Contracting with Heterogeneous Externalities[*]

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Abstract

We model situations in which a principal offers a set of contracts to a group of agents to participate in a project (such as a social event or a commercial activity). Agents' benefits from participation depend on the identity of other participating agents. We assume multilateral externalities and characterize the optimal contracting scheme. We show that the optimal contracts' payoff relies on a ranking of the agents, which can be described as arising from a tournament among the agents (similar to ones carried out by sports associations). Rather than simply ranking agents according to a measure of popularity, the optimal contracting scheme makes use of a more refined two-way comparison between the agents. Using the structure of the optimal contracts we derive results on the principal's revenue extraction and the role of the level of externalities' asymmetry.

Keywords: Bilateral contracting, heterogeneous externalities, mechanism design

1 Introduction

The success of economic ventures often depends on the participation of a group of agents among which externalities prevail. Very often these externalities are heterogeneous in the sense that when agents are making their participation choices they are considering not only how many agents are expected to participate but, more importantly, *who* is expected to participate. The focus of this paper is the implications of heterogeneous externalities in a bilateral contracting environment. This emphasis on heterogeneous externalities allows capturing a realistic ingredient of bilateral contracts, which are affected by the complex relationships between the agents.

Consider first a few bilateral contracting examples. An owner of a mall needs to convince store owners to lease stores. Standardization agency succeeds in introducing a new standard if it manages to attract a group of firms to adopt the new standard. A raider makes tender offers to major shareholders in a target firm. The raider's success hinges on gathering enough shares to gain control. Throwing a party or organizing a conference are yet other examples; their success depends on the participation of the invited guests.

Such contracting environments generate externalities that are rarely symmetric. In a mall, a small store substantially gains from the presence of an anchor store (such as a national brand name), while the opposite externality, induced by the small store, has hardly any effect. The recruitment of a senior star to an academic department can easily attract a young assistant professor to apply to that department, but not the other way around. The adoption of a new standard proposed by a standardization agency induces externalities among the adopting firms but the level of benefits for a given firm crucially depends on the identity of the other adopting firms.

We explore a project initiated by a certain party (henceforth a principal), whose

success depends on the participation of other agents. The principal provides incentive contracts to convince them to participate (incentives could be discounts, gifts, or other benefits). The goal is to design these contracts optimally in view of the prevailing heterogeneous externalities between the agents. Any set of participating agents generates some revenue for the principal, and the principal attempts to maximize his revenue net of the cost of the optimal contracting scheme.

The profit maximization problem can be separated into two stages: the *selection stage*, in which the principal selects the target audience for the venture, and the *participation stage*, in which the principal introduces a set of contracts to induce the participation of the selected group. Clearly, these two stages are related. To work out the overall solution we solve backward by first characterizing the optimal contracts that induce the participation of a given group, which in turn will enable solving optimally the selection part of the problem. Our focus in the paper is on the second stage, the characterization of the optimal contracts for a general set of agents.

The heterogeneous externalities among agents are described in our model by a matrix whose entry $w_i(j)$ represents the extent to which agent i benefits from joint participation with agent j. A contracting scheme is a vector of rewards (offered by the principal) that sustains agents' participation at minimal cost (or maximal total extraction) to the principal. In characterizing the optimal contracts we will focus on the following questions: 1. What is the hierarchy of incentives across agents as a function of the externalities; i.e., who should be getting higher-powered incentives for participation? 2. How does the structure of externalities affect the principal's cost of sustaining the group's participation?

We show that the optimal contracts are determined by a virtual popularity tournament among the agents. In this tournament, we say that agent i *beats* agent j if agent j's benefit from i's participation is greater than i's benefit from

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j's participation. This binary relation is described by a directed graph. We use basic graph theory arguments to characterize the optimal contracts and show that success in the virtual tournament ranks agents according to the payoffs they receive in the optimal contracting scheme.

The idea that agents who induce relatively stronger positive externalities receive higher payoffs is supported by an empirical paper by Gould et al. (2005). They demonstrate how externalities between stores in malls affect contracts offered by the mall's owners. As in our model, stores are heterogeneous in the externalities they induce on each other. Anchor stores generate large positive externalities by attracting most of the customer traffic to the mall, and therefore increase the sales of non-anchor stores. The most noticeable characteristic of mall contracts is that most anchor stores either do not pay any rent or pay only trivial amounts. On average, anchor stores occupy over 58% of the total leasable space in the mall and yet pay only 10% of the total rent collected by the mall's owner.

We point out that since our optimal contracts are derived by means of a virtual tournament our results are surprisingly connected to the literature on two quite distinct topics: 1. ranking sports teams based on tournament results, which has been discussed in the Operations Research literature, and 2. ranking candidates based on the outcome of binary elections. It turns out that Condorcet's (1785) solution to the voting problem as well as the methods proposed by the Operations Research literature to the first problem are closely related to our solution of the participation problem.

A key characteristic of the structure of externalities in a certain group of agents is the level of asymmetry between the pairs of agents, which we show to reduce the principal's cost. Put differently, the principal gains whenever the bilateral benefits between any two agents are distributed more asymmetrically (less mutually). Such greater asymmetry allows the principal more leverage in exploiting the externalities

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to lower costs. This observation has an important implication on the principal's choice of group for the initiative in the selection stage.

This work is part of an extensive literature on multi-agent contracting in which externalities arise between the agents and is akin to various applications introduced in the literature[1]. Most of the literature assumes that externalities depend on the volume of aggregate trade, and not on the identity of the agents. Our emphasis on heterogeneous externalities allows us to capture a more realistic ingredient of the contracting environment, which is affected by the complex relationships between the agents. Heterogeneous externalities were used in Jehiel and Moldovanu (1996) and Jehiel, Moldovanu, and Stachetti (1996), which consider the sale of a single indivisible object by the principal to multiple heterogeneous agents using auctions, when the utilities of the agents depend on which agent ultimately receives the good. Jehiel and Moldovanu (1999) introduce resale markets and consider the implications of the identity of the initial owner of the good on the final consumer.

Our general approach is closely related to the seminal papers by Segal (1999, 2003) on contracting with externalities. These papers present a generalized model for the applications mentioned above as well as others. We add to these paper by considering the implications of heterogeneous externalities. Our paper is also related to the incentive schemes investigated by Winter (2004) in the context of organizations. While we provide a solution for *partial implementation*, in which agents' participation is achieved in a Nash equilibrium, we follow Segal (2003)

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[1]To give a few examples, these applications include vertical contracting models (Katz and Shapiro 1986a; Kamien, Oren, and Tauman 1992) in which the principal supplies an intermediate good to N identical downstream firms (agents), which then produce substitute consumer goods; employment models (Levin 2002) in which a principal provides wages to induce effort in a joint production of a group of workers; exclusive dealing models (Rasmusen, Ramseyer, and Wiley 1991; Segal and Whinston 2000) in which the principal is an incumbent monopolist who offers exclusive dealing contracts to N identical buyers (agents) in order to deter the entry of a rival; acquisition for monopoly models (Lewis 1983; Kamien and Zang 1990; Krishna 1993) in which the principal makes acquisition offers to N capacity owners (agents); and network externalities models (Katz and Shapiro 1986b).

and Winter (2004) in that we concentrate on situations in which the principal cannot coordinate agents on his preferred equilibrium. That is, we mainly consider contracts that sustain agents' participation as a unique Nash equilibrium; i.e., *full implementation* is achieved. Indeed, recent experimental papers (see Brandt and Cooper 2005) indicate that in an environment of positive externalities agents typically are trapped in the bad equilibrium of no-participation.

We demonstrate that our analysis is valid in more general settings. We consider situations in which agents' choices are sequential and show that our solution is important when the principal is interested in a dominant strategies solution.[2] We show that the analysis remains valid when we allow the externalities to affect agents' outside options, as well as with more complicated contingent contracts. Finally, we consider more general externality structures. In particular, we allow externalities to be both negative and positive, and provide the conditions under which the solution for the mixed externalities participation problem is a simple joint solution of the separated negative and positive participation problems. Also, we consider the case of a non-additive externality structure.

The rest of the paper is organized as follows. In Section 2 we provide a simple two-agent example to illustrate some of the key results in the paper. We introduce the general model in Section 3 and Section 4 provides the solution for a participation problem with positive externalities between the agents. In particular, we derive the ranking of incentives in the optimal contracting scheme by forming a virtual popularity tournament between the agents and explore how the externality structure affects the principal's costs. In Section 5 we consider several extensions of the model, in which we demonstrate that our results apply in more general settings. In Section 6 we demonstrate how this model can be used to solve selection problems. Section 7 concludes. Proofs are presented in the Appendix.

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[2]In situations of complicated backward induction reasoning, dominant strategies can be useful.

2 A Simple Two-agent Example

To illustrate some of the key ideas in this paper let's consider a simple two-agent example. Suppose a principal would like to attract agents 1 and 2 to take part in his initiative by offering agent $i \in \{1, 2\}$ a contract that pays v_i if he participates. Let's assume the agents have identical outside options in case they decline the principal's offer of $c > 0$. Furthermore, the decision to participate induces an externality on the other agent. If agent 1 participates, agent 2's benefit (loss) is $w_2(1)$. Equivalently, if agent 2 participates agent 1's benefit (loss) is $w_1(2)$. The agents will choose to participate if the payoff from the principal and the benefit/loss from other participating agents, taken together, is greater than the outside option.

