) = U\left(c^{\mathcal{M},l} \mid h\right)$. If $\Pi\left(c^{\mathcal{M},l} \mid l\right) < 0$, than considering a firm i such that $P_i^{\mathcal{M},l} > 0$. By the same construction used in the proof of Part 4, firm i can strictly improve by just offering $c^{\mathcal{M},h} + (\varepsilon, \varepsilon)$ for $\varepsilon > 0$ sufficiently small.

In case $\Pi\left(c^{\mathcal{M},l} \mid l\right) = 0$, we know that the certainty equivalent of this allocation, for type l, is profitable. This means that $\Pi\left(u^{-1}\left(U\left(c^{\mathcal{M},l} \mid l\right)\right)\}l\right) > 0$. Therefore, any firm can guarantee strictly positive profits by offering (c, c), where $c = u^{-1}\left(U\left(c^{\mathcal{M},l} \mid l\right)\right) + \varepsilon$, for $\varepsilon > 0$ sufficiently small so that $\Pi\left((c, c) \mid l\right) > 0$. This is a contradiction. $\square$

8.3 Proof of Lemma 12

In order to use the integration formula presented in Lemma 12, we need to bound variation of the integrand. This is what we do in this section. We have chosen to present this results separately because of their technical nature and the fact that they do not add novel intuition to the arguments presented before.

Lemma 16. *Let* $C \equiv \left\{c \in \mathbb{R}_+^2 \mid U(c \mid l) \geq u(p_l),\ c_1 \geq c_0 \text{ and } \Pi(c \mid h) \geq 0\right\}$. *Then*

$$\sup_{c,c' \in C} \frac{\Pi(c \mid h) - \Pi(c' \mid h)}{\|c - c'\|} < \infty,$$

i.e., the restriction of $\Pi(\cdot \mid h)$ *to* C *is Lipschitz.*

Proof. Follows from compactness of C and continuous differentiability of function $\Pi(\cdot \mid h)$. □

Now we are interested in the absolute continuity of h. Since we already know that a pure equilibrium might exist, we won't be able to prove that h is absolutely continuous. However, we will show that there (at most) one problematic point, namely the pure Rothschild-Stiglitz h-type utility level $U\left(c^{RS,h} \mid h\right)$. The utility distribution h will be absolutely continuous away from this point. This will allow us to use the integral formula for utility levels above this 'problematic' one.

Lemma 17. *Function* $H^h(\cdot)$ *has bounded variation except (possibly) for the lowest h-type utility provided. If* $\sup_{U,U' \in \mathbb{R}} \frac{H^h(U) - H^h(U')}{U - U'} = \infty$, *then there exists a unique* $\bar{c} \in supp(\Phi)$ *such that:*
(i) $\frac{H^h(U_n) - H^h(U'_n)}{U_n - u'_n} \to \infty$ *implies* $\lim_n U_n = \lim_n U'_n = U(\bar{c} \mid h)$ *and*
(ii) $\mu(u_l < U(\bar{c} \mid l)) = \mu(u_h < U(\bar{c} \mid h)) = 0$.

Proof. Assume that U is not a mass point of $H^h(\cdot)$ and $c \in supp(\Phi)$ such that $U(c \mid h) = U$. Then optimality implies that a deviation to offer $c^{\varepsilon} = (c_1 + \varepsilon, c_0 - \xi(\varepsilon))$ with $\xi(\cdot)$ defined implicitly as in equation (4). Let us consider deviation $c^{\Delta} = c^{\varepsilon(\Delta)}$ such that $U\left(c^{\varepsilon(\Delta)} \mid h\right) = U + \Delta$. Since the deviation is non-profitable we have that

$$H^h(U)\,\Pi(c \mid h) \geq H^h(U + \Delta)\,\pi\left(c^{\Delta} \mid h\right)$$

which can be rewritten as

$$\frac{H^h(U + \Delta) - H^h(U)}{\Delta} \leq H^h(U)\left[\frac{\Pi(c \mid h) - \Pi\left(c^{\Delta} \mid h\right)}{\Delta}\right]\frac{1}{\Pi\left(c^{\Delta} \mid h\right)}.$$

Then, if $\frac{H^h(U_n) - H^h(U'_n)}{U_n - u'_n} \to \infty$, by compactness of the support of Φ, U_n and U'_n are generated by c_n and c'_n in the support of Φ and $\lim_n c_n = \lim_n c'_n = \bar{c}$. From the equation above we know that $\Pi(\bar{c} \mid h) = 0$. This means that the offers considered make almost zero profits from the h types, therefore they must make almost zero losses from the l-type, i.e.,

$$\mu(u_l < U(c_n \mid l)) \to^n 0. \tag{14}$$

Then we know that offer $\bar{c} - (\varepsilon, \varepsilon)$ is certainly never accepted by the l-type, therefore since firms might make zero profits we know that

$$\mu\left(u_h < U\left(\overline{c} \mid h\right)\right) = 0. \tag{15}$$

Now to the uniqueness of $\overline{c}$. Assume there are two such $\overline{c} \neq \overline{\overline{c}}$ (that are limits of such sequences), then both of them lie in the zero profit line for the h-type. This means that $U\left(\overline{c} \mid l\right) \neq U\left(\overline{\overline{c}} \mid l\right)$, and then (14) and (15) cannot hold for both. $\square$

Now we finally conclude with the desired condition.