Suppose first that the externalities $w_1(2)$ and $w_2(1)$ are strictly positive. Simple contracts that induce the participation of both agents as a Nash equilibrium are such that agent 1 is offered $c - w_1(2)$ and agent 2 is offered $c - w_2(1)$. However, these contracts are not satisfactory as an additional equilibrium exits in which neither agent participates. We refer to such contracts as *partial implementation* contracts, as additional equilibria exist in addition to full participation. In order to sustain the participation of both agents in a unique equilibrium, it is necessary to provide at least one agent, say agent 1, with his entire outside option c. In this case, agent 1 will participate even if agent 2 declines. Given agent 1's participation, it is sufficient to offer agent 2 only $c - w_2(1)$ to induce his participation, as $w_2(1) > 0$. Hence the contracts $(c, c - w_2(1))$, while more expensive than the partial implementation, induce participation in a unique equilibrium. We refer to such contracts as *full implementation* contracts, and we will consider full implementation contracts for the rest of the example.

Let's assume further that externalities are symmetric, hence $w_1(2) = w_2(1) > 0$. In this case, the decision of which agent is to receive a higher payoff is arbitrary, as the cost of both contract sets $(c, c - w_2(1))$ and $(c - w_1(2), c)$ is identical.

Suppose now that externalities are asymmetric, say, $w_1'(2) = w_1(2) + \varepsilon$ and $w_2'(1) = w_2(1) - \varepsilon$, when $\varepsilon > 0$, so that $w_1'(2) > w_2'(1)$. Note that the sum of externalities remains unchanged. In this case, clearly, the principal would prefer to offer agent 2 a higher payoff as the payments in $(c - w_1'(2), c)$ are lower than the alternative full implementation contracts $(c, c - w_2'(1))$. To get the cheaper full implementation contracts, the principal exploits the fact that agent 1 favors 2 more than agent 2 favors 1, and thus gives preferential treatment to agent 2 by providing him with a higher incentive. We will later provide a general result, and demonstrate that the set of full implementation contracts that minimize the principal's cost is based on these bilateral relationships between the agents.

This simple example also demonstrates that the principal benefits from higher asymmetry between agents' externalities (i.e., lower mutuality). Note that the principal's optimal cost in the full implementation is $2c - w_1'(2) = 2c - w_1(2) - \varepsilon$. This observation is extended later in the paper. Moreover, we show that the cost difference between the more expensive full implementation contracting scheme and the partial implementation is decreasing with the level of asymmetry. In this example, the difference between the two types of contracting schemes is simply $w_2(1) - \varepsilon$. Therefore, the level of asymmetry is a significant consideration both at the agents' selection stage and at the decision of whether to use a partial or full implementation contracting scheme.

3 The Model

A participation problem is given by a triple (N, w, c) where N is a set of n agents. The agents' decision is binary: participate in the initiative or not. The structure of externalities w is an $n \times n$ matrix specifying the bilateral externalities between the agents. An entry $w_i(j)$ represents the added value from participation in the

initiative of agent i when agent j participates. Agents gain no additional benefit from their own participation, so $w_i(i) = 0$. Agents' preferences are additively separable; i.e., agent i's utility from participating jointly with a group of agents M is $\sum_{j \in M} w_i(j)$ for every $M \subseteq N$. In one of the extensions we consider a model in which agents' preferences are non-additive; i.e., externalities are defined over all subsets of agents in group N.

We assume that the externality structure w is fixed and exogenous. Also, c is the vector of the outside options of the agents. For simplicity, and with a slight abuse of notation, we assume that outside option is constant and equals to c for all agents. In the extensions section we demonstrate that our results hold also when the outside options are affected by the participation choices of the agents.

We assume that contracts offered by the principal are simple and descriptive in the sense that the principal cannot provide payoffs that are contingent on the participation behavior of other agents. Many of the examples discussed above seem to share this feature. Based on the data used by Gould et al. (2005) which includes contractual provisions of over 2,500 stores in 35 large shopping malls in the US, there is no evidence that contracts make use of such contingencies. The theoretical foundation for the absence of such contracts is beyond the scope of this paper. One possible explanation is the complexity of such contracts. In Section 5 we demonstrate that our analysis remains valid even if we allow contingencies to be added to the contracts.

The set of contracts offered by the principal can be described as an incentives vector $v = (v_1, v_2, ..., v_n)$ by which agent i receives a payoff of v_i if he decides to participate and zero otherwise. v_i is not constrained in sign and the principal can either pay or charge the agents but he cannot punish agents for not participating (limited liability). Given a contracting scheme v, agents face a normal form game

$G(v)$.[34]Each agent has two strategies in the game: participation or abstention. For a given set M of participating agents, each agent $i \in M$ earns $\sum_{j \in M} w_i(j) + v_i$ and each agent $j \notin M$ earns c,his outside option.

4 Contracting with Positive Externalities

Positive externalities are likely to arise in many contracting situations. Network goods, opening stores in a mall and attracting customers, contributing to public goods, are a few such examples. In this section we consider situations in which agents benefit in various degrees from the participation of the other agents in the group. Suppose that $w_i(j) > 0$ for all $i, j \in N$, such that $i \neq j$. In this case, agents are more attracted to the initiative as the set of participants grows. We demonstrate how an agent's payment is affected by the externalities that she induces on others as well as by the externalities that others induce on her. We will also refer to how changes in the structure of externalities affect the principal's welfare.

As a first step toward characterizing the optimal full implementation contracts, we show in Proposition 1 that an optimal contracting scheme is part of a general set of contracts characterized by the *divide-and-conquer*[5] property. This set of contracts is constructed by ranking agents in an arbitrary fashion, and by offering each agent a reward that would induce him to participate under the belief that all

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[3]We view the participation problem as a reduced form of the global optimization problem faced by the principal, which involves both the selection of the optimal group for the initiative and the design of incentives. Specifically, let U be a (finite) universe of potential participants. For each $N \subseteq U$ let $v^*(N)$ be the total payment made in an optimal mechanism that sustains the participation of the set of agents N. The principal will maximize the level of net benefit she can guarantee herself which is given by the following optimization problem: $\max_{N \subseteq U}[u(N) - v^*(N)]$, where $u(N)$ is the principal's gross benefit from the participation of the set N of agents and is assumed to be strictly monotonic with respect to inclusion; i.e., if $T \subsetneq S$, then $u(T) < u(S)$.

[4]In the extensions section we also consider the case of a sequential offers game.

[5]See Segal (2003) and Winter (2004) for a similarly structured optimal incentive mechanism in a setting of homogeneous externalities.

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the agents who precede him in the ranking participate and all subsequent agents abstain. Due to positive externalities, "later" agents are induced to participate (implicitly) by the participation of others and thus can be offered smaller (explicit) incentives. More formally, the *divide-and-conquer* (*DAC*) contracts have the following structure:

$$v = (c, c - w_{i_2}(i_1), c - w_{i_3}(i_1) - w_{i_3}(i_2), ..., c - \sum_k w_{i_n}(i_k))$$

where $\varphi = (i_1, i_2, ..., i_n)$ is an arbitrary order of agents. We say that v is a DAC contracting scheme with respect to the ranking φ. The following proposition, which is similar to the analysis in Segal (2003, subsection 4.1.1) provides a necessary condition for the optimal contracts.

Proposition 1 *If v is an optimal full implementation contracting scheme then it is a divide-and-conquer contracting scheme.*

Note that given contracting scheme v, agent i_1 has a dominant strategy in the game $G(v)$ to participate.[6] Given the strategy of agent i_1, agent i_2 has a dominant strategy to participate as well. Agent i_k has a dominant strategy to participate provided that agents i_1 to i_{k-1} participate as well. Therefore, contracting scheme v sustains full participation through an iterative elimination of dominated strategies.

4.1 Optimal Ranking

The optimal contracting scheme satisfies the divide-and-conquer property with the ranking that minimizes the principal's payment. The optimal ranking is determined by a virtual popularity tournament among the agents, in which each agent is "challenged" by all other agents. The results of the matches between all

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[6] Since rewards take continuous values we assume that if an agent is indifferent then he chooses to participate.

pairs of agents are described by a *simple* and *complete*[7] directed graph $G(N, A)$, when N is the set of nodes and A is the set of arcs. N represents the agents, and $A \subset N \times N$ represents the results of the matches, which is a binary relation on N. We refer to such graphs as *tournaments.*[8] More precisely, the set of arcs in tournament $G(N, A)$ is as follows:

(1) $w_i(j) < w_j(i) \iff (i, j) \in A$

(2) $w_i(j) = w_j(i) \iff (i, j) \in A$ *and* $(j, i) \in A$

The interpretation of a directed arc (i, j) in the tournament G is that agent j values mutual participation with agent i more than agent i values mutual participation with agent j. We simply say that agent i *beats* agent j whenever $w_i(j) < w_j(i)$. In the case of a two-sided arc, i.e., $w_i(j) = w_j(i)$, we say that agent i is *even* with agent j and the match ends in a tie.

In characterizing the optimal contracts we distinguish between the case in which the tournament is cyclic and acyclic. We say that a tournament is *cyclic* if there exists at least one node v for which there is a directed path starting and ending at v, and *acyclic* if no such path exists for all nodes.[9] The solution for cyclic tournaments relies on the acyclic solution, and therefore the acyclic tournament is a natural first step.

4.2 Optimal Ranking for Acyclic Tournaments

A ranking φ is said to be *consistent* with tournament $G(N, A)$ if for every pair $i, j \in N$ if i is ranked before j in φ, then i beats j. In other words, if agent i is ranked higher than agent j in a consistent ranking, then agent j values agent i more than agent i values j. We start with the following graph theory lemma:

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[7]A directed graph $G(N, A)$ is *simple* if $(i, i) \notin A$ for every $i \in N$ and *complete* if for every $i, j \in N$ at least $(i, j) \in A$ or $(j, i) \in A$.

[8]We allow that (i, j) and (j, i) are both in A.

[9]By definition, if $(i, j) \in A$ and $(j, i) \in A$, then the tournament is cyclic.

Lemma 1 *If tournament $G(N, A)$ is acyclic, then there exists a unique ranking that is consistent with $G(N, A)$.*

We refer to the unique consistent ranking proposed in Lemma 1 as the *tournament ranking*.[10] In the tournament ranking, each agent's location in the tournament ranking is determined by the number of his wins. Hence, the agent ranked first is the agent who won all matches and the agent ranked last lost all matches. As we demonstrate later, there may be multiple solutions when tournament $G(N, A)$ is cyclic. Proposition 2 provides the solution for participation problems with acyclic tournaments, and shows that the solution is unique.

Proposition 2 *Let (N, w, c) be a participation problem for which the corresponding tournament $G(N, A)$ is acyclic. Let φ be the tournament ranking of $G(N, A)$. The optimal full implementation contracting scheme is given by the DAC with respect to φ.*

The intuition behind Proposition 2 is based on the notion that if agents $i, j \in N$ satisfy $w_i(j) < w_j(i)$ then the principal is able to reduce the cost of incentives by $w_j(i)$, rather than by only $w_i(j)$, by giving preferential treatment to i and placing him higher in the ranking. Applying this notion to all pairs of agents minimizes the principal's total payment to the agents, since it maximizes the inherent value of the participants from the participation of the other agents.

The optimal contracting scheme can be viewed as follows. First the principal pays the outside option c for each one of his agents. The winner of each match in the virtual tournament is the agent who imposes a higher externality on his competitor. The loser of each match pays the principal an amount equal to the benefit that he gets from mutually participating with his competitor. The total amount

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[10]The tournament ranking is actually the ordering of the nodes in the unique hamiltonian path of tournament $G(N, A)$.

paid depends on the size of bilateral externalities and not merely on the number of winning matches. However, the higher agent i is located in the tournament, the lower is the total amount paid to the principal.

An intuitive solution for the participation problem is to reward agents according to their level of popularity in the group, such that the most popular agents would be the most rewarded. A possible interpretation of popularity in our context would be the sum of externalities imposed on others by participation, i.e., $\sum_{j=1}^{n} w_j(i)$. However, as we have seen, agents' ranking in the optimal contracting scheme is determined by something more refined than this standard definition of popularity. Agent i's position in the ranking depends on the set of peers that value agent i's participation more than i values theirs. This two-way comparison may result in a different ranking than the one imposed by a standard definition of popularity. This can be illustrated in the following example in which agent 3 is ranked first in the optimal contracting scheme despite being less "popular" than agent 1.

Example 1 *Consider a group of four agents with an identical outside option $c = 20$. The externality structure of the agents is given by matrix w, as shown in Figure 1. The tournament G is acyclic and the tournament ranking is $\varphi = (3, 1, 2, 4)$. Consequently, the optimal contracts set is $v = (20, 17, 14, 10)$, which is the divide-and-conquer contracting scheme with respect to the tournament ranking. Note that agent 3 who is ranked first is not the agent who has the maximal $\sum_{j=1}^{n} w_j(i)$.*
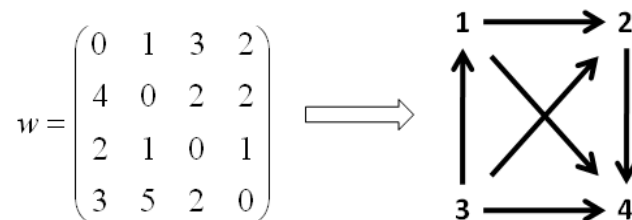

$$w = \begin{pmatrix} 0 & 1 & 3 & 2 \\ 4 & 0 & 2 & 2 \\ 2 & 1 & 0 & 1 \\ 3 & 5 & 2 & 0 \end{pmatrix}$$

Figure 1

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The derivation of the optimal contracting scheme requires the rather elaborate step of constructing the virtual tournament. However, it turns out that a substantially simpler formula can derive the cost of the optimal contracts. Two terms play a role in this formula: the first measures the aggregate level of externalities, i.e., $K_{agg} = \sum_{i\,j} w_i(j)$; the second measures the bilateral asymmetry between the agents, i.e., $K_{asym} = \sum_{i<j} |w_i(j) - w_j(i)|$. Hence, K_{asym} stands for the extent to which agents induce mutual externalities on each other. The smaller the value of K_{asym} the higher the degree of mutuality of the agents. Proposition 3 shows that the cost of the optimal contracting scheme is additive and declining in these two measures.

Proposition 3 *Let (N, w, c) be a participation problem and V_{full} be the principal's cost of the optimal full implementation contracts. If the corresponding tournament $G(N, A)$ is acyclic then $V_{full} = n \cdot c - \frac{1}{2} \left(K_{agg} + K_{asym} \right).$*

An interesting consequence of Proposition 3 is that for a given level of aggregate externalities, the principal's payment is decreasing with a greater level of asymmetry among the agents, as stated in Corollary 3.1.

Corollary 3.1 *Let (N, w, c) be a participation problem with an acyclic tournament. Let V_{full} be the principal's cost of the optimal full implementation contracts. For a given level of aggregate externalities, V_{full} is strictly decreasing with the asymmetry level of the externalities within the group of agents.*

The intuition behind this result is related to the virtual tournament discussed above. In each match the principal extracts "fines" from the losing agents. It is clear that these fines are increasing with the level of asymmetry (assuming $w_i(j) + w_j(i)$ is kept constant). Hence, a higher level of asymmetry allows the principal more leverage in exploiting the externalities. This observation has important implications for the principal's selection stage.

Consider the comparison between the optimal full and partial implementation contracts, where in the latter the principal suffices with the existence of a full participation equilibrium, not necessarily unique. With partial implementation, the cost for the principal in the optimal contracting scheme is substantially lower. More specifically, in the least costly contracting scheme that induces full participation, each agent i receives $v_i = c - \sum_j w_i(j)$. However, these contracts entail a no-participation equilibrium as well; hence coordination is required. The total cost of the partial implementation contracts is $V_{partial} = n \cdot c - \sum_{i\,j} w_i(j)$ and the principal can extract the full revenue generated by the externalities.

Our emphasis on full implementation is motivated by the fact that under most circumstances the principal cannot coordinate the agent to play his most-preferred equilibrium. Brandts and Cooper (2005) report experimental results that speak to this effect. Agents' skepticism about the prospects of the participation of others trap the group in the worst possible equilibrium even when the group is small. Nevertheless, one might be interested in evaluating the cost of moving from partial to full implementation. The following corollary points out that for a given level of aggregate externalities, the premium is decreasing with the level of asymmetry. Hence, the asymmetry level is an important factor in the choice between partial and full implementation contracting schemes.

Corollary 3.2 *Let* (N, w, c) *be a participation problem with a corresponding acyclic tournament. Let* V_{full} *be the principal's cost of the optimal full implementation contracts and let* $V_{partial}$ *be the equivalent partial implementation contracts. For a given level of aggregate externalities,* $V_{full} - V_{partial}$ *is strictly decreasing with the level of asymmetry.*

We say that a participation problem is *symmetric* if the asymmetry level is $K_{asym} = 0$ (when $w_i(j) = w_j(i)$ for all pairs); then the cost of moving from

partial to full implementation is the most expensive. The other extreme case is when the externalities are always one-sided; i.e., for each pair of agents $i, j \in N$ satisfies that either $w_i(j) = 0$ or $w_j(i) = 0$.[11] In this case, the additional cost is zero and full implementation is as expensive as partial implementation.

It is worth noting that increasing the aggregate level of externalities will not necessarily increase the principal extraction of revenue in the optimal contracting scheme. For example, in an asymmetric two-person problem raising slightly the externality that the less attractive agent induces on the other one will not change the principal revenue.[12] From the perspective of the agents, their reward is *not* a continuous increasing function of the externalities they impose on the others. However, it is possible that a slight change in these externalities may increase rewards significantly, since a minor change in externalities may change the optimal ranking and thus affect agents' payoffs.

The asymmetric case nicely contrasts with the symmetric case, where the principal's surplus increases with any slight increase of the externalities. With partial implementation, which allows the principal full extraction of surplus, the principal revenue is sensitive to the values of externalities whether the problem is symmetric or asymmetric.

4.3   Optimal Ranking of Cyclic Tournaments

In the previous section we demonstrated that the optimal full implementation contracts are derived from a virtual tournament among the agents in which agent i beats agent j if $w_i(j) < w_j(i)$. However, the discussion was based on the tournament being acyclic. If the tournament is cyclic, the choice of the optimal DAC

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[11]Since this section deals with positive externalities, assume that $w_i(j) = \varepsilon$ or $w_j(i) = \varepsilon$ when ε is very small.

[12]It can be shown that in an n-person asymmetric problem one can raise the externalities in half of the matrix's entries (excluding the diagonal) without affecting the principal surplus extraction.

contracting scheme (i.e., the optimal ranking) is more delicate since Lemma 1 does not hold. Any ranking is prone to inconsistencies in the sense that there must be a pair i, j such that i is ranked above j although j beats i in the tournament. To illustrate this point, consider a three-agent example where agent i beats j, agent j beats k, and agent k beats i. The tournament is cyclic and any ranking of these agents necessarily yields inconsistencies. For example, take the ranking $\{i, j, k\}$, which yields an inconsistency involving the pair (k, i) since k beats i and i is ranked above agent k. This applies to all possible rankings of the three agents.

The inconsistent ranking problem is similar to problems in sports tournaments, which involve bilateral matches that may turn out to yield cyclic outcomes. Various sports organizations (such as the National Collegiate Athletic Association - NCAA) nevertheless provide rankings of teams/players based on the cyclic tournament outcome. Extensive literature in Operations Research suggests solution procedures for determining the "minimum violation ranking" (e.g., Kendall 1955, Ali et al. 1986, Cook and Kress 1990, and Coleman 2005) that selects the ranking for which the number of inconsistencies is minimized. It can be shown that this ranking is obtained as follows. Take the cyclic (directed) graph obtained by the tournament and find the smallest set of arcs such that reversing the direction of these arcs results in an acyclic graph. The desired ranking is taken to be the consistent ranking (per Lemma 1) with respect to the resulting acyclic graph.[13]

One may argue that this procedure can be improved by assigning weights to arcs in the tournament depending on the score by which team i beats team j and then look for the acyclic graph that minimizes the total weighted inconsistencies. In fact this approach goes back to Condorcet's (1785) classical voting paper in which he proposed a method for ranking multiple candidates. In the voting game, the set of nodes is the group of candidates, the arcs' directions are the results of

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[13]Multiple rankings may result from this method.

pairwise votings, and the weights are the plurality in the votings. The solution to our problem follows the same path. In our framework arcs are not homogeneous and so they will be assigned weights determined by the difference in the bilateral externalities. As in Condorcet's voting paper, we will look for the set of arcs such that their reversal turns the graph into an acyclic one. While Young (1988) characterized Condorcet's method axiomatically, our solution results from a completely different approach, i.e., the design of optimal incentives to maximize revenues.

Formally, we define the weight of each arc $(i, j) \in A$ by $t(i, j) = w_j(i) - w_i(j)$. Note that weights are always non-negative as an arc (i, j) refers to a situation in which j favors i more than i favors j. Hence $t(i, j)$ refers to the extent of the one-sidedness of the externalities between the pairs of agents. If an inconsistency in the ranking arises due to an arc (i, j), then this implies that agent j precedes agent i despite the fact that i beats j. Relative to consistent rankings, inconsistencies generate additional costs for the principal. More precisely, the principal has to pay an additional $t(i, j)$ when inconsistency is due to arc $(i, j) \in A$. To illustrate this point, consider a two-agent example in which agent 1 beats agent 2. In the consistent ranking $\phi_1 = \{1, 2\}$ the payment vector is $v_1 = \{c, c - w_2(1)\}$. If an inconsistency arises, i.e., the ranking is $\phi_2 = \{2, 1\}$ then the payment is $v_2 = \{c, c - w_1(2)\}$ and the principal has to pay an additional cost of $w_2(1) - w_1(2)$ since $w_1(2) < w_2(1)$. In other words, the fact that inconsistencies arise in a ranking prevents the principal from fully exploiting the externalities between the agents, as inconsistencies increase the payment relative to the consistent ranking. Therefore the principal's goal would be to select a ranking with the least costly inconsistencies.

For each subset of arcs $S = \{(i_1, j_1), (i_2, j_2), ..., (i_k, j_k)\}$ we define $t(S) = \sum_{(i,j) \in S} t(i, j)$, which is the total weight of the arcs in S. For each graph G and subset of arcs S we denote by G_{-S} the graph obtained from G by reversing the

arcs in the subset S. Consider a cyclic graph G and let S^* be a subset of arcs that satisfies the following:

(1) G_{-S^*} is acyclic.

(2) $t(S^*) \leq t(S)$ for all S such that G_{-S^*} is acyclic.

Then, G_{-S^*} is the acyclic graph obtained from G by reversing the set of arcs with the minimal total weight, and S^* is the set of pairs of agents that satisfies inconsistencies in the tournament ranking of G_{-S^*}. Proposition 4 shows that the optimal ranking of G is the tournament ranking of G_{-S^*} since the additional cost from inconsistencies, $t(S^*)$, is the lowest.

Proposition 4 *Let* (N, w, c) *be a participation problem with a cyclic tournament* G. *Let* φ *be the tournament ranking of* G_{-S^*}. *Then, the optimal full implementation contracts are the DAC with respect to* φ.

In the following example we demonstrate how the optimal contracts are obtained in the case of cyclic tournaments with positive externalities.

Example 2 *Consider a group of four agents each having identical outside option* $c = 20$. *The externality structure and the equivalent cyclic tournament are demonstrated in Figure 2. The reversal of the arcs of both subsets* $S_1^* = \{(2, 4)\}$, $S_2^* = \{(1, 2), (3, 4)\}$ *provide acyclic graphs* $G_{-S_1^*}$ *and* $G_{-S_2^*}$ *with minimal weights. The corresponding tournament rankings are* $\varphi_1 = (4, 3, 1, 2)$ *and* $\varphi_2 = (3, 2, 4, 1)$. *Hence, the optimal contracts are* $v_1 = (20, 13, 13, 12)$ *and* $v_2 = (20, 16, 10, 12)$. *Note that the total cost for the principal, 58, is identical in these two contracting schemes.*

$$w = \begin{pmatrix} 0 & 1 & 2 & 5 \\ 2 & 0 & 4 & 2 \\ 1 & 2 & 0 & 7 \\ 3 & 4 & 6 & 0 \end{pmatrix} \implies$$
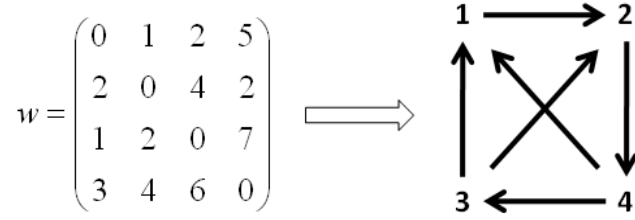

Figure 2

In the symmetric case, the principal cannot exploit the externalities among the agents, as $K_{asym} = 0$, and the total payment made by the principal is identical for all rankings. This can be seen to follow from Proposition 4 as well by noting that the tournament has two-way arcs connecting all pairs of agents, and $t(i,j) = 0$ for all i, j and $t(S)$ is uniformly zero. An intriguing feature of the symmetric case is that all optimal contracting schemes are discriminative in spite of the fact that all agents are identical.

Corollary 4.1 *When the externality structure w is symmetric then all DAC contracts are optimal.*

We can now provide the analogue version of Proposition 3 for the cyclic case. In this case, the optimal ranking has an additional term $K_{cyclic} = t(S^*)$ representing the cost of making the tournament acyclic, i.e., the cost the principal needs to bear due to the inconsistencies.

Proposition 5 *Let (N, w, c) be a participation problem. Let V_{full} be the principal's optimal cost of a full implementation contract. Then $V_{full} = n \cdot c - \frac{1}{2}(K_{agg} + K_{asym}) + K_{cyclic}$.*

Corollary 3.1 still holds for pairs of agents that are not in S^*. More specifically, if we increase the level of asymmetry between pairs of agents that are outside of S^*,

we reduce the total expenses that the principal incurs in the optimal contracting scheme.

5   Extensions

In this section we discuss the implications of the assumptions we made so far. We demonstrate that the optimal contracts remain optimal if we assume sequential participation choices when the principal desires to implement participation in a subgame perfect equilibrium with the property that each player has a dominant strategy on the subgame that he/she plays. In addition, we show that even when outside option is affected by the agents' participation choices, the construction of the optimal contracts remains unchanged. We demonstrate that when contracts can be contingent on the participation of a subset of the agents, then the optimal contracts are closely related to the analysis above. Finally, we show that our analysis is valid in more general setups in which externalities can be either negative or positive. Moreover, the solution is also relevant to non-additive externality structures.

5.1   Sequential Participation Decisions

We first point out that our analysis applies to any sequential game except for the one of perfect information, i.e., when each player is fully informed about all the participation decisions of his predecessors. Indeed, this extreme case of perfect information is a strong assumption as agents rarely possess the participation decisions of *all* their predecessors. Any partial information environment implies that some actions are taken simultaneously, and therefore the divide-and-conquer contracting scheme and the virtual tournament apply. Nevertheless, it is interesting to point out that our analysis is also relevant to the extreme case of perfect infor-

mation. Consider a game in which players have to decide sequentially about their participation based on a given order. Suppose that the principal wishes to implement the full participation in a subgame perfect equilibrium with the additional requirement that each player has a dominant strategy on the subgame in which he/she has to play.[14] It is easily verified that the optimal contracting scheme in this framework is the DAC applied to the order of moves; i.e., the first moving player is paid c and the last player is paid $c - \sum_{j \in N} w_i(j)$. Under this contracting scheme each player has a dominant strategy on each subgame. Assume now that the principal can control the order of moves (which he can do by making the offers sequentially and setting a deadline on agents' decisions). Then the optimal sequential contracting scheme is exactly identical to the one provided in previous sections for the simultaneous case. If the principal suffices with a standard subgame perfect equilibrium (without the strategy dominance condition), then the optimal contracting scheme will allow him to extract more and he will pay $c - \sum_{j \in N} w_i(j)$ to all agents.

5.2 Participation-dependent Outside Options

In many situations non-participating agents are affected by the participation choices of other agents. Consider the case of a corporate raider who needs to acquire the shares of N identical shareholders to gain control (similar to Grossman and Hart 1980). If the raider is enhancing the value of the firm when he holds a larger stake in the firm, then selling shareholders impose positive externalities on non-participating agents. If the raider gains private benefits from the firm which will decrease its value, then selling shareholders induce negative externalities on the non-participating agents.

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[14]Such a requirement may reflect the principal's concern that a player will fail to apply complex backward induction reasoning

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In this section we consider the case in which the agents' outside option is partly determined by the agents who choose to participate. For a given group of agents $P \subseteq N$ who participate, we define the outside option of non-participants as $c + \eta \sum_{j \in P} w_i(j)$. In the former analysis we assumed $\eta = 0$.[15] Segal (2003) defines externalities as increasing (decreasing) when an agent is more (less) eager to participate when more agents participate. In our setup, eagerness to participate is identity-dependent. When $\eta \leq 1$, we say that agents are more eager to participate when highly valued agents are choosing to participate. If $\eta > 1$, the benefits of non-participation outweigh the benefits of participation when highly valued agents choose to participate; hence agents are less eager to participate. In Segal's terminology, the former case is equivalent to *increasing* externalities, while the latter is equivalent to *decreasing* externalities.

Following the analysis of Proposition 1, if v is an optimal full implementation contracting scheme then it is easy to verify that under the current setup, v is a DAC of the form:

$$v = (c, c - (1 - \eta)w_{i_2}(i_1), ..., c - (1 - \eta)\sum_k w_{i_n}(i_k))$$

where $\varphi = (i_1, i_2, ..., i_n)$ is an arbitrary ranking. In this setup, the only change relative to Proposition 1 is the existence η. This leads to the following proposition:

Proposition 6 *Let (N, w, c^*) be a participation problem where $c_i^* = c + \eta \sum_{j \in P} w_i(j)$ and $P \subseteq N$ is a group of participating agents. Let $G(N, A)$ be the equivalent tournament. The optimal full implementation contracts are given as follows:*

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[15]The following analysis can be generalized by specifying an externalities matrix q that defines agents' benefits from participating agents, when they do not participate. It can be shown that in such a case our analysis remains unchanged. However, for similicity we choose to use the simpler and more intuitive outside option form of $c + \eta \sum_{j \in C} w_i(j)$.

(1) for $\eta < 1$, DAC contracts with respect to the optimal ranking.[16]

(2) for $\eta = 1$, DAC contracts with respect to any ranking.

(3) for $\eta > 1$, DAC contracts with respect to the optimal ranking of G_{-N}.

A few interesting observations arise. First, when $\eta = 1$, the benefit from participation is identical to the benefit of non-participation and thus incentives do not rely on externalities. Second, when $\eta < 1$, the benefits of participation outweigh the benefits of staying out; the optimal ranking is identical to the one outlined in Proposition 4. The contracting scheme provides lower incentives for the agents who are more eager to participate when other agents participate. When $\eta > 1$, agents benefit more from non-participation. The optimal ranking is determined with respect to G_{-N}, the graph obtained from G by reversing all the arcs. Agents who benefit more from joint participation should be ranked higher. The lower they are ranked, the more costly will be the rewards necessary to induce their participation, as their value from non-participation is increasing when valuable agents choose to participate.

5.3   Contingent Contracts

Our model assumes that the principal cannot write contracts that make a payoff to an agent contingent on the participation of other agents. With such contracts the principal can extract the total surplus from positive externalities among the agents.[17] We find such contracts not very descriptive. Based on the data used by Gould et al. (2005) which consists of contractual provisions of over 2,500

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[16]As described in Section 4.

[17]One possible contracting scheme is to offer agent i a participation reward of $v_i = c - \sum_{j \in N} w_i(j)$ if each of the other agents participates, and a reward of $v_i = c$ if the any of the contingencies is violated. Such contracts will sustain full participation as a unique Nash equilibrium, and the principal extracts the entire surplus.

stores in 35 large shopping malls in the US, there is no evidence that contracts make use of such contingencies. Shopping malls are a natural environment for contingent contracting; the fact that these contracts are still not used makes it likely that in other, more complicated settings, such contracts are exceptional as well. The theoretical foundation for the absence of such contracts is beyond the scope of this paper. However, one possible reason for this absence is the complexity of such contracts, especially in environments where participation involves long-term engagement and may be carried out by different agents at different points in time. We point out that if partial contingencies are used, i.e., participation is contingent upon a subset of the group, our model and its analysis remain valid. Specifically, for each player i, let $T_i \subseteq N$ be the contingency set, i.e., the set of agents whose participation choice can appear in the contract with agent i. Let $T = (T_1, T_2, ..., T_n)$ summarize the contingency sets in the contracts. The optimal contracts under the contingency sets are closely related to the original optimal contract (when contingencies are not allowed). More precisely, Let w be the original matrix of externalities. Denote by w^T the matrix of externalities obtained from w by replacing $w_i(j)$ with zero whenever $j \in T_i$. Lemma 6.1 in the Appendix shows that the optimal full implementation contracting scheme is as follows: agent i gets c if one of the agents $j \in T_i$ is not participating; i.e., the contingency requirement is violated.[18] If all agents in T_i participate, then agent i gets the payoff $v_i(N, w^T, c) - \sum_{j \in T_i} w_i(j)$, where $v_i(N, w^T, c)$ is the payoff for agent i for the participation problem (N, w^T, c) under no-contingencies (as developed in Section 4).

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[18]In fact, the principal can offer lower payments to the agents in case of contingencies' violations, by exploiting the participation of other agents. However, these off-equilibrium payments do not affect the principal's payment in the full participation equilibrium.

5.4 Mixed Externalities Structure

So far we have limited our discussion to environments in which agents' participation positively affects the willingness of other agents to participate. However, in many situations this is not the case, such as in environments of congestion. Traffic, market entry, and competition among applicants all share the property that the larger the number of agents who participate, the lower the utility of each participant is. The heterogenous property in our framework seems quite descriptive in some of these examples. In the context of competition it is clear that a more qualified candidate/firm induces a larger negative externality. It is also reasonable to assume, at least for some of these environments, that the principal desires a large number of participants in spite of the negative externalities that they induce on each other.

In Proposition 7 we demonstrate that in order to sustain full participation as a unique Nash equilibrium under negative externalities the principal has to fully compensate all agents for the participation of the others.

Proposition 7 *Let (N, w, c) be a participation problem with negative externalities. Then optimal full implementation contracts v are given by $v_i = c + \sum_{i \neq j} |w_i(j)|$, and v is unique.*

Naturally, real-world multi-agent contracting problems may capture both positive and negative types of externalities. In social events, individuals may highly benefit from some of the invited guests, while preferring to avoid others. In a mall, the entry of a new store will benefit some stores as it attracts more customers, but impose negative externalities on its competitors.

Our analysis of the mixed externalities case is based on the following binary relation. We say that an agent i is *non-averse* to agent j if $w_i(j) \geq 0$, and we write it as $i \succeq j$. We will assume that $\succeq$ is symmetric and transitive, i.e., $i \succeq j \implies$

$j \succeq i$ and if $i \succeq j$ and $j \succeq k$ then $i \succeq k$. Note that this assumption does not imply any constraint on the magnitude of the externalities, but just on their sign. While the symmetry and transitivity of the *non-averse* relation seem rather intuitive assumptions, not all strategic environments satisfy them. These assumptions are particularly relevant to environments where the selected population is partitioned into social, ethnic, or political groups with animosity potentially occurring only between groups but not within groups. We analyze a specific example of this sort of environment in Section 6.

It turns out that the optimal solution of participation problems with symmetry and transitivity of the *non-averse* relation is derived by a decomposition of the participation problem into two separate participation problems: one that involves only positive externalities, and the other that involves only negative externalities. This is done by simply decomposing the externalities matrix into a negative and a positive matrix. In the following proposition we show that the *decomposition contracting scheme,* a contract set which is the sum of the two optimal contracts of the two decomposed participation problems, is the optimal contracting scheme for the mixed externalities participation problem.

Proposition 8 *Consider a participation problem* (N, w, c) *. Let* (N, w^+, c) *be a participation problem such that* $w_i^+(j) = w_i(j)$ *if* $w_i(j) \geq 0$ *and* $w_i^+(j) = 0$ *if* $w_i(j) < 0$*, and let* u^+ *be the optimal full implementation contracts of* (N, w^+, c)*. Let* $(N, w^-, 0)$ *be a participation problem such that* $w_i^-(j) = w_i(j)$ *if* $w_i(j) < 0$ *and* $w_i^-(j) = 0$ *if* $w_i(j) \geq 0$*, and let* u^- *be the optimal full implementation contracts of* $(N, w^-, 0)$*. Then, the decomposition contracting scheme* $v = u^+ + u^-$ *induces a unique full participation equilibrium. Moreover, if agents satisfy symmetry and transitivity with respect to the non-averse relation,* v *is the optimal contracting scheme.*

Proposition 8 shows that the virtual popularity tournament discussed in earlier sections plays a central role also in the mixed externalities case as it determines payoffs for the positive component of the problem. When symmetry and transitivity hold, the principal can exploit the positive externalities to reduce payments. In this tournament i beats j whenever (1) $w_j(i) \geq 0$ and $w_j(i) \geq 0$, and (2) $w_j(i) > w_i(j)$. Note that under the *non-averse* assumptions, the principal provides complete compensation for the agents who suffer from negative externalities, as with the negative externalities case. Finally, it is easy to show that equivalently to Proposition 5, the principal's cost of achieving full implementation in a mixed externalities setting is equivalent to the positive externalities setup, except that now the principal has to add the compensation for the negative externalities.

5.5   Non-additive Preferences

We propose here an extension of the model in which agents' preferences are non-additive. A participation problem is described by a group of agents N, and their outside option is equal to c as noted previously. We assume a general externality structure, which is given by non-additive preferences of the agents over all subsets of agents in the group N. More specifically, for each i, $u_i : 2^{N\setminus\{i\}} \longrightarrow R$. The function $u_i(S)$ stands for the benefit of agent i from the participation with the subset $S \subseteq N$. We normalize $u(\emptyset) = 0$. The condition of positive externalities reads now: for each i and subsets S, T such that $T \subset S$ we have $u_i(S) \geq u_i(T)$.

Arguments similar to those used in Proposition 1 show that the optimal contracting scheme that sustains full participation as a unique equilibrium also satisfies the divide-and-conquer property. Hence, to construct the optimal contracts it is necessary to construct the optimal ranking of the agents.

Consider a three-agent example, with the following order $\phi = \{i_1, i_2, i_3\}$. The payoff vector in a DAC contracting scheme with ranking ϕ is $\{c, c - u_{i_2}(i_1), c -

$u_{i_3}(i_1, i_2)\}$. Hence, the optimal order would maximize the intrinsic value of participation of other agents, $u_{i_2}(i_1) + u_{i_3}(i_1, i_2)$. More generally, the principal has to choose ϕ to solve the following optimization problem:

$$\max_{\phi} \sum_{j=2}^{n} u_{i_j}(i_1, ..., i_{j-1})$$

We say that agent i beats j if for all $S \subset N$ such that $i, j \notin S$ we have $u_i(S \bigcup j) - u_i(S) < u_j(S \bigcup i) - u_j(S)$.[19] Intuitively, i beats j if i's marginal contribution to the utility of j is greater than j's marginal contribution to the utility of i, regardless of subset S at which marginal contributions are being calculated. Assuming this binary relation to be complete (and not necessarily transitive) enables us to construct a complete directed graph $G(N, A)$ when N is the set of nodes (which represent the agents), and A is the set of arcs that are defined in the following way: if agent i beats j then $(i, j) \in A$. The following result is based on similar arguments to those used in Proposition 2.

Proposition 9 *Let (N, c) be a participation problem with non-additive preferences, for which the corresponding directed graph $G(N, A)$ is complete and acyclic. Let φ be the tournament ranking of $G(N, A)$. The optimal full implementation contracts of (N, c) is given by the DAC contracts with respect to φ.*

The framework presented here is more general than the separable additive preferences in that the marginal contribution of agent i to the utility of agent j is not constant as assumed in the additive separable case, but depends on the set of other agents who participate in the initiative. Nevertheless, the general structure of the solution remains unchanged.

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[19]With $S = \emptyset$ we get the condition we had with the additively separable preferences.

6   Group Identity and Selection

In this section we consider special externality structures to demonstrate how the selection stage can be incorporated, once we have solved the participation problem. Assume that the externalities take values of 0 or 1. We interpret it as an environment in which an agent either benefits from the participation of his peer or gains no benefit. We provide three examples of group identities in which the society is partitioned into two groups and agents have hedonic preferences over members in these groups. We demonstrate how the optimal contracting scheme proposed in previous sections may affect the selection of the agents in the planning of the initiative.

(1) **Segregation** - agents benefit from participating with their own group's members and enjoy no benefit from participating with members from the other group. More specifically, consider the two groups B_1 and B_2 such that for each $i, j \in B_k$, $k = 1, 2$, we have $w_i(j) = 1$. Otherwise, $w_i(j) = 0$.

(2) **Desegregation**[20] - agents benefit from participating with the other group's members and enjoy no benefit from participating with members of their own group. More specifically, consider the two groups B_1 and B_2 such that for each $i, j \in B_k$, $k = 1, 2$, we have $w_i(j) = 0$. Otherwise, $w_i(j) = 1$.

(3) **Status** - the society is partitioned into two status groups, high and low. Each member of the society benefits from participating with each member of the high-status group and enjoys no benefit from participating with members of the low-status group. Formally, let B_1 be the high status group and set $w_i(j) = 1$ if and only if $j \in B_1$ (otherwise $w_i(j) = 0$).

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[20] An example could be a singles party.

Proposition 10 *Let (N, w, c) be a participation problem. Let n_1 and n_2 be the number of agents selected from groups B_1 and B_2, respectively, such that $n_1 + n_2 = n$. Denote by $v(n_1, n_2)$ the principal cost of incentivizing agents under the optimal contracts given that the group composition is n_1 and n_2. The following holds:*

1) under Segregation $v(n_1, n_2)$ is decreasing with $\mid n_1 - n_2 \mid$.

2) under Desegregation $v(n_1, n_2)$ is increasing with $\mid n_1 - n_2 \mid$.

3) under Status $v(n_1, n_2)$ is decreasing with n_1.

In the case of Segregation, the principal's cost of incentives is increasing with the mixture of groups; hence in the selection stage the principal would prefer to give precedence to one group over the other. In the Desegregation case the principal's cost is declining with mixture; hence in the selection stage the principal would like to balance between members of the groups. In the Status case the cost is declining with the number of agents recruited from B_1, which will be strongly preferred to members from B_2.

7   Conclusion

In this paper we analyzed a bilateral contracting framework in which externalities are heterogeneous. Introducing a complicated structure of heterogeneous externalities allowed us to explore a few aspects of the multi-agent contracting environments that are not apparent in the homogeneous case. These include the impact of externalities asymmetry on payments, the implications of externality structure on the hierarchy of incentives, and the effect of variations in structures of externalities on the principal's payments and agents' rewards.

More specifically, greater asymmetry between the agents' benefits reduced the principal's payment in the full implementation problem. This is an important im-

plication for the selection stage of the initiative. In addition, externalities asymmetry turns out to play a role also in the selection between partial and full implementation, as it affects the premium required to sustain full participation as a unique equilibrium. Greater asymmetry decreases this premium, and thus makes full implementation more likely.

The hierarchy of incentives is determined by a ranking that results from a virtual popularity tournament. In the simplest case, an agent i is ranked above agent j if agent i benefits less from the joint participation with agent j than agent j's benefit from agent i. We demonstrated that this ranking of incentives is different from the standard ranking that is based on agents' popularity.

We provided a few comparative statics of changes in externality structures. In an asymmetric participation problem increasing the positive externalities that some players enjoy from the participation of some other players will not necessarily increase the principal extraction of revenue in the optimal contracting scheme. This is an important consideration also in the stage of forming the group of agents. In addition, we show that from the agents' perspective a slight change in externalities can lead to a substantial impact on the rewards due to changes in ranking. Hence, there is a discontinuity in the principal's payment to induce participation.

This discontinuity in rewards may suggest a preliminary game in which agents invest effort to increase the positive externalities that they induce on others. For example, agents can invest in their social skills to make themselves more attractive guests at social events. A firm may invest to increase its market share in order to improve its ranking position in an acquisition game. Under certain circumstances such an investment may turn out to be quite attractive as we have seen that a slight change in externalities may result in a substantial gain, due to a change in the ranking. Our analysis contributes to the understand how to make such strategic investment profitable.

The preliminary game on externalities can be thought of as a network formation game similar to the ones discussed in the network formation literature (see Jackson 2003 for a comprehensive survey). Specifically, consider a selection[21] of an optimal contracting scheme function that maps each matrix of externalities onto a payoff vector $\Gamma : w \to \pi$ (payoffs for agents include both the transfer from the principal as well as the intrinsic benefits from participation). One can think of the matrix of externalities as a generalized network in the sense that it specifies the intensity[22] of arcs, in contrast to standard networks which only specify whether a link exists. If we assume that agents can increase bilateral externalities according to a given cost function then the externalities become endogenous in the model. The new game will now have two stages. The first stage is a network formation game (which determines the externalities) and the second stage is the participation game. The analysis of such a game is beyond the scope of this paper but seems to be a natural next step.

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[21]We refer to selection because the optimal contracting scheme may not be unique.

[22]For such models, see Calvo, Lasaga, and van den Nouweland (1999), Calvo-Armengol and Jackson (2001, 2001b), Goyal and Moraga (2001), and Page, Wooders, and Kamat (2001).

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Appendix

Proof of Proposition 1 Let $v = (v_{i_1}, v_{i_2}, ..., v_{i_n})$ be an optimal full implementation contracting scheme of the participation problem (N, w, c). Hence, v generates full participation as a unique Nash equilibrium. Since no-participation is not an equilibrium, at least a single agent, say i_1, receives a reward weakly higher than his outside option c. Otherwise, a no-participation equilibrium exists. Due to the optimality of v his payoff would be exactly c. Agent i_1 chooses to participate under any profile of other agents' decisions. Given that agent i_1 participates and an equilibrium of a single participation is not feasible, at least one other agent, say i_2, must receive a reward weakly greater than $c - w_{i_2}(i_1)$. Since v is the optimal contracting scheme, i_2's reward cannot exceed $c - w_{i_2}(i_1)$, and under any profile of decisions i_2 will participate. Applying this argument iteratively on the first $k-1$ agents, at least one other agent, henceforth i_k, must get a payoff weakly higher than $c - \sum_{j=1}^{k-1} w_{i_k}(j)$, but again, since v is optimal, the payoff for agent k must be equal to $c - \sum_{j=1}^{k-1} w_{i_k}(j)$. Hence, the optimal contracting scheme v must satisfy the *divide-and-conquer* property with respect to a ranking φ. ∎

Proof of Lemma 1 We will demonstrate that there is a single node with $n-1$ outgoing arcs. Since the tournament is a complete, directed, and acyclic graph

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there cannot be two such nodes. If such a node does not exist, then all nodes in G have both incoming and outgoing arcs. Since the number of nodes is finite, we get a contradiction to G being acyclic. We denote this node as i_1 and place its corresponding agent first in the ranking (hence this agent beats all other agents). Now let us consider a subgraph $G(N^1, A^1)$ which results from the removal of node i_1 and its corresponding arcs. Graph $G(N^1, A^1)$ is directed, acyclic, and complete and, therefore, following the previous argument, has a single node that has exactly $n-2$ outgoing arcs. We denote this node as i_2, and place its corresponding agent at the second place in the ranking. Note that agent i_1 beats agent i_2 and therefore the ranking is consistent so far. After the removal of node i_2 and its arcs we get subgraph $G(N^2, A^2)$ and consequently node i_3 is the single node that has $n-3$ outgoing arcs in subgraph $G(N^2, A^2)$. Following this construction, we can easily observe that the ranking $\varphi = (i_1, i_2, ..., i_n)$ is consistent among all pairs of agents and due to its construction is also unique. ∎

Proof of Proposition 2 According to Proposition 1 the optimal contracting scheme satisfies the DAC property. Hence the optimal contracting scheme is derived from constructing the optimal ranking and is equivalent to minimizing the sum of incentives, V_{full} :

$$
\begin{aligned}
V_{full} &= \min_{(j_1, j_2, ..., j_n)} \left[n \cdot c - \left\{ \sum_{k=1}^{1} w_{j_1}(j_k) + \sum_{k=1}^{2} w_{j_2}(j_k) + ... + \sum_{k=1}^{n} w_{j_n}(j_k) \right\} \right] \\
&= \max_{(j_1, j_2, ..., j_n)} \left[\sum_{k=1}^{1} w_{j_1}(j_k) + \sum_{k=1}^{2} w_{j_2}(j_k) + ... + \sum_{k=1}^{n} w_{j_n}(j_k) \right]
\end{aligned}
$$

Since no externalities are imposed on nonparticipants, the outside options of the agents have no role in the determination of the optimal contracting scheme. We will show that the ranking that solves the maximization problem of the principal is the tournament ranking. Let us assume, without loss of generality, that the

tournament ranking φ is the identity permutation: hence $\varphi(i) = i$, and $W_\varphi = \sum_{k=1}^{2} w_2(k) + ... + \sum_{k=1}^{n} w_n(k)$, when W_φ is the principal's revenue extraction. By contradiction, assume that there exists $\varphi \neq \sigma$ such that $W_\varphi \leq W_\sigma$. First, assume that σ is obtained from having two *adjacent* agents i and j in φ trade places such that i precedes j in φ and j precedes i in σ. By Lemma 1, agent i beats agent j; thus $W_\sigma = W_\varphi - w_j(i) + w_i(j)$ and $W_\sigma < W_\varphi$.

Note that since φ is the tournament ranking, agent 1 beats all agents, agent 2 beats all agents but agent 1, and so on. Now consider unconstrained $\sigma = \{i_1, ..., i_n\}$ such that $\varphi \neq \sigma$. If agent 1 is not located first, by a sequence of adjacent swaps $(1, i_j)$, we move agent 1 to the top of the ranking. In each of the substitutions agent 1 beats i_j. Next, if agent 2 is not located at the second place, by a sequence of adjacent substitutions $(2, i_j)$, we move agent 2 to the second place. Again, agent 2 beats all agents i_j. The process ends in at most n stages and produces the desired order φ. As demonstrated, any adjacent substitution results in a higher extraction, and therefore $W_\sigma < W_\varphi$. Therefore, the DAC contracting scheme with respect to the tournament ranking is unique and optimal. ∎

Proof of Proposition 3 Without loss of generality, assume that the tournament ranking φ is the identity permutation. Hence, under the optimal contracting scheme, the principal's payment is $V_{full} = n \cdot c - \left[\sum_{j=1}^{1} w_1(j) + ... + \sum_{j=1}^{n} w_n(j) \right]$. Denote $s_i(j) = [w_i(j) + w_j(i)]$ and $a_i(j) = [w_i(j) - w_j(i)]$. We can represent K_{agg} and K_{asym} in the following manner: $K_{agg} = \sum_{i\,j} w_i(j) = \sum_{i<j} (w_i(j) + w_j(i)) = \sum_{i<j} s_i(j)$ and $K_{asym} = \sum_{i<j} |a_i(j)|$. Since $w_i(j) = \frac{1}{2}(s_i(j) + a_i(j))$ we can rewrite the principal's payment as

$$
\begin{aligned}
V_{full} &= n \cdot c - \frac{1}{2} \left[\sum_{j=1}^{1} \{s_1(j) + a_1(j)\} + ... + \sum_{j=1}^{n} \{s_n(j) + a_n(j)\} \right] \\
&= n \cdot c - \frac{1}{2} \left(\sum_{i>j} s_i(j) + \sum_{i>j} a_i(j) \right)
\end{aligned}
$$

Note that $s_i(j) = s_j(i)$ and $a_i(j) = -a_j(i)$. In addition $a_i(j) > 0$ when $i > j$ as the tournament is acyclic and ranking is consistent. Therefore, $V_{full} = n \cdot c - \frac{1}{2} \left(\sum_{i<j} s_i(j) - \sum_{i<j} |a_i(j)| \right) = n \cdot c - \frac{1}{2} \left(K_{agg} + K_{asym} \right)$. ∎

Proof of Corollary 3.2 The result follows immediately from Proposition 3, where we show that $V_{full} = n \cdot c - \frac{1}{2} \sum_{i \, j} w_i(j) - \frac{1}{2} \sum_{i<j} |w_i(j) - w_j(i)|$, and from $V_{partial} = n \cdot c - \sum_{i \, j} w_i(j)$. Taken together, the two yield $V_{full} - V_{partial} = \frac{1}{2} \sum_{i \, j} w_i(j) - \frac{1}{2} \sum_{i<j} |w_i(j) - w_j(i)| = \frac{1}{2} \left(K_{agg} - K_{asym} \right)$. ∎

Proof of Proposition 4 Let $G(N, A)$ be a cyclic graph. Consider a subset of arcs S such that G_{-S} is acyclic, and the tournament ranking of G_{-S} is $\varphi = (j_1, j_2, ..., j_n)$. The payment of the principal V_{full} under the DAC contracting scheme with respect to φ is

$$V_{full} = n \cdot c - \left\{ \sum_{k=1}^{1} w_{j_1}(j_k) + \sum_{k=1}^{2} w_{j_2}(j_k) + ... + \sum_{k=1}^{n} w_{j_n}(j_k) \right\}$$

. Note that each $(i, j) \in S$ satisfies an inconsistency in tournament ranking φ. More specifically, if $(i, j) \in S$, then i beats j, and agent j is positioned above agent i. In addition, $w_i(j) = w_j(i) - t(i, j)$, where $w_i(j) < w_j(i)$ and $t(i, j) > 0$. Consider the following substitution: if $(i, j) \in S$ then $w_i(j) = \hat{w}_j(i) - t(i, j)$; otherwise $w_i(j) = \hat{w}_i(j)$. This allows us to rewrite the principal's payment as $V_{full} = n \cdot c - \left\{ \sum_{k=1}^{1} \hat{w}_{j_1}(j_k) + ... + \sum_{k=1}^{n} \hat{w}_{j_n}(j_k) \right\} + t(S)$. Note that $\hat{w}_i(j) = \max(w_i(j), w_j(i))$. Therefore, different rankings affect only the level of $t(S)$, as the first two terms in V_{full} remain indifferent to variations in the ranking. This implies that the subset S with the lowest $t(S)$ brings V_{full} to a minimum. Hence, the optimal contracting scheme is the DAC with respect to the tournament ranking of G_{-S^*}. ∎

Proof of Proposition 5 As demonstrated in Proposition 4, the optimal payment of the principal is the DAC contracting scheme with respect to the tourna-

ment ranking of G_{-S^*}. According to Proposition 4, this can be written as $V_{full} = n \cdot c - \left\{ \sum_{k=1}^{1} \widehat{w}_{j_1}(j_k) + ... + \sum_{k=1}^{n} \widehat{w}_{j_n}(j_k) \right\} + t(S)$ when $\widehat{w}_i(j) = \max(w_i(j), w_j(i))$. Following the argument of Proposition 3, denote $s_i(j) = [\widehat{w}_i(j) + \widehat{w}_j(i)]$ and $a_i(j) = [\widehat{w}_i(j) - \widehat{w}_j(i)]$ and the principal's payment is $V_{full} = n \cdot c - \frac{1}{2} \left(\sum_{i<j} s_i(j) + \sum_{i<j} |a_i(j)| \right) + t(S) = n \cdot c - \frac{1}{2} \left(K_{agg} + K_{asym} \right) + K_{cyclic}$. ∎

Proof of Proposition 6 The cost of a full implementation contracting scheme is simply $V_{full} = nc - (1 - \eta) \sum_i \sum_{j<i} w_i(j)$. If $\eta = 1$, then the cost does not depend on the externalities. If $\eta < 1$, the minimal cost is obtained by selecting a ranking that maximizes $\sum_i \sum_{j<i} w_i(j)$. This is equivalent to the tournament ranking outlined in Proposition 4. If $\eta > 1$, the minimal cost is obtained by selecting a tournament that minimizes $\sum_i \sum_{j<i} w_i(j)$. Note that

$$\min \sum_i \sum_{j<i} w_i(j) = \max \sum_i \sum_{j<i} -w_i(j) = \max \sum_i \sum_{j<i} q_i(j)$$

when matrix q is defined by $q_i(j) = -w_i(j)$. Denote G^Q the corresponding tournament of matrix q and G^W the corresponding tournament of matrix w. Because $q_i(j) = -w_i(j)$, G^Q is received from G^W by inverting all arcs. Due to Proposition 4 we can define ϕ as the optimal ranking that maximizes $\sum_i \sum_{j<i} q_i(j)$. This ranking minimizes $\sum_i \sum_{j<i} w_i(j)$. Therefore, the optimal ranking when $\eta > 1$ is the one with respect to G_{-N}, the graph obtained by reversing the arcs in graph G^W. ∎

Lemma 6.1 *Let (N, w, c) be a participation problem and $T = (T_1, ..., T_n)$ define the contingency sets. Define w^T to be such that $w_i^T(j) = w_i(j)$ if $j \notin T_i$ and $w_i^T(j) = 0$ otherwise. Let φ be the optimal ranking of the participation problem (N, w^T, c), and $v(N, w^T, c)$ the corresponding DAC payment vector. The optimal full implementation contracts set of (N, w, c) is such that it provides c for agent i if contingencies T_i are violated, and $v_i = v_i(N, w^T, c) - \sum_{j \in T_i} w_i(j)$ otherwise.*

Proof of Lemma 6.1 Since externalities are positive, contingencies allow the principal to reduce payments. In particular, when exploiting all contingencies allowed in T, the contracting scheme that sustains a unique full participation Nash equilibrium offers each agent i a reward $v_i = c - \sum_{j \in T_i} w_i(j)$ if contingencies are met, and c if they are violated. If for all agents $T_i = N/\{i\}$, then full extraction of surplus is possible as a unique equilibrium. However, if only partial contingencies are allowed, i.e., for some agents $T_i \subset N/\{i\}$ then the principal can perform even better than in the contracts outlined above.

Let's define $\hat{w}_i(j) = w_i(j)$ if $j \notin T_i$ and $\hat{w}_i(j) = 0$ otherwise. Consider an arbitrary ranking of agents $\varphi = \{1, 2, ..., n\}$ in which the first agent is offered $v_1 = c - \sum_{j \in T_1} w_1(j)$ if contingencies are met, and c otherwise. Agent 1 will choose to participate. Given the participation of agent 1, we can offer agent 2 the following payment: $v_2 = c - \hat{w}_2(1) - \sum_{j \in T_2} w_2(j)$ if contingencies are met, and c otherwise. Hence, agent 2 will agree to participate given the participation of agent 1. Following the same argument, we could offer the last agent in the ranking $v_n = c - \sum_{i=1}^{n-1} \hat{w}_n(i) - \sum_{j \in T_n} w_n(j)$. This set of contracts will sustain full participation as a unique Nash equilibrium.

The optimal full implementation contracting scheme is thus achieved by obtaining the ranking of agents that will maximize $\sum_i \sum_{j>i} \hat{w}_i(j)$. Given our definition of $\hat{w}_i(j)$, this is equivalent to finding the optimal ranking of agents in the problem (N, w^T, c) when $w_i^T(j) = w_i(j)$ if $\notin T_i$ and $w_i^T(j) = 0$ otherwise. In other words, in the optimal full implementation contracting scheme, the payment for participation for each agent will be $v_i = v_i(N, w^T, c) - \sum_{j \in T_i} w_i(j)$ if contingencies are met, and c otherwise. ∎

Proof of Proposition 7 Given contracting scheme v, participation is a dominant strategy for all agents, under the worst-case scenario in which all other agents participate since $u_i = \sum_{i=1}^{n} w_i(j) + v_i = c$ for every $i \in N$. To show that

v is optimal, consider a contracting scheme m for which $m_i < v_i$ for some agents and $m_i = v_i$ for the rest. By contradiction, assume full participation equilibrium holds under contracting scheme m. Consider an agent i for which $m_i < v_i$. If all other agents are participating, then agent i's best response is to abstain since $u_i = \sum_{i=1}^n w_i(j) + m_i < c$. Hence, v is a unique and optimal contracting scheme. ∎

Proof of Proposition 8 See complementary note.

Proof of Proposition 9 Since the optimal contracting scheme is a DAC, it is a result of the following optimization problem:

$$\max_{(j_1, j_2, \dots, j_n)} \left[u_{j_2}(j_1) + u_{j_3}(j_1, j_2) + \dots + u_{j_n}(j_1, \dots, j_{n-1}) \right]$$

Assume, without loss of generality, that the tournament ranking φ is the identity permutation; hence $\varphi(i) = i$, and $W_\varphi = u_2(1) + u_3(1, 2) + \dots + u_n(1, \dots, n-1)$. W_φ is the principal's revenue extraction. By contradiction, assume that there exists a different ranking denoted by σ such that $W_\varphi \le W_\sigma$. First, assume that σ is obtained from having two *adjacent* agents i and j ($j = i + 1$) in φ trade places such that i precedes j in φ (hence i beats j) and j precedes i in σ. Therefore, $\sigma = \{1, \dots, i - 1, j, i, \dots, n\}$. First note that all the agents that appear after j in order φ earn the same payoff in the DAC contracting scheme of both φ and σ. The same holds also for all the agents who appear before i in the order φ. So the cost of the DAC contracting schemes with respect to φ and σ differs only in terms of the payoff of agents i and j, and we get that

$$W_\sigma = W_\varphi + A$$

when $A = [u_i(1, \dots, i - 1, j) - u_i(1, \dots, i - 1)] - [u_j(1, \dots, i - 1, i) - u_j(1, \dots, i - 1)]$. The term A compares the marginal contribution of i relative to the marginal

contribution of j, given a subset $S = \{1, ..., i-1\}$. Therefore, $A < 0$, which entails $W_\sigma < W_\varphi$.

Note that since φ is the tournament ranking, agent 1 beats all agents, agent 2 beats all agents except agent 1, and so on. Now consider $\sigma = \{i_1, ..., i_n\}$ such that $\varphi \neq \sigma$. If agent 1 is not located first, by a sequence of adjacent swaps $(1, i_j)$, we move agent 1 to the top of the ranking. In each of the substitutions agent 1 beats i_j. Next, if agent 2 is not located at the second place, by a sequence of adjacent substitutions $(2, i_j)$, we move agent 2 to the second place. Again, agent 2 beats all agents i_j. The process ends in at most n stages and produces the desired order φ. As demonstrated, any adjacent substitution results in a higher extraction, and therefore $W_\sigma < W_\varphi$. Therefore, the DAC contracting scheme with respect to the tournament ranking is unique and optimal. ■

Proof of Proposition 10 In both segregated and desegregated environments the externality structure is symmetric and, following Corollary 5.1, all rankings are optimal. Consider first the segregated environment. Since all rankings are optimal, a possible optimal contracting scheme is $v = (c, ..., c - (n_1 - 1), c, ..., c - (n_2 - 1))$. Hence, the optimal payment for the principal is $v(n_1, n_2) = n \cdot c - \sum_{l=1}^{n_1-1} l - \sum_{k=1}^{n_2-1} k = n \cdot c - \frac{n_1(n_1-1)}{2} - \frac{(n-n_1)(n-n_1-1)}{2}$. Assuming that $v(n_1, n_2)$ is continuous with n_1 then $\frac{\partial v(n_1, n_2)}{\partial n_1} = n - 2n_1$ and maximum is achieved at $n_1^* = n_2^* = \frac{n}{2}$, and the cost of incentivizing is declining with $|n_1 - n_2|$. In the desegregated example, a possible optimal contracting scheme is $v = (c, ..., c, c - n_1, ..., c - n_1)$. Therefore, the principal's payment is $v(n_1, n_2) = n \cdot c - (n - n_1) \cdot n_1$. Again, let us assume that $v(n_1, n_2)$ is continuous with n_1, in which case solving $\frac{\partial v(n_1, n_2)}{\partial n_1} = 2n_1 - n = 0$ results in the minimum payment for the principal in the desegregated environment being received at $n_1^* = n_2^* = \frac{n}{2}$, and the cost of incentivizing is increasing with $|n_1 - n_2|$. In a status environment, since group B_1 is the more esteemed group, all agents from B_1 beat all agents from B_2; therefore agents from

B_1 should precede the agents from B_2 in the optimal ranking. A possible optimal ranking is $\varphi = \{i_1, .., , i_{n_1}, j_1, ..., j_{n_2}\}$ when $i_k \in B_1$, $j_m \in B_2$ and $1 \leq k \leq n_1$, $1 \leq m \leq n_2$. Therefore, a possible optimal contracting scheme is $v = (c, c - 1..., c - (n_1 - 1), c - n_1, ..., c - n_1)$. The principal's payment is $v(n_1, n_2) = n \cdot c - \sum_{l=1}^{n_1-1} l - n_2 \cdot n_1 = n \cdot c - \frac{n_1(n_1-1)}{2} - (n - n_1)n_1 = \frac{1}{2}n_1 - nn_1 + \frac{1}{2}n_1^2 + cn$. Again, assuming that $v(n_1, n_2)$ is continuous with n_1, $\frac{\partial v(n_1, n_2)}{\partial n_1} = n_1 + \frac{1}{2} - n = 0$ and the minimal payment is achieved at $n_1^* = n - \frac{1}{2}$. Note that $V(n_1 = n) = V(n_1 = n-1)$. Therefore, the best scenario for the principal is when $n_1 = n$. Alternatively, the cost of incentivizing is decreasing with n_1. ∎