Corollary 6. *The restriction of $H^h\left(\cdot\right)$ to $\left(\underline{u}, \infty\right]$ is absolutely continuous.*

Proof. From Lemma 17, it follows that the restriction of $H^h\left(\cdot\right)$ to $\left[\underline{u} + \varepsilon, \overline{u}\right]$ is Lipschitz, for any $\varepsilon > 0$ small. Therefore $H^h\left(\cdot\right)$ restricted to this domain is absolutely continuous. Consider a set $A \subseteq \left(\underline{u}, \overline{u}\right]$ of zero Lebesgue measure. Absolute continuity of the restrictions of $H^h\left(\cdot\right)$, we know that $\int \left(\mathbf{1}A\right)\left(\mathbf{1}\left[\underline{u} + \frac{1}{n}, \overline{u}\right]\right) dH^h = 0$ for any $n \in \mathbb{N}$. By dominated convergence, we know that

$$\int \left(\mathbf{1}A\right)\left(\mathbf{1}\left(\underline{u}, \overline{u}\right]\right) dH^h = 0.$$

$\square$

This means that we can find a Radon-Nikodym derivative $h : D_0 \to \mathbb{R}_+$ such that for any measurable function $f : \mathbb{R} \to \mathbb{R}_+$

$$\int f\left(z\right) dH^h\left(z\right) = \int_{\underline{u}}^{\overline{u}} f\left(z\right) h\left(z\right) dz + H^h\left(\underline{u}\right) f\left(\underline{u}\right).$$

Finally, we can conclude with the proof of Lemma 12.

For any Lipschitz differentiable function $f : D_0 \to \mathbb{R}$, $U \in D_0$ and $\Delta > 0$, we want to compute

$$H^h\left(U + \Delta\right) f\left(U + \Delta\right) - H^h\left(U\right) f\left(U\right). \tag{16}$$

The first part is given by

$$\int h\left(z\right) f\left(U + \Delta\right)\left(\mathbf{1}\left(\underline{u}, U + \Delta\right]\right) dz + H^h\left(\underline{u}\right) f\left(U + \Delta\right),$$

while the second term can be written as

$$\int h(z)\, f(U)\, (\mathbf{1}(\underline{u}, U])\, dz + H^h(\underline{u})\, f(U)$$
$$= \int h(z)\, f(U)\, (\mathbf{1}(\underline{u}, U])\, dz + H^h(\underline{u})\, f(U) \pm \int h(z)\, f(U+\Delta)\, (\mathbf{1}(\underline{u}, U])\, dz$$
$$= \int h(z)\, f(U+\Delta)\, (\mathbf{1}(\underline{u}, U])\, dz - \int h(z)\, [f(U+\Delta) - f(U)]\, (\mathbf{1}(\underline{u}, U])\, dz + H^h(\underline{u})\, f(U),$$

and by using Fubini's theorem it reduces to

$$\int h(z)\, f(U+\Delta)\, (\mathbf{1}(\underline{u}, U])\, dz - \int \left[f'(\omega)\, (\mathbf{1}(U, U+\Delta](\omega))\right] d\omega \left[H^h(U) - H^h(\underline{u})\right] + H^h(\underline{u})\, f(U).$$

Therefore, the final difference in (16) reduces to

$$\int h(z)\, f(U+\Delta)\, (\mathbf{1}(U, U+\Delta])\, dz + \int \left[f'(\omega)\, (\mathbf{1}(U, U+\Delta](\omega))\right] d\omega \left[H^h(U) - H^h(\underline{u})\right]$$
$$+ H^h(\underline{u})\, [f(U+\Delta) - f(U)]$$
$$= \int_U^{U+\Delta} \left[h(z)\, f(U+\Delta) + H^h(U)\, f'(z)\right] dz.$$

The integral in Lemma 12 uses this fact applied to the case in which $f(U) = \Pi(c(U_l, U) \mid h)$, with $c(U_l, U)$ being the contract that delivers utility levels U_l and U_h, i..e, it is defined implicitly as the solution to $U(c \mid l) = U_l$ and $U(c \mid h) = U_h$.

